Almost–Periodic Attractors for a Class of Nonautonomous Reaction–Diffusion Equations on $\mathbb{R}^N$

II. Codimension–One Stable Manifolds+

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Abstract. In this paper, we initiate an investigation of the stability properties of a one–parameter family $\{\hat{u}\}$ of spatially homogeneous, time almost–periodic classical solutions to a class of nonautonomous semilinear parabolic initial value problems with Neumann boundary conditions on bounded regions Ω of $\mathbb{R}^N$. In particular, for $p \in (N,\infty)$ and for every $\hat{u} \in \{\hat{u}\}$, we construct in the Sobolev space $H^{2,p}(\Omega,\mathbb{R})$ a codimension–one local stable manifold of classical solutions of small amplitude, which thereby all stabilize exponentially rapidly around $\hat{u}$. Our method of investigation exploits the Banach algebra structure of $H^{2,p}(\Omega,\mathbb{R})$, and mainly rests upon the construction of fixed point solutions to certain nonlinear integral equations in weighted Banach spaces of exponentially decaying $H^{2,p}(\Omega,\mathbb{R})$–valued maps. The class of equations which we analyze here contains in particular Fisher's type reaction–diffusion equations of population genetics. The results of this paper are thereby complementary to those of [14] and [15].

1. Introduction and Outline.

This is the second of a series of articles devoted to the analysis of stabilization phenomena for certain classical solutions to real semilinear parabolic Neumann boundary value problems of the form

$$\left\{\begin{array}{lll} u_t(x,t) = \Delta u(x,t)+s(t)g(u(x,t)), & (x,t)\in\Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \underset{\sim}{n}}(x,t) = 0 & , (x,t)\in\partial\Omega\times\mathbb{R}^+ \end{array}\right\} \quad (1.1)$$

([14]–[17]). In equations (1.1), Ω denotes an open bounded connected subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N \in [2,\infty) \cap \mathbb{N}^+$, while Δ stands for Laplace's operator in the x–variables. Furthermore, $s : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the restriction to $\mathbb{R}^+$ of a Bohr almost–periodic function on $\mathbb{R}$ which we shall also denote by s, while $g \in \mathscr{C}^{(1)}(\mathbb{R},\mathbb{R})$ possesses at least two zeroes u_0 and u_1 such that $g(u) > 0$ for every $u \in (u_0,u_1)$, with the property that $g'(u_0) > 0$ and $g'(u_1) < 0$. Finally, $\mathrm{Ran}(u)$ denotes the range of u and $\underset{\sim}{n}$ stands for the normalized outer normal vector to $\partial\Omega$. In [15], we proved that for every (suitably defined) classical solution $(x,t) \longrightarrow u(x,t)$ to problem (1.1), there exists a classical time almost–periodic solution $t \longrightarrow \hat{u}(t)$ to the initial value problem

$$\left\{\begin{array}{l} \hat{u}'(t) = s(t)g(\hat{u}(t)), \ t\in\mathbb{R} \\ \mathrm{Ran}(\hat{u}) \subseteq [u_0,u_1] \\ \hat{u}(0) = \hat{\nu} \in [u_0,u_1] \end{array}\right\} \quad (1.2)$$

such that $u(t) - \hat{u}(t) \longrightarrow 0$ as $t \longrightarrow \infty$, strongly in the Sobolev space $H^{2,p}(\Omega,\mathbb{R})$ for some $p \in (N,\infty)$, where $u(t)(x) = u(x,t)$ for every $x \in \overline{\Omega}$. In fact, we proved this result under the additional hypotheses

$$\lim_{u \to u_0} G(u) = -\infty \tag{1.3}$$

$$\lim_{u \to u_1} G(u) = +\infty \tag{1.4}$$

where G denotes any primitive of $1/g$ over the open interval (u_0,u_1), and under the condition that s be Hölder continuous on $\mathbb{R}^+$. Furthermore, we distinguished two very different cases; in the first one we assumed that the primitive of s is itself almost–periodic, in which case we proved that every attractor $\hat{u}$ is a classical almost–periodic solution to (1.2) of the form

$$\hat{u}(t) = G^{-1}\left\{\int_0^t d\xi s(\xi) + G(\hat{\nu})\right\} \tag{1.5}$$

where $\hat{\nu} \in (u_0,u_1)$, and where G^{-1} denotes the monotone inverse of G. In the second case, we assumed that the primitive of s is not almost–periodic and moreover that its time average $\mu_B(s)$ satisfies $\mu_B(s) < 0$ (resp. $\mu_B(s) > 0$). Under such conditions, we proved that the (global) attractor is given by the equilibrium solution $\hat{u} = u_0$ (resp. $\hat{u} = u_1$). With the exception of the case $\hat{u} = u_{0,1}$, those results of [15] thus left entirely open the question of the stability properties of the one–parameter family of functions $\{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$, since for a given $\hat{\nu} \in (u_0,u_1)$ it is a priori still conceivable that the corresponding solution $\hat{u}$ of (1.2) attracts no classical solution of (1.1).

In this paper, we initiate an investigation of the stability properties of the one–parameter family $\{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$. In Section 2, assuming that the primitive of s is also almost–periodic, our purpose is to prove that for $p \in (N,\infty)$, for every $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ and under further restrictions concerning the regularity of s and g , there exists in $H^{2,p}(\Omega,\mathbb{R})$ a smooth one–codimensional local stable manifold of classical solutions of small amplitude to Problem (1.1). We also prove that those solutions stabilize around $\hat{u}$ exponentially rapidly, with a rate determined by the largest negative eigenvalue of some appropriate realization of Laplace's operator in $H^{2,p}(\Omega,\mathbb{C})$. In order to accomplish this, we first analyze the initial–boundary value problem

$$\left\{\begin{array}{lll} u_t(x,t) = \Delta u(x,t)+s(t)g(u(x,t)) , & (x,t) \in \Omega\times\mathbb{R}^+ \\ u(x,0) = \hat{\mu}(x) \in (u_0,u_1) , & x \in \bar{\Omega} \\ \frac{\partial u}{\partial \underline{n}} (x,t) = 0 , & (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{1.6}$$

rather than (1.1). We then transform (1.6) into a suitable dynamical system on $H^{2,p}(\Omega,\mathbb{R})$, a space which becomes a commutative Banach algebra with respect to the usual pointwise operations and an appropriate norm [1]. We finally exploit the Banach algebra structure of $H^{2,p}(\Omega,\mathbb{R})$ to carry out the stable manifold construction without any growth conditions on g , upon invoking fixed point arguments in a weighted Banach space of exponentially decaying maps defined from $\mathbb{R}_0^+$ into $H^{2,p}(\Omega,\mathbb{R})$. By using some regularity arguments and the strong parabolic maximum principle, we can then easily prove that our construction provides the classical solutions to problem (1.1) that we seek. Methodologically, the techniques of this section are in fact complementary to those used in [7], [8] and [10]–[13] for the analysis of some hyperbolic problems. In Section 3, we assume that the primitive of s is not almost–periodic and moreover that is time average $\mu_B(s)$ satisfies $\mu_B(s) < 0$ (resp. $\mu_B(s) > 0$). Under further restrictions on s and g and using techniques similar to those of Section 2, we then prove that the two equilibria u_0 and u_1 are exponentially asymptotically stable with the decay rate $r_{u_0} = g'(u_0)\mu_B(s) < 0$ (resp. $r_{u_1} = g'(u_1)\mu_B(s) < 0$). In Section 4 we apply the results of Sections 2 and 3 to Fisher's type equations of population genetics. Finally, Section 5 is devoted to some remarks and to the discussion of some open problems while Appendices A, B and C are devoted to proving some more technical facts of the theory. For a short announcement of the above results, we refer the reader to [16].

At this point, it is worth observing that the stabilization processes discussed in this article have a very different physical origin in the case where s has an almost–periodic primitive, than they do have when s has a time average different from zero. In fact, our methods of proof will show that in the first case, the convergence of the solutions of small amplitude to the attractors $\{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ is mainly governed by the diffusion process. This is in sharp contrast to the second case, where the convergence to the equilibria u_0 and u_1 is governed by the reaction process. In the third and last part of this work [17], we shall complete our stability analysis of the attractors $\{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ through the construction of a one–parameter family of one–dimensional local center manifolds corresponding to the fact that zero is an eigenvalue of Laplace's operator in (1.1). The combination of that analysis with comparison arguments based on parabolic maximum principles will then show that the above remark is not merely limited to the particular solutions constructed in this article, but applies to the stabilization process of <u>every</u> classical solution to Problem (1.1). We still refer the reader to [15] for further references regarding the origins of the problems investigated here.

2. A One–Parameter Family of Codimension–One Stable Manifolds associated with Problem (1.1).

Let $\mathbb{R}_B$ be the Bohr compactification of the real line ([4] , [6] , [9]). Whenever convenient we shall identify a real Bohr almost–periodic function s with its unique uniformly continuous extension on $\mathbb{R}_B$, that is $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$; in this case we have

$$s(t) \sim \sum_{k \in \mathbb{N}^+} s_k \exp[i\Lambda_k t] \tag{2.1}$$

for every $t \in \mathbb{R}$, where

$$s_k = \mu_B(s\chi_k) = \lim_{\ell \to \infty} \ell^{-1} \int_0^\ell d\xi s(\xi)\chi_k(\xi) \tag{2.2}$$

and $t \longrightarrow \chi_k(t) = \exp[-i\Lambda_k t]$ for each k . In particular,

$$\mu_B(s) = \lim_{\ell \to \infty} \ell^{-1} \int_0^\ell d\xi s(\xi) \tag{2.3}$$

denotes the time average of s . It then follows from a classic criterion of Bohr that $t \longrightarrow \int_0^t d\xi s(\xi) \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$ if, and only if $t \longrightarrow \int_0^t d\xi s(\xi) = 0(1)$ as $|t| \longrightarrow \infty$ [4].

Under this condition and under the hypotheses concerning g described in the preceding section, we proved in [15] that Problem (1.2) possesses the one–parameter family of almost–periodic $\mathscr{C}^{(1)}$–solutions $\{\hat{u}\}_{\hat{\nu} \in (u_0,u_1)}$ given by relation (1.5); in addition, we proved that each one of those solutions remains uniformly bounded away from the equilibria u_0 and u_1, and that every Fourier exponent of $\hat{u}$ is a finite linear combination with integer coefficients of the Fourier exponents of s. In order to construct a local stable manifold for every $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$, we now proceed to define the notion of classical solution to Problems (1.1) and (1.6). Let $[N/2]$ be the integer part of $N/2$; throughout the remaining part of this paper, we shall assume that Ω has a $\mathscr{C}^{5+[N/2]}$–boundary in the sense of [1], in such a way that Ω lies only on one side of $\partial\Omega$, and that it satisfies the interior ball condition for every $x \in \partial\Omega$. We note that we have assumed the boundary $\partial\Omega$ to be more regular than in [15] (compare with the proof of Lemma 2.1 below). We shall also write $\bar{\Omega}$ for the compact closure of Ω, and denote by $\mathscr{C}^{2,1}(\Omega\times\mathbb{R}^+,\mathbb{R})$ the set consisting of all functions $z \in \mathscr{C}(\Omega\times\mathbb{R}^+,\mathbb{R})$ such that $(x,t) \longrightarrow \partial_t^\gamma D^\alpha z(x,t) \in \mathscr{C}(\Omega\times\mathbb{R}^+,\mathbb{R})$ for all $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{N}^N$, $\gamma \in \mathbb{N}$, satisfying $\sum_{j=1}^N \alpha_j + 2\gamma \leq 2$. In a similar way we define $\mathscr{C}^{1,0}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ as the set consisting of all $z \in \mathscr{C}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ with the property that $D^\alpha z \in \mathscr{C}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ for all $\alpha \in \mathbb{N}^N$ such that $\sum_{j=1}^N \alpha_j \leq 1$. Now fix $p \in (N,\infty)$; we then have the following

<u>Definition 2.1</u>. A function $u \in \mathscr{C}^{2,1}(\Omega\times\mathbb{R}^+,\mathbb{R}) \cap \mathscr{C}(\bar{\Omega}\times\mathbb{R}_0^+,\mathbb{R}) \cap \mathscr{C}^{1,0}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ is said to be a classical solution to Problem (1.1) (resp. (1.6)) if the following conditions are satisfied:

(C_1) There exists $\theta \in (0,1]$ and, for every $\tau \in \mathbb{R}^+$, a function $c \in L^p(\Omega,\mathbb{R})$ such that $|u(x,t)-u(x,t')| \leq c(x)|t-t'|^\theta$ for every $x \in \Omega$ and every $t,t' \in [\tau,\infty)$.

(C_2) $x \longrightarrow u(x,t) \in \mathscr{C}^{(2)}(\bar{\Omega},\mathbb{R})$ for every $t \in \mathbb{R}^+$.

(C_3) $(x,t) \longrightarrow u_t(x,t) \in \mathscr{C}(\bar{\Omega}\times\mathbb{R}^+,\mathbb{R})$ and in fact $t \longrightarrow u_t(x,t) \in \mathscr{C}(\mathbb{R}^+,\mathbb{R})$ uniformly in $x \in \bar{\Omega}$.

(C_4) u satisfies relations (1.1) (resp. 1.6) identically.

It is then clear that every classical solution to Problem (1.1) is also a classical solution to Problem (1.6), and that conversely every classical solution to (1.6) is a classical solution to (1.1) by the strong parabolic maximum principle. Now let u be any classical solution to problem (1.6), pick $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ and define $y(x,t) = u(x,t) - \hat{u}(t)$ for every $(x,t) \in \bar{\Omega}\times\mathbb{R}_0^+$. Upon using relations (1.2) and (1.6) we then conclude that y is a classical solution to the initial–boundary value problem

$$\left\{\begin{array}{lll} y_t(x,t) = \Delta y(x,t)+s(t)g'(\hat{u}(t))y(x,t)+s(t)g_{\hat{u}}(t,y(x,t)), & (x,t)\in\Omega\times\mathbb{R}^+ \\ y(x,0) = \hat{\mu}(x) - \hat{\nu} & ,x\in\bar{\Omega} \\ \frac{\partial y}{\partial n}(x,t) = 0 & ,(x,t)\in\partial\Omega\times\mathbb{R}^+ \end{array}\right\} \tag{2.4}$$

where

$$g_{\hat{u}}(t,y(x,t)) = g(\hat{u}(t)+y(x,t))-g(\hat{u}(t))-g'(\hat{u}(t))y(x,t)$$

Clearly, the stability analysis of $\hat{u}$ is then reduced to the stability analysis of the trivial solution of the partial differential equation in (2.4). Let $H^{2,p}(\mathbb{C}) = H^{2,p}(\Omega,\mathbb{C})$ be the usual Sobolev space consisting of all complex L^p–functions z with L^p–distributional derivatives $D^\alpha z$ for $|\alpha| \in [0,2]$, equipped with the norm

$$z \longrightarrow \|z\|_{2,p} = \Big\{ \sum_{|\alpha|\in[0,2]} \|D^\alpha z\|_p^p \Big\}^{1/p} \tag{2.5}$$

where $\|\cdot\|_p$ denotes the usual L^p–norm. For $\beta \in (0,1-p^{-1}N]$, let

$\mathscr{C}^{1,\beta}(\mathbb{C}) = \mathscr{C}^{1,\beta}(\bar{\Omega},\mathbb{C})$ be the Banach space of all complex Hölder continuous functions on $\bar{\Omega}$ with Hölderian derivatives $D^\alpha z$ of exponent β for $|\alpha| \in [0,1]$ and the norm

$$\|z\|_{1,\beta} = \|z\|_{1,\infty} + \max_{|\alpha| \in [0,1]} \sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |D^\alpha z(x) - D^\alpha z(y)|$$

$$= \max_{|\alpha| \in [0,1]} \sup_{x\in\bar{\Omega}} |D^\alpha z(x)| + \max_{|\alpha| \in [0,1]} \sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |D^\alpha z(x) - D^\alpha z(y)| \qquad (2.6)$$

Recall that there exists the continuous embedding

$$H^{2,p}(\mathbb{C}) \longrightarrow \mathscr{C}^{1,\beta}(\mathbb{C}) \qquad (2.7)$$

and that $H^{2,p}(\mathbb{C})$ is a commutative Banach algebra with respect to the usual pointwise operations and a norm equivalent to (2.5) [1]. We denote by $\Delta_{p,\mathscr{N}}$ the $L^p(\mathbb{C})$–realization of Laplace's operator on the domain $Dom(\Delta_{p,\mathscr{N}}) = H^{2,p}_{\mathscr{N}}(\mathbb{C})$, where

$$H^{2,p}_{\mathscr{N}}(\mathbb{C}) = \left\{ z \in H^{2,p}(\mathbb{C}) : \frac{\partial z}{\partial \underline{n}}(x) = 0 \,,\; x \in \partial\Omega \right\} \qquad (2.8)$$

It follows from the standard methods of [5] that $\Delta_{p,\mathscr{N}}$ is the infinitesimal generator of a compact holomorphic contraction $\mathscr{C}^{(0)}$–semigroup on $L^p(\mathbb{C})$; in addition, $\Delta_{p,\mathscr{N}}$ has a discrete point spectrum, namely $\sigma(\Delta_{p,\mathscr{N}}) = \{\lambda_k\}_{k\in N^+} \cup \{0\}$ where $\{\lambda_k\}_{k\in N^+} \subset \mathbb{R}^-$, the eigenvalues λ_k have finite multiplicities and $\lambda_k \longrightarrow -\infty$ as $k \longrightarrow \infty$ [15]. Let $\rho(\Delta_{p,\mathscr{N}})$ be the resolvent set of $\Delta_{p,\mathscr{N}}$ and fix $\lambda_0 \in \rho(\Delta_{p,\mathscr{N}}) \cap \mathbb{R}$; we first renorm $H^{2,p}_{\mathscr{N}}(\mathbb{C})$ with

$$z \longrightarrow \left\|z\right\|_{\lambda_0,2,p} = \left\|(\lambda_0-\Delta_{p,\mathcal{N}})z\right\|_p \tag{2.9}$$

It then follows from the closed graph theorem and standard elliptic theory that the norm (2.9) is equivalent to that defined by (2.5). According to the above remarks and without restricting the generality, we shall thus assume that $H^{2,p}_{\mathcal{N}}(\mathbb{C})$ is a Banach algebra with respect to the usual pointwise operations and the norm (2.9). This Banach algebra will henceforth be denoted by $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$. Our first preparatory result states the existence of a diffusion semigroup on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$ whose properties are identical to those of the semigroup generated by $\Delta_{p,\mathcal{N}}$.

