Discriminant and the existence of Hermite-Einstein metrics in vector bundles on a Del Pezzo surface

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MPI 96-85

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June 30, 1996

Abstract

A Del Pezzo surface X over $\boldsymbol{C}$ is either a projective plane $\boldsymbol{P^2}$ or a quadric Q or a blow up of t, $t < 9$, generic points in $\boldsymbol{P^2}$. Let $\mathcal{F}$ be a topological vector bundle on X with $\operatorname{rk}\mathcal{F} = r$, $c_1(\mathcal{F}) = c_1$, $c_2(\mathcal{F}) = c_2$ and hence with the discriminant

$$\Delta_{\mathcal{F}} = \frac{1}{r}\left(c_2 - \frac{r-1}{2r}c_1^2\right).$$

In this paper we show that sufficient conditions for the existence of an Hermite-Einstein metric in $\mathcal{F}$ can be obtained via an inequality on the discriminant of $\mathcal{F}$. Namely if $b_2(X) \le 3$ then

$$\Delta_{\mathcal{F}} \ge 1$$

*The research described in the paper was partially supported by INTAS grant.

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is the sufficient condition and if $4 \leq b_2(X) \leq 7$ then the condition

$$\Delta_{\mathcal{F}} \geq \frac{(b_2-1)(b_2-2)}{2}, \text{ where } b_2 = b_2(X),$$

is sufficient for the existence of an Hermite-Einstein metric in $\mathcal{F}$.

1 Introduction

We consider vector bundles on a 2-dimensional compact complex manifold X which is a Del Pezzo surface. Del Pezzo surfaces are algebraic and their classification over $\boldsymbol{C}$ is well known. A Del Pezzo surface X over $\boldsymbol{C}$ is either a projective plane $\boldsymbol{P^2}$ or a quadric Q or a blow up t generic points in $\boldsymbol{P^2}$ where $t < 9$.

The question of the existence of an Hermite-Einstein metric in a topological vector bundle $\mathcal{F}$ over X that we are dealing with can too be reformulated as algebraic. By the famous Kobayashi-Hitchin correspondence theorem ([LT]) the existence of such a metric in $\mathcal{F}$ is equivalent to the existence of a stable algebraic vector bundle structure in $\mathcal{F}$, or to the existence of an algebraic stable vector bundle F such that its rank and Chern classes coincide with those of $\mathcal{F}$.

We will deal with the question in the latter form and whence our task is to determine weither it is a stable bundle F among algebraic bundles on X with given rank r and Chern classes c_1, c_2 (with given Chern datum in the terminology of [R2]). We consider here the stability defined by the anticanonical embedding.

We need to mention that as the stability considerations presume that there is chosen an embedding of X in a projective space so in order to define an Hermite-Einstein metric in a vector bundle one has to choose a metric on a base X first. Given a projective embedding one can induce the metric from the Fubini-Study metric of the projectic space and exactly this procedure is used in the Kobayashi-Hitchin correspondence.

The question about stable bundles with given Chern data has been solved completely for $X = \boldsymbol{P^2}$ in a seminal paper [DL]. Under additional condition $\Delta \neq 1/2$ it was also solved for $X = Q$ a quadric surface in [R1, R2] and for $X = \boldsymbol{P}^2_{<1>}$ (a blow up of one point) in [R2].

It was established in [R2] that in order to obtain a stable vector bundle with a required topological invariants on a Del Pezzo surface it is sufficient to construct an exceptional system with a certain set of properties. We follow this approach here. The study of the properties of exceptional systems on X will be the main content of this paper.[1]

Let us fix some notations. As usual Pic$X$ is the Picard group of line bundles which is the same as the Neron-Severi group for a Del Pezzo surface. The first Chern class provide an isomorphism

$$c_1 : \mathrm{Pic}X \to \mathrm{Cl}\, X \subset \mathtt{H}^2(X,\, \boldsymbol{Z})$$

of Pic X onto the subgroup of algebric classes in $\mathtt{H}^2(X,\, \boldsymbol{Z})$. The latter subgroup is also identified with the divisor class group by means of Poincare duality. The intersection pairing for divisors corresponds to the multiplication of cohomology classes as soon as we identify $\mathtt{H}^4(X,\, \boldsymbol{Z})$ with $\boldsymbol{Z}$. This identification also permit us to treat the second Chern classes as integer numbers.

