INDUCED REPRESENTATIONS OF DOUBLE AFFINE HECKE ALGEBRAS AND APPLICATIONS

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In this paper we apply the main results about the structure of double affine Hecke algebras from [C1,C2] (see [C6] for the proofs) to its induced representations. The technique is based on rather standard facts from the theory of affine Weyl groups and the matrix Demazure - Lusztig operators from [C3] There are close connections with the Macdonald theory [M1,M2] and the approach from [H,O].

As an application, we establish the difference counterpart of Theorem 4.6 from [C5] (the isomorphism between matrix Calogero-Sutherland eigenvalue problems and affine Knizhnik-Zamolodchikov equations generalizing the main theorem from [Ma]). Its scalar version (announced in [C1]) gives the equivalence of the generalized Macdonald eigenvalue problems and the corresponding quantum (difference) affine KZ equations. The latter are directly related to the Smirnov- Frenkel- Reshetikhin equations.

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1. Affine root systems.

Let $R=\{\alpha\}\subset \mathbf{R}^n$ be a root system of type $A,B,...,F,G$ with respect to a euclidean form (z,z') on $\mathbf{R}^n\ni z,z'$. We fix the set R_+ of positive roots ($R_-=-R_+$), the corresponding simple roots $\alpha_1,...,\alpha_n$, and their dual counterparts $a_1,...,a_n, a_i=\alpha_i^\vee$, where $\alpha^\vee=2\alpha/(\alpha,\alpha)$. The fundamental weights $\beta_1,...,\beta_n$ and the dual fundamental weights $b_1,...,b_n$ are determined from the relations $(\beta_i,a_j)=\delta_i^j=(\alpha_i,b_j)$ for the Kronecker delta. We will also introduce the lattices

$$Q=\oplus_{i=1}^n\mathbf{Z}\alpha_i\subset P=\oplus_{i=1}^n\mathbf{Z}\beta_i,\ A=\oplus_{i=1}^n\mathbf{Z}a_i\subset B=\oplus_{i=1}^n\mathbf{Z}b_i,$$

and $Q_\pm,P_\pm,A_\pm,B_\pm$ for $\mathbf{Z}_\pm=\{m\in\mathbf{Z},\pm m\geq 0\}$ instead of $\mathbf{Z}$. (In the standard notations, $B=P^\vee, P_+=P^{++},\beta_i=\omega_i$ etc.) Later on,

$$\begin{aligned}\nu_\alpha &= (\alpha,\alpha),\ \nu_i = \nu_{\alpha_i},\ \nu_R=\{\nu_\alpha,\alpha\in R\},\\ \rho_\nu &= (1/2)\sum_{\nu_\alpha=\nu}\alpha = \sum_{\nu_i=\nu}\beta_i,\ \text{ for }\ \alpha\in R_+.\end{aligned}\tag{1.1}$$

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The vectors $\tilde{\alpha} = [\alpha, k] \in \mathbf{R}^n \times \mathbf{R} \subset \mathbf{R}^{n+1}$ for $\alpha \in R, k \in \mathbf{Z}$ form the *affine root system* $R^a \supset R$ ($z \in \mathbf{R}^n$ are identified with $[z, 0]$). We add $\alpha_0 \stackrel{def}{=} [-\theta, 1]$ to the simple roots for the *maximal root* $\theta \in R$. The corresponding set R^a_+ of positive roots coincides with $R_+ \cup \{[\alpha, k], \alpha \in R, k > 0\}$.

We will use the Dynkin diagram Γ and its affine completion Γ^a with $\{\alpha_j, 0 \leq j \leq n\}$ as the vertices ($m_{ij} = 2, 3, 4, 6$ if α_i and α_j are joined by 0,1,2,3 laces respectively). The set of the indices of the images of α_0 by all the automorphisms of Γ^a will be denoted by O ($O = \{0\}$ for E_8, F_4, G_2). Let $O^* = r \in O, r \neq 0$.

Without going into detail, we mention that $(\theta^\vee, \alpha) \leq 1$ for $\theta \neq \alpha \in R_+$. More precisely, $\theta = \sum_i \beta_i$, where $m_{i0} > 2$. The multiplicity (b_r, α) of the roots α_r in arbitrary $\alpha \in R_+$ is also not more than 1 for $r \in O^*$, $(b_r, \theta) = 1$ (see [B,C4]).

Given $\tilde{\alpha} = [\alpha, k] \in R^a$, $b \in B$, let

$$s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)] \text{ for } \tilde{z} = [z, \zeta] \in \mathbf{R}^{n+1}. \quad (1.2)$$

The *affine Weyl group* W^a is the span $< s_{\tilde{\alpha}} >$. It is generated by the simple reflections $s_j = s_{\alpha_j}, 0 \leq j \leq n$, and can be represented as the semi-direct product $W \ltimes A'$ of its subgroups $W = < s_\alpha, \alpha \in R_+ >$ and $A' = \{a', a \in A\}$, where

$$a' = s_\alpha s_{[\alpha,1]} = s_{[-\alpha,1]} s_\alpha \text{ for } a = \alpha^\vee.$$

The *extended Weyl group* W^b generated by W and B' (instead of A') is isomorphic to $W \ltimes B'$:

$$(wb')([z, \zeta]) = [w(z), \zeta - (z, b)] \text{ for } w \in W, b \in B. \quad (1.3)$$

DEFINITION 1.1.

i) Given $b_+ \in B_+$, let

$$\omega_{b_+} = w_0 w_0^+ \in W, \ \pi_{b_+} = b'_+ (\omega_{b_+})^{-1} \in W^b, \ \omega_i = \omega_{b_i}, \pi_i = \pi_{b_i}, \quad (1.4)$$

where w_0 (respectively, w_0^+) is the longest element in W (respectively, in W_{b_+} generated by s_i preserving b_+) relative to the set of generators $\{s_i\}$ for $i > 0$.

ii) If b is arbitrary then there exist unique elements $w \in W$, $b_+ \in B_+$ such that $b = w(b_+)$ and $(\alpha, b_+) \neq 0$ if $(-\alpha) \in R_+ \ni w(\alpha)$. We set

$$\omega_b = \omega_{b_+} w^{-1}, \ \pi_b = w \pi_{b_+}. \quad (1.5)$$

□

We will discuss general properties of $\{\omega_b, \pi_b\}$ later. Now we only note that the elements $\pi_r, r \in O$, leave Γ^a invariant and form a group denoted by Π, which is isomorphic to B/A by the natural projection $\{b_r \to \pi_r\}$. As to $\{\omega_r\}$, they preserve the set $\{-\theta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (\omega_r)^{-1}(-\theta)$ distinguish the indices $r \in O^*$. These elements are important because (due to [B,V]):

$$W^b = \Pi \ltimes W^a, \text{ where } \pi_r s_i \pi_r^{-1} = s_j \text{ if } \pi_r(\alpha_i) = \alpha_j. \tag{1.6}$$

To go further we need the notion of length and its geometric interpretation. Given $\nu \in \nu_R$, $r \in O^*$, $\tilde{w} \in W^a$, and a reduced decomposition $\tilde{w} = s_{j_l} \ldots s_{j_2}\, s_{j_1}$ with respect to $\{s_j, 0 \le j \le n\}$, we call $l = l(\hat{w})$ the *length* of $\hat{w} = \pi_r \tilde{w} \in W^b$ and introduce the sets

$$\lambda(\hat{w}) = \{\tilde{\alpha}^1 = \alpha_{j_1},\ \tilde{\alpha}^2 = s_{j_1}(\alpha_{j_2}),\ \tilde{\alpha}^3 = s_{j_1}s_{j_2}(\alpha_{j_3}), \ldots, \tilde{\alpha}^l = \tilde{w}^{-1}s_{j_l}(\alpha_{j_l})\},$$

$$\lambda_\nu(\hat{w}) = \{\tilde{\alpha}^m,\ \nu(\tilde{\alpha}^m) = \nu(\tilde{\alpha}_{j_m}) = \nu\} \text{ for } \nu([\alpha, k]) \overset{def}{=} \nu_\alpha,\ 1 \le m \le l. \tag{1.7}$$

One has: $l = \sum_\nu l_\nu$, where $l_\nu = l_\nu(\hat{w}) = |\lambda_\nu(\hat{w})|$ denotes the corresponding number of elements.