<u>Lemma 2.1</u>. Let $\Delta_{\mathcal{N}}$ be the $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$–realization of Laplace's operator on the domain $\mathrm{Dom}(\Delta_{\mathcal{N}}) = \{z \in H^{4,p}_{\mathcal{N}}(\mathbb{C}) : \Delta_{p,\mathcal{N}} z \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})\}$. Then $\Delta_{\mathcal{N}}$ is the infinitesimal generator of a compact holomorphic contraction $\mathscr{C}^{(0)}$–semigroup on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$. In addition, $\Delta_{\mathcal{N}}$ has a discrete pure point spectrum, namely $\sigma(\Delta_{\mathcal{N}}) = \sigma_p(\Delta_{\mathcal{N}}) = \{\lambda_k\}_{k\in\mathbb{N}^+} \cup \{0\}$, where $\{\lambda_k\}_{k\in\mathbb{N}^+} \subset \mathbb{R}^-$, the λ_k's have finite multiplicities and $\lambda_k \longrightarrow -\infty$ as $k \longrightarrow \infty$.

<u>Proof</u>. Let $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ be the $\mathscr{C}^{(0)}$–semigroup generated by $\Delta_{p,\mathcal{N}}$ on $L^p(\mathbb{C})$; since $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is holomorphic, we have $W_{\Delta_{p,\mathcal{N}}}(t)(L^p(\mathbb{C})) \subseteq H^{2,p}_{\mathcal{N}}(\mathbb{C})$ for every $t \in \mathbb{R}^+$ by the smoothing property, so that $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ leaves $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$ globally invariant. Let $\{W(t)\}_{t\in\mathbb{R}_0^+}$ be the restriction of $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ to $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$. We claim that $\{W(t)\}_{t\in\mathbb{R}_0^+}$ is the desired semigroup with the infinitesimal

generator $\Delta_{\mathscr{N}}$; it is indeed easily verified that $\{W(t)\}_{t\in\mathbb{R}_0^+}$ is a $\mathscr{C}^{(0)}$–semigroup on $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{C})$. In addition

$$\left\|W(t)z\right\|_{\lambda_0,2,p} = \left\|(\lambda_0-\Delta_{p,\mathscr{N}})W_{\Delta_{p,\mathscr{N}}}(t)z\right\|_p \leq$$

$$\leq \left\|(\lambda_0-\Delta_{p,\mathscr{N}})z\right\|_p = \left\|z\right\|_{\lambda_0,2,p}$$

for every $t \in \mathbb{R}_0^+$ and every $z \in H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{C})$, so that $\{W(t)\}_{t\in\mathbb{R}_0^+}$ is a contraction semigroup. Using similar graph–norm arguments we can prove that the compactness and the holomorphy of $\{W(t)\}_{t\in\mathbb{R}_0^+}$ follow from the corresponding properties of $\left\{W_{\Delta_{p,\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$. Now let $z \in \mathrm{Dom}(\Delta_{\mathscr{N}})$; then $\hat{z} = (\lambda_0-\Delta_{p,\mathscr{N}})z \in H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{C})$, so that we obtain

$$\left\|t^{-1}(W(t)z-z) - \Delta_{p,\mathscr{N}}z\right\|_{\lambda_0,2,p} =$$

$$= \left\|t^{-1}(W_{\Delta_{p,\mathscr{N}}}(t)\hat{z}-\hat{z}) - \Delta_{p,\mathscr{N}}\hat{z}\right\|_p \longrightarrow 0 \tag{2.10}$$

as $t \longrightarrow 0$. Hence $z \in \hat{D}$, the domain of the infinitesimal generator of $\{W(t)\}_{t\in\mathbb{R}_0^+}$.

Conversely let $z \in \hat{D}$, set $\hat{z} = (\lambda_0-\Delta_{p,\mathscr{N}})z$ and $\tilde{z} = \operatorname*{s-lim}_{t\to 0} t^{-1}(W(t)z-z)$ in $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{C})$; then

$$\left\|t^{-1}(W_{\Delta_{p,\mathcal{N}}}(t)\hat{z}-\hat{z})-(\lambda_0-\Delta_{p,\mathcal{N}})\tilde{z}\right\|_p =$$

$$= \left\|t^{-1}(W(t)z-z)-\tilde{z}\right\|_{\lambda_0,2,p} \longrightarrow 0 \qquad (2.11)$$

as $t \to 0$. Hence $\hat{z} \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$ so that $z \in H^{4,p}_{\mathcal{N}}(\mathbb{C})$ by elliptic regularity, which implies that $z \in \mathrm{Dom}(\Delta_{\mathcal{N}})$ since $\Delta_{p,\mathcal{N}}z = \lambda_0 z-\hat{z} \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$. In addition we have

$$\Delta_{p,\mathcal{N}}\hat{z} = (\lambda_0-\Delta_{p,\mathcal{N}})\Delta_{p,\mathcal{N}}z \qquad (2.12)$$

from the definition of $\hat{z}$, as well as

$$\Delta_{p,\mathcal{N}}\hat{z} = (\lambda_0-\Delta_{p,\mathcal{N}})\tilde{z} \qquad (2.13)$$

which follows from relation (2.11). Since $\lambda_0 \in \rho(\Delta_{p,\mathcal{N}})$, we infer from relations (2.12) and (2.13) that $\tilde{z} = \Delta_{p,\mathcal{N}}z$. We conclude from this that $\hat{D} = \mathrm{Dom}(\Delta_{\mathcal{N}})$, and that the infinitesimal generator of $\{W(t)\}_{t\in\mathbb{R}_0^+}$ is $\Delta_{p,\mathcal{N}}$ restricted to $\mathrm{Dom}(\Delta_{\mathcal{N}})$. Finally, let $\varphi \in \mathrm{Dom}(\Delta_{\mathcal{N}})$ be an eigenfunction of $\Delta_{\mathcal{N}}$ corresponding to the eigenvalue λ; then a fortiori $\varphi \in H^{2,2}_{\mathcal{N}}(\mathbb{C})$ and is trivially an eigenfunction of the corresponding $L^2(\mathbb{C})$–realization of the Laplacian associated with the same eigenvalue. Conversely, let $\varphi \in H^{2,2}_{\mathcal{N}}(\mathbb{C})$ be a (generalized) eigenfunction of the $L^2(\mathbb{C})$–realization of Laplace's operator. Since Ω has a $\mathscr{C}^{5+[N/2]}$–boundary, it follows from standard elliptic regularity theory that $\varphi \in H^{5+[N/2],2}_{\mathcal{N}}(\mathbb{C}) \longrightarrow \mathscr{C}^4(\overline{\Omega},\mathbb{C})$. Hence $\varphi \in H^{4,p}_{\mathcal{N}}(\mathbb{C})$ and $\Delta_{p,\mathcal{N}}\varphi = \lambda\varphi \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$, so that $\varphi \in \mathrm{Dom}(\Delta_{\mathcal{N}})$ and is an eigenfunction of $\Delta_{\mathcal{N}}$ associated with the same eigenvalue. In addition, the remaining spectral properties of $\Delta_{\mathcal{N}}$

are identical to those of $\Delta_{p,\mathcal{N}}$. ■

From now on we shall write $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ for the real component of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{C})$ and $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ for the restriction of the diffusion semigroup of Lemma 2.1 on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. Our next preparatory result states that $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ enjoys properties of exponential dichotomies on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ similar to those of $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ on $L^p(\mathbb{R})$. In fact, define the operators P and Q on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ by

$$P = I_{\lambda_0,2,p} - Q$$

$$Qz = |\Omega|^{-1} \int_{\Omega} dx z(x) \tag{2.14}$$

where $I_{\lambda_0,2,p}$ denotes the identity operator. It is then easily verified that P and Q are projection operators on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. We write $|||\cdot|||_{\infty,\lambda_0,2,p}$ for the usual operator norm on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. We then have the following counterpart of Proposition 2.3 of [15].

Lemma 2.2. The diffusion semigroup $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ leaves Ran P globally invariant; moreover, if λ_1 denotes the largest negative eigenvalue of $\Delta_{\mathcal{N}}$, there exists a positive constant c_1 depending on N, p, λ_1 and the geometry of Ω, such that the estimates

$$\left|\left|\left| W_{\Delta_{\mathcal{N}}}(t)P \right|\right|\right|_{\infty,\lambda_0,2,p} \leq c_1 \exp[\lambda_1 t]$$

and (2.15)

$$\left|\left|\left| \Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t)P \right|\right|\right|_{\infty,\lambda_0,2,p} \leq c_1 t^{-1} \exp[\lambda_1 t]$$

hold for every $t \in \mathbb{R}^+$. Finally, $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ leaves Ran Q pointwise invariant; that is,

$$W_{\Delta_{\mathcal{N}}}(t)z = z \tag{2.16}$$

for every $t \in \mathbb{R}_0^+$ and every $z \in \text{Ran } Q$.

Proof. We first show that relation (2.15) follows from relation (2.35) in Proposition 2.3 of [15] through an appropriate graph–norm argument. Since the first estimate (2.15) also holds for $t = 0$ we may assume that $t \in \mathbb{R}^+$ throughout; then from relation (2.35) of [15] we get

$$\left\| W_{\Delta_{\mathcal{N}}}(t)Pz \right\|_p \leq c_1 \exp[\lambda_1 t] \left\| z \right\|_p \tag{2.17}$$

for every $z \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, since on this space $\left\{W_{\Delta_{p,\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ and $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ coincide. We further notice that on $\text{Dom}(\Delta_{\mathcal{N}}) \cap H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, we have $(\lambda_0-\Delta_{\mathcal{N}})P = P(\lambda_0-\Delta_{\mathcal{N}})$, $\Delta_{p,\mathcal{N}} = \Delta_{\mathcal{N}}$ and $\Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t) = W_{\Delta_{\mathcal{N}}}(t)\Delta_{\mathcal{N}}$; moreover, $W_{\Delta_{\mathcal{N}}}(t)P$ leaves $\text{Dom}(\Delta_{\mathcal{N}}) \cap H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ globally invariant. Therefore, upon using relation (2.17), we obtain

$$\left\|W_{\Delta_{\mathscr{N}}}(t)Pz\right\|_{\lambda_0,2,p} = \left\|(\lambda_0-\Delta_{p,\mathscr{N}})W_{\Delta_{\mathscr{N}}}(t)Pz\right\|_p =$$

$$= \left\|W_{\Delta_{\mathscr{N}}}(t)P(\lambda_0-\Delta_{p,\mathscr{N}})z\right\|_p \leq$$

$$\leq c_1\exp[\lambda_1 t]\left\|(\lambda_0-\Delta_{p,\mathscr{N}})z\right\|_p = c_1\exp[\lambda_1 t]\left\|z\right\|_{\lambda_0,2,p} \tag{2.18}$$

for every $z \in \mathrm{Dom}(\Delta_{\mathscr{N}}) \cap H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. Inequality (2.15) then follows by extending the validity of (2.18) by a density argument to every $z \in H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. The proofs of the remaining statements of the lemma are identical to those of the corresponding statements in Proposition 2.3 of [15]. ■

In relation with our stability analysis of the trivial solution of equation (2.4), we now observe that the linearized part of (2.4) also contains an almost–periodic perturbation to Laplace's operator. Accordingly, we next investigate a related family of evolution operators on $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. Pick $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$; on $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$, define the two–parameter family of operators $\left\{U_{\hat{u}}(t,r)\right\}_{t\geq r\geq 0}$ by

$$U_{\hat{u}}(t,r) = \exp\left[\int_r^t d\eta s(\eta)g'(\hat{u}(\eta))\right]W_{\Delta_{\mathscr{N}}}(t-r) \tag{2.19}$$

Our next result states that in fact the family $\left\{U_{\hat{u}}(t,r)\right\}_{t\geq r\geq 0}$ enjoys the same exponential decay properties as the diffusion semigroup $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$.

Proposition 2.1. Let $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$ be such that $t \longrightarrow \int_0^t d\eta s(\eta) = 0(1)$ as $|t| \longrightarrow \infty$. Let $g \in \mathscr{C}^{(1)}(\mathbb{R},\mathbb{R})$ be such that there exists $u_{0,1} \in \mathbb{R}$ with $g(u_0) = g(u_1) = 0$ and $g(u) > 0$ for each $u \in (u_0,u_1)$, in such a way that $g'(u_0) > 0$ and $g'(u_1) < 0$. Let G be the primitive of $1/g$ over the open interval (u_0,u_1) and assume that it satisfies relations (1.3) and (1.4). Pick $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$. Then the two–parameter family (2.19) generates a compact family of evolution operators on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. Moreover, the following conclusions hold:

(A) The evolution system $\left\{U_{\hat{u}}(t,r)\right\}_{t\geq r\geq 0}$ decays exponentially on Ran P ; that is, there exists a constant $c_2 \in \mathbb{R}^+$ uniform in t and r such that the estimate

$$\left|\left|\left|U_{\hat{u}}(t,r)P\right|\right|\right|_{\infty,\lambda_0,2,p} \leq c_2\exp[\lambda_1(t-r)] \tag{2.20}$$

holds for every $t \geq r \geq 0$.

(B) The evolution system $\left\{U_{\hat{u}}(t,r)\right\}_{t\geq r\geq 0}$ remains uniformly bounded on Ran Q ; that is, there exists a constant $c_3 \in \mathbb{R}^+$ uniform in t and r such that

$$\left|\left|\left|U_{\hat{u}}(t,r)Q\right|\right|\right|_{\infty,\lambda_0,2,p} \leq c_3 \tag{2.21}$$

(C) The restriction of each one of the operators $\left\{U_{\hat{u}}(t,r)\right\}_{t\geq r\geq 0}$ to Ran Q is invertible.