By K_X we denote the canonical class of X that means the divisor class of the canonical line bundle on X which is the same as the bundle of complex holomorphic differential 2-forms of X.

We keep also the following notations from [R2]:

$$\mathtt{M}_X = \boldsymbol{Z} \times \mathrm{Cl}\, X \times \boldsymbol{Z},$$

$$\mathtt{M}_X^+ = \boldsymbol{N} \times \mathrm{Cl}\, X \times \boldsymbol{Z}.$$

$$\chi(A,B) = \sum(-1)^i \dim \mathrm{Ext}^i(A,B),$$

$$r_F = r(F) = \mathrm{rk} F,$$

$$\nu_F = \frac{c_1(F)}{r(F)} \in \mathrm{Cl}\, X \otimes \boldsymbol{Q},$$

$$\nu_{A,B} = \nu_B - \nu_A,$$

$$\mu(F) = \nu(F) \cdot (-K_X),$$

$$p_F = p(F) = \left(\frac{1}{2}c_1^2 - c_2\right)(F) \in \frac{1}{2}\boldsymbol{Z},$$

[1]The author would like to express his gratitude to Max-Planck-Institute for the stimulating atmosphere and gospitality.

$$\Delta_F = \left(\frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2)\right)(F).$$

An element $c = (r, c_1, c_2) \in \mathtt{M}_X$ is said to be the Chern datum for a vector bundle or for an algebraic coherent sheaf F when $r = rk(F)$, $c_i = c_i(F)$ with the corresponding notation:

$$c = (r, c_1, c_2) = Chd(F).$$

It is convenient to consider the Chern data set $\mathtt{M}_X$ as an abelian group in a way that

$$Chd(F_1 \oplus F_2) = Chd(F_1) + Chd(F_2).$$

This results for a Del Pezzo surface in an isomorphism

$$\mathtt{M}_X = \mathtt{K}_0(X)$$

where $\mathtt{K}_0(X)$ is the Grothendieck group for algebraic coherent sheaves on X. It is important to mention that the functions ν, μ, p, π, Δ and χ depend only on Chern data of sheaves which are their arguments so we consider those functions as functions on $\mathtt{M}_X$ or on $\mathtt{K}_0(X)$ when it is convenient.

The Riemann-Roch theorem adjusted for sheaves on a Del Pezzo surface states that:

$$\chi(A,B) = r_A r_B + \frac{1}{2}(-K_X)(r_A c_1(B) - r_B c_1(A)) + r_A p_B + r_B p_A - c_1(A)\cdot c_1(B). \quad (1)$$

If $r_A \neq 0$, $r_B \neq 0$ then this can be rewritten:

$$\chi(A,B) = r_A r_B \left(\frac{\nu_{A,B}\cdot(\nu_{A,B} - K_X)}{2} + 1 - \Delta_A - \Delta_B\right). \quad (2)$$

Let us denote by $\rho = (-K_X)/2$ and rewrite it once more:

$$\chi(A,B) = r_A r_B \left(\frac{(\nu_{A,B} - \rho)^2}{2} - \frac{\rho^2}{2} + 1 - \Delta_A - \Delta_B\right). \quad (3)$$

The Serry duality theorem for sheaves on a smooth surface can be stated here in a form ([DL]):

$$\mathrm{Ext}^i(A,B)^* = \mathrm{Ext}^{2-i}(B, A\otimes K_X).$$

We often use it in this form through the text.

Let us remind definitions and properties of exceptional sheaves on X ([G], [KO], [R1], [R3]).

Definition 1.1 *A sheaf E is called exceptional if $Hom(E,E) = \boldsymbol{k}$ and $Ext^i(E,E) = 0$ for $i > 0$.*

Thus $\chi(E,E) = 1$ and it implies that

$$\Delta_E = \frac{1}{2}\left(1 - \frac{1}{r(E)^2}\right) \quad \text{in particular} \quad 0 < \Delta_F < 1/2.$$

Let $\mathtt{Ex}_X$ denotes the set of Chern data for exceptional vector bundles (locally free exceptional sheaves) on X. Clearly $\mathtt{Ex}_X \subset \mathtt{M}_X^+$.