To see that these sets do not depend on the choice of the reduced decomposition we will use the following (affine) action of W^b on $z \in \mathbf{R}^n$:

$$\begin{aligned} (wb')\langle z \rangle &= w(b + z),\ w \in W, b \in B, \\ s_{\tilde{\alpha}}\langle z \rangle &= z - ((z, \alpha) + k)\alpha^\vee,\ \tilde{\alpha} = [\alpha, k] \in R^a, \end{aligned} \tag{1.8}$$

and the affine Weyl chamber:

$$C^a = \bigcap_{j=0}^{n} L_{\alpha_j},\ L_{\tilde{\alpha}} = \{z \in \mathbf{R}^n,\ (z, \alpha) + k > 0\}. \tag{1.9}$$

PROPOSITION 1.2.

$$\begin{aligned} \lambda_\nu(\hat{w}) &= \{\tilde{\alpha} \in R^a,\ \hat{w}^{-1}\langle C^a \rangle \not\subset L_{\tilde{\alpha}},\ \nu(\tilde{\alpha}) = \nu\} \\ &= \{\tilde{\alpha} \in R^a,\ l_\nu(\hat{w}s_{\tilde{\alpha}}) < l_\nu(\hat{w})\}. \end{aligned} \tag{1.9}$$

□

As to the latter condition, a direct calculation shows that

$$\begin{aligned} &l(\ \hat{w}s_{\tilde{\alpha}\{1\}} \ldots s_{\tilde{\alpha}\{p\}}\ ) > l(\ \hat{w}s_{\tilde{\alpha}\{1\}} \ldots s_{\tilde{\alpha}\{p+1\}}\ ), \text{ if} \\ &\tilde{\alpha}\{q\} \overset{def}{=} \tilde{\alpha}^{m_q},\ l \ge m_1 > m_2 > \ldots > m_p > m_{p+1} \ge 1. \end{aligned} \tag{1.10}$$

Vice versa, an arbitrary sequence of positive roots $\tilde{\alpha}\{1\}, \tilde{\alpha}\{2\}, \ldots$ satisfying the consequent conditions (1.10) for $p = 0, 1, \ldots$ can be obtained by the above construction (i.e. belongs to $\lambda_\nu(\hat{w})$ and corresponds to a certain reduced decomposition of $\tilde{w}$). We will not use this fact and only mention that it results from the following rather standard proposition.

PROPOSITION 1.3. *(see e.g. [C4], Proposition 1.4).*
Each of the following conditions for $x, y \in W^b$ is equivalent to the relation $l_\nu(xy) = l_\nu(x) + l_\nu(y)$:

$$a)\ \lambda_\nu(xy) = \lambda_\nu(y) \cup y^{-1}(\lambda_\nu(x)),\ b)\ y^{-1}(\lambda_\nu(x)) \subset R_+^a$$
$$c)\ \lambda_\nu(y) \subset \lambda_\nu(xy),\ d)\ y^{-1}(\lambda_\nu(x)) \subset \lambda_\nu(xy). \tag{1.11}$$

□

Now everything is prepared to motivate the construction of $\{\pi_b\}$.

THEOREM 1.4.
i) In the above notations,

$$\lambda(b') = \{\tilde\alpha, \alpha \in R_+, (b, \alpha) > k \geq 0\} \cup \{\tilde\alpha, \alpha \in R_-, (b, \alpha) \geq k > 0\}, \tag{1.12}$$
$$\lambda(\pi_b^{-1}) = \{\tilde\alpha, -(b, \alpha) > k \geq 0\}, \quad \textit{where}\ \tilde\alpha = [\alpha, k] \in R_+^a, b \in B. \tag{1.13}$$

ii) If $\hat w \in b'W$ (i.e. $\hat w\langle 0\rangle = b$) then $\hat w = \pi_b w$ for $w \in W$ such that $l(\hat w) = l(\pi_b) + l(w)$. Given $b \in B$, this property (valid for any $\hat w$ taking 0 to b) determines π_b uniquely.

Proof. Formula (1.12) is verified directly (see Proposition 1.6, b) from [C4]). By the way, it gives the useful formulas (cf. [L1], 1.4) :

$$l_\nu(b') = \sum_\alpha |(b, \alpha)|, \quad \text{where } |\ | = \text{abs. value}, \alpha \in R_+, \nu_\alpha = \nu \in \nu_R,$$
$$l_\nu(b'_+) = 2(b, \rho_\nu), \quad \text{when } b \in B_+. \tag{1.14}$$

One can follow the same proposition (assertion a)) to check that

$$\lambda(\omega_{b'_+}) = \{\alpha \in R_+, (b_+, \alpha) > 0\} \ \text{ for }\ b_+ \in B_+. \tag{1.15}$$

It proves (1.13) for B_+ due to Proposition 1.3, a) and the relation $\lambda(\hat w^{-1}) = -\hat w\langle\lambda(\hat w)\rangle$ (resulting from Proposition 1.2).

Let $b = w(b_+)$ for positive b_+ and $w \in W$. We can multiply w on the right by elements preserving b_+ (i.e. belonging to W_{b_+}). If the length of w is the least possible, then $\lambda(w)$ does not contain roots $\alpha \in R_+$ orthogonal to b_+ (Proposition 1.2) and w is defined uniquely. This condition is from Definition 1.1, ii).

Setting $b = \pi\omega$ for $\omega \in W$, where $\pi \in W$ has the least possible length $l(\pi)$, we are going to calculate $\lambda(\omega)$ and $\lambda(\pi^{-1})$.

The set $\lambda(\pi)$ containes only roots $\tilde\alpha = [\alpha, k]$ with $k > 0$. Otherwise we could find in this set a root from R_+ and apply the second formula from (1.9) to reduce π by the corresponding reflection from W. Hence, $w^{-1}(\lambda(\pi)) \subset R_+^a$ and the decomposition $b = \pi\omega$ satisfies condition (1.11). Moreover, $w^{-1}(\lambda(\pi))$ contains all the elements from $\lambda(b)$ with $k > 0$ (since $w \in W$ – use (1.11) again). It is enough to calculate $\lambda(\omega)$ because $\lambda(b)$ is already known. We will arrive at the same formula (1.15) (but now for ω and $b \in B$). Applying (1.11) after the passage to $-b$, we obtain precisely (1.13) for $\lambda(\pi^{-1})$.

Let us calculate $\lambda(\omega_b)$ and $\lambda(\pi_b^{-1})$. Thanks to formula (1.15) for b_+ and the properties of w (see above) we have the embedding $\lambda(w) \subset \lambda(\omega_{b_+})$. Hence the decomposition $\omega_{b_+} = \omega_b w$ satisfies conditions (1.11) and

$$\begin{aligned}\lambda(\omega_b) &= w(\lambda(\omega_{b_+}) \setminus \lambda(w)) = w(\lambda(\omega_{b_+})) \cap R_+ \\ &= w(\{\alpha \in R, (\alpha, b_+) > 0\}) \cap R_+ = \{\alpha' \in R_+, (\alpha', b) > 0, \}.\end{aligned}$$

Here one can use Proposition 1.3 with the relation $\lambda(w) = \{\alpha \in R_+, w(\alpha) \in R_-\}$ resulting directly from (1.9). We see that (abstact) ω defined above and ω_b from (1.5) coincide (they have the same λ-sets). It gives the coincidence of π and π_b, formula (1.13), and statement ii). As for the latter, if $\hat{w}\langle 0\rangle = b$, then $\hat{w} = \pi_b w', w' \in W$. However we know that $l(\pi_b w') = l(\pi_b) + l(w')$ for any $w' \in W$. □

We set

$$c \preceq b, b \succeq c \quad \text{for} \quad b, c \in B \quad \text{if} \quad b - c \in A_+, \tag{1.16}$$

and use $\prec, \succ$ respectively if $b \neq c$. Given $b \in B$, let $b_+ = w_+^{-1}(b) \in B_+$ for w_+ from Definition 1.1. The sets

$$\begin{aligned}\sigma^\vee(b) &\stackrel{def}{=} \{g \in B, w(c) \preceq b_+ \quad \text{for} \quad \text{any } w \in W\}, \\ \sigma_0^\vee(b) &\stackrel{def}{=} \{c \in B, w(c) \prec b_+ \quad \text{for} \quad \text{any } w \in W\}\end{aligned} \tag{1.17}$$

are W-invariant (which is evident) and convex. The latter means that if $c, c^* = c + r\alpha^\vee \in \sigma^\vee(b) (\in \sigma_0^\vee(b))$ for $\alpha \in R, r \in \mathbf{Z}_+$, then

$$\{c,\ c + \alpha^\vee, ..., c + (r-1)\alpha^\vee,\ c^*\} \subset \sigma^\vee(b) (\subset \sigma_0^\vee(b)). \tag{1.18}$$

Really, $w(c + r'\alpha^\vee)$ for $0 < r' < r$ is always between $w(c), w(c^*)$ for any w with respect to the ordering '$\prec$' and therefore belongs to (1.17) because $w(c), w(c^*)$ do.