Proof. The first part of the statement follows immediately from relation (2.19) and the compactness of $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$. As for the proof of estimate (2.20), it is sufficient to show that

$$(t,r) \longrightarrow \exp\left[\int_r^t d\eta s(\eta) g'(\hat{u}(\eta))\right]$$

is bounded since we already know that estimate (2.15) holds. We first notice that the differential equation in (1.2) implies the relation

$$\int_0^t d\eta s(\eta) g'(\hat{u}(\eta)) = \int_0^t d\eta \frac{d}{d\eta} \mathrm{Ln}(g(\hat{u}(\eta))) = \mathrm{Ln}\left[\frac{g(\hat{u}(t))}{g(\hat{\nu})}\right] \tag{2.22}$$

It then follows from (2.22), the almost–periodicity of $t \longrightarrow g(\hat{u}(t))$ and statement (C) of Proposition 2.1 of [15] that

$$(t,r) \longrightarrow \exp\left[\int_r^t d\eta s(\eta) g'(\hat{u}(\eta))\right] = \frac{g(\hat{u}(t))}{g(\hat{u}(r))} \tag{2.23}$$

remains uniformly bounded as the product of Bohr almost–periodic functions. This proves statement (A). The proof of statement (B) is then immediate since, by relation (2.16), we obtain

$$U_{\hat{u}}(t,r)Q = \exp\left[\int_r^t d\eta s(\eta) g'(\hat{u}(\eta))\right] Q \tag{2.24}$$

as an operator equality on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. As for the proof of conclusion (C), we note that the equality

$$U_{\hat{u}}(t,r)Q = \exp\left[\int_r^t d\eta s(\eta) g'(\hat{u}(\eta))\right] I_{\lambda_0,2,p} \tag{2.25}$$

holds on Ran Q . ■

Remark. We note that if we choose $y_0 \in \mathrm{Dom}(\Delta_{\mathcal{N}}) \cap H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}) \cap \mathrm{Ran}\, P$, then the $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ – valued function $y(t) = U_{\hat{u}}(t,0)y_0$ provides a classical solution (in the sense of the theory of evolution equations on Banach spaces) to the equation

$$y'(t) = (\Delta_{\mathcal{N}} + s(t)g'(\hat{u}(t)))y(t) \tag{2.26}$$

which decays exponentially rapidly as $t \longrightarrow \infty$. On the other hand, if $y_0 \in \mathrm{Ran}\, Q$ then $t \longrightarrow y(t)$ provides an almost–periodic classical solution to (2.26). This observation thus suggests that we identify Ran P with the codimension–one stable manifold associated with equation (2.26), and Ran Q with its one–dimensional center manifold. While nonlinear versions of Ran Q will be constructed in [17], our purpose in the remaining part of this section is to construct a local nonlinear version of Ran P associated with the initial value problem

$$\left\{\begin{array}{l} y'(t)=(\Delta_{\mathcal{N}}+ s(t)g'(\hat{u}(t)))y(t)+s(t)\hat{g}_{\hat{u}}(t,y(t)), t \in \mathbb{R}^+ \\ y(0) = \hat{\mu} - \hat{\nu} \end{array}\right\} \tag{2.27}$$

on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, corresponding to equation (2.4). In relation (2.27), we have $\hat{g}_{\hat{u}}(t,\cdot) : H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}) \longrightarrow H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, and this map will be properly defined and analyzed in Proposition 2.2 below. It is in the proof of Proposition 2.2 and in the related Appendix A that the Banach algebra structure of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ will be used for the first time in a crucial way. The precise result is the following

Proposition 2.2. Assume that s and g satisfy the hypotheses of Proposition 2.1 and pick $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$. Assume in addition that $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$; for $z \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ and $t \in \mathbb{R}$, define

$$\hat{g}_{\hat{u}}(t,z) = g\circ(\hat{u}(t)+z)-g\circ\hat{u}(t)-(g'\circ\hat{u}(t))z \tag{2.28}$$

Then $\hat{g}_{\hat{u}}(t,\cdot) \in \mathscr{C}^{(2)}(H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}), H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}))$ for every $t \in \mathbb{R}$. Moreover, for $j = 0,1,2$ there exist non decreasing mappings $\Phi^{(j)}_{\hat{u}} : \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+_0$ such that the following estimates hold uniformly in $t \in \mathbb{R}$ and for all $z,h,k \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$:

$$(E_1) \qquad \left\|\hat{g}_{\hat{u}}(t,z)\right\|_{\lambda_0,2,p} \leq \Phi^{(0)}\left[\left\|z\right\|_{\lambda_0,2,p}\right] \tag{2.29}$$

$$(E_2) \qquad \left\|D\hat{g}_{\hat{u}}(t,z)h\right\|_{\lambda_0,2,p} \leq \Phi^{(1)}\left[\left\|z\right\|_{\lambda_0,2,p}\right]\left\|h\right\|_{\lambda_0,2,p} \tag{2.30}$$

$$(E_3) \qquad \left\|D^2\hat{g}_{\hat{u}}(t,z)(h,k)\right\|_{\lambda_0,2,p} \leq \Phi^{(2)}\left[\left\|z\right\|_{\lambda_0,2,p}\right]\left\|h\right\|_{\lambda_0,2,p}\left\|k\right\|_{\lambda_0,2,p} \tag{2.31}$$

In the above expressions, $D\hat{g}_{\hat{u}}(t,\cdot)$ and $D^2\hat{g}_{\hat{u}}(t,\cdot)$ stand for the first and the second Fréchet derivative of $\hat{g}_{\hat{u}}(t,\cdot)$, respectively.

Proof. The fact that $z \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ implies $\hat{g}_{\hat{u}}(t,z) \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ for every $t \in \mathbb{R}$ follows immediately from relation (2.28), the embedding (2.7) and the smoothness of g. The proof of the fact that $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ implies $\hat{g}_{\hat{u}}(t,\cdot) \in \mathscr{C}^{(2)}(H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}),H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}))$ is given in Appendix A. There we establish the relations

$$D\hat{g}_{\hat{u}}(t,z)h = (g'\circ(\hat{u}(t)+z)-g'\circ\hat{u}(t))h \tag{2.32}$$

and

$$D^2\hat{g}_{\hat{u}}(t,z)(h,k) = g''\circ(\hat{u}(t)+z)hk \tag{2.33}$$

valid for every $z,h,k \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ in the sense of pointwise multiplication in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. It remains to prove estimate (E_1), (E_2) and (E_3). In order to establish (2.29), we have to estimate the L^p–norm of $(\lambda_0-\Delta_{p,\mathcal{N}})\hat{g}_{\hat{u}}(t,z)$. We first write

$$\begin{aligned}(\lambda_0-\Delta_{p,\mathcal{N}})\hat{g}_{\hat{u}}(t,z) &= \\ &= \lambda_0(g\circ(\hat{u}(t)+z)-g\circ\hat{u}(t)) - g'\circ\hat{u}(t)(\lambda_0-\Delta_{p,\mathcal{N}})z \\ &\quad -g'\circ(\hat{u}(t)+z)\Delta_{p,\mathcal{N}}z - g''\circ(\hat{u}(t)+z)\,|\nabla z|^2 = \qquad (2.34)\\ &= \lambda_0(g\circ(\hat{u}(t)+z) - g\circ\hat{u}(t) - g'\circ(\hat{u}(t)+z)z) - g'\circ\hat{u}(t)(\lambda_0-\Delta_{p,\mathcal{N}})z \\ &\quad + g'\circ(\hat{u}(t)+z)(\lambda_0-\Delta_{p,\mathcal{N}})z - g''\circ(\hat{u}(t)+z)\,|\nabla z|^2\end{aligned}$$

We then proceed to estimate the L^p–norm of each term in (2.34). In order to simplify the notation somewhat, we omit all of the irrelevant positive multiplicative constants in the formulae that follow; this includes in particular all of the embedding constants. With this in mind consider the first term in (2.34) and write momentarily $\tau(t,x) = \hat{u}(t) + z(x)$; then

$$|\tau(x,t)| \leq |\hat{u}(t)| + \|z\|_{1,\infty} \leq a_1 + \|z\|_{\lambda_0,2,p} \tag{2.35}$$

uniformly in t and x, because of the boundedness of $\hat{u}$ and embedding (2.7); in relation (2.35), a_1 denotes some positive constant. We then infer the estimate

$$\|g\circ(\hat{u}(t)+z)\|_p \leq \sup_{t\in\mathbb{R}} \max_{x\in\bar{\Omega}} |g\circ(\hat{u}(t)+z(x))| \leq$$

$$\leq \sup_{|\tau|\in[0,a_1+\|z\|_{\lambda_0,2,p}]} |g(\tau)| \tag{2.36}$$

This leads us to define $\psi_{1,\hat{u}} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ by $\psi_{1,\hat{u}}(\xi) = \sup_{|\tau|\in[0,a_1+\xi]} |g(\tau)|$; clearly, $\psi_{1,\hat{u}}$ is non decreasing and we have

$$\|g\circ(\hat{u}(t)+z)\|_p \leq \psi_{1,\hat{u}}\left(\|z\|_{\lambda_0,2,p}\right)$$

uniformly in $t \in \mathbb{R}$. The second term in (2.34) is handled in a similar way. As for the third one, we may write

$$\left\| g' \circ (\hat{u}(t)+z)z \right\|_p \leq \sup_{t\in\mathbb{R}} \max_{x\in\overline{\Omega}} |g' \circ (\hat{u}(t)+z(x))| \left\| z \right\|_p \leq$$

$$\leq \sup_{t\in\mathbb{R}} \max_{x\in\overline{\Omega}} |g' \circ (\hat{u}(t)+z(x))| \left\| z \right\|_{\lambda_0,2,p} \tag{2.37}$$

because of embedding (2.7); we then argue as above to conclude that the estimate

$$\left\| g' \circ (\hat{u}(t)+z)z \right\|_p \leq \left\| z \right\|_{\lambda_0,2,p} \phi_{2,\hat{u}} \left[\left\| z \right\|_{\lambda_0,2,p} \right] \tag{2.38}$$

holds with some non decreasing function $\phi_{2,\hat{u}} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ uniform in t. The remaining terms can be handled in a similar fashion, upon using the definition of $\left\| \cdot \right\|_{\lambda_0,2,p}$ and embedding (2.7). This proves estimate (E_1). As for the proof of inequalities (2.30) and (2.31), we start from relations (2.32) and (2.33). From the method of Appendix A we already know that $g' \circ (\hat{u}(t)+z) - g' \circ \hat{u}(t) \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ along with $g'' \circ (\hat{u}(t)+z)$. Invoking the Banach algebra properties of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, we then obtain from (2.32) and (2.33) the inequalities

$$\left\| D\hat{g}_{\hat{u}}(t,z)h \right\|_{\lambda_0,2,p} \leq \left\| g' \circ (\hat{u}(t)+z) - g' \circ \hat{u}(t) \right\|_{\lambda_0,2,p} \left\| h \right\|_{\lambda_0,2,p} \tag{2.39}$$

and

$$\left\| D^2\hat{g}_{\hat{u}}(t,z)(h,k) \right\|_{\lambda_0,2,p} \leq \left\| g'' \circ (\hat{u}(t)+z) \right\|_{\lambda_0,2,p} \left\| h \right\|_{\lambda_0,2,p} \left\| k \right\|_{\lambda_0,2,p} \tag{2.40}$$

In order to prove estimate (E_2) and (E_3), it is thus sufficient to show that the inequalities

$$\left\| g' \circ (\hat{u}(t)+z) - g' \circ \hat{u}(t) \right\|_{\lambda_0,2,p} \leq \Phi_{\hat{u}}^{(1)}\left[\left\| z \right\|_{\lambda_0,2,p}\right] \tag{2.41}$$

and

$$\left\| g'' \circ (\hat{u}(t)+z) \right\|_{\lambda_0,2,p} \leq \Phi_{\hat{u}}^{(2)}\left[\left\| z \right\|_{\lambda_0,2,p}\right] \tag{2.42}$$

hold for some nondecreasing functions $\Phi_{\hat{u}}^{(1,2)} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$. But this follows from considerations entirely similar to those entering the proof of estimate (E_1). ∎

In order to construct a codimension–one stable manifold associated with the trivial solution of equation (2.27), we now convert the initial value problem (2.27) into an integral equation on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. The interplay between the estimates of Proposition 2.2 and those of Proposition 2.1 is here essential. We begin with the following

Definition 2.2. As before let λ_1 be the largest negative eigenvalue of $\Delta_{\mathcal{N}}$; we denote by Y_{λ_1} the set of all continuous maps $y : \mathbb{R}_0^+ \longrightarrow H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ such that

$$\left\| y \right\|_{\lambda_1} = \sup_{t\in\mathbb{R}_0^+} \left\| y(t) \right\|_{\lambda_0,2,p} \exp\left[-\lambda_1 t\right] < \infty \tag{2.43}$$

It is clear that Y_{λ_1} becomes a real Banach space with respect to the usual pointwise operations and the weighted norm (2.43).

The conversion of equation (2.27) into an equivalent integral equation will be proved for decaying solutions of the following kind.

Definition 2.3. Let $y : \mathbb{R}_0^+ \longrightarrow H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$; we say that y is an exponentially decaying classical solution to equation (2.27) if the following three requirements are satisfied:

(R_1) $\quad y \in Y_{\lambda_1}$

(R_2) $\quad y \in \mathscr{C}(\mathbb{R}_0^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})) \cap \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$

(R_3) $\quad y(t) \in \mathrm{Dom}(\Delta_{\mathscr{N}}) \cap H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$ and satisfies equation (2.27) identically on $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$ for every $t \in \mathbb{R}^+$.

The basic conversion result is then the following

Proposition 2.3. Let s and g satisfy the hypotheses of Proposition 2.1. In addition, assume that $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ and that s be locally Hölder continuous on $\mathbb{R}^+$. Pick $\hat{u} \in \{\hat{u}\}_{\hat{\nu} \in (u_0,u_1)}$ and let $y \in Y_{\lambda_1}$ be such that $y \in \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$. Then y is an exponentially decaying classical solution to equation (2.27) if, and only if, the integral equation

$$y(t) = U_{\hat{u}}(t,0)Py(0) + \int_0^t d\xi U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y(\xi))$$

$$-\int_t^\infty d\xi U_{\hat{u}}(t,\xi)s(\xi)(I_{\lambda_0,2,p}-P)\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.44}$$

holds for every $t \in \mathbb{R}_0^+$. In the first two terms of relation (2.44), $\left\{U_{\hat{u}}(t,\xi)\right\}_{t \geq \xi \geq 0}$ is given by relation (2.19), while in the third one we have defined

$$U_{\hat{u}}(t,\xi)z = U_{\hat{u}}^{-1}(\xi,t)z = \exp\left[\int_{\xi}^{t} d\eta s(\eta)g'(\hat{u}(\eta))\right]z \tag{2.45}$$

for $\xi \geq t \geq 0$ and every $z \in \operatorname{Ran} Q$, according to Statement (C) of Proposition 2.1 and relation (2.25).

<u>Proof</u>. Let y be an exponentially decaying classical solution to equation (2.27) and write $y_P(t) = Py(t)$, $y_Q(t) = Qy(t)$ for each $t \in \mathbb{R}_0^+$. We then infer from (2.27) that the equations

$$y_P'(t) = (\Delta_{\mathcal{N}} + s(t)g'(\hat{u}(t)))y_P(t) + s(t)P\hat{g}_{\hat{u}}(t,y(t)) \tag{2.46}$$

and

$$y_Q'(t) = s(t)g'(\hat{u}(t))y_Q(t) + s(t)Q\hat{g}_{\hat{u}}(t,y(t)) \tag{2.47}$$

hold for each $t \in \mathbb{R}_0^+$, since P and Q are continuous operators on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ and since $\Delta_{\mathcal{N}}P = P\Delta_{\mathcal{N}}$, $\Delta_{\mathcal{N}}Q = Q\Delta_{\mathcal{N}} = 0$ on $\operatorname{Dom}(\Delta_{\mathcal{N}}) \cap H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. We now prove that equation (2.46) implies the relation

$$y_P(t) = U_{\hat{u}}(t,0)Py(0) + \int_0^t d\xi U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.48}$$

for every $t \in \mathbb{R}_0^+$, while equation (2.47) implies

$$y_Q(t) = -\int_t^\infty d\xi U_{\hat{u}}(t,\xi)s(\xi)(I_{\lambda_0,2,p}-P)\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.49}$$

with the absolute convergence of the integral in (2.49). This will prove relation (2.44) since $y(t) = y_P(t) + y_Q(t)$. In order to derive (2.48) from (2.46), it is sufficient to show that

$$\xi \longrightarrow U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y(\xi)) \in L^1((0,t),H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})) \tag{2.50}$$

for then (2.48) follows from a standard argument involving the variation of constants. But statement (2.50) is a simple consequence of inequalities (2.20) and (2.31), for

$$\begin{aligned} \xi \longrightarrow \left\| U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y(\xi))\right\|_{\lambda_0,2,p} &\le \\ \le c_1 \exp[\lambda_1(t-\xi)] \left\|s\right\|_{\infty,\mathbb{R}} \left\|\hat{g}_{\hat{u}}(\xi,y(\xi))\right\|_{\lambda_0,2,p} &\le \qquad (2.51) \\ \le c_1 \exp[\lambda_1(t-\xi)] \left\|s\right\|_{\infty,\mathbb{R}} \int_0^1 d\sigma(1-\sigma)\Phi^{(2)}\left[\left\|\sigma y(\xi)\right\|_{\lambda_0,2,p}\right] \left\|y(\xi)\right\|^2_{\lambda_0,2,p} \\ \le c_1 \exp[\lambda_1(t-\xi)] \left\|s\right\|_{\infty,\mathbb{R}} \Phi^{(2)}\left[\left\|y\right\|_{\lambda_1} \exp[\lambda_1\xi]\right] \left\|y\right\|^2_{\lambda_1} \exp[2\lambda_1\xi] \\ \le c_1 \exp[\lambda_1 t] \left\|s\right\|_{\infty,\mathbb{R}} \Phi^{(2)}\left[\left\|y\right\|_{\lambda_1}\right] \left\|y\right\|^2_{\lambda_1} \exp[\lambda_1\xi] \in L^1((0,t),\mathbb{R}) \end{aligned}$$