It is well known that if $c \in \mathtt{M}_X^+$ and $\Delta(c) < 1/2$ then a stable vector bundle F with $Chd(F) = c$ exists if and only if $c \in \mathtt{Ex}_X$.

For $c \in \mathtt{M}_X^+$ with $\Delta(c) \geq 1/2$ a more complicated necessary condition for the existence of a stable F, $Chd(F) = c$, is known ([DL], [R1], [R2]). It was called DL-condition in [R1]. We remind it below adding the condition $\Delta > 1/2$ because the case when $\Delta = 1/2$ is more subtle and we do not consider it here.

Definition 1.2 *An element $c \in \mathtt{M}_X^+$ is said to satisfy **DL**-condition if:*

1. *$\chi(c,e) \leq 0$ for any $e \in \mathtt{Ex}_X$ such that $r(e) < r(c)$ and*

$$\mu(c) \geq \mu(e) \geq \mu(c) - K_X^2 \,, \tag{4}$$

2. *$\Delta(c) > \frac{1}{2}$.*

Proposition 1.3 *If $\Delta(c) \geq 1$ then **DL**-condition is valid for c.*

Proof. By the Riemann-Roch theorem (3)

$$\frac{1}{r_c r_e}\chi(c,e) = \frac{(\nu_{c,e} - \rho)^2}{2} - \frac{\rho^2}{2} + 1 - \Delta(c) - \Delta(e). \tag{5}$$

It is possible to write $\nu_{c,e} = a\rho + \lambda$ where $\lambda \cdot \rho = 0$. This implies

$$\nu_{c,e} \cdot \rho = \frac{1}{2}(\mu(c) - \mu(e)) = a\,\rho^2 = a\,\frac{K_X^2}{4}.$$

On the other hand (4) shows that

$$0 \leq \mu(c) - \mu(e) \leq K_X^2.$$

Hence $0 \leq a \leq 2$ or $|a-1| \leq 1$. Now $\lambda^2 \leq 0$ by the Hodge index theorem and thus

$$(\nu_{c,e} - \rho)^2 = (a-1)^2\,\rho^2 + \lambda^2 \leq (a-1)\,\rho^2 \leq \rho^2.$$

This permit us to conclude that

$$\frac{1}{r_c r_e}\chi(c,e) \leq \frac{\rho^2}{2} - \frac{\rho^2}{2} + 1 - \Delta(c) - \Delta(e) \leq -\Delta(e) \leq 0. \ \square$$

It was shown that **DL**-condition is also sufficient when $X = \boldsymbol{P}^2$ ([DL]), $X = Q$ ([R1], [R2]), or $X = \boldsymbol{P}^2_{<1>}$ ([R2]). We have to mention that the results of these papers are formulated as conditions for the existence of a stable sheaf but the proofs permit one to conclude that if $r > 1$ then it exists not only a stable sheaf but a stable vector bundle for the same Chern datum.

If $r = 1$ then the situation is different. The only rank 1 vector bundles are line bundles and their discriminants are equal to 0. For rank 1 and a positive discriminant one can have a stable sheaf but not a stable vector bundle.

Thus we have got a corollary.

Corollary 1.4 *Suppose that $X = \boldsymbol{P}^2, Q$ or $\boldsymbol{P}^2_{<1>}$ and that the metric on X is induced by the anticanonical embedding. Let $\mathcal{F}$ be a topological vector bundle on X. If $rk\,\mathcal{F} > 1$ and $\Delta_{\mathcal{F}} \geq 1$ then there exists an Hermite-Einstein metric in $\mathcal{F}$.*

Here is the main result of the paper.

Theorem 1.1 *Let X be a complex surface which is made by blowing up t, $2 \leq t \leq 6$, generic points in $\boldsymbol{P}^2$ and let it be provided with a metric induced* *by the anticanonical projective embedding. Suppose $\mathcal{F}$ is a topological vector bundle on X with rk $\mathcal{F} > 1$ and*

$$\Delta_{\mathcal{F}} \geq \frac{t(t-1)}{2}.$$

Then it exists an Hermite-Einstein metric in $\mathcal{F}$.