For the sake of completeness, we will check another well known property of $\sigma^\vee(b)$. It contains the orbit $W(b)$. If $w(b) \preceq b_+$ and $l(ws_\alpha) > l(w)$ for $\alpha \in R_+$, then $w(\alpha) \in R_+$ and $ws_\alpha(b_+) = w(b_+ - (b_+, \alpha)\alpha^\vee) \preceq b_+$. Hence we can argue by induction.

PROPOSITION 1.5.

i) *Given $\hat{w} \in W^b, \tilde{\alpha} \in \lambda(\hat{w})$, let $b = \hat{w}\langle 0\rangle, \hat{w}_* = \hat{w}s_{\tilde{\alpha}}, b_* = \hat{w}_*\langle 0\rangle$. Then $b_* \in \sigma^\vee(b)$. If $b \in B_+$ and $b_* \neq b$, then $b_* \in \sigma_0^\vee(b)$.*

ii) *In the above hypotheses, $\ell(\hat{w}) > \ell(b'_+)$ if $b_+ \neq b$, and*

$$\ell(\hat{w}_*) < \ell(\hat{w}) \ \text{ if } \ b_* \neq b, \ \text{ where } \ \ell(\hat{w}) = \ell(b') \stackrel{def}{=} l(\pi_b). \tag{1.19}$$

iii) *Let $\hat{w}_* = s_{\tilde{\alpha}\{p\}} ... s_{\tilde{\alpha}\{1\}}\hat{w}$,where we take any sequence (1.10) for $\hat{w}^{-1}$ (instead of $\hat{w}$) such that $\ell(s_{\tilde{\alpha}\{1\}}\hat{w}) < \ell(\hat{w})$. Then $\ell(\hat{w}_*) < \ell(\hat{w})$ and $\hat{w}_*\langle 0\rangle \neq b$.*

Proof. One has: $\lambda(\tilde{w}^{-1}) \subset \{\tilde{\alpha} = [\alpha, k] \in R^a_+, -(b,\alpha) \geq k \geq 0\}$ (use (1.9)). Hence,

$$b_* = s_{\tilde{\alpha}}\langle b\rangle = b - ((b,\alpha) + k)\alpha^\vee$$

is between b and $s_\alpha(b)$ with respect to the odering '$\preceq$'. If $b \in B_+$ (i.e. $b = b_+$) and $b_* \neq b$, then $\alpha \in R_-, k > 0$, and $b \prec b_* \prec s_\alpha(b)$. It completes i). Assertions ii) and iii) follow directly from the definitions of π_b and $\ell(\ )$. □

2. Double affine Hecke algebras.

Let us fix $\delta \in \mathbf{C}^*$ which is not a root of unity and $\{q_\nu \in \mathbf{C}^*, \ \nu \in \nu_R\}$. The notations are from Sec.1. We denote the least common order of the elements of Π by m ($m = 2$ for D_{2k}, otherwise $m = |\Pi|$) and set

$$\Delta \ = \ \delta^m \ , \ q_{\tilde{\alpha}} = q_{\nu(\tilde{\alpha})}, \ q_j = q_{\alpha_j}, \ \text{ where } \ \tilde{\alpha} \in R^a, 0 \leq j \leq n. \tag{2.1}$$

Let us put formally $x_i = exp(\beta_i), \ x_\beta = exp(\beta) = \prod_{i=1}^n x_i^{k_i}$ for $\beta = \sum_{i=1}^n k_i\beta_i$, and introduce the algebra $\mathbf{C}[x] = \mathbf{C}[x_\beta]$ of polynomials in terms of $x_i^{\pm 1}$. We will also use

$$X_{\tilde{\beta}} \ = \ \prod_{i=1}^n X_i^{k_i}\delta^{mk} \ \text{ if } \ \tilde{\beta} = [\beta, k], \ \beta = \sum_{i=1}^n k_i\beta_i \in P, \ mk \in \mathbf{Z}, \tag{2.2}$$

where $\{X_i\}$ are independent variables which act in $\mathbf{C}[x]$ naturally:

$$X_{\tilde{\beta}}(p(x)) \ = \ x_{\tilde{\beta}}p(x), \ \text{ where } \ x_{\tilde{\beta}} \stackrel{def}{=} x_\beta\delta^{mk}; \ p(x) \in \mathbf{C}[x]. \tag{2.3}$$

The elements $\tilde{w} \in W^b$ act in $\mathbf{C}[x]$, $\mathbf{C}[X] = \mathbf{C}[X_\beta]$ by the formulas:

$$\tilde{w}(x_{\tilde{\beta}}) \ = \ x_{\tilde{w}(\tilde{\beta})}, \ \ \tilde{w}X_{\tilde{\beta}}\tilde{w}^{-1} \ = \ X_{\tilde{w}(\tilde{\beta})}. \tag{2.4}$$

In particular (we will use this in the sequel):

$$\pi_r(x_\beta) = x_{\omega_r^{-1}(\beta)}\delta^{m(\beta,b_{r^*})} \text{ for } \alpha_{r^*} \stackrel{def}{=} \pi_r^{-1}(\alpha_0),\ r \in O^*. \tag{2.5}$$

DEFINITION 2.1. *(see [C1,C2])*

The double affine Hecke algebra $\mathfrak{H}$ *is generated by the elements* $T_j,\ 0 \le j \le n\}$, *pairwise commutative* $\{X_\beta,\ \beta \in P\}$, *and the group* Π, *satisfying the following relations (depending on* δ, q*) :*

(o) $(T_j - q_j)(T_j + q_j^{-1}) = 0,\ 0 \le j \le n;$

(i) $T_iT_jT_i ... = T_jT_iT_j ...,\ m_{ij}$ *factors on each side;*

(ii) $\pi_r T_i \pi_r^{-1} = T_j$ *if* $\pi_r(\alpha_i) = \alpha_j;$

(iii) $T_iX_\beta T_i = X_\beta X_{\alpha_i}^{-1}$ *if* $(\beta, a_i) = 1,\ 1 \le i \le n;$

(iv) $T_0^{-1}X_\beta T_0^{-1} = X_{s_0(\beta)} = X_\beta X_\theta^{-1}\Delta$ *if* $(\beta,\theta^\vee) = 1;$

(v) $T_iX_\beta = X_\beta T_i$ *if* $(\beta, a_i) = 0$, *where* $a_0 = \theta^\vee;$

(vi) $\pi_r X_\beta \pi_r^{-1} = X_{\pi_r(\beta)} = X_{\omega_r^{-1}(\beta)}\delta^{m(b_{r^*},\beta)},\ r \in O^*.$