In order to establish relation (2.51), we have successively used relation (2.20), the second–order Taylor expansion

$$\hat{g}_{\hat{u}}(\xi,y(\xi)) = \int_0^1 ds(1-s)D^2\hat{g}_{\hat{u}}(\xi,sy(\xi))y^2(\xi) \tag{2.52}$$

around the origin of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, estimate (2.31) and relation (2.43) through the monotonicity properties of $\Phi^{(2)}$ and the notation $\|s\|_{\infty,\mathbb{R}} = \sup_{\xi\in\mathbb{R}} |s(\xi)|$. Hence relation (2.48) holds. We now prove that (2.47) implies (2.49). To this end, define

$$z(t) = U_{\hat{u}}(0,t)y_Q(t) = \exp\left[-\int_0^t d\eta s(\eta)g'(\hat{u}(\eta))\right]y_Q(t) \tag{2.53}$$

for every $t \in \mathbb{R}_0^+$ according to definition (2.45). We then obtain $y_Q(t) = U_{\hat{u}}(t,0)z(t)$, from which we infer that the relation

$$y_Q'(t) = U_{\hat{u}}(t,0)z'(t) + s(t)g'(\hat{u}(t))U_{\hat{u}}(t,0)z(t) \tag{2.54}$$

holds for each $t \in \mathbb{R}_0^+$. Comparing equation (2.54) with (2.47), solving for $z'(t)$ and integrating over $[t,\hat{t}]$ for some fixed $\hat{t} \in (t,\infty)$, we obtain

$$U_{\hat{u}}(t,0)z(\hat{t}) = y_Q(t) + \int_t^{\hat{t}} d\xi U_{\hat{u}}(t,\xi)s(\xi)Q\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.55}$$

where $\left\{U_{\hat{u}}(t,\xi)\right\}_{\xi\geq t\geq 0}$ is given by relation (2.45). For any fixed $t \in \mathbb{R}_0^+$, we now can prove that

$$\xi \longrightarrow \left\| U_{\hat{u}}(t,\xi)s(\xi)Q\hat{g}_{\hat{u}}(\xi,y(\xi))\right\|_{\lambda_0,2,p} \in L^1((t,\infty),\mathbb{R}_0^+)$$

upon invoking the boundedness of $(t,\xi) \longrightarrow U_{\hat{u}}(t,\xi)$ on Ran Q and an argument similar to that leading to estimate (2.51). This and classic results of integration theory now imply the absolute convergence of the integral in relation (2.55), with

$$U_{\hat{u}}(t,0)z(\infty) = y_Q(t) + \int_t^\infty d\xi U_{\hat{u}}(t,\xi)s(\xi)Q\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.56}$$

and $z(\infty) = s\text{-}\lim_{t\to\infty} z(t)$. In order to show that (2.49) holds, it remains to prove that $z(\infty) = 0$. But this is immediate, for

$$\left\|z(t)\right\|_{\lambda_0,2,p} \leq 0(1)\left|\left|\left|Q\right|\right|\right|_{\infty,\lambda_0,2,p}\left\|y\right\|_{\lambda_1}\exp[\lambda_1 t] \longrightarrow 0$$

as $t \longrightarrow \infty$. This proves the only if part of the proposition. Conversely, assume that relation (2.44) holds for every $t \in \mathbb{R}_0^+$ and define

$$y_1(t) = U_{\hat{u}}(t,0)Py(0) + \int_0^t d\xi U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.57}$$

$$y_2(t) = \int_t^\infty d\xi U_{\hat{u}}(t,\xi)s(\xi)Q\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.58}$$

Clearly, $t \longrightarrow U_{\hat{u}}(t,0)Py(0)$ is continuously differentiable on $\mathbb{R}^+$ and belongs to $Dom(\Delta_{\mathcal{N}}) \cap H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ for every $t \in \mathbb{R}^+$. The same property holds true for the second term in (2.57); in fact, invoking the remark immediately following the proof of Proposition A.1 in Appendix A, we have $\xi \longrightarrow \hat{g}_{\hat{u}}(\xi,y(\xi)) \in \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}))$; the continuous differentiability of the second term in (2.57) then follows from the local Hölder property of s, the fact that $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is a holomorphic semigroup and minor modifications of the standard arguments of [3] and [5]. Moreover, $y_1(t) \in Dom(\Delta_{\mathcal{N}}) \cap H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ and

$$y_1'(t) = (\Delta_{\mathcal{N}} + s(t)g'(\hat{u}(t)))y_1(t) + s(t)P\hat{g}_{\hat{u}}(t,y(t)) \tag{2.59}$$

for every $t \in \mathbb{R}^+$. Since $y \in \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R}))$ by hypothesis, it follows in the same way that $t \longrightarrow y_2(t) = y_1(t) - y(t)$ is continuously differentiable on $\mathbb{R}^+$ with $y_2(t) \in Dom(\Delta_{\mathcal{N}}) \cap H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ and

$$y_2'(t) = (\Delta_{\mathcal{N}} + s(t)g'(\hat{u}(t)))y_2(t) - s(t)Q\hat{g}_{\hat{u}}(t,y(t)) \tag{2.60}$$

for every $t \in \mathbb{R}^+$. Equation (2.27) is then obtained upon subtracting relation (2.60) from relation (2.59). ∎

While it is conceivable to analyze relation (2.44) as an integral equation on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, it is now our intention to interpret it as a fixed point equation on Y_{λ_1}. To this end, define

$$F_{0,\hat{u}}(y)(t) = U_{\hat{u}}(t,0)Py(0) \tag{2.61}$$

$$F_{1,\hat{u}}(y)(t) = \int_0^t d\xi U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.62}$$

$$F_{2,\hat{u}}(y)(t) = -\int_t^\infty d\xi U_{\hat{u}}(t,\xi)s(\xi)(I_{\lambda_0,2,p}-P)\hat{g}_{\hat{u}}(\xi,y(\xi)) \tag{2.63}$$

It follows from easy considerations that $t \longrightarrow F_{j,\hat{u}}(y)(t) \in \mathscr{C}(\mathbb{R}_0^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ for each $y \in Y_{\lambda_1}$. In addition, it follows from the arguments used in the first part of the proof of Proposition 2.3 that $F_{\hat{u},j}$ maps Y_{λ_1} into itself for every j (for instance, estimate (2.51) proves immediately that $y \in Y_{\lambda_1}$ implies that $F_{1,\hat{u}}(y) \in Y_{\lambda_1}$). Equation (2.44) may thus be read as the fixed point equation

$$y = \sum_{j=0}^{2} F_{j,\hat{u}}(y) \tag{2.64}$$

on Y_{λ_1}. With the results of Propositions (2.2), (2.3) and relation (2.64), the structure of our theory thus becomes identical to that developed in ([10]–[13]) for the analysis of some hyperbolic problems. We are thereby in a position to invoke the methods developed in those articles to solve equation (2.64) in small balls of Y_{λ_1}. In this way, we get the following local stable manifold theorem for equation (2.27), which is the main result of this section.

<u>Theorem 2.1</u>. Let s and g satisfy all of the hypotheses of Proposition 2.3. Let $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ and choose $\rho_0 \in \mathbb{R}^+$ in such a way that

$I_{\hat{\nu}}(\rho_0) = \{\nu \in \mathbb{R} : |\nu-\hat{\nu}| < \rho_0\} \subsetneq (u_0,u_1)$. Let $c \in \mathbb{R}^+$ denote the embedding constant corresponding to the embedding (2.7). Then there exist constants $\hat{\epsilon}_0 \in (0,\infty)$, $\hat{k}_0 \in [1,\infty)$ and, for each $\epsilon \in (0,\hat{\epsilon}_0)$, an open spherical neighborhood $\mathscr{N}_{(2\hat{k}_0)^{-1}_\epsilon}$ of radius $(2\hat{k}_0)^{-1}_\epsilon$ centered at the origin of $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$, such that the following statements hold:

(A) For every $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}_\epsilon} \cap \operatorname{Ran} P$, there exists a unique $y_{\hat{u}}(\eta) \in \mathscr{N}_\epsilon$ such that $Py_{\hat{u}}(\eta) = \eta$, and a unique function
$t \longrightarrow y_{\hat{u}}(t,\eta) \in \mathscr{C}(\mathbb{R}^+_0,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})) \cap \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$
which provides a classical solution to the Cauchy problem

$$\left\{\begin{array}{l} y'(t)=(\Delta_{\mathscr{N}}+ s(t)g'(\hat{u}(t)))y(t)+s(t)\hat{g}_{\hat{u}}(t,y(t)),\ t \in \mathbb{R}^+ \\ y(0) = y_{\hat{u}}(\eta) \end{array}\right\} \tag{2.65}$$

Moreover, the inequality

$$\left\|y_{\hat{u}}(\eta)\right\|_{\lambda_0,2,p} < c^{-1}\rho_0 \tag{2.66}$$

holds.

(B) The exponential decay estimate

$$\left\|y_{\hat{u}}(t,\eta)\right\|_{\lambda_0,2,p} < \epsilon \exp[\lambda_1 t] \tag{2.67}$$

holds for every $t \in \mathbb{R}^+_0$.

(C) There exists a codimension–one $\mathscr{C}^{(1)}$–manifold $\mathscr{M}^{loc}_{s,\hat{\nu}} \subset H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$, tangent to Ran P at the origin, namely

$$\mathscr{M}^{loc}_{s,\hat{\nu}} = \left\{ y_{\hat{u}}(\eta)\ ,\ \eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}_{\epsilon}} \cap \operatorname{Ran} P \right\} \tag{2.68}$$

Proof. Upon using the result of Proposition 2.2, we first easily get for $\hat{g}_{\hat{u}}(t,\cdot)$ an estimate analogous to relation (3.27) in Lemma 3.1 of [13]. This combined with relations (2.61), (2.62), (2.63) and with inequalities (2.20), (2.21) of Proposition 2.2 then leads immediately to estimates for the $F_{j,\hat{u}}$'s which are identical to those of Proposition 3.3 of [13]. By a nearly verbatim adaptation of the proof of Theorem 3.1. of [13], we therefore conclude that there exists $\epsilon_0 \in (0,\infty)$, $\hat{k}_0 \in [1,\infty)$ and, for $\epsilon \in (0,\epsilon_0]$, an open spherical neighborhood $\mathscr{N}_{(2\hat{k}_0)^{-1}_{\epsilon}}$ of radius $(2\hat{k}_0)^{-1}_{\epsilon}$ centered at the origin of $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$, such that for every $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}_{\epsilon}} \cap \operatorname{Ran} P$ the nonlinear mapping

$$y \longrightarrow F_{\hat{u}}(y,\eta) = U_{\hat{u}}(t,0)\eta + \sum_{j=1}^{2} F_{j,\hat{u}}(y) \tag{2.69}$$

becomes a contraction in the ball

$$S_{\lambda_1}(\epsilon) = \left\{ y \in Y_{\lambda_1} : \left\|y\right\|_{\lambda_1} \leq \epsilon \right\} \tag{2.70}$$

Now define $\hat{\epsilon}_0 = \min(\epsilon_0, c^{-1}\rho_0)$, choose $\epsilon \in (0,\hat{\epsilon}_0)$ and let us carry out the above construction for such a restricted set of ϵ's. Then the mapping $F_{\hat{u}}(\cdot,\eta)$ defined by relation (2.69) possesses a unique fixed point $y_{\hat{u}}(\cdot,\eta) \in S_{\lambda_1}(\epsilon)$. In order to prove statement (A), we first notice that $t \longrightarrow y_{\hat{u}}(t,\eta) \in \mathscr{C}(\mathbb{R}^+_0, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ by definition of Y_{λ_1}; in addition, it follows from relation (2.69) that the equation

$$y_{\hat{u}}(t,\eta) = U_{\hat{u}}(t,0)\eta + \int_0^t d\xi U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta))$$
$$- \int_t^{\infty} d\xi U_{\hat{u}}(t,\xi)s(\xi)(I_{\lambda_0,2,p}-P)\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta)) \tag{2.71}$$

holds for every $t \in \mathbb{R}_0^+$. Set $y_{\hat{u}}(\eta) = y_{\hat{u}}(0,\eta)$; clearly $y_{\hat{u}}(\eta) \in \mathscr{N}_\epsilon$ and $Py_{\hat{u}}(\eta) = \eta$, the latter relation being a consequence of relation (2.71) with $t = 0$. Since it follows from Appendix B that $t \longrightarrow y_{\hat{u}}(t,\eta) \in \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$, we conclude that (2.65) holds as a consequence of relation (2.71) because of Proposition 2.3. Finally, relation (2.66) follows from the definition of $\hat{\epsilon}_0$ and the choice of ϵ's . This proves statement (A). Statement (B) follows immediately from Definition 2.2. The proof of Statement (C) follows from a direct adaptation of the proof of the corresponding statement in Theorem 3.2 of [8]. ∎

Remark. The necessity of having inequality (2.66) may at first look rather mysterious; in fact, its role is elucidated by translating the content of Theorem 2.1 back into the context of Problem (1.1) or (1.6). The precise result is the following statement which is a simple consequence of Theorem 2.1.

Corollary 2.1. Let s and g satisfy the same hypotheses as in Theorem 2.1. Fix $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ and let ρ_0, c, $\hat{\epsilon}_0$, $\hat{k}_0$ be as in Theorem 2.1; for $\epsilon \in (0,\hat{\epsilon}_0)$ and $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}_\epsilon} \cap \operatorname{Ran} P$, let $y_{\hat{u}}(\cdot,\eta)$ be the classical solution to problem (2.65) which satisfies estimates (2.66) and (2.67). On $\overline{\Omega} \times \mathbb{R}_0^+$, define the function $(x,t) \longrightarrow u(x,t,\eta) = y_{\hat{u}}(t,\eta)(x) + \hat{u}(t)$. Then the following conclusions hold:

(A) For every $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}_\epsilon} \cap \operatorname{Ran} P$, the function $u(\cdot,\cdot,\eta)$ is a classical solution to Problem (1.1) in the sense of Definition 2.1 for every $p \in (N,\infty)$. In addition, $x \longrightarrow u(x,t,\eta) \in \mathscr{C}^{3,\beta}(\overline{\Omega},\mathbb{R})$ for each $t \in \mathbb{R}^+$ and each $\beta \in (0,1-p^{-1}N]$. Finally, if $\eta_1 \neq \eta_2$, the function $u(\cdot,\cdot,\eta_1)$ is not identically equal to $u(\cdot,\cdot,\eta_2)$.

(B) There exist positive constants $c_{4,5}$ depending only on N, p and the geometry of Ω such that the following exponential decay estimates hold for every $t \in \mathbb{R}_0^+$ and every $\beta \in (0,1-p^{-1}N]$:

$$\sup_{x\in\overline{\Omega}} |u(x,t,\eta)-\hat{u}(t)| \le c_4\epsilon \exp[\lambda_1 t] \tag{2.72}$$

$$\sup_{x\in\overline{\Omega}} |\nabla u(x,t,\eta)| \le c_5\epsilon \exp[\lambda_1 t] \tag{2.73}$$

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |u(x,t,\eta)-u(y,t,\eta)| \le c_4\epsilon \exp[\lambda_1 t] \tag{2.74}$$

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |\nabla u(x,t,\eta)-\nabla u(y,t,\eta)| \le c_5\epsilon \exp[\lambda_1 t] \tag{2.75}$$

Remarks. (1) We first note that relations (2.72), (2.73), (2.74) and (2.75) immediately imply relations (2.10), (2.11), (2.12) and (2.13) of [15]. However, we shall prove in [17] that the converse statement is not true: in general, an arbitrary classical solution to problem (1.1) stabilizes only polynomially rapidly around $\hat{u}$; this is related to the fact that there exists a one–dimensional center manifold around $\hat{u}$ since zero is an eigenvalue of $\Delta_{\mathscr{N}}$. Relations (2.72), (2.73), (2.74) and (2.75) thus only reflect a codimension–one exponential stability of $\hat{u}$.

(2) We stress the fact that the method of investigation of this paper is essentially different from the philosophy of [15]; in that paper, we started with any classical solution u to problem (1.1) and proved that there exists a $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ such that relations (2.10), (2.11), (2.12) and (2.13) of [15] hold. In contrast, here we start with any $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ and prove that there exists a smooth codimension–one stable manifold of classical solutions to (1.1) which satisfy relations (2.72), (2.73), (2.74) and (2.75). This complementarity of the two approaches will be exploited in [17].

(3) Relations (2.72), (2.73), (2.74) and (2.75) show that the exponential stabilization of the solutions $u(\cdot,\cdot,\eta)$ around $\hat{u}$ is essentially governed by the diffusion process in (1.1) through the largest negative eigenvalue of $\Delta_{\mathscr{N}}$. This was not a priori obvious since in equation (2.65), Laplace's operator is perturbed by an almost–periodic function coming from the reaction term.

(4) From the definition of $(x,t) \longrightarrow u(x,t,\eta)$ in the preceding corollary, we note that the set of initial configurations for Problem (1.6) may be written as $u(\cdot,0,\eta) = y_{\hat{u}}(\eta) + \hat{\nu}$ where $\hat{\nu} \in (u_0,u_1)$. According to Statement (C) of Theorem 2.1, those configurations thus also generate a smooth codimension–one manifold in $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$ parametrized by $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}_{\epsilon}} \cap \text{Ran}\, P$. Since the construction of Theorem 2.1 can be repeated for each $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$, we obtain a one–parameter family $\left\{\mathscr{M}^{loc}_{s,\hat{\nu}}\right\}_{\hat{\nu}\in(u_0,u_1)}$ of such manifolds indexed by $\hat{\nu} \in (u_0,u_1)$.