Clearly one can reformulate the restrictions on the type on Del Pezzo surfaces in the theorem and the corollary as inequalities on the second Betti number $b_2(X)$. It is $b_2(X) \leq 2$ for the corollary and $3 \leq b_2(X) \leq 7$ for the theorem. Of course $t = b_2(X) - 1$.

The proof of the theorem depends on the results from [R2] that connect the existence of stable sheaves with the construction of exceptional systems on X satisfing certain properties. We will start the next section with the reminding of the definitions and the properties.

2 Exceptional systems

Definition 2.1 *The system of sheaves $E_0, \ldots, E_m$ is called exceptional system if the sheaves E_i are exceptional and for any $i < j$ and all q*

$$Ext^q(E_j, E_i) = 0.$$

An exceptional system on X is called complete if its image in the Grothendieck group $K_0(X)$ provides a $\boldsymbol{Z}$-base for the group.

Let us denote by I the ordered set $[0, \ldots, m]$ of the indices of the system in question.

The papers [KO], [R3] are good references for results on exceptional systems on Del Pezzo surfaces. In particular for our surface X an exceptional system $E_0, \ldots, E_m$ is complete if and only if $m = t + 2$.

Given an element $c \in K_0(X)$ we can write

$$c = \sum_{i \in I} n_i [E_i] \tag{6}$$

One can express the coordinates n_i with the help of a "dual exceptional system" where we have to distinguish between the left and right duals and permit for "the signs $\{\varepsilon_i\}$ to appear in the picture".

Definition 2.2 *The exceptional system $E'_m, \ldots, E'_0$ is called left dual to the system $E_0, \ldots, E_m$ if*

$$\chi(E'_j,\, E_i) = \delta_{ij}\, \varepsilon_i \tag{7}$$

where $\varepsilon_i = +1, -1$. *Then the coordinates $\{n_i\}$ in (6) are given by the formula:*

$$n_i = \varepsilon_i\, \chi(E'_i,\, c)$$

.

It is known ([G], [KO]) that for an exceptional system on a Del Pezzo surface the left dual system always exits and it is uniquelly defined together with its signs by the initial system.

Let us fix the notations $L_1, \ldots, L_t$ for the divisors that are preimages of the blown up points and the notation H for the divisor class which is equal to the preimage of a generic line in $\boldsymbol{P}^2$.

Elements $L_1, \ldots, L_t, H$ constitute a $\boldsymbol{Z}$-base for Cl X and their intersection numbers are as follows:

$$H \cdot H = 1, \quad H \cdot L_i = 0,$$
$$L_i \cdot L_i = -1, \quad L_i \cdot L_j = 0.$$

As usual we denote by $\mathcal{O}(D)$ a line bundle with the first Chern class (or a divisor class) D.

Lemma 2.3 *The system*

$$E_0 = \mathcal{O},\ E_1 = \mathcal{O}(L_1),\ \ldots,\ E_t = \mathcal{O}(L_t),\ E_{m-1} = \mathcal{O}(H),\ E_m = \mathcal{O}(2H) \tag{8}$$

where $m = t + 2$, is a complete exceptional system on X.

We leave it to the reader to prove the lemma.

We would like to find the left dual for this system. Let $\mathcal{O}(n)_{L_i}$ denote the sheaf which is trivial outside L_i and coincides with a line bundle $\mathcal{O}(n)$ on L_i. Also let us fix the notation $T(D)$ for the preimage onto X of the tangent vector bundle on $\boldsymbol{P}^2$ twisted by a line bundle on X with the first Chern class D.

Lemma 2.4 *The system*

$$\mathcal{O}(2H),\ T(H),\ \mathcal{O}_{L_1},\ \ldots,\ \mathcal{O}_{L_t},\ \mathcal{O}(3H - \sum L_k) \tag{9}$$

is the left dual system to (8) and the corresponding signs are

$$\varepsilon_m = +1,\ \varepsilon_{m-1} = -1,\ \varepsilon_t = +1,\ \ldots, \varepsilon_1 = +1,\ \varepsilon_0 = +1.$$

We would like also leave this as an exercise to the reader. Let us mention that the paper [G] contains the general rule for constructing the dual systems on Del Pezzo surfaces.