□

Given $\tilde{w} \in W^a, r \in O$, the product

$$T_{\pi_r\tilde{w}} \stackrel{def}{=} \pi_r \prod_{k=1}^{l} T_{i_k}, \text{ where } \tilde{w} = \prod_{k=1}^{l} s_{i_k}, l = l(\tilde{w}), \tag{2.6}$$

does not depend on the choice of the reduced decomposition (because $\{T\}$ satisfy the same "braid" relations as $\{s\}$ do). Moreover,

$$T_{\tilde{v}}T_{\tilde{w}} = T_{\tilde{v}\tilde{w}} \text{ whenever } l(\tilde{v}\tilde{w}) = l(\tilde{v}) + l(\tilde{w}) \text{ for } \tilde{v}, \tilde{w} \in W^b, \tag{2.7}$$

which follows from (2.6) and relations (ii). In particular, we arrive at the pairwise commutative operators (use (2.7) and (1.14)):

$$Y_b = \prod_{i=1}^{n} Y_i^{k_i} \text{ if } b = \sum_{i=1}^{n} k_i b_i \in B, \text{ where } Y_i \stackrel{def}{=} T_{b_i'}. \tag{2.8}$$

PROPOSITION 2.2.

$$\begin{aligned} T_i^{-1}Y_bT_i^{-1} &= Y_bY_{a_i}^{-1} \text{ if } (b,\alpha_i) = 1, \\ T_iY_b &= Y_bT_i \text{ if } (b,\alpha_i) = 0,\ 1 \le i \le n. \end{aligned} \tag{2.9}$$

Proof(cf. [L1], 2.7). We will deduce these relations from (i)-(ii). It suffices to check that

$$T_i^{-1}Y_iT_i^{-1} = Y_iY_{a_i}^{-1},\ T_iY_j = Y_jT_i \text{ for } 1 \le i \ne j \le n. \tag{2.10}$$

Applying (1.15) to $\tilde{b} = s_i(b_i) = b_i - a_i$, we see that $l(\tilde{b}') = \sum_{\alpha\in R_+} |(b_i, s_i(\alpha))| = l(b_i') - 2$, since $s_i(\alpha) \in R_+$ for $\alpha \in R_+ \setminus \{\alpha_i\}$. Hence formula (2.7) works for the triple decomposition $b_i' = s_i\tilde{b}s_i$. If $j \neq i$, then $\alpha_j \notin \lambda(b_i')$ (see (1.12)) and $l(b_i's_j) = l(b_i') + 1$. Now we only have to use the commutativity of b_i and s_j. □

Let $\mathcal{H}_Y$ be the *affine Hecke algebra* generated over $\mathbf{C}$ by $\{T_i, 1 \leq i \leq n\}$ and pairwise commutative $\{Y_i\}$ satisfying relations (o,i) from Definition 2.1 (for $1 \leq i, j \leq n$) and (2.10). Because δ is not a root of unity we can identify $\mathcal{H}_Y$ with the corresponding subalgebra of $\mathfrak{H}$. It results from Theorem 2.3, [C6], which gives that an arbitrary element $H \in \mathfrak{H}$, can be uniquely represented as follows:

$$H = \sum_{b\in B, w\in W} h_{b,w} Y_b T_w = \sum_{\hat{w}\in W^b} h_{\hat{w}} T_{\hat{w}}, \tag{2.11}$$

where $h_{b,w}, h_{\hat{w}}$ belong to $\mathbf{C}[X]$ (are Laurent polynomials in $\{X_1, ..., X_n\}$).

In particular, we have another description of $\mathcal{H}_Y$. It is generated by $T_j, 0 \leq j \leq n$ and Π with the defining relations (o-ii).

Let us fix a finite dimensional representation V of $\mathcal{H}_Y$:

$$\zeta : \mathcal{H}_Y \rightarrow End_{\mathbf{C}}(V). \tag{2.12}$$

The *matrix Demazure-Lusztig operators* (see [C5])

$$\hat{T}_j = \zeta(T_j)s_j + (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1}(s_j - 1),\ 0 \leq j \leq n, \tag{2.13}$$

act in the space $V[x]$ of polynomials in $\{x_\beta\}$ with the coefficients from V. They generalize the scalar operators from [KL, KK, C1]. In particular,

$$\hat{T}_0 = \zeta(T_0)s_0 + (q_0 - q_0^{-1})(\Delta X_\theta^{-1} - 1)^{-1}(s_0 - 1),$$
$$\text{where } s_0(X_i) = X_i X_\theta^{-(\beta_i,\theta^\vee)} \Delta^{(\beta_i,\theta^\vee)}.$$

It is worth mentioning that W^b acts only on $\{x\}$ commuting with the action of $\zeta(\mathcal{H}_Y)$ on the coefficients (from V).

THEOREM 2.3.

The map $\hat{\zeta}(T_j) = \hat{T}_j$, $\hat{\zeta}(X_\beta) = X_\beta$ (see (2.3)), $\hat{\zeta}(\pi_r) = \zeta(\pi_r)\pi_r$ (see (2.5)) can be uniquely extended to a faithful homomorphism $\hat{\zeta}$ (depending on $\{\delta \in \mathbf{C}^* \ni q\}$) from $\mathfrak{H}$ to the algebra of linear endomorphisms of $V[x]$. The resulting module coincides with the induced (=universal) $\mathfrak{H}$-module $\hat{V}$ generated by V with the action of $\mathcal{H}_Y$ via (2.12).

Proof. The decomposition from (2.11) identifies $\hat{V}$ with $V[x]$. Given $H \in \mathfrak{H}, \beta \in P$, and $v \in V$, the induced action is as follows:

$$H(vx_\beta) \stackrel{def}{=} \sum_{b\in B, w\in W} h'_{b,w}(x)\zeta(Y_b T_w)(v), \quad \text{where}$$
$$HX_\beta = \sum_{b\in B, w\in W} h'_{b,w}(X) Y_b T_w. \tag{2.14}$$

In particular, $\{X_\beta\}$ and Π operate naturally (see (2.3), (2.5)). As to the formulas for the action of $\{T_j\}$, the coincidence with (2.13) was checked in [C3] (Theorem 2.1) when $j > 0$. The reasoning for T_0 is the same.

The induced representation is faithful. To see this we may extend $\mathbf{C}[X]$ to the field $\mathbf{C}(X)$ of rational functions of X_β replacing $\mathfrak{H}$ by

$$\mathfrak{H}' = \oplus_{\hat{w}\in W^b} \mathbf{C}(X) T_{\hat{w}} = \oplus_{\hat{w}\in W^b} \mathbf{C}(X)\Phi_{\hat{w}}, \quad \text{where}$$
$$\Phi_{s_j} = T_j + (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1}, \ 0 \le j \le n, \quad \Phi_{\pi_r} = \pi_r, \ r \in O,$$
$$\Phi_{\hat{v}\hat{w}} = \Phi_{\hat{v}}\Phi_{\hat{w}} \quad \text{whenever} \quad l(\hat{v}\hat{w}) = l(\hat{v}) + l(\hat{w}), \hat{v}, \hat{w} \in W^b. \tag{2.15}$$

This algebra acts in $V(x) = V \otimes \mathbf{C}(x)$ (formulas (2.14) remain the same). The elements $\Phi_{\hat{w}}$ are well-defined and (see [C3], Proposition 1.2) satisfy the following relations:

$$\Phi_{\hat{w}} X_\beta = X_{\hat{w}(\beta)} \Phi_{\hat{w}}, \ \beta \in B. \tag{2.16}$$

If the induced action of $H = \sum_{\hat{w}\in W^b} h_{\hat{w}}(X)\Phi_{\hat{w}}$ is zero, then (use (2.14-16)) the same holds true for $\Phi_{\hat{w}}$ with $h_{\hat{w}} \neq 0$. However $\Phi_{\hat{w}}$ are invertible in $\mathfrak{H}'$. □

Thanks to formulas (2.15) we can introduce the set $\phi_{\hat{w}}, \hat{w} \in W^b$, such that

$$\phi_{\hat{v}\hat{w}} = \phi_{\hat{v}} \hat{v}(\phi_{\hat{w}}) \ \text{ if } \ l(\hat{v}\hat{w}) = l(\hat{v}) + l(\hat{w}), \quad \text{where} \quad \hat{v}(\ ) = \hat{v}(\ )\hat{v}^{-1}, \tag{2.17}$$
$$\phi_{s_j} = \zeta(T_j) + (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1}, \ 0 \le j \le n, \ \phi_{\pi_r} = \zeta(\pi_r), r \in O. \tag{2.18}$$