We now can give the

Proof of Corollary 2.1. We start by proving that $u(\cdot,\cdot,\eta) \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}_0^+,\mathbb{R}) \cap \mathscr{C}^{(1)}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$; we first note that $y_{\hat{u}}(t_0,\eta)$ is uniformly continuous on $\overline{\Omega}$ for each fixed $t_0 \in \mathbb{R}_0^+$, because of embedding (2.7) and the compactness of $\overline{\Omega}$; in addition, the same embedding (2.7) implies that $t \longrightarrow y_{\hat{u}}(t,\eta)(x) \in \mathscr{C}(\mathbb{R}_0^+,\mathbb{R})$ uniformly in $x \in \overline{\Omega}$. These two properties combined prove the joint continuity $(x,t) \longrightarrow y_{\hat{u}}(t,\eta)(x) \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}_0^+,\mathbb{R})$ through the triangle inequality. In a similar way we have $(x,t) \longrightarrow (y_{\hat{u}})_t(t,\eta)(x) \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$, since $y'_{\hat{u}}(t_0,\eta)$ is also uniformly continuous on $\overline{\Omega}$ for each $t_0 \in \mathbb{R}^+$ by equation (2.65) and embedding (2.7), and since $t \longrightarrow y'_{\hat{u}}(t,\eta) \in \mathscr{C}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$. A similar argument holds to prove that $(x,t) \longrightarrow (y_{\hat{u}})_{x_j}(t,\eta)(x) \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$ for each $j \in \{1,\dots,N\}$. We conclude thereby that $u(\cdot,\cdot,\eta) = y_{\hat{u}}(\cdot,\eta)(\cdot) + \hat{u}(\cdot) \in \mathscr{C}(\overline{\Omega}\times\mathbb{R}_0^+,\mathbb{R}) \cap \mathscr{C}^{(1)}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$. In order to prove the required regularity of Definition 2.1, it remains to show that $u_{x_i,x_j}(\cdot,\cdot,\eta) \in \mathscr{C}(\Omega\times\mathbb{R}^+,\mathbb{R})$ for every $i,j \in \{1,\dots,N\}$. Since $y_{\hat{u}}(t,\eta) \in H^{4,p}_{\mathscr{N}}(\mathbb{R})$ for $t \in \mathbb{R}^+$ by construction, we already know that $u(\cdot,t,\eta) \in \mathscr{C}^{3,\beta}(\overline{\Omega},\mathbb{R}) \longrightarrow \mathscr{C}^2(\overline{\Omega},\mathbb{R})$ for every $t \in \mathbb{R}^+$ and every $\beta \in (0,1-p^{-1}N]$. The fact that $u_{x_i,x_j}(\cdot,\cdot,\eta) \in \mathscr{C}(\Omega\times\mathbb{R}^+,\mathbb{R})$ then follows from standard parabolic regularity theory ([2], [18]) and we conclude that $u(\cdot.\cdot,\eta) \in \mathscr{C}^{2,1}(\Omega\times\mathbb{R}^+,\mathbb{R}) \cap \mathscr{C}(\overline{\Omega}\times\mathbb{R}_0^+,\mathbb{R}) \cap \mathscr{C}^{1,0}(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$. It remains to prove that conditions (C_1)–(C_4) of Definition 2.1 hold. From Appendix C we know that $y_{\hat{u}}(\cdot,\eta)$ is globally Hölder continuous on every interval of the form $[\tau,\infty)$ and for every $\tau \in \mathbb{R}^+$. From the definition of $u(\cdot,\cdot,\eta)$, embedding (2.7) and the fact that $\hat{u}$ is Lipschitz continuous on $\mathbb{R}$, we infer the existence of a constant $c_6 \in \mathbb{R}^+$ and the existence of some $\theta \in (0,1)$ such that the estimate

$$
\begin{aligned}
|u(x,t,\eta)-u(x,t',\eta)| &\le |y_{\hat u}(t,\eta)(x)-y_{\hat u}(t',\eta)(x)| + |\hat u(t)-\hat u(t')| \\
&\le c\left\|y_{\hat u}(t,\eta)-y_{\hat u}(t',\eta)\right\|_{2,p} + |\hat u(t)-\hat u(t')| \le c_6|t-t'|^{\theta}
\end{aligned}
\tag{2.76}
$$

holds for every $x \in \bar\Omega$, every $t,t' \in [\tau,\infty)$ and every $\tau \in \mathbb{R}^+$. This proves that (C_1) holds. Condition (C_2) has already been proved; condition (C_3) follows easily from the fact that $t \longrightarrow y'_{\hat u}(t,\eta) \in \mathscr{C}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$. As for condition (C_4), we first observe that inequality (2.66) and embedding (2.7) imply that $y_{\hat u}(\eta)(x) + \hat\nu \in I_{\hat\nu}(\rho_0)$ for every $x \in \bar\Omega$; we then conclude from equation (2.65) that the function $(x,t) \longrightarrow u(x,t,\eta)$ satisfies the initial boundary value problem

$$
\begin{cases}
u_t(x,t) = \Delta u(x,t)+s(t)g(u(x,t)), & (x,t)\in\Omega\times\mathbb{R}^+ \\
u(x,0) = y_{\hat u}(\eta)(x)+\hat\nu \in (u_0,u_1), & x \in \bar\Omega \\
\frac{\partial u}{\partial \underline{n}}(x,t) = 0, & (x,t)\in\partial\Omega\times\mathbb{R}^+
\end{cases}
\tag{2.77}
$$

But since s is uniformly bounded in t and since g is smooth, it follows from the strong parabolic maximum principle that $u(x,t,\eta) \in (u_0,u_1)$ for every $(x,t) \in \bar\Omega\times\mathbb{R}^+_0$, and hence that $u(\cdot,\cdot,\eta)$ is a classical solution to Problem (1.1). The fact that $x \longrightarrow u(x,t,\eta) \in \mathscr{C}^{3,\beta}(\bar\Omega,\mathbb{R})$ for each $t \in \mathbb{R}^+$ and each $\beta \in (0,1-p^{-1}N]$ has already been proved. Finally, assume that $\eta_{1,2} \in \mathscr{N}_{(2\hat k_0)^{-1}\epsilon} \cap \operatorname{Ran} P$ with $\eta_1 \neq \eta_2$; if we had $u(x,t,\eta_1) = u(x,t,\eta_2)$ on $\bar\Omega \times \mathbb{R}^+_0$, then $y_{\hat u}(t,\eta_1) = y_{\hat u}(t,\eta_2)$ would hold for each $t \in \mathbb{R}^+_0$. In particular, this would imply that $y_{\hat u}(0,\eta_1) = y_{\hat u}(0,\eta_2)$, so that $Py_{\hat u}(0,\eta_1) = \eta_1 = Py_{\hat u}(0,\eta_2) = \eta_2$ would follow from Statement (A) of Theorem 2.1, thereby contradicting the hypotheses $\eta_1 \neq \eta_2$. This proves Statement (A). As for conclusion (B), it follows immediately from relation (2.67) and embedding (2.7). ■

Remarks. (1) Without any further conditions on η, the above method does not allow one to construct classical solutions to Problem (1.1) which satisfy condition (C_1) of Definition 2.1 with $\theta = 1$ (compare with the proof of Proposition C.1 of Appendix C). However, if $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}\epsilon} \cap \text{Ran } P \cap \text{Dom}(\Delta_{\mathscr{N}})$ and if s is globally Hölder continuous on $\mathbb{R}^+$, a much stronger result holds: the classical solutions of Corollary 2.1 satisfy

$$|u(x,t,\eta)-u(x,t',\eta)| \leq c|t-t'| \tag{2.78}$$

for every $x \in \bar{\Pi}$, for some $c \in \mathbb{R}^+$ and for every $t,t' \in \mathbb{R}^+$ (and not merely for $t,t' \in [\tau,\infty)$ for each $\tau \in \mathbb{R}^+$). This follows immediately from estimate (2.76) and Proposition C.2 of Appendix C. It is precisely the global stabilization properties of classical solutions satisfying the conditions of Definition 2.1 with (C_1) replaced by (2.78) which were discussed in [15].

(2) It is not possible to reiterate the above construction if $\hat{u} = u_{0,1}$. In fact, the classical solutions to Problem (1.1) remain uniformly bounded away from u_0 and u_1 when $t \longrightarrow \int_0^t d\eta s(\eta) = 0(1)$ as $|t| \longrightarrow \infty$ [15].

In the next section, we investigate the stability properties of the two equilibria u_0 and u_1 when $\mu_B(s) \neq 0$.

3. On the Exponential Stability of the Two Equilibria u_0 and u_1.

In this section we prove that under certain restrictions on the selection function s, there exist classical solutions of small amplitude to Problem (1.1) which converge exponentially rapidly to u_0 or u_1. We also show that the corresponding rates of decay are determined solely by $g'(u_0), g'(u_1)$ and $\mu_B(s)$, and thus that they do not depend on any spectral property of Laplace's operator. Throughout this section, we still assume that Ω and $\partial\Omega$ satisfy the same geometric conditions as before, and that the notion of classical solution is the same as in Section 2 . We begin with the description of the exponential dichotomies of the compact families of evolution operators $\left\{U_{u_{0,1}}(t,r)\right\}_{t\geq r\geq 0}$ defined by

$$U_{u_0}(t,r) = \exp\left[g'(u_0)\int_r^t d\eta s(\eta)\right] W_{\Delta_{\mathscr{N}}}(t-r) \tag{3.1}$$

and

$$U_{u_1}(t,r) = \exp\left[g'(u_1)\int_r^t d\eta s(\eta)\right] W_{\Delta_{\mathscr{N}}}(t-r) \tag{3.2}$$

In relations (3.1) and (3.2), $\left\{W_{\Delta_{\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is the diffusion semigroup of Lemma 2.1. We remark that expressions (3.1) and (3.2) correspond to relation (2.19) when $\hat{u} = u_{0,1}$.

Proposition 3.1. Let $s \in \mathscr{C}(\mathbb{R}_B,\mathbb{R})$ be such that $\mu_B(s) \neq 0$ and assume that $t \longrightarrow \int_0^t d\eta \hat{s}(\eta) = 0(1)$ as $|t| \longrightarrow \infty$, where we have defined $\hat{s} = s-\mu_B(s)$. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable at u_0 and u_1 in such a way that $g'(u_0) > 0$ and $g'(u_1) < 0$. Set $r_{u_0} = g'(u_0)\mu_B(s)$ and $r_{u_1} = g'(u_1)\mu_B(s)$. Then there exists $c_7 \in \mathbb{R}^+$ such that the following two conclusions hold:

(A) If $\mu_B(s) < 0$, then the estimate

$$\left|\left|\left|U_{u_0}(t,r)\right|\right|\right|_{\infty,\lambda_0,2,p} \leq c_7\exp\left[r_{u_0}(t-r)\right] \tag{3.3}$$

holds for every $t \geq r \geq 0$.

(B) If $\mu_B(s) > 0$, then the estimate

$$\left|\left|\left|U_{u_1}(t,r)\right|\right|\right|_{\infty,\lambda_0,2,p} \leq c_7 \exp\left[r_{u_1}(t-r)\right] \qquad (3.4)$$

holds for every $t \geq r \geq 0$.

Proof. Write $s = \mu_B(s) + \hat{s}$ in relation (3.1); relation (3.3) then follows from the facts that $\hat{s}$ has an almost–periodic primitive and that $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t \in \mathbb{R}_0^+}$ is a contraction semigroup on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. The proof of estimate (3.4) is of course similar. ∎

Remarks. (1) The hypothesis concerning $\hat{s}$ in Proposition 3.1 is satisfied whenever s is periodic, and for a wide class of almost–periodic functions such as for instance $s(t) = 1 + \cos(\omega_1 t) + \cos(\omega_2 t)$ where $\{\omega_1, \omega_2\} \subset \mathbb{R}/\{0\}$ is rationally independent. However, it fails to hold for instance for $s(t) = 1 + \sum_{k=1}^{\infty} k^{-2} \exp\left[ik^{-2}t\right]$, since the primitive of $t \longrightarrow \hat{s}(t) = \sum_{k=1}^{\infty} k^{-2} \exp\left[ik^{-2}t\right]$ is unbounded.

(2) In contrast to the estimates of Proposition 2.1, estimates (3.3) and (3.4) hold on the whole of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, irrespective of the fact that $0 \in \sigma_p(\Delta_{\mathcal{N}})$. In fact, the nature of the spectrum of $\Delta_{\mathcal{N}}$ plays no role in the considerations that follow.

In order to investigate the stability properties of u_0 and u_1, we may now proceed along the lines of Section 2; the relevant initial value problems on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ are then

$$\begin{Bmatrix} y'(t) = (\Delta_{\mathscr{N}} + s(t)g'(u_0))y(t) + s(t)\hat{g}_{u_0}(y(t)) \\ y(0) = \hat{\mu} - u_0 \end{Bmatrix} \tag{3.5}$$

and

$$\begin{Bmatrix} y'(t) = (\Delta_{\mathscr{N}} + s(t)g'(u_1))y(t) + s(t)\hat{g}_{u_1}(y(t)) \\ y(0) = \hat{\mu} - u_1 \end{Bmatrix} \tag{3.6}$$

In relations (3.5) and (3.6), we have defined

$$\hat{g}_{u_{0,1}}(z) = g \circ (u_{0,1} + z) - g'(u_{0,1})z \tag{3.7}$$

for every $z \in H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. Converting first equations (3.5) and (3.6) into appropriate integral equations when g is sufficiently smooth and using then fixed point arguments similar to those of Section 2, we obtain the following statement which is the main result of this section.

Theorem 3.1. Let s satisfy the hypotheses of Proposition 3.1. Assume in addition that s is locally Hölder continuous on $\mathbb{R}^+$. Let $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ be such that there exist $u_{0,1} \in \mathbb{R}$ with the property that $g(u_0) = g(u_1) = 0$, $g(u) > 0$ for every $u \in (u_0,u_1)$ and $g'(u_0) > 0$, $g'(u_1) < 0$. Let $c \in \mathbb{R}^+$ denote the embedding constant corresponding to the embedding (2.7). Then there exist constants $\hat{\epsilon}_1 \in (0,\infty)$, $\hat{k}_1 \in [1,\infty)$ and, for each $\epsilon \in (0,\hat{\epsilon}_1)$, an open spherical neighborhood $\mathscr{N}_{(2\hat{k}_1)^{-1}\epsilon}$ of radius $(2\hat{k}_1)^{-1}\epsilon$ centered at the origin of $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$, such that the following statements hold:

(A) If $\mu_B(s) < 0$, then for every $\eta \in \mathscr{N}^+_{(2\hat{k}_1)^{-1}\epsilon} = \left\{\eta \in \mathscr{N}_{(2\hat{k}_1)^{-1}\epsilon} : \eta > 0 \text{ on } \Pi\right\}$,

there exists a unique function

$t \longrightarrow y_{u_0}(t,\eta) \in \mathscr{C}(\mathbb{R}^+_0, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})) \cap \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$

which provides a classical solution to the Cauchy problem (3.5) with $y(0) = \eta$. Moreover, the exponential decay estimate

$$\left\|y_{u_0}(t,\eta)\right\|_{\lambda_0,2,p} \leq \epsilon \exp[r_{u_0} t] \tag{3.8}$$

holds for every $t \in \mathbb{R}^+_0$.

(B) If $\mu_B(s) > 0$, then for every $\eta \in \mathscr{N}^-_{(2\hat{k}_1)^{-1}\epsilon} = \left\{\eta \in \mathscr{N}_{(2\hat{k}_1)^{-1}\epsilon} : \eta < 0 \text{ on } \Pi\right\}$,

there exists a unique function

$t \longrightarrow y_{u_1}(t,\eta) \in \mathscr{C}(\mathbb{R}^+_0, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})) \cap \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$

which provides a classical solution to the Cauchy problem (3.6) with $y(0) = \eta$. Moreover, the exponential decay estimate

$$\left\|y_{u_1}(t,\eta)\right\|_{\lambda_0,2,p} \leq \epsilon \exp[r_{u_1} t] \tag{3.9}$$

holds for every $t \in \mathbb{R}^+_0$.

In addition, the inequality

$$\left\|\eta\right\|_{\lambda_0,2,p} < c^{-1}(u_1-u_0) \tag{3.10}$$

holds in both cases.

We omit the proof of Theorem 3.1 since it is essentially a repetition of the arguments of

Section 2 through Proposition 3.1. We simply observe that inequality (3.10) comes about in suitably restricting the set of admissible ϵ's , as we did in the proof of Theorem 2.1 to obtain inequality (2.66). We also emphasize the fact that for any $\eta \in \mathscr{N}^{\pm}_{(2\hat{k})^{-1}\epsilon}$, the corresponding initial condition is $y_{u_{0,1}}(0,\eta) = \eta$, and not a more complicated function of η as in Section 2. This is because of the fact that the exponential dichotomies of the evolution operators $\left\{U_{u_{0,1}}(t,r)\right\}_{t\geq r\geq 0}$ hold on the whole of $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$.

Remarks. (1) We can easily verify that the sets $\mathscr{N}^{\pm}_{(2\hat{k}_1)^{-1}\epsilon}$ are (non empty and) open in the $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$–topology. In order to see this let $\mathscr{C}(\mathbb{R})$ be the usual Banach space consisting of all real continuous functions on Π equipped with the uniform norm. Since there exists the continuous embedding $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}) \longrightarrow \mathscr{C}(\mathbb{R})$, and since $\mathscr{N}_{(2\hat{k}_1)^{-1}\epsilon}$ is an open ball in $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$, it is then sufficient to prove that the set of all positive (resp. negative) continuous functions on Π is open in $\mathscr{C}(\mathbb{R})$. This fact is easily verified for, if $f_0 \in \mathscr{C}(\mathbb{R})$ with $f_0 > 0$ on Π, then there exists $R_0 > 0$ such that $f_0 \geq R_0$ as a consequence of the compactness of Π . It is thus clear that for every $\epsilon \in (0,R_0)$, the open ball of radius ϵ centered at f_0 in the $\mathscr{C}(\mathbb{R})$–topology consists exclusively of positive functions on Π . The sets $\mathscr{N}^{\pm}_{(2\hat{k}_1)^{-1}\epsilon}$ thus provide $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$–smooth manifolds of small initial data associated with the exponentially decaying solutions $y_{u_0}(\cdot,\eta)$ and $y_{u_1}(\cdot,\eta)$.