Suppose it is given an element $\mathbf{c} \in \mathtt{M}_X^+$. According to Theorem 7.1 from [R2] in order to get a stable vector bundle F with $Chd(F) = \mathbf{c}$ we have to fix an expression of $-K_X$ as a sum of lines

$$-K_X = \sum P_s \tag{10}$$

and to find a complete exceptional system $E_0, \ldots, E_m$ of vector bundles together with the decomposition of its indices in two subsets $I = I^- \sqcup I^+$ satisfying the following properties.

Gls sheaves $\mathcal{H}om(E_i, E_j)$ for $i \in I^-$ and $j \in I^+$ are generated by global sections.

Hm $\mathrm{Ext}^q(E_i, E_j) = 0$ for $i < j$ and $q \neq 0$.

R1 $\mathrm{Ext}^1(E_i, E_j(-P)) = 0$ for $i \in I^-$ and $j \in I^+$ and for any line $P = P_s$ from (10).

R2 $\mathrm{Ext}^2(E_i, E_j(-P)) = 0$ for either $i, j \in I^-$ or $i, j \in I^+$ and for any line $P = P_s$ from (10).

Iq The coordinates $\varepsilon_i \chi(E_i', \mathbf{c})$ of $\mathbf{c}$ are ≥ 0 for $i \in I^+$ and ≤ 0 for $i \in I^-$.

Then F can be constructed via an exact sequence:

$$0 \longrightarrow \bigoplus_{i \in I^-} |n_i|\, E_i \xrightarrow{\Phi} \bigoplus_{i \in I^+} |n_i|\, E_i \longrightarrow F \longrightarrow 0$$

where Φ has to be a generic morphism of the bundles (and this way one can get a versal family for such F).

We know that for surfaces under consideration $-K_X = 3H - \sum_{k=1,\dots,t} L_k$ and H, $H - L_p$, $H - L_{p_1} - L_{p_2}$ are lines. As $t \leq 6$ in our case so we can choose the decomposition (10) using only the lines of the above mentioned type. Let us fix one for the following.

Now we are ready to start the proof of Theorem 1.1.

Proof of Theorem 1.1. What we are to do is to provide for any element c from the theorem an exceptional system and sets I^+, I^- such that the above conditions are satisfied. Let us mention that the first four conditions **Gls** , **Hm**, **R1**, and **R2** do not depend on c so it is practical to check them first.

Proposition 2.5 *For any divisor D the system $E_0(D), \dots, E_m(D)$, where $E_0, \dots, E_m$ is defined by (8), satisfies **Gls** , **Hm**, **R1**, and **R2** for the cases:*

(a): $I^- = \{0, 1, \dots, t\}$, $I^+ = \{m-1, m\}$;

(b): $I^- = \{0, 1, \dots, t, m-1\}$, $I^+ = \{m\}$.

Proof. Because of equalities

$$\begin{aligned} \mathcal{H}om(A(D), B(D)) &= \mathcal{H}om(A, B), \\ \mathrm{Ext}^q(A(D), B(D)) &= \mathrm{Ext}^q(A, B) \end{aligned} \tag{11}$$

nothing depends on D and we can suppose that $D = 0$.

The condition **Gls** is obviously valid and the rest depends on computations of cohomologies. It is well known that for vector bundles:

$$\mathrm{Ext}^q(A, B) = \mathrm{H}^q(X, A^* \otimes B)$$

hence for **Hm** we are to check that sheaves

$$\mathcal{O},\ \mathcal{O}(L_i),\ \mathcal{O}(H),\ \mathcal{O}(2H),\ \mathcal{O}(H - L_i),\ \mathcal{O}(2H - L_i)$$

have trivial higher cohomologies. Similarly in order to check **R1** one needs to prove triviality of 1-cohomologies for line bundles

$$\mathcal{O}(H - P),\ \mathcal{O}(2H - P),\ \mathcal{O}(H - L_i - P),\ \mathcal{O}(2H - L_i - P),$$

and the triviality of 2-cohomologies for sheaves

$$\begin{array}{cc} \mathcal{O}(L_i - P), & \mathcal{O}(-L_i - P), \\ \mathcal{O}(H - P), & \mathcal{O}(-H - P), \\ \mathcal{O}(H - L_i - P), & \mathcal{O}(-H + L_i - P) \end{array}$$

have to be proved for **R2**.