Arbitrary element $\hat{H} \stackrel{def}{=} \hat{\zeta}(H)$, $H \in \mathfrak{H}$, has the unique representation

$$\hat{H} = \sum_{b\in B, w\in W} g_{b,w} b' w, \quad \text{where} \quad g_{b,w} \in (End_{\mathbf{C}} V)(X). \tag{2.19}$$

PROPOSITION 2.4.
i) Given $b \in B$ and $\hat{w} = \pi_b \omega, \omega \in W$,

$$\hat{T}_{\hat{w}} = \phi_{\pi_b} \pi_b \hat{T}_\omega + \sum_{b_*, w\in W} g_{b_*,w}\, b'_* w, \tag{2.20}$$

summed over $b_* \in \sigma^\vee(b)$ such that $\ell(b') > \ell(b'_*)$.

ii) If $b \in B_-$, then $\pi_b = b'$ and

$$Y_b \;=\; \phi_{b'} b' + \sum_{b_*, w \in W} g_{b_*,w}\, b'_* w, \quad b \neq b_* \in \sigma^\vee(b), \tag{2.21}$$

where we omit the condition $\ell(b') > \ell(b'_*)$ because it is valid for any $b \neq b_* \in \sigma^\vee(b)$ (Theorem 1.4).

Proof. Following [C4], let

$$F_j(\tilde\alpha) \;=\; \zeta(T_j) + (q_{\tilde\alpha} - q_{\tilde\alpha}^{-1})(X_{\tilde\alpha} - 1)^{-1}(1 - s_{\tilde\alpha}), \; \tilde\alpha \in R^a, 0 \le j \le n. \tag{2.22}$$

Given a reduced decomposition $\hat w = \pi_b \omega = \pi_r s_{j_1} \cdots s_{j_l}$, where $l = l(\hat w), r \in O$,

$$\hat T_{\hat w} = F_{\hat w} \hat w \stackrel{def}{=} \zeta(\pi_r) F_{j_1}(\tilde\alpha(1)) F_{j_2}(\tilde\alpha(2)) \cdots F_{j_l}(\tilde\alpha(l)) \hat w \;\text{ for}$$
$$\tilde\alpha(1) = \pi_r \alpha_{j_1}, \tilde\alpha(2) = \pi_r s_{j_1}(\alpha_{j_2}), \tilde\alpha(3) = \; \pi_r s_{j_1} s_{j_2}(\alpha_{j_3}), \ldots . \tag{2.23}$$

These roots constitute the set $\lambda(\hat w^{-1})$ (see (1.7)). The set $\{F_{\hat w}\}$ satisfies the cocycle relations from (2.17). We may assume here that $\pi_b = \pi_r s_{j_1} \cdots s_{j_\ell}$, $\ell = l(\pi_b)$. If the terms with $s_{\tilde\alpha}$ from $F_{\tilde\alpha^p}$ such that $p \le \ell$ are omitted, then the resulting product coincides with the leading term of (2.20) (compare (2.18) and (2.22)). Any other terms contribute to to the elements $g_{b_*,w} b'_* w$ with $b'_* \neq b$ (see Proposition 1.5).

Let us consider now $b \in B_-$. Since $Y_b = \hat T^{-1}_{-b'}$, we have to inverse the product

$$\hat T_{-b'} = (-b') G_{j_l}(\tilde\alpha(l)) \cdots G_{j_1}(\tilde\alpha(1)) \pi_r^{-1} \;\text{ for }\; b' = \pi_r s_{j_1} \cdots s_{j_l},$$
$$G_j(\tilde\alpha) \;=\; \zeta(T_j) + (q_{\tilde\alpha} - q_{\tilde\alpha}^{-1})(X_{\tilde\alpha}^{-1} - 1)^{-1}(1 - s_{\tilde\alpha}), \; l = l(b), \tag{2.24}$$

and use that

$$G_j^{-1}(\tilde\alpha) \;=\; \zeta(T_j) + (q_{\tilde\alpha} - q_{\tilde\alpha}^{-1})((X_{\tilde\alpha} - 1)^{-1} + (X_{\tilde\alpha}^{-1} - 1)^{-1} s_{\tilde\alpha}). \tag{2.25}$$

Ignoring the terms with $\{s\}$, we arrive at (2.21). □

3. Difference operators. The algebra of W-invariant elements in the $\mathbf{C}[Y]$ is denoted by $\mathbf{C}[Y]^W$. We will use that $\mathbf{C}[Y]^W$ is the center of $\mathcal{H}_Y$. The same of course holds for $\mathbf{C}[X]^W$ and $\mathcal{H}_X$. This property is due to Bernstein (see e.g. [L1], [C3]).

Let $\{\varphi_{\hat w}\}$ be the set obeying (2.17) for any $\hat v, \hat w$ (regardless of the lengths) and normalized as follows:

$$\varphi_{s_j} = (q + (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1})^{-1} \phi_{s_j}, \; \varphi_{\pi_r} \;=\; \zeta(\pi_r). \tag{3.1}$$

We introduce the corresponding action of W^b on $\hat{z} \in V(x) = V \otimes \mathbf{C}(x)$ and $\hat{g} \in End_{\mathbf{C}}(V(x))$:

$$\hat{w}^{\#}(\hat{z}) \stackrel{def}{=} \varphi_{\hat{w}}\hat{w}(\hat{z}),\ \hat{w}^{\#}(\hat{g}) \stackrel{def}{=} \varphi_{\hat{w}}\hat{w}\hat{g}\hat{w}^{-1}\varphi_{\hat{w}}^{-1}. \tag{3.2}$$

Let $W_{\#} \subset W^b_{\#} = \{\hat{w}^{\#},\ \hat{w} \in W^b\}$, $V[x]^{W_{\#}}$ be the subspace of $W_{\#}$-invariants. The $W_{\#}$-invariance of $\hat{z}$ means that $\hat{T}_i(\hat{z}) = q_i\hat{z}$ for $1 \le i \le n$, because

$$\hat{T}_j \;=\; q_j s_j^{\#} \;+\; (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1}(s_j^{\#} - 1),\ 0 \le j \le n. \tag{3.3}$$

Given arbitrary element $H \in \mathcal{H}_Y$, its $\hat{\zeta}$-image can be uniquely represented in the form :

$$\hat{H} \;=\; \sum_{w\in W, b\in B} g_{b,w} b' w^{\#},\quad \text{where}\ \ g_{b,w} \in (End_{\mathbf{C}}V)(X). \tag{3.4}$$

The rational functions $g_{b,w}$ are regular at the points

$$\diamondsuit \stackrel{def}{=} (X_1 = \ldots = X_n = 0),\ \bowtie \stackrel{def}{=} (X_1 = \ldots = X_n = \infty).$$

Indeed, $\{(X_{\tilde{\alpha}} - 1)^{-1}\}$ (from (2.11) etc.) are well-defined at these points either for positive or for negative $\tilde{\alpha} \in R^a$.

Let us introduce the *difference Harish-Chandra homomorphism*:

$$\chi(\sum_{w\in W, b\in B} g_{b,w} b' w^{\#}) \;=\; \sum_{w,b} g_{b,w}(\diamondsuit) b'. \tag{3.5}$$

PROPOSITION 3.1.

$$\chi(\hat{Y}_b) \;=\; \zeta(Y_b) b',\ b \in B. \tag{3.6}$$

Proof. Let us start with $b \in B_-$. It follows from formula (2.21), that the χ-value of the leading term of Y_b gives exactly (3.5). Really, $\alpha \in R_+$ for all $X^{-1}_{[\alpha,k]}$ in the formula for $\phi_{b'}$ (see (2.23)). Hence

$$\chi(\phi_{b'}) = \zeta(\pi_r T_{j_1}^{-1} \cdots T_{j_l}^{-1}) = \zeta(Y_{-b}^{-1}) = \zeta(Y_b)$$

for a reduced decomposition $b' = \pi_r s_{j_1} \cdots s_{j_l}$. Any other $g_{b_*,w}$ (corresponding to $b_* \neq b$) will contain at least one factor $(X^{-1}_{[\alpha,k]} - 1)^{-1}$ for positive α. Its value at $\diamondsuit$ is zero.