(2) The conclusions of Statements (A) and (B) already hold if the subsets $\mathscr{N}^{\pm}_{(2\hat{k}_1)^{-1}\epsilon}$ are replaced by $\mathscr{N}_{(2\hat{k}_1)^{-1}\epsilon}$. In this case, it is the open ball $\mathscr{N}_{(2\hat{k}_1)^{-1}\epsilon}$ which provides a manifold of small initial data associated with the exponentially decaying

solutions $y_{u_0}(\cdot,\eta)$ and $y_{u_1}(\cdot,\eta)$. However, without the additional sign constraints of Statements (A) and (B) and without inequality (3.10), it is not possible to garantee that the functions $u(\cdot,\cdot,\eta) = y_{u_0}(\cdot,\eta) + u_0$ and $u(\cdot,\cdot,\eta) = y_{u_1}(\cdot,\eta) + u_1$ satisfy the range condition in (1.1). In fact, the role of these additional constraints is clarified in the following

Corollary 3.1. Let s and g satisfy the same hypotheses as in Theorem 3.1. Let c, $\hat{\epsilon}_1$, $\hat{k}_1$ be as in that theorem; for $\epsilon \in (0,\hat{\epsilon}_1)$ and $\eta \in \mathcal{N}^+_{(2\hat{k}_1)^{-1}\epsilon}$, let $y_{u_0}(\cdot,\eta)$ be the classical solution to problem (3.5) with $y(0,\eta) = \eta$ which satisfies estimates (3.8) and (3.10) when $\mu_B(s) < 0$. On $\overline{\Omega} \times \mathbb{R}^+_0$, define the function $(x,t) \longrightarrow u(x,t,\eta) = y_{u_0}(t,\eta)(x) + u_0$. Then the following conclusions hold:

(A) For every $\eta \in \mathcal{N}^+_{(2\hat{k}_1)^{-1}\epsilon}$, the function $u(\cdot,\cdot,\eta)$ is a classical solution to Problem (1.1) in the sense of Definition 2.1 for every $p \in (N,\infty)$. In addition, $x \longrightarrow u(x,t,\eta) \in C^{3,\beta}(\overline{\Omega},\mathbb{R})$ for each $t \in \mathbb{R}^+$ and each $\beta \in (0,1-\beta^{-1}N]$. Finally, if $\eta_1 \neq \eta_2$, the function $u(\cdot,\cdot,\eta_1)$ is not identically equal to $u(\cdot,\cdot,\eta_2)$.

(B) There exist positive constants $c_{8,9}$ depending only on N,p and the geometry of Ω such that the following exponential decay estimates hold for every $t \in \mathbb{R}^+_0$ and every $\beta \in (0,1-p^{-1}N]$:

$$\sup_{x\in\overline{\Omega}} |u(x,t,\eta)-u_0| \leq c_8\epsilon \exp[r_{u_0}t] \tag{3.11}$$

$$\sup_{x\in\overline{\Omega}} |\nabla u(x,t,\eta)| \leq c_9\epsilon \exp[r_{u_0}t] \tag{3.12}$$

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |u(x,t,\eta)-u(y,t,\eta)| \leq c_8 \epsilon \exp[r_{u_0} t] \tag{3.13}$$

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} |x-y|^{-\beta} |\nabla u(x,t,\eta)-\nabla u(y,t,\eta)| \leq c_9 \epsilon \exp[r_{u_0} t] \tag{3.14}$$

(C) Identical statements hold for the function $(x,t) \longrightarrow u(x,t,\eta) = y_{u_1}(t,\eta)(x) + u_1$ when $\mu_B(s) > 0$ and $\eta \in \bar{\mathcal{N}}_{(2\hat{k}_1)^{-1}\epsilon}$, with u_1 and r_{u_1} replacing u_0 and r_{u_0} in estimates (3.11)–(3.14).

Proof. The above statements follow from Theorem 3.1 exactly as Corollary 2.1 follows from Theorem 2.1. We simply note that because of inequality (3.10) and embedding (2.7) we get $|\eta(x)| < u_1 - u_0$ for every $x \in \bar{\Omega}$, which implies that $u(x,0,\eta) = \eta(x) + u_0 \in (u_0,u_1)$ if $\eta > 0$ on $\bar{\Omega}$. In a completely similar way $u(x,0,\eta) = \eta(x) + u_1 \in (u_0,u_1)$ if $\eta < 0$ on $\bar{\Omega}$, so that the range condition in (1.1) is satisfied in both cases by the strong parabolic maximum principle. ■

Remarks. (1) We first note that relations (3.11)–(3.14) imply relations (3.1) and (2.12)–(2.13) of [15]. We shall in fact prove in [17] that if s satisfies the condition of Theorem 3.1, and if s is globally Hölder continuous on $\mathbb{R}^+$, then every classical solution to (1.1) converges to u_0 or u_1 exponentially rapidly with a rate of decay determined by r_{u_0} or r_{u_1}.

(2) Relations (3.11)–(3.14) show that the exponential stabilization of the solutions $u(\cdot,\cdot,\eta)$ around u_0 or u_1 is essentially governed by the reaction–selection process in (1.1), in contrast to the results of Section 2.

(3) A remark similar to that immediately following the proof of Corollary 2.1 can be made concerning Condition (C_1) of Definition 2.1; in particular, if η is chosen sufficiently

regular and if s is globally Hölder continuous on $\mathbb{R}^+$, then the classical solutions of Corollary 3.1 are globally Lipschitz continuous in the time variable on $\mathbb{R}^+$.

In the next section, we discuss several examples.

4. The Role of Reaction–Diffusion Processes in Some Examples from Population Genetics.

It is instructive to reconsider some of the examples of Section 4 of [15] in light of the preceding results. We begin with the following

Example 4.1. Consider the problem

$$\left.\begin{cases} u_t(x,t)=\Delta u(x,t)+(\cos(\omega_1 t)+\cos(\omega_2 t))u(x,t)(1-u(x,t))(\alpha u(x,t)+(1-\alpha)(1-u(x,t)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subset (0,1) \\ \frac{\partial u}{\partial n}(x,t) = 0 \qquad\qquad\qquad\qquad (x,t) \in \partial\Omega\times\mathbb{R}^+ \end{cases}\right\} \quad (4.1)$$

where $\alpha \in (0,1)$ and where $\{\omega_1,\omega_2\} \subset \mathbb{R}/\{0\}$ is rationally independent. Here we have $g(u) = u(1-u)(\alpha u+(1-\alpha)(1-u))$ with $u_0 = 0$, $u_1 = 1$ and $s(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$. We can easily verify that all of the hypotheses of Theorem 2.1 or of Corollary 2.1 are satisfied. We can then conclude that every attractor $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(0,1)}$ is quasiperiodic. In addition, given any $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(0,1)}$, there exists a codimension–one manifold of classical solutions to (4.1) which stabilize around $\hat{u}$ in the sense of relations (2.72)–(2.75). In this example, the role of the diffusion process is thus predominant.

Example 4.2. Conclusions entirely similar to those of the preceding example hold for the boundary value problem

$$\left.\begin{cases} u_t(x,t) = \Delta u(x,t)+\sin(\omega t)\sin(\pi u(x,t)) & ,\ (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) & \\ \frac{\partial u}{\partial n}(x,t) = 0 & ,(x,t) \in \partial\Omega\times\mathbb{R}^+ \end{cases}\right\} \quad (4.2)$$

where $g(u) = \sin(\pi u)$, $u_0 = 0$ and $u_1 = 1$; here $s(t) = \sin(\omega t)$ with $\omega \in \mathbb{R}/\{0\}$, and all of the attractors are time periodic with period $\tau = 2\pi\left|\omega\right|^{-1}$.

We conclude with the following

Example 4.3. Consider the problem

$$\left.\begin{cases} u_t(x,t)=\Delta u(x,t)+(\cos(\omega_1 t)+\cos(\omega_2 t)\pm 1)u(x,t)(1-u(x,t))\exp[-u(x,t)] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (0,1) \\ \frac{\partial u}{\partial n}(x,t) = 0, \qquad\qquad\qquad\qquad\qquad\quad (x,t)\in \partial\Omega\times\mathbb{R}^+ \end{cases}\right\} \quad (4.3)$$

where $\{\omega_1,\omega_2\}$ is as in Example 4.1. Here $g(u) = u(1-u)\exp[-u]$ so that $u_0 = 0$ and $u_1 = 1$. Moreover, $s(t) = \cos(\omega_1 t) + \cos(\omega_2 t)\pm 1$, with $\mu_B(s) = \pm 1$. It is then easily checked that all of the hypotheses of Theorem 3.1 and of Corollary 3.1 are satisfied, so that Problem (4.3) possesses classical solutions converging to $u_0 = 0$ if $\mu_B(s) = -1$, and to $u_1 = 1$ if $\mu_B(s) = 1$. In the first case the rate of decay is $r_{u_0} = -1$, while in the second case we have $r_1 = -e^{-1}$. In both cases the reaction–selection process is primarily responsible for the stabilization phenomenon.

We refer the reader to the references of [15] for more information concerning the significance of Examples (4.1)–(4.3) in population genetics.

5. Concluding Remark and Formulation of an Open Problem.

In relation with the developments of the preceding sections, the major open problem concerns Neumann boundary value problems in which the selection function exhibits a spatial structure. Those boundary value problems are of the form

$$\left.\begin{cases} u_t(x,t)=\Delta u(x,t)+s(x,t)g(u(x,t)), & (x,t) \in \Omega\times\mathbb{R}^+ \\ \mathrm{Ran}(u) \subseteq (u_0,u_1) & \\ \frac{\partial u}{\partial \nu}(x,t) = 0 & ,(x,t) \in \partial\Omega\times\mathbb{R}^+ \end{cases}\right\} \tag{5.1}$$

where the selection function depends explicitly on $x \in \overline{\Omega}$ in such a way that $t \longrightarrow s(x,t)$ is Bohr almost–periodic for each $x \in \overline{\Omega}$, and that $x \longrightarrow s(x,t)$ is smooth on $\overline{\Omega}$ for every $t \in \mathbb{R}^+$. In (5.1) we assume that g satisfies the same hypotheses as in Section 2. Define $\bar{s}(t) = \max_{x\in\overline{\Omega}} s(x,t)$ and $\underline{s}(t) = \min_{x\in\overline{\Omega}} s(x,t)$. If $t \longrightarrow \int_0^t d\eta\bar{s}(\eta) = 0(1)$ as $|t| \longrightarrow \infty$, it is possible to show that every classical solution to (5.1) stabilizes around a spatially homogeneous, time almost–periodic solution to the initial value problem

$$\left.\begin{cases} \hat{u}'(t)=\bar{s}(t)\, g(\hat{u}(t)) \quad , & t \in \mathbb{R} \\ \mathrm{Ran}(u) \subseteq [u_0,u_1) & \\ \hat{u}(0) = \hat{\nu} \in [u_0,u_1) & \end{cases}\right\} \tag{5.2}$$

On the other hand, if $\mu_B(\bar{s}) < 0$ (resp. $\mu_B(\underline{s}) > 0$), then every classical solution to (5.1) converges to the equilibrium $u_0 = 0$ (resp. $u_1 = 1$). While these statements easily follow from the methods of [15], it is important to note that a local geometric theory similar to that developed in Sections 2 and 3 does not exist at the present time. It is thereby impossible to determine how fast the above stabilization processes develop, and to specify their physical nature. Due to the presence of a spatial structure in s, the physical origins of the stabilization phenomena for the solutions to (5.1) are presumably more complicated than just the combination of reaction–diffusion processes. In this context, the major open problem consists in developing an invariant manifold theory for nonautonomous parabolic problems such as (5.1).

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Appendix A. On the Fréchet Differentiability of $\hat{g}_{\hat{u}}(t,\cdot)$ on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$.

In this appendix we complete the proof of Proposition 2.2 by proving the following

Proposition A.1. Assume that s and g satisfy the hypotheses of Proposition 2.2. Pick $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ and define $\hat{g}_{\hat{u}}(t,\cdot)$ by relation (2.28) for every $t \in \mathbb{R}$. Then $\hat{g}_{\hat{u}}(t,\cdot) \in \mathscr{C}^{(2)}(H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}),H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$.

Proof. For simplicity we shall write $g^{(n)}$ for the n^{th}–derivative of g. From relation (2.28) we obtain

$$\hat{g}_{\hat{u}}(t,z+h) - \hat{g}_{\hat{u}}(t,z) =$$

$$= g \circ (\hat{u}(t)+z+h) - g \circ (\hat{u}(t)+z) - (g^{(1)} \circ \hat{u}(t))h \tag{A.1}$$

for every $h,z \in H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. We now prove that we in fact have

$$\hat{g}_{\hat{u}}(t,z+h)-\hat{g}_{\hat{u}}(t,z) =$$

$$(g^{(1)}\circ(\hat{u}(t)+z)-g^{(1)}\circ\hat{u}(t))h + h^2\int_0^1 ds(1-s)g^{(2)}\circ(\hat{u}(t)+z+sh) \tag{A.2}$$

on $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. From relation (A.1) and for each $x \in \Pi$, we obtain

$$\hat{g}_{\hat{u}}(t,z+h)(x) - \hat{g}_{\hat{u}}(t,z)(x) =$$

$$= g(\hat{u}(t)+z(x)+h(x))-g(\hat{u}(t)+z(x))-g'(\hat{u}(t))h(x) = \tag{A.3}$$

$$= (g^{(1)}(\hat{u}(t)+z(x))-g^{(1)}(\hat{u}(t)))h(x)+h^2(x)\int_0^1 ds(1-s)g^{(2)}(\hat{u}(t)+z(x)+sh(x)).$$

In order to derive relation (A.2) from relation (A.3), it is thus sufficient to prove that $g^{(1)}\circ(\hat{u}(t)+z)-g^{(1)}\circ\hat{u}(t) \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ along with $g^{(2)}\circ(\hat{u}(t)+z)$ whenever $z \in H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. It is clear that these two functions belong to $L^p(\mathbb{R})$ and satisfy Neumann's boundary condition. As for their partial derivatives, we obtain

$$(g^{(1)}\circ(\hat{u}(t)+z)-g^{(1)}\circ\hat{u}(t))_{x_j} = (g^{(2)}\circ(\hat{u}(t)+z))z_{x_j} \tag{A.4}$$

$$(g^{(1)}\circ(\hat{u}(t)+z)-g^{(1)}\circ\hat{u}(t))_{x_i,x_j} = (g^{(2)}\circ(\hat{u}(t)+z))z_{x_i,x_j}+(g^{(3)}\circ(\hat{u}(t)+z))z_{x_i}z_{x_j} \tag{A.5}$$

in the first case, and

$$(g^{(2)}\circ(\hat{u}(t)+z))_{x_j} = (g^{(3)}\circ(\hat{u}(t)+z))z_{x_j} \tag{A.6}$$

$$(g^{(2)}\circ(\hat{u}(t)+z))_{x_i,x_j} = (g^{(3)}\circ(\hat{u}(t)+z))z_{x_i,x_j}+(g^{(4)}\circ(\hat{u}(t)+z))z_{x_i}z_{x_j} \tag{A.7}$$

in the second case, for each $i,j \in \{1,\dots,N\}$. From the smoothness of g, the properties of $\hat{u}$ and embedding (2.7), we then infer that $(g^{(1)}\circ(\hat{u}(t)+z)-g^{(1)}\circ\hat{u}(t))_{x_j} \in L^p(\mathbb{R})$ along with the second–order derivatives given by (A.5). The conclusion for (A.6) and (A.7) is similar, so that relation (A.2) holds. Now, define $D\hat{g}_{\hat{u}}(t,z)$ by

$$D\hat{g}_{\hat{u}}(t,z)(h) = (g^{(1)}\circ(\hat{u}(t)+z)-g^{(1)}\circ\hat{u}(t))h \tag{A.8}$$

where pointwise multiplication in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ is meant on the right–hand side of (A.8). From the Banach algebra properties of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, it follows that $D\hat{g}_{\hat{u}}(t,z)$ is a linear bounded operator on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ for each z, with its uniform operator norm bounded above by

$$\left|\left|\left| D\hat{g}_{\hat{u}}(t,z) \right|\right|\right|_{\infty,\lambda_0,2,p} \le \left\| g^{(1)}\circ(\hat{u}(t)+z)-g^{(1)}\circ\hat{u}(t) \right\|_{\lambda_0,2,p} \tag{A.9}$$

We now wish to prove that $z \longrightarrow D\hat{g}_{\hat{u}}(t,z)$ is continuous on $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ for every $t \in \mathbb{R}$. Assume that $z_n \longrightarrow z$ strongly in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$; it is then sufficient to prove that $g^{(1)}\circ(\hat{u}(t)+z_n) \longrightarrow g^{(1)}\circ(\hat{u}(t)+z)$ strongly in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ according to relations (A.8) and (A.9). From the smoothness of g and embedding (2.7), it is already clear that $g^{(1)}\circ(\hat{u}(t)+z_n) \longrightarrow g^{(1)}\circ(\hat{u}(t)+z)$ strongly in $L^p(\mathbb{R})$. By the definition of the norm (2.9), it thus remains to prove that $\Delta_{p,\mathcal{N}}(g^{(1)}\circ(\hat{u}(t)+z_n)) \longrightarrow \Delta_{p,\mathcal{N}}(g^{(1)}\circ(\hat{u}(t)+z))$ strongly in $L^p(\mathbb{R})$. To this end, write momentarily $f_{\hat{u}}(t) = \hat{u}(t)+z$ and $f_{\hat{u},n}(t) = \hat{u}(t) + z_n$. We have the identity

$$\Delta_{p,\mathcal{N}}(g^{(1)}\circ(\hat{u}(t)+z_n)) - \Delta_{p,\mathcal{N}}(g^{(1)}\circ(\hat{u}(t)+z)) =$$

$$= \Delta_{p,\mathcal{N}}(g^{(1)}\circ f_{\hat{u},n}(t)) - \Delta_{p,\mathcal{N}}(g^{(1)}\circ f_{\hat{u}}(t)) = \tag{A.10}$$

$$= (g^{(2)}\circ f_{\hat{u},n}(t))(\Delta_{p,\mathcal{N}}f_{\hat{u},n}(t)-\Delta_{p,\mathcal{N}}f_{\hat{u}}(t)) + \Delta_{p,\mathcal{N}}f_{\hat{u}}(t)(g^{(2)}\circ f_{\hat{u},n}(t)-g^{(2)}\circ f_{\hat{u}}(t))$$

$$+(g^{(3)}\circ f_{\hat{u},n}(t))(\left|\nabla f_{\hat{u},n}(t)\right|^2-\left|\nabla f_{\hat{u}}(t)\right|^2) + \left|\nabla f_{\hat{u}}(t)\right|^2(g^{(3)}\circ f_{\hat{u},n}(t)-g^{(3)}\circ f_{\hat{u}}(t))$$