The task is relatively easy as these are the cohomololies of line bundles. We leave it to the reader to make the computations with the help of the following lemmas.

Lemma 2.6 *(a). Let* $A = (-pH + \sum a_i L_i)$ *and* $p > 0$ *then* $H^0(X, \mathcal{O}(A)) = 0$.
(b). Let $B = (-p'H + \sum a_i L_i)$ *and* $p' \leq 2$ *then* $H^2(X, \mathcal{O}(B)) = 0$.

Lemma 2.7 *Let* $L = L_i$.
(a). If $A \cdot L \geq 0$ *and* $H^1(X, \mathcal{O}(A)) = 0$ *then*

$$H^1(X, \mathcal{O}(A + L)) = 0.$$

(b). If $B \cdot L < 0$ *and* $H^1(X, \mathcal{O}(B)) = 0$ *then*

$$H^1(X, \mathcal{O}(B - L)) = 0.$$

(c). If $B \cdot L = 0$, $H^1(X, \mathcal{O}(B)) = 0$, *and the restriction induces an epimorphism* $H^0(X, \mathcal{O}(B)) \to H^0(L, \mathcal{O})$ *then*

$$H^1(X, \mathcal{O}(B - L)) = 0.$$

Proof of the lemmas. Lemma 2.6.(a) follows from the fact that H has no fixed components and the intersection

$$(-pH + \sum a_i L_i) \cdot H = -p$$

is negative. Lemma 2.6.(b) follows from Lemma 2.6.(a) and the Serre duality.

All the statements of Lemma 2.7 are consequences of the long exact sequence of cohomologies for the restriction exact sequence of shehaves

$$0 \longrightarrow \mathcal{O}(-L) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_L \longrightarrow 0$$

tensored either by $\mathcal{O}(A + L)$ or by $\mathcal{O}(B)$. □

More difficult is the proving of the last property **Iq**. It will be done through the nest section.

3 Geometry of inequalities

We have to prove the statement **Iq** for the system that is the shift by D of the one defined by (8). Clearly the left dual system to $E_0(D), \dots, E_m(D)$ is also the shift by D of the dual to the initial system (one can easily conclude this from (11)).

Hence we need to prove that given c from the theorem one can find D such that

$$\begin{aligned} \varepsilon_i \chi(E'_i(D), c) = \varepsilon_i \chi(E'_i, c(-D)) \geq 0 \text{ for } i \in I^+, \\ \varepsilon_i \chi(E'_i(D), c) = \varepsilon_i \chi(E'_i, c(-D)) \leq 0 \text{ for } i \in I^-, \end{aligned} \tag{12}$$

where $\{E'_i,\}$ and $\{\varepsilon_i\}$ are defined in Lemma 2.4 and the sets I^+, I^- are either (a) or (b) from Proposition 2.5.

The difference between (a) and (b) cases amounts to a move of an index $m-1$ from I^+ to I^- thus to a change of the one of the inequalities to the opposite. Because we have to unite the solutions of these sets of inequalities, we can combine them instead as the following:

$$\begin{aligned} \varepsilon_i \chi(E'_i(D), c) = \varepsilon_i \chi(E'_i, c(-D)) \quad \geq 0 \quad \text{for } i = m, \\ \varepsilon_i \chi(E'_i(D), c) = \varepsilon_i \chi(E'_i, c(-D)) \quad \leq 0 \quad \text{for } i = t, \dots, 1, 0. \end{aligned} \tag{13}$$