The case of positive b formally follows from this consideration, since $Y_b = Y_{-b}^{-1}$. The direct reasoning is not difficult as well. One has (see (3.3) and (2.17)):

$$\chi(\phi_{\pi_b}\pi_b\hat{T}_{\omega_b}) = \chi(\phi_{b'}b'(\phi_{\omega_b}\omega_b)^{-1})\prod_\nu q_\nu^{l_\nu(\omega_b)} = \zeta(Y_b)b'$$

(here we will meet $\tilde{\alpha} = [\alpha, k]$ only with $\alpha \in R_-$). Any other terms contribute to the coefficients $g_{b_*,w}$ with $b_* \neq b$ and come from the s-parts of the products (cf. (1.10)):

$$F(\tilde{\alpha}\{1\})\cdots F(\tilde{\alpha}\{p\})b', \quad \text{where} \quad \tilde{\alpha}\{1\} = \tilde{\alpha}(m_1),\ldots, \quad 1 \leq m_1 < \ldots < m_p \leq l.$$

Moreover, $m_1 \leq \ell$, which gives the factor $(X_{[\alpha,k]} - 1)^{-1}$ for $\tilde{\alpha}\{1\} = [\alpha, k], \alpha \in R_-$. Its value at $\diamondsuit$ is 0.

Turning to arbitrary $b \in B$, let $b = b_+ + b_-$, where $b_\pm \in B_\pm$. Then (see (2.8)), $Y_b = Y_{b_+}Y_{b_-}$, and we can use the relations (obtained above)

$$g_{b_*,w}(\diamondsuit) = 0 \quad \text{for} \quad b \in B_\pm, \ w \in W, \ b_* \neq b,$$

to complete the proof. □

Given any element $A = \sum_{w\in W, b\in B} g_{b,w}b'w^\#$, where $g_{b,w} \in (End_{\mathbf{C}}V)(X)$, set

$$A_{red} \stackrel{def}{=} \sum_{w\in W, b\in B} g_{b,w}b', \ L_H = \hat{H}_{red}, \ H \in \mathfrak{H}. \tag{3.7}$$

We note that $\{L_H\}$ act in $V[x]$, because to erase $\{w^\#\}$ means to replace each $\hat{T}_i$ by q_i (see (3.3)). The restrictions of L_H and $\hat{H}$ on $V[x]^{W_\#}$ and their χ-values (see Proposition 3.1) coincide.

THEOREM 3.3.

Let us denote the algebra generated by $\{T_i, 1 \leq i \leq n\}$ by **H**. *The reduction map L is an algebraic homomorphism on the centralizer $\mathfrak{H}^{\mathbf{H}}$ of* **H** *in $\mathfrak{H}$. Given $H \in \mathfrak{H}^{\mathbf{H}}$, L_H is $W_\#$-invariant (i.e $w^\# L_H(w^\#)^{-1} = L_H$ for all $w \in W$) and preserves $V[x]^{W_\#}$. Operators L_H for $H \in \mathcal{H}_Y^{\mathbf{H}}$ commute with the operators $\{L_F, \ F \in \mathbf{C}[Y]^W\}$.*

Proof. The reduction procedure is trivial exactly on the left ideal in $End_{\mathbf{C}}V(x)$ generated by the elements $\{\hat{T}_i - q_i, 1 \leq i \leq n\}$. The multiplication on the right by $\hat{H}$ leaves this ideal invariant.. Hence $(A\hat{H})_{red} = A_{red}L_H$ for any A from (3.7). Moreover, we see that $w^\#(\hat{H})_{red}(w^\#)^{-1} = (w^\#\hat{H}(w^\#)^{-1})_{red} = (w^\#\hat{H})_{red} = w^\#_{red}L_H = L_H$ (cf. [C5], Theorem 2.4). The commutativity of L_H with $\{L_F\}$ for $H \in \mathcal{H}_Y^{\mathbf{H}}$ is clear because $\{F\}$ are central in $\mathcal{H}_Y$. □

PROPOSITION 3.4.

Given $b \in B_+$, let $P_b = \sum_{w\in W/W_b} Y_{w(b)}$, where W_b is the stabilizer of b in W. Then

$$N_b \stackrel{def}{=} L_{P_b} = [N_b] + \sum_{b_*} g_{b_*}\, b'_*, \quad \text{where } b_* \in \sigma_0^\vee(b),$$

$$[N_b] = \sum_{w\in W/W_b} \prod_{\tilde\alpha\in\lambda(b)} \frac{q_{\tilde\alpha} X_{w(\tilde\alpha)} - q_{\tilde\alpha}^{-1}}{X_{w(\tilde\alpha)} - 1} \varphi_{w(-b')} w(-b)'. \tag{3.8}$$

If $r \in O^*$ then $\sigma_0^\vee(b) = \emptyset$ and $N_{b_r} = [N_{b_r}]$.

Proof. The term with $-b'$ in the operator $\hat P_b$ can come only from $\hat Y_{-b}$, which follows from (2.20) and (2.21). The $W_\#$-invariance of $N_b = (\hat P_b)_{red}$ gives that

$$[N_b] = \sum_{w\in W/W^b} w^\#(\phi_{(-b')}) w(-b)',$$

$$w^\#(\phi_{(-b')}) = \prod_{\tilde\alpha\in\lambda(b)} \frac{q_{\tilde\alpha} X_{w(\tilde\alpha)} - q_{\tilde\alpha}^{-1}}{X_{w(\tilde\alpha)} - 1} \varphi_{w(-b')}.$$

□

This theorem generalizes Theorem A.3. from [C4] (the construction of Macdonald's operators for A_n via affine Hecke algebras). The operators N_{b_r} coincide with the operators corresponding to (the minuscule wheights) $\{b_r\}$ from [M2] when ζ is the following character:

$$\sigma(T_j) = q_j,\ \sigma(\pi_r) = 1, \quad \text{where } 0 \le j \le n,\ r \in O. \tag{3.9}$$

The construction holds when the reduction procedure is defined for $\{\varphi_w w \in W\}$, multiplied by any cocycle on W with the values in the centralizer of $\zeta(\mathcal{H}_Y)$. It will be used in the next section.

Without going into detail we demonstrate some other properties of the operators under consideration. Let us introduce the *shift operator* by the formula $\mathcal{G} = \mathcal{X}^{-1}\mathcal{Y}$, where

$$\mathcal{X} = \prod_{\alpha\in R_+} (q_\alpha X_\alpha^{1/2} - q_\alpha^{-1} X_\alpha^{-1/2}),\ \mathcal{Y} = \prod_{\alpha\in R_+} (q_\alpha^{-1} Y_{\alpha^\vee}^{1/2} - q_\alpha Y_{\alpha^\vee}^{-1/2}).$$

There will be no $X^{1/2}, Y^{1/2}$ in the final formulas. Elements $\mathcal{X}$, $\mathcal{Y}$ belong to $\mathbf{C}[X]$, $\mathbf{C}[Y]$ respectively. The following proposition in the scalar case is from [C6].

PROPOSITION 3.5.
The operator $\hat{G} \stackrel{def}{=} \hat{\mathcal{G}}_{red}$ *preserves* $V[x]^{W_\#}$ *and is* $W_\#$*-invariant. Moreover,* $N_b(q\delta^{m/2})\ \hat{G}(q) \ = \ \hat{G}(q)\ N_b(q)$ *for* $b \in B$*, where we write* $N_b(q)$ *and so on to show the dependence on* $q = \{q_\nu\}$*.*

□

Let

$$\gamma \succeq \beta, \beta \preceq \gamma \ \text{ for } \ \beta, \gamma \in P \quad \text{if} \quad \gamma - \beta \in Q_+.$$

This ordering is dual to (1.16). The cone corresponding to $\beta \in P$ (the counterpart of $\sigma^\vee(b)$) will be denoted by $\sigma(\beta)$. The proof of the next statement repeats the proof of Proposition 3.6 from [C6].