Since $f_{\hat{u},n}(t) \longrightarrow f_{\hat{u}}(t)$ strongly in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ for each $t \in \mathbb{R}$, it follows from the smoothness of g and embedding (2.7) that every term in (A.10) converges strongly to zero in $L^p(\mathbb{R})$. Thus $z \longrightarrow D\hat{g}_{\hat{u}}(t,z)$ is continuous. In order to conclude that $D\hat{g}_{\hat{u}}(t,\cdot)$ is the smooth Fréchet derivative of $\hat{g}_{\hat{u}}(t,\cdot)$, it remains to prove that

$$\|h\|^{-1}_{\lambda_0,2,p}\, h^2 \int_0^1 ds(1-s)g^{(2)}\circ(\hat{u}(t)+z+sh) \longrightarrow 0 \tag{A.11}$$

as $h \longrightarrow 0$ in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$. Write

$$r(z,h) = h^2 \int_0^1 ds(1-s)g^{(2)}\circ(\hat{u}(t)+z+sh) \tag{A.12}$$

By the Banach algebra properties of $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, we obtain the estimate

$$\|h\|^{-1}_{\lambda_{0,2,p}}\|r(z,h)\|_{\lambda_{0,2,p}} \le \|h\|_{\lambda_{0,2,p}}\int_0^1 ds\left\|g^{(2)}\circ(\hat{u}(t)+z+sh)\right\|_{\lambda_{0,2,p}} \tag{A.13}$$

In order to prove (A.11) from (A.13), it is thus sufficient to show that $h \longrightarrow \int_0^1 ds\left\|g^{(2)}\circ(\hat{u}(t)+z+sh)\right\|_{\lambda_0,2,p}$ remains bounded as $h \longrightarrow 0$. In fact we prove a stronger result, namely that

$$\lim_{h\to 0}\int_0^1 ds\left\|g^{(2)}\circ(\hat{u}(t)+z+sh)\right\|_{\lambda_0,2,p} = \left\|g^{(2)}\circ(\hat{u}(t)+z)\right\|_{\lambda_0,2,p} \tag{A.14}$$

This in turn follows from the fact that $g^{(2)}\circ(\hat{u}(t)+z+sh) \longrightarrow g^{(2)}\circ(\hat{u}(t)+z)$ strongly in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$, uniformly in $s \in (0,1)$ as $h \longrightarrow 0$. In order to see this, write $f_{\hat{u},sh}(t) = \hat{u}(t)+z+sh$ and $f_{\hat{u}}(t) = \hat{u}(t)+z$. Clearly, $f_{\hat{u},sh}(t) \longrightarrow f_{\hat{u}}(t)$ strongly in

$H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ uniformly in s as $h \longrightarrow 0$; since $g^{(n)}$ is continuous for $n = 2,3,4$, we infer from this that

$$g^{(2)} \circ f_{\hat{u},sh}(t) \longrightarrow g^{(2)} \circ f_{\hat{u}}(t) \tag{A.15}$$

$$g^{(3)} \circ f_{\hat{u},sh}(t) \longrightarrow g^{(3)} \circ f_{\hat{u}}(t) \tag{A.16}$$

$$g^{(4)} \circ f_{\hat{u},sh}(t) \longrightarrow g^{(4)} \circ f_{\hat{u}}(t) \tag{A.17}$$

uniformly on Π, uniformly in $s \in (0,1)$ as $h \longrightarrow 0$. It now follows from (A.15) that $\lambda_0 g^{(2)} \circ f_{\hat{u},sh}(t) \longrightarrow \lambda_0 g^{(2)} \circ f_{\hat{u}}(t)$ strongly in $L^p(\mathbb{R})$, with the same uniformity in s. Since

$$\left\| g^{(2)} \circ f_{\hat{u},sh}(t) - g^{(2)} \circ f_{\hat{u}}(t) \right\|_{\lambda_0,2,p} = \left\| (\lambda_0 - \Delta_{p,\mathcal{N}})(g^{(2)} \circ f_{\hat{u},sh}(t) - g^{(2)} \circ f_{\hat{u}}(t)) \right\|_p$$

it remains to prove that $\Delta_{p,\mathcal{N}}(g^{(2)} \circ f_{\hat{u},sh}(t)) \longrightarrow \Delta_{p,\mathcal{N}}(g^{(2)} \circ f_{\hat{u}}(t))$ strongly in $L^p(\mathbb{R})$ uniformly in $s \in (0,1)$ as $h \longrightarrow 0$. To this end we note as above that the relation

$$\Delta_{p,\mathcal{N}}(g^{(2)} \circ f_{\hat{u},sh}(t)) - \Delta_{p,\mathcal{N}}(g^{(2)} \circ f_{\hat{u}}(t)) = \tag{A.18}$$

$$= (g^{(3)} \circ f_{\hat{u},sh}(t))(\Delta_{p,\mathcal{N}} f_{\hat{u},sh}(t) - \Delta_{p,\mathcal{N}} f_{\hat{u}}(t)) + \Delta_{p,\mathcal{N}} f_{\hat{u}}(t)(g^{(3)} \circ f_{\hat{u},sh}(t) - g^{(3)} \circ f_{\hat{u}}(t))$$

$$+ (g^{(4)} \circ f_{\hat{u},sh}(t))\left(\left| \nabla f_{\hat{u},sh}(t) \right|^2 - \left| \nabla f_{\hat{u}}(t) \right|^2 \right) + \left| \nabla f_{\hat{u}}(t) \right|^2 (g^{(4)} \circ f_{\hat{u},sh}(t) - g^{(4)} \circ f_{\hat{u}}(t))$$

holds. Invoking relations (A.16), (A.17) and the strong convergence $f_{\hat{u},sh}(t) \longrightarrow f_{\hat{u}}(t)$ in $H^{2,p}_{\lambda_0,\mathcal{N}}(\mathbb{R})$ uniformly in s, we can now conclude that each term in relation (A.18) converges to zero strongly in $L^p(\mathbb{R})$ with the desired uniformity in $s \in (0,1)$. This proves relation (A.14), and hence that $D\hat{g}_{\hat{u}}(t,\cdot)$ is the smooth Fréchet derivative of $\hat{g}_{\hat{u}}(t,\cdot)$ for every $t \in \mathbb{R}$. While the above arguments require $g \in \mathscr{C}^{(4)}(\mathbb{R},\mathbb{R})$, we note that the hypothesis $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$ allows one to carry out the above steps once more to prove that

$$D^2\hat{g}_{\hat{u}}(t,z)(h,k) = g^{(2)}\circ(\hat{u}(t)+z)hk \tag{A.19}$$

is the smooth Fréchet derivative of $D\hat{g}_{\hat{u}}(t,\cdot)$. ∎

Remarks. (1) Using relation (2.28) and arguments similar to those of the above proof, it is possible to show that if $\xi \longrightarrow y(\xi) \in \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$, then $\xi \longrightarrow \hat{g}_{\hat{u}}(\xi,y(\xi)) \in \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ as well. This fact was used in the second part of the proof of Proposition 2.3.

(2) A proof identical to that of Proposition A.1 also shows that if $g \in \mathscr{C}^{(5)}(\mathbb{R},\mathbb{R})$, then $\hat{g}_{u_{0,1}} \in \mathscr{C}^{(2)}(H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}),H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ where $\hat{g}_{u_{0,1}}$ is defined by relation (3.7). This was used implicitly in the proof of Theorem 3.1.

Appendix B. Proof of the Continuous Differentiability of the Fixed Point Solution to Equation (2.71).

The main purpose of this appendix is to complete the proof of Theorem 2.1.

Proposition B.1. Let s and g satisfy all of the hypotheses of Theorem 2.1. Let $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$ and let $y_{\hat{u}}(\cdot,\eta)$ be the fixed point solution to equation (2.71). Then $y_{\hat{u}}(\cdot,\eta) \in \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ for every η as specified in Theorem 2.1.

We begin our discussion by splitting equation (2.71) into the sum of the three terms

$$F_{0,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) = U_{\hat{u}}(t,0)Py_{\hat{u}}(0,\eta) \tag{B.1}$$

$$F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) = \int_0^t d\xi U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta)) \tag{B.2}$$

$$F_{2,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) = -\int_t^\infty d\xi U_{\hat{u}}(t,\xi)s(\xi)Q\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta)) \tag{B.3}$$

according to the notation introduced in relations (2.61), (2.62) and (2.63). We first notice that $t \longrightarrow F_{j,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) \in \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ for $j=0$ and $j=2$. For $j=0$ this is an immediate consequence of relation (2.19) and the fact that $\left\{W_{\Delta_{\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is a holomorphic semigroup. For $j=2$ we first invoke relation (2.45) to rewrite (B.3) as

$$F_{2,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) = -\exp\left[\int_0^t d\eta s(\eta)g'(\hat{u}(\eta))\right] \times$$

$$\times \int_t^\infty d\xi \exp\left[-\int_0^\xi d\eta s(\eta)g'(\hat{u}(\eta))\right]s(\xi)Q\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta)) \tag{B.4}$$

The result then follows from the absolute convergence of the integral and the fact that

$$\exp\left[-\int_0^\xi d\eta s(\eta)g'(\hat{u}(\eta))\right]s(\xi)Q\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta)) \in$$
$$\in L^1((t,\infty),H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))\cap \mathscr{C}([t,\infty),H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$$

for every $t \in \mathbb{R}^+$. The remaining part of this appendix is therefore devoted to proving that $t \longrightarrow F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) \in \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$. We begin with the following

Lemma B.1. The functions $\xi \longrightarrow y_{\hat{u}}(\xi,\eta)$ and $\xi \longrightarrow \hat{g}_{\hat{u}}(\xi, y_{\hat{u}}(\xi,\eta))$ are both locally Hölder continuous on $\mathbb{R}^+$.

Proof. In the first case it is sufficient to prove that $t \longrightarrow F_{j,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t)$ is locally Hölder continuous on $\mathbb{R}^+$ for each j because of relation (2.71) (Note that $F_{j,\hat{u}}(y_{\hat{u}}(\cdot,\eta))$ is uniformly bounded in t for each j since $F_{j,\hat{u}}(y_{\hat{u}}(\cdot,\eta)) \in Y_{\lambda_1}$ by construction). Because of the remark preceding the statement of Lemma B.1, we already know that the statement holds true for $j = 0$ and $j = 2$. We complete the proof of the first part of the Lemma in showing that $t \longrightarrow F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t)$ is locally Hölder continuous on $\mathbb{R}^+$. Fix $t_0 \in \mathbb{R}^+$ arbitrarily and choose $T \in \mathbb{R}^+$ in such a way that $t_0 \in (0,T)$. Since $y_{\hat{u}}(\cdot,\eta) \in \mathscr{C}(\mathbb{R}_0^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ by construction it follows from relation (2.28) and the smoothness of g that $\xi \longrightarrow \hat{g}_{\hat{u}}(\xi, y_{\hat{u}}(\xi,\eta)) \in \mathscr{C}(\mathbb{R}_0^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$. Now write

$$F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) = \exp\left[\int_0^t d\eta s(\eta) g'(\hat{u}(\eta))\right] \times \int_0^t d\xi W_{\Delta_{\mathscr{N}}}(t-\xi)\varphi_{\hat{u}}(\xi) \quad \text{(B.5)}$$

according to relation (2.19), where we have defined

$$\varphi_{\hat{u}}(\xi) = \exp\left[-\int_0^\xi d\eta s(\eta) g'(\hat{u}(\eta))\right] s(\xi) P \hat{g}_{\hat{u}}(\xi, y_{\hat{u}}(\xi,\eta)) \quad \text{(B.6)}$$

It follows from the remark immediately preceding (B.5) and from relation (B.6) that $\varphi_{\hat{u}} \in L^q((0,T), H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ for every $q \in (1,\infty)$ since

$$\left\|\varphi_{\hat{u}}(\xi)\right\|_{\lambda_0,2,p} \leq 0(1)\left\|\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta))\right\|_{\lambda_0,2,p} \in L^q((0,T),\mathbb{R}^+)$$

Since $\left\{W_{\Delta_{\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is holomorphic it then follows from the standard Hölder estimates of [5] that $t \longrightarrow \int_0^t d\xi W_{\Delta_{\mathscr{N}}}(t-\xi)\varphi_{\hat{u}}(\xi)$ is Hölder continuous on $[0,T]$, and hence a fortiori locally so around t_0. This and relation (B.5) then imply that $F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))$ is locally Hölder continuous on $\mathbb{R}^+$. We conclude that the latter property holds true for $y_{\hat{u}}(\cdot,\eta)$ because of relation (2.71). As for the second part of the lemma we first notice that we have

$$\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta)) = g\circ(\hat{u}(\xi)+y_{\hat{u}}(\xi,\eta))-g\circ\hat{u}(\xi)-(g'\circ\hat{u}(\xi))y_{\hat{u}}(\xi,\eta) \tag{B.7}$$

from relation (2.28). We next observe that each term in (B.7) is uniformly bounded in ξ in the $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$–topology, and that the last two terms on the right–hand side are trivially locally Hölder continuous on $\mathbb{R}^+$ because of the corresponding property for $\xi \longrightarrow \hat{u}(\xi)$ and $\xi \longrightarrow y_{\hat{u}}(\xi,\eta)$. It thus remains to prove that the property holds true for $\xi \longrightarrow g\circ(\hat{u}(\xi)+y_{\hat{u}}(\xi,\eta))$. To this end, write momentarily $z(\xi) = \hat{u}(\xi) + y_{\hat{u}}(\xi,\eta)$; upon using the mean–value theorem and the Banach algebra properties of $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$ as in the proof of Proposition A.1 we get

$$\left\|g\circ z(\xi)-g\circ z(\xi')\right\|_{\lambda_0,2,p} \leq \left\|z(\xi)-z(\xi')\right\|_{\lambda_0,2,p} \times \tag{B.8}$$

$$\times \int_0^1 ds \left\|g' \circ (z(\xi')+s(z(\xi)-z(\xi'))\right\|_{\lambda_0,2,p}$$

for every $\xi,\xi' \in \mathbb{R}^+$. Now fix $\xi,\xi' \in \mathbb{R}^+$ and define $\chi_{\hat{u}}(\xi,\xi')$ by the relation

$$z(\xi')+\mathscr{A}(z(\xi)-z(\xi')) = \hat{u}(\xi)+\chi_{\hat{u}}(\xi,\xi') \tag{B.9}$$

Clearly $\chi_{\hat{u}}(\xi,\xi') \in H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$; it then follows from relation (2.41) of Proposition 2.2 and from the triangle inequality that

$$\left\| g' \circ (\hat{u}(\xi)+\chi_{\hat{u}}(\xi,\xi')) \right\|_{\lambda_0,2,p} \le \Phi^{(1)}_{\hat{u}} \left[\left\| \chi_{\hat{u}}(\xi,\xi') \right\|_{\lambda_0,2,p} \right] + \left\| g' \circ \hat{u}(\xi) \right\|_{\lambda_0,2,p} \tag{B.10}$$

But $\xi \longrightarrow \hat{u}(\xi)$ and $\xi \longrightarrow y_{\hat{u}}(\xi,\eta)$ are uniformly bounded in ξ with respect to the $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$–topology so that there exist positive constants c_{10} and c_{11} such that $\left\| \chi_{\hat{u}}(\xi,\xi') \right\|_{\lambda_0,2,p} \le c_{10}$ and $\left\| g' \circ \hat{u}(\xi) \right\|_{\lambda_0,2,p} \le c_{11}$. Combining this with the fact that $\Phi^{(1)}_{\hat{u}}$ is nondecreasing in relation (B.10) and inserting then the resulting estimate into relation (B.8) we obtain

$$\left\| g \circ z(\xi) - g \circ z(\xi') \right\|_{\lambda_0,2,p} \le c_{12} \left\| z(\xi)-z(\xi') \right\|_{\lambda_0,2,p} \tag{B.11}$$

for some $c_{12} \in \mathbb{R}^+$ and every $\xi,\xi' \in \mathbb{R}^+$. But from the first part of the proof and the definition of z we infer that $\xi \longrightarrow z(\xi)$ is locally Hölder continuous on $\mathbb{R}^+$. Then the same property holds true for $\xi \longrightarrow g \circ z(\xi)$. ■

It is now easy to complete the

Proof of Proposition B.1. It remains to prove that $t \longrightarrow F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) \in \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$. We first infer from relation (B.6), the fact that s is locally Hölder continuous on $\mathbb{R}^+$, the second statement of Lemma B.1 and the boundedness of the functions involved that $\varphi_{\hat{u}}$ is locally Hölder continuous on $\mathbb{R}^+$. Since $\left\{W_{\Delta_{\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}_0^+}$ is a holomorphic semigroup, we then conclude that the convolution $t \longrightarrow \int_0^t d\xi W_{\Delta_{\mathscr{N}}}(t-\xi)\varphi_{\hat{u}}(\xi)$ is continuously differentiable on $\mathbb{R}^+$. This and relation (B.5) then lead to the desired conclusion. ■

Remark. Similar arguments can be used to prove the $\mathscr{C}^{(1)}$–regularity of the solutions $y_{u_{0,1}}(\cdot,\eta)$ in Theorem 3.1.