The changing c for $c(-D)$ influences neither rank nor discriminant but only $\nu(c - D) = \nu(c) - D$. It convenient for us to use (r, ν, Δ) as a coordinates in $\mathtt{M}_X^+$. This way we identify $\mathtt{M}_X^+$ with a subset in $\boldsymbol{Z} \times (\mathrm{Cl}\, X \otimes \boldsymbol{Q}) \times \boldsymbol{Q}$. Also we make it more explicit by means of the following isomorphism:

$$\boldsymbol{Z} \times (\mathrm{Cl}\, X \otimes \boldsymbol{Q}) \times \boldsymbol{Q} = \boldsymbol{Z} \times \boldsymbol{Q}^{t+1} \times \boldsymbol{Q}$$

where $\nu \in \mathrm{Cl}\, X \otimes \boldsymbol{Q}$ is mapped to $(x, y_1, \dots, y_t)$ such that

$$\nu = xH - y_1 L_1 - \dots - y_t L_t.$$

Let us mention first that

$$r(\mathcal{O}_{L_i}) = 0, \; c_1(\mathcal{O}_{L_i}) = L_i, \; \nu(\mathcal{O}_{L_i}) = \frac{1}{2}.$$

The Riemann-Roch theorem (1),(3) and Lemma 2.4 permit us to write inequalities (13) in the following form.

Lemma 3.1 *Let* $\nu(c) - D - \rho = (x+2)H - y_1L_1 - \cdots - y_tL_t$ *and*

$$\delta(c) = 2(\Delta(c) + \frac{\rho^2}{2} - 1) = 2\Delta(c) - \frac{t-1}{4}.$$

Then the inequalities (13) are the same as

$$\begin{cases} x^2 - y_1^2 - \ldots - y_t^2 & \geq \delta(c) \\ -y_t - \frac{1}{2} \leq 0, \ldots, -y_0 - \frac{1}{2} & \leq 0 \\ (x-1)^2 - (y_1-1)^2 - \ldots - (y_t-1)^2 & \leq \delta(c) \end{cases} \tag{14}$$

This is just a straight computation with the Riemann-Roch formalas.

Let us denote the set of solutions for (14) in $\boldsymbol{Q}^{t+1}$ by S. Of course it depends on $\delta = \delta(c)$ as well as the inequalities do. Now to finish with the proof of the theorem it is sufficient to establish the following proposition.

Proposition 3.2 *If* $\Delta(c)$ *satisfies the conditions of Theorem 1.1 then*

$$S + \boldsymbol{Z}^{t+1} = \boldsymbol{Q}^{t+1}.$$

Obviously it is sufficient to prove the same substituting for S a subset in the solution set S, or a subset in an integer shift of S, or a union of such subsets.

Proof of the proposition. The crucial is the following lemma.

Lemma 3.3 *Let* (a_1, b_1) *be the positive solution of the equations:*

$$\begin{cases} x^2 - t\,y^2 & = \delta \\ (x-1)^2 - t(y-1)^2 & = \delta \end{cases} \tag{15}$$

and (a_0, b_0) *be the solution with* $a_0 > 0$ *of the equations:*

$$\begin{cases} x^2 - t\,y^2 & = \delta \\ y + \frac{1}{2} & = 0. \end{cases} \tag{16}$$

Suppose a set M *in* $\boldsymbol{Q}^{t+1}$ *is defined by the inequalities:*

$$\{b_0 - a_0 \leq y_i - x \leq b_1 - a_1 \quad where\ i = 1, \ldots, t\}, \tag{17}$$

and $u = (1, 1, \ldots, 1) \in \boldsymbol{Q}^{t+1}$. *Then*

$$M \subset S + \boldsymbol{Z}\,u = \bigcup_{n \in \boldsymbol{Z}} (S + n\,u).$$

Proof of the lemma. Let us use the notations

$$s_0 = (a_0, b_0, \ldots, b_0), \qquad s_1 = (a_1, b_1, \ldots, b_1).$$

Clearly points s_0, $(s_1 - u)$, $s_0 + u$, s_1 belong to the boundary of S. We can say even more. Let B_0 and B_1 be the regions on the boubdary of S defined as follows:

$$B_0 = S \bigcap \{x^2 - y_1^2 - \ldots - y_t^2 = \delta\} \bigcap \{-1/2 \leq y_i \leq (b_1 - 1)\},$$

$$B_1 = S \bigcap \{(x-1)^2 - (y_1 - 1)^2 - \ldots - (y_t - 1)^2 = \delta\} \bigcap \{+1/2 \leq y_i \leq b_1\}.$$