PROPOSITION 3.6.
Operators $\{\hat{H}, H \in \mathcal{H}_Y\}$ *preserve the space* $\oplus_{\gamma\in\sigma(\beta)} V x_\gamma$ *for arbitrary* $\beta \in P$*.*

□

4. AQKZ and the isomorphism. Let us extend the action of $\mathbf{C}[X]$ and W^b (see (2.3), (2.4)) from $\mathbf{C}[x]$ to the algebra $\mathbf{C}\{x\}$ of meromorphic functions of $x_1, \ldots, x_n$. Let $\Psi \in (End_{\mathbf{C}}V)\{x\} \stackrel{def}{=} End_{\mathbf{C}}V \otimes \mathbf{C}\{x\}$ be a solution of the *affine quantum KZ equation (AQKZ)*:

$$(b')^\#(\Psi) \ = \ \Psi \ \text{ where } \ b \in B. \qquad (4.1)$$

This system of difference equations is self-consistent because $\{b\}$ are pairwise commutative. If V is finite dimensional and $|\delta| \neq 1$, one can follow [A] to check that it has an invertible solution (q is arbitrary). This solution is holomorphic where $x_\beta \neq \delta^k$ for all $\beta \in B, k \in \mathbf{Z}$ and unique up to B'-invariant $Aut_{\mathbf{C}}V$-valued functions of x as the right factors.

We will assume further that Ψ exists and is invertible. The equivalent statement is that the $\mathfrak{H}$-module $V\{x\}$ is isomorphic to the direct sum of the $\mathfrak{H}$-modules with trivial $\{\varphi_{\hat{w}}, \hat{w} \in W^b\}$ (i.e. coresponding to $\zeta = \sigma$ for the character from (3.9)). When Ψ satisfies (4.1) for all $\hat{w} \in W^b$ the equivalence is clear. Otherwise it is necessary to introduce the monodromy cocycle (see below) and to use the proper version of Hilbert Theorem 90 (see [C4], Corollary 3.3).

The *monodromy matrices* $\{C_{\hat{w}}\}$ and the corresponding actions of $\hat{w} \in W^b$ on $\hat{g} \in (End_{\mathbf{C}}V)\{x\}$ are as follows:

$$\hat{w}^*(\hat{g}) = \hat{w}(\hat{g})C_{\hat{w}}, \ \hat{w}^\flat(\hat{g}) = \hat{w}^\#(\hat{g})C_{\hat{w}} \ \text{ for } \ C_{\hat{w}} = \Psi^{-1}\hat{w}^\#(\Psi). \qquad (4.2)$$

The $\flat$-action can be uniquely determined from the relations

$$s_j^\flat = \varphi_{s_j} s_j^*,\ 0 \le j \le n,\ \ \pi_r^\flat = \zeta(Y_{b_r} T_{\omega_r}^{-1})\pi_r,\ r \in O,$$
$$\varphi_{s_j} = \frac{\zeta(T_j) + (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1}}{q_j + (q_j - q_j^{-1})(X_{\alpha_j} - 1)^{-1}},\ \ \hat{u}^\flat \hat{w}^\flat = (\hat{u}\hat{w})^\flat. \tag{4.3}$$

Actually the restriction of C to W is enough to know : $C_{b'w} = C_w$, where $C_{b'} = 1$. Moreover, $C_{uw} = C_u\ u(C_w)$ and $b'(C_w) = C_w$ for $u, w \in W, b \in B$ (see [C4], Theorem 3.2). The function Ψ is $\flat$-invariant with respect to the entire W^b.

Let us modify Theorem 2.3 to construct the following operators. Given a reduced decomposition $\hat{w} = s_{j_1} ... s_{j_l} \pi_r$,

$$\bar{\sigma}^*(T_{\hat{w}}) \stackrel{def}{=} \prod_{m=1}^{l} \left(q_{j_m}^{-1} s_{j_m}^* + \frac{q_{j_m}^{-1} - q_{j_m}}{X_{\alpha_{j_m}} - 1}(s_{j_m}^* - 1)\right) \pi_r^*. \tag{4.4}$$

They can be obtained for the character σ from (3.9) taken as ζ, after the substitution $s_j \to s_j^*, \pi_r \to \pi_r^*$, and $q \to q^{-1}$.

PROPOSITION 4.1. *Let* $\bar{\sigma}^*(Y_b) = \sum_{w \in W, c \in B} g_{c,w} c' w^*$ *for proper* $g_{c,w} \in \mathbf{C}(X)$. *Then*

$$\zeta(Y_{-b})\Psi = \bar{\sigma}^*(Y_b)(\Psi) = Red(\bar{\sigma}^*(Y_b))(\Psi) \ \ \text{where } b \in B, \tag{4.5}$$
$$Red(\sum_{w \in W, c \in B} g_{c,w} c' w^*) \stackrel{def}{=} \sum_{w \in W, c \in B} g_{c,w} c' \varphi_w^{-1}. \tag{4.6}$$

Proof. It suffices to check (4.6) for $b \in B_+$. If $b = s_{j_1} ... s_{j_l} \pi_r$ then $Y_{-b} = \pi_r^{-1} T_{j_l}^{-1} ... T_{j_1}^{-1}$. We can now use the relations

$$\zeta(T_j^{-1})\Psi = \left(q_j^{-1} s_j^* + \frac{q_j^{-1} - q_j}{X_{\alpha_j} - 1}(s_j^* - 1)\right)(\Psi), \tag{4.7}$$

that are equivalent to $s_j^\flat(\Psi) = \Psi$, and replace T_j^{-1} by $\bar{\sigma}^*(T_j)$ one after another. We may do this because the latter operators are scalar and commute with the action of $\zeta(\mathcal{H}_Y)$ on (the coefficients of) $V[x]$. The order of the indices becomes opposit after this procedure. As to $\zeta(\pi_r^{-1})$, it goes to π_r^*, since $\zeta(\pi_r)\pi_r^* = \pi_r^\flat$ (see (2.18), (4.2)). The reduction *Red* of $\bar{\sigma}^*(Y_b)$ is possible because $w^\flat(\Psi) = \Psi$. □

Let us fix a $\mathbf{H}$- module U and a $\mathbf{H}$-morphism $\tau : V \to U$. We denote the corresponding homomorphism $\mathbf{H} \to End_{\mathbf{C}} U$ alternately by ξ and $\tau\zeta$. Set

$$\bar{\sigma}^*(P_b) = \sum_{w \in W, c \in B} g_{c,w} c' w^*,\ b \in B_+,$$
$$M_b^* \stackrel{def}{=} Red_\tau(\bar{\sigma}^*(P_b)) \stackrel{def}{=} \sum_{w \in W, c \in B} g_{c,w} c' \tau(\varphi_w^{-1}). \tag{4.8}$$

The operation Red_τ eliminates the authomorphisms $\tau(\varphi_w)w^*$ on the right. We emphasize that operators $\bar{\sigma}^*(P_b)$ are scalar and $\{w^*\}$ act on $\{X_\beta\}$ naturally (as $\{w\}$ do). Hence we can omit $*$ in $\hat{\sigma}^*$ when applying Red and Red_τ. In particular, M_b (constructed for the standard action of W) coincide with M_b^*. Thus we deal with a certain direct generalization of (3.8) for scalar ζ. Let us reformulate Theorem 3.3 and Proposition 3.4 in this special case.

THEOREM 4.2.

i) The matrix difference operators $M_b, b \in B_+$, are pairwise commutative, W_ξ-invariant with respect to the action $\{w \to w_\xi \stackrel{def}{=} \tau(\varphi_w)w\}$, and preserve $U[x]^{W_\tau}$. Their leading terms are as follows:

$$M_b = \sum_{w\in W/W_b}\prod_{\tilde{\alpha}\in\lambda(b)} \frac{q_{\tilde{\alpha}}^{-1}X_{w(\tilde{\alpha})} - q_{\tilde{\alpha}}}{X_{w(\tilde{\alpha})} - 1} w_\xi(-b)' + \sum g_{b_*} b'_*, \qquad (4.9)$$

where $b_* \in \sigma_0^\vee(b)$, $g_{b_*} \in (End_{\mathbf{C}}U)(X)$, $w_\xi(b) = \tau(\varphi_w)w(b)\tau(\varphi_w)^{-1}$.

ii) Let Ψ be a solution of AQKZ from (4.1). Then $\psi = \tau(\Psi z)$ satisfies the relations

$$M_b(\psi) = \zeta(P_{-b})\psi \quad \text{for} \quad b \in B_+. \qquad (4.10)$$

where z belongs to the space $V\{x\}^{B'}$ of the V-valued functions that are B'-periodic with respect to the action from (2.4).

Proof. The reduction procedure Red_τ acts trivially on the left ideal in $End_{\mathbf{C}}V(x)$ generated by the elements $\{\bar{\sigma}(T_i) - \xi(T_i^{-1}), 1 \le i \le n\}$. The multiplication on the right by $\bar{\sigma}(P_b)$ preserves this ideal because $\bar{\sigma}(P_b)$ is scalar and P_b is $\mathbf{H}$-invariant. Then we may follow the proof of Theorem 3.3. Formula (4.9) is a straightforward version of (3.8).

To check the last statement, we substitute P_{-b} for Y_{-b} in (4.5), then place $\{\varphi_w w^*\}$ on the right in $\bar{\sigma}^*(P_b)$, erase them thanks to the $\flat$-invariance of Ψ, apply everything to z, and afterwards take τ. □

The main aplication of the theorem is when U co-induces V. To define the latter we will use the spaces $U^o = Hom_{\mathbf{C}}(U, \mathbf{C})$, $V^o = Hom_{\mathbf{C}}(V, \mathbf{C})$ equipped with the action

$$(T_{j_1}...T_{j_l}\pi_r(g))(z) \stackrel{def}{=} g(\pi_r^{-1}T_{j_l}...T_{j_1}(z)),\ 0 \le j \le n, r \in O,$$

of the corresponding Hecke algebra on linear functions $g(z)$ from either U^o or V^o.

Starting with a finite dimensional U and a homomorphism $\xi : \mathbf{H} \to End_{\mathbf{C}}U$, we introduce the space $U^o[y]$ for $\{y_\beta\}$ satisfying relations (2.3)-(2.4) with Y instead of X, and set

$$T_i^\vee = \xi(T_i)s_i + (q_i - q_i^{-1})(Y_{\alpha_i}^{-1} - 1)^{-1}(s_i - 1),\ 1 \le i \le n. \qquad (4.11)$$

These operators and $\{Y_b\}$ acting in $U^o[y]$ give the $\mathcal{H}_Y$-module isomorphic to the induced module generated by U^o (cf. (2.13), Theorem 2.3, and [C3]). We fix a set $\lambda = \{\lambda_1, ..., \lambda_n\} \in \mathbf{C}^*$ and consider the quotient $U^o_\lambda[y]$ of $U^o[y]$ by the (central) relations $P_b(y_1, ..., y_n) = P_b(\lambda_1, ..., \lambda_n)$ for all $b \in B$ in the setup of (3.8). Finally, $V \stackrel{def}{=} (U^o_\lambda[y])^o$ with the structure of a $\mathcal{H}_Y$-module as above. The dimension of V is $|W|\ \dim_{\mathbf{C}} U$.

This module has the natural projection $\tau : V \to U$ that is a **H**- homomorphism. The image of its arbitrary proper $\mathcal{H}_Y$-submodule V' ($\neq V$) with respect to τ is non-zero. Indeed, if $\tau(V') = 0$ then there exists a proper $\mathcal{H}_Y$-submodule in $U^o[y]$ containing U^o, which is impossible because U^o generates $U^o[y]$. There are connections of co-induced modules with induced ones and other related constructions which will not be discussed here (see [C5] for the scalar case).

THEOREM 4.3.

Let Ψ be the solution of AQKZ from (4.1). Then the map $\tau : (\Psi z) \to \psi = \tau(\Psi z)$ from Theorem 4.2 is an isomorphism of the space of the solutions $\{\Psi z\}$ of AQKZ in the above co-induced V and the space of solutions of the following U-valued system of difference equations:

$$M_b(\psi) \ = \ P_{-b}(\lambda_1, ..., \lambda_n)\psi \ \text{ for } \ b \in B_+. \tag{4.12}$$

Proof.† Formula (4.12) results from (4.10). If $\tau(\Psi z) = 0$ (identically) then it holds true for $Y_b \Psi z$ and $T_i \Psi z$ for any $b \in B$ and $1 \leq i \leq n$. The latter follows from the **H**-invariance of τ. As to Y_b, we can use (4.5) because $Red(\bar{\sigma}^*(Y_b))$ is a scalar difference operator preserving the (constant linear) relation $\tau(\Psi z) = 0$. We see that Ψz generates a $\mathcal{H}_Y$-submodule of V with zero projection onto U, which is impossible.

The dimension d of the space of solutions of (4.12) over $\mathbf{C}\{x\}^{B'}$ is not greater than $|W|\ \dim_{\mathbf{C}} U$. One can use (4.9) or the formulas $\chi(\bar{\sigma}^*(Y_b)) = \sigma(Y_{-b})b'$ to check this (here $-b$ appeared because we have to replace q by q^{-1}). We proved that τ is injective in the space of solutions of (4.1) in $V\{x\}$ (coinciding with the dimension of V). Hence $d = |W|\ \dim_{\mathbf{C}} U$ and we have the required isomorphism. □

Formula (4.9) gives explicit expressions for the operators $M_{b_r}, r \in O^*$ (coinciding with their leading terms). Let us put down the formulas for M_{b_i} in the case of A_2.

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†*Recently the author received the work by S.Kato "R matrix arising from Hecke algebras and its application to Macdonald's difference operators", containing a direct proof of a certain version of Theorem 3.4 from [C4] (see also [C2]) in the case of Macdonald's operators . In the above notations, he proved (4.12) for $\xi = \sigma$ and minuscule (and certain similar) wheights.*

Here $R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$, $\alpha_1 = 2\beta_1 - \beta_2$, $\alpha_2 = 2\beta_2 - \beta_1$, $s_i(\beta_i) = \beta_{3-i} - \beta_i$ (the same relations hold for $\{a_i, b_i\}$). One has: $X_{k_1\beta_1+k_2\beta_2} = X_1^{k_1}X_2^{k_2}$, $(-b')(X_\beta) = \delta^{2(b,\beta)}X_\beta$. Setting

$$f(\alpha) = (qX_\alpha - q^{-1})/(X_\alpha - 1),\ f^+(\alpha) = (qX_\alpha^{-1} - q^{-1})/(X_\alpha^{-1} - 1),\ q \in \mathbf{C}^*,$$
$$F_i(\alpha) = \frac{X_\alpha T_i - T_i^{-1}}{qX_\alpha - q^{-1}},\ F_i^+(\alpha) = \frac{\delta^2 X_\alpha^{-1} T_i - T_i^{-1}}{q\delta^2 X_\alpha^{-1} - q^{-1}}, \qquad (4.13)$$

we arrive at the following formula:

$$M_{b_1} = f^+(\alpha_1)f^+(\alpha_1+\alpha_2)(-b_1') + f(\alpha_1)f^+(\alpha_2)F_1(\alpha_1)F_1^+(\alpha_1)(b_1' - b_2') +$$
$$f(\alpha_1+\alpha_2)f(\alpha_2)F_2(\alpha_2)F_1(\alpha_1+\alpha_2)F_1^+(\alpha_1+\alpha_2)F_2^+(\alpha_2)(b_2'). \qquad (4.14)$$

To obtain M_{b_2} it is necessary to switch the indices 1 and 2. Here $\{T_i, i = 1, 2\}$ are the generators of $\mathbf{H}$ in an arbitrary representation.‡

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‡*This summer, the paper "Yang-Baxter equation in long range interacting systems" by D.Bernard, M.Gaudin, F.D.M.Haldane, and V.Pasquier was distributed, where (at the end) the authors applied the operation* Red_τ *to the operators* Y *of type* A_n *from the Appendix of [C4] (formula (A.5)). It was mention (without discussion) that the properties of the corresponding operators are analogous to those in the differential case. The explicit formulas for* M_{b_r} *(the hamiltonians) were not obtained (see ((4.9) and (4.14) above).*

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