We see that

$$s_0 \in B_0 \qquad s_1 \in B_1,$$

B_0 belongs to the lower part and B_1 to the upper part of the boundary of S, and they differ on "a shift by u":

$$B_0 + u = B_1.$$

The set in between $\{B_0 + [0,1]\, u\}$ belongs to S. Hence we conclude that

$$B_0 + \boldsymbol{Q}u \subset S + \boldsymbol{Z}\, u.$$

But, as M is a parallelotop with the opposite vertices s_0, s_1 by (17), it is not difficult to check, and we leave it to the reader, that

$$M \subset (B_0 + \boldsymbol{Q}u).$$

So we have got the lemma. □

Now all what we need is to evaluate "the size of M" as it states the lemma below.

Lemma 3.4 *If* $(b_1 - a_1) - (b_0 - a_0) \geq 1$ *then* $M + \boldsymbol{Z}^{t+1} = \boldsymbol{Q}^{t+1}$.

It follows immidiately from (17).

The lemma below gives us the estimation that finishes the proofs of the proposition and the theorem.

Lemma 3.5 $(b_1 - a_1) - (b_0 - a_0) \geq 1$ *if and only if* $\Delta(c) \geq t(t-1)/2$.

Proof of the lemma. The defining equations (15) for (a_1, b_1) are equivalent to:

$$\begin{cases} a_1^2 - t\,b_1^2 &= \delta \\ t\,b_1 - a_1 &= \frac{t-1}{2}. \end{cases} \tag{18}$$

The second equation in (18) can be rewritten as

$$a_1 - b_1 = \frac{t-1}{t}\left(a_1 - \frac{1}{2}\right).$$

Let $a_1 = z + 1/2$, then $b_1 = z/t + 1/2$ and we get an equation for z:

$$\left(z + \frac{1}{2}\right)^2 - t\left(\frac{z}{t} + \frac{1}{2}\right)^2 = \delta$$

Simplifying we come to:

$$\left(\frac{t-1}{t}\right) z^2 = 2\Delta$$

where $\Delta = \Delta(c)$. So we have

$$z = \sqrt{\frac{t}{t-1}}\,\sqrt{2\Delta}. \tag{19}$$

One the other hand it follows from (16) that

$$a_0 = \sqrt{2\Delta + \frac{1}{4}}, \qquad b_0 = -\frac{1}{2}$$

As a result we have got

$$(b_1 - a_1) - (b_0 - a_0) = -\frac{t-1}{t}\sqrt{\frac{t}{t-1}}\,\sqrt{2\Delta} + \frac{1}{2} + \sqrt{2\Delta + \frac{1}{4}}.$$

The desired inequality

$$\frac{1}{2} + \sqrt{2\Delta + \frac{1}{4}} - \sqrt{\frac{(t-1)}{t}}\,\sqrt{2\Delta} \geq 1$$

is equivalent to

$$\sqrt{2\Delta + \frac{1}{4}} \geq \frac{1}{2} + \sqrt{\frac{(t-1)}{t}}\sqrt{2\Delta}.$$

Taking the second power of the both sides and we derive the equivalent inequality

$$\frac{1}{t}2\Delta \geq \sqrt{\frac{(t-1)}{t}}\sqrt{2\Delta}.$$

But this is nothing but

$$\Delta \geq \frac{t(t-1)}{2}. \square$$

4 Remarks

There are other exceptional systems that satisfy conditions **Gls** , **Hm**, **R1**, **R2** . For each of them one can find some subset of c in $\mathtt{M}_X^+$ that satisfy **Iq**. But it is difficult to understand what a set in $\mathtt{M}_X^+$ would be obtained as a whole.

It is natural to conjecture that just **DL**-condition is sufficient for any Del Pezzo surface, or, at least, that $\Delta \geq 1$ is sufficient.

There are known results about existence of stable sheaves or, in other words, about non-emptiness of moduli spaces, for some other surfaces. As soon as we know, they could be reformulated as $\Delta \geq a$ but the number a on the right hand side usually depends on something more than just the surface. What special about Del Pezzo is that here the right hand side constant is the same for any rank $\mathcal{F}$.

For the opposite question of emptiness of the moduli space there is a uniform sufficient condition for any surface. It is $\Delta < 0$ the well known Bogomolov inequality.

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