Appendix C. On the Global Hölder–Lipschitz Continuity of the Fixed Point Solution to Equation (2.71).

While the result of the preceding section implies that $y_{\hat{u}}(\cdot,\eta)$ is locally Lipschitz continuous on $\mathbb{R}^+$, we prove in this appendix that the fixed point solution to equation (2.71) is in fact globally Hölder continuous on every interval of $\mathbb{R}^+$ located at a positive distance of the origin. The precise result is the following

Proposition C.1. Let s and g satisfy all of the hypotheses of Theorem 2.1. Let $\hat{u} \in \{\hat{u}\}_{\hat{\nu}(u_0,u_1)}$ and let $y_{\hat{u}}(\cdot,\eta)$ be the fixed point solution to equation (2.71) for some $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}\epsilon} \cap \operatorname{Ran} P$, where $\hat{k}_0$ and ϵ are as in Theorem 2.1. Then $y_{\hat{u}}(\cdot,\eta)$ is globally Hölder continuous on every interval of the form $[\tau,\infty)$, for every $\tau \in \mathbb{R}^+$.

Proof. Again we write

$$y_{\hat{u}}(t,\eta) = \sum_{j=0}^{2} F_{j,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) \tag{C.1}$$

for every $t \in \mathbb{R}^+$, where the $F_{j,\hat{u}}(y_{\hat{u}}(\cdot,\eta))$'s are given by relations (B.1), (B.2) and (B.3). Since

$$F_{0,\hat{u}}(y_{\hat{u}}(\cdot,\eta))(t) = \exp\left[\int_0^t d\eta s(\eta) g'(\hat{u}(\eta))\right] W_{\Delta_{\mathscr{N}}}(t)\eta \tag{C.2}$$

and since $\left\{W_{\Delta_{\mathscr{N}}}(t)\right\}_{t \in \mathbb{R}_0^+}$ is a holomorphic semigroup, it is clear that $F_{0,\hat{u}}(y_{\hat{u}}(\cdot,\eta))$ is globally Lipschitz continuous on every interval of the form $[\tau,\infty)$ where $\tau \in \mathbb{R}^+$. The same conclusion holds true for $F_{2,\hat{u}}(y_{\hat{u}}(\cdot,\eta))$, since the time derivative of the second factor on the right–hand side of (B.4) is uniformly bounded on $\mathbb{R}^+$ and since $F_{2,\hat{u}}(y_{\hat{u}}(\cdot,\eta)) \in \mathscr{C}^{(1)}(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$. It remains to prove that $F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))$ is globally Hölder continuous on every interval of the form $[\tau,\infty)$ for $\tau \in \mathbb{R}^+$. It is then sufficient to prove that $t \longrightarrow \int_0^t d\xi W_{\Delta_{\mathscr{N}}}(t-\xi)\varphi_{\hat{u}}(\xi)$ is globally Hölder continuous on $\mathbb{R}^+$, since $F_{1,\hat{u}}(y_{\hat{u}}(\cdot,\eta))$ can be written as in (B.5) and (B.6). To this end we first observe that $\varphi_{\hat{u}} \in L^q(\mathbb{R}^+, H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ for every $q \in (1,\infty)$, since

$$\left\|\varphi_{\hat{u}}(\xi)\right\|^q_{\lambda_0,2,p} \leq 0(1)\left\|\hat{g}_{\hat{u}}(\xi, y_{\hat{u}}(\xi,\eta))\right\|^q_{\lambda_0,2,p} \leq$$

$$\leq 0(1)\left\{\Phi^{(2)}\left[\left\|y_{\hat{u}}(\xi,\eta)\right\|_{\lambda_0,2,p}\right]\right\}^q \left\|y_{\hat{u}}(\xi,\eta)\right\|^{2q}_{\lambda_0,2,p} \leq$$

$$\leq 0(1)\exp[2\lambda_1 q\xi] \in L^1(\mathbb{R}^+,\mathbb{R}^+)$$

for such q's, upon using successively relations (B.6), (2.52), estimate (2.31), the monotonicity of $\Phi^{(2)}$ and relation (2.43). Since $\left\{W_{\Delta_{\mathscr{N}}}(t)\right\}_{t\in\mathbb{R}^+}$ is a contraction semigroup, we then note that all of the estimates of the proof of Theorem 4.3.1 of [5] remain virtually unchanged. We conclude that the convolution $t \longrightarrow \int_0^t d\xi W_{\Delta_{\mathscr{N}}}(t-\xi)\varphi_{\hat{u}}(\xi)$ is globally Hölder continuous on $\mathbb{R}^+$, and hence on every interval of the form $[\tau,\infty)$ with $\tau \in \mathbb{R}^+$. The conclusion then follows from relation (C.1).■

The result of Proposition C.1 was used in the proof of Corollary 2.1 to show that the classical solutions constructed there satisfy condition (C1) of Definition 2.1 for some $\theta \in (0,1)$.

A much stronger result holds true if in addition $\eta \in \text{Dom}(\Delta_{\mathscr{N}})$. In fact, in this case we have the following

<u>Proposition C.2.</u> Let s and g satisfy all of the hypotheses of Theorem 2.1. In addition, assume that s be globally Hölder continuous on $\mathbb{R}^+$. Let $\hat{u} \in \{\hat{u}\}_{\hat{\nu}\in(u_0,u_1)}$, $\eta \in \mathscr{N}_{(2\hat{k}_0)^{-1}\epsilon} \cap \text{Ran } P \cap \text{Dom } (\Delta_{\mathscr{N}})$ and let $y_{\hat{u}}(\cdot,\eta)$ be the corresponding fixed point solution to equation (2.71). Then $y_{\hat{u}}(\cdot,\eta)$ is globally Lipschitz continuous on $\mathbb{R}^+$.

<u>Proof.</u> Since $y_{\hat{u}}(\cdot,\eta) \in \mathscr{C}^{(1)}(\mathbb{R}^+,H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R}))$ by Appendix B, it is sufficient to prove that the derivative of $y_{\hat{u}}(\cdot,\eta)$ is uniformly bounded on $\mathbb{R}^+$. Since $y_{\hat{u}}(\cdot,\eta) \in Y_{\lambda_1}$ and since $\hat{u}$ is uniformly bounded in t, the function $t \longrightarrow s(t)g'(\hat{u}(t))y_{\hat{u}}(t,\eta)$ is bounded in the $H^{2,p}_{\lambda_0,\mathscr{N}}(\mathbb{R})$–topology for every $t \in \mathbb{R}^+$. The same statement holds true for $t \longrightarrow s(t)\hat{g}_{\hat{u}}(t,y_{\hat{u}}(t,\eta))$, since

$$\left\|s(t)\hat{g}_{\hat{u}}(t,y_{\hat{u}}(t,\eta))\right\|_{\lambda_0,2,p} \leq \left\|s\right\|_{\infty,\mathbb{R}}\Phi^{(0)}\left[\left\|y_{\hat{u}}(t,\eta)\right\|_{\lambda_0,2,p}\right] \leq$$

$$\leq \left\|s\right\|_{\infty,\mathbb{R}}\Phi^{(0)}\left[\left\|y_{\hat{u}}(\cdot,\eta)\right\|_{\lambda_1}\right] \tag{C.3}$$

upon using estimate (2.29) and relation (2.43). It then follows from equation (2.65) that $t \longrightarrow y'_{\hat{u}}(t,\eta)$ is bounded if, and only if, $t \longrightarrow \Delta_{\mathscr{N}} y_{\hat{u}}(t,\eta)$ is bounded. In order to prove this last statement we first project equation (2.65) onto the subspaces Ran P and Ran Q. We obtain

$$Py'_{\hat{u}}(t,\eta) = (\Delta_{\mathscr{N}}+s(t)g'(\hat{u}(t))Py_{\hat{u}}(t,\eta)+s(t)P\hat{g}_{\hat{u}}(t,y_{\hat{u}}(t,\eta)) \tag{C.4}$$

and

$$Qy'_{\hat{u}}(t,\eta) = s(t)g'(\hat{u}(t))Qy_{\hat{u}}(t,\eta)+s(t)Q\hat{g}_{\hat{u}}(t,y_{\hat{u}}(t,\eta)) \tag{C.5}$$

since $\Delta_{\mathscr{N}}Q = Q\Delta_{\mathscr{N}} = 0$ on $H^{4,p}_{\mathscr{N}}(\mathbb{R})$. From relation (C.5) and the above remarks it follows that $t \longrightarrow Qy'_{\hat{u}}(t,\eta)$ is bounded on $\mathbb{R}^+$. According to relation (C.4) it then remains to prove that $t \longrightarrow \Delta_{\mathscr{N}}Py_{\hat{u}}(t,\eta)$ is bounded on $\mathbb{R}^+$. In order to accomplish this we start once again from the integral equation (2.71) which we project onto Ran P. We obtain

$$Py_{\hat{u}}(t,\eta) = U_{\hat{u}}(t,0)\eta + \int_0^t d\xi U_{\hat{u}}(t,\xi)s(\xi)P\hat{g}_{\hat{u}}(\xi,y_{\hat{u}}(\xi,\eta)) \tag{C.6}$$

since $\eta \in$ Ran P and since $PQ = QP = 0$. From relation (C.6), (B.2), (B.5) and the standard arguments of [3] and [5], we then get

$$\Delta_{\mathcal{N}} P y_{\hat{u}}(t,\eta) = \exp\left[\int_0^t d\eta s(\eta) g'(\hat{u}(\eta))\right] \Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t)\eta +$$

$$+ \exp\left[\int_0^t d\eta s(\eta) g'(\hat{u}(\eta))\right] \int_0^t d\xi \Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t-\xi)\{\varphi_{\hat{u}}(\xi) - \varphi_{\hat{u}}(t)\} +$$

$$+ \exp\left[\int_0^t d\eta s(\eta) g'(\hat{u}(\eta))\right] (W_{\Delta_{\mathcal{N}}}(t)\varphi_{\hat{u}}(t) - \varphi_{\hat{u}}(t)) \tag{C.7}$$

Since $\eta \in \mathrm{Dom}(\Delta_{\mathcal{N}})$ and since $\left\{W_{\Delta_{\mathcal{N}}}(t)\right\}_{t \in \mathbb{R}_0^+}$ is a contraction semigroup we have

$$\left\|\Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t)\eta\right\|_{\lambda_0,2,p} = \left\|W_{\Delta_{\mathcal{N}}}(t)\Delta_{\mathcal{N}}\eta\right\|_{\lambda_0,2,p} \leq \left\|\Delta_{\mathcal{N}}\eta\right\|_{\lambda_0,2,p}$$

uniformly in $t \in \mathbb{R}^+$ so that the first term of (C.7) is bounded on $\mathbb{R}^+$. The same conclusion holds true for the third term of (C.7) since $\varphi_{\hat{u}}$ is bounded on $\mathbb{R}^+$, by relations (B.6), (2.29) and (2.43). It remains to prove that the second term of (C.7) is bounded on $\mathbb{R}^+$, which is equivalent to proving that

$$t \longrightarrow \int_0^t d\xi \Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t-\xi)\{\varphi_{\hat{u}}(\xi) - \varphi_{\hat{u}}(t)\} = 0(1)$$

on $\mathbb{R}^+$. In order to accomplish this, we first notice that $\varphi_{\hat{u}}$ is globally Hölder continuous on $\mathbb{R}^+$. To see that we simply reiterate the argument given in the proof of Proposition (C.1) to conclude that since $\eta \in \mathrm{Dom}(\Delta_{\mathcal{N}})$, $y_{\hat{u}}(\cdot,\eta)$ is globally Hölder continuous on $\mathbb{R}^+$ (and not merely on $[\tau,\infty)$ for every $\tau \in \mathbb{R}^+$); it then follows that $\xi \longrightarrow \hat{g}_{\hat{u}}(\xi, y_{\hat{u}}(\xi,\eta))$ is globally Hölder continuous on $\mathbb{R}^+$ through relation (B.11). This immediately implies the global Hölder continuity of $\varphi_{\hat{u}}$ through relation (B.6) and the global Hölder continuity of s. We conclude the argument by observing that $\varphi_{\hat{u}} = P\varphi_{\hat{u}} \in \mathrm{Ran}\, P$, which gives

$$\int_0^t d\xi \Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t-\xi)\{\varphi_{\hat{u}}(\xi)-\varphi_{\hat{u}}(t)\} =$$

$$= \int_0^t d\xi \Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t-\xi) P\{\varphi_{\hat{u}}(\xi)-\varphi_{\hat{u}}(t)\} \tag{C.8}$$

Invoking then the second estimate (2.15) of Lemma 2.2 and the global Hölder continuity of $\varphi_{\hat{u}}$, we obtain after a simple change of variables the estimate

$$\left\|\int_0^t d\xi \Delta_{\mathcal{N}} W_{\Delta_{\mathcal{N}}}(t-\xi)\{\varphi_{\hat{u}}(\xi)-\varphi_{\hat{u}}(t)\}\right\|_{\lambda_{0,2,p}} \leq$$

$$\leq 0(1) \int_0^t d\xi \exp[\lambda_1 \xi]\xi^{\gamma-1} \tag{C.9}$$

for some $\gamma \in (0,1)$. But the last integral in (C.9) is $0(1)$ on $\mathbb{R}^+$. Hence $t \longrightarrow \Delta_{\mathcal{N}} P y_{\hat{u}}(t,\eta)$ remains bounded on $\mathbb{R}^+$ by relation (C.7). ∎

Remark. Similar results can be proved for the fixed point solutions $y_{u_{0,1}}(\cdot,\eta)$ of Section 3.

Bibliographical References

1. Adams, R.A.: Sobolev Spaces. New York, London: Academic Press 1975.

2. Friedman, A.: Partial Differential Equations of Parabolic Type. Englewood Cliffs: Prentice Hall, Inc. 1964.

3. Kato, T.: Perturbation Theory for Linear Operators. Berlin, Heidelberg, New York: Springer Verlag 1980.

4. Levitan, B.M., Zhikov, V.V.: Almost–Periodic Functions and Differential Equations. London, New York: Cambridge Univ. Press 1982.

5. Pazy A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Math. Sciences, Vol. 44. Berlin, Heidelberg, New York, Springer Verlag 1983.

6. Rudin, W.: Fourier Analysis on Groups. New York: Wiley 1960.

7. Scarpellini, B., Vuillermot, P.A.: Variétés Stables et Instables pour Certaines Equations des Ondes Semilinéaires sur $\mathbb{R}^2$. C.R. Acad. Sci. Paris Série I Math. 306, 33–36 (1988).

8. Scarpellini, B., Vuillermot, P.A.: Smooth Manifolds for Semilinear Wave Equations on $\mathbb{R}^2$. On the Existence of Almost–Periodic Breathers. J. Diff. Equations 77, 123–166 (1989).

9. Shubin, M.A.: Differential and Pseudodifferential Operators in Spaces of Almost-Periodic Functions. Math. USSR–Sb. 24, 547–573 (1974).

10. Vuillermot, P.A.: Variétés Lisses Associées à Certains Systèmes Dynamiques et Solitons Périodiques pour les Equations de Klein–Gordon Nonlinéaires sur $\mathbb{R}^2$. C.R. Acad. Sci. Paris, Série I, Math. 307, 639–642 (1988).

11. Vuillermot, P.A.: Problèmes de Cauchy Multipériodiques et Solitons Quasipériodiques pour les Equations de Klein–Gordon Nonlinéaires sur $\mathbb{R}^2$. C.R. Acad. Sci. Paris, Série I, Math. 308, 215–218 (1989).

12. Vuillermot, P.A.: Quasiperiodic Soliton Solutions to Nonlinear Klein–Gordon Equations on $\mathbb{R}^2$. Math. Z. 203, 2, 235–253 (1990).

13. Vuillermot, P.A.: Periodic Soliton Solutions to Nonlinear Klein–Gordon Equations on $\mathbb{R}^2$. Differential and Integral Equations 3, 3, 541–570 (1990).

14. Vuillermot, P.A.: Attracteurs Presque–Périodiques pour une Classe d'Equations Paraboliques Nonlinéaires du Type Réaction–Diffusion sur $\mathbb{R}^N$. C.R. Acad. Sci. Paris, Série I, Math. in press (1990).

15. Vuillermot, P.A.: Almost–Periodic Attractors for a Class of Nonautonomous Reaction-Diffusion Equations on $\mathbb{R}^N$, I. Global Stabilization Processes. J. Diff. Equations, in press (1991).

16. Vuillermot, P.A.: Variétés Stables de Codimension Un pour une Classe d'Equations Paraboliques Semilinéaires Presque–Périodiques sur $\mathbb{R}^N$. C.R. Acad. Sci. Paris, Série I, Math. to be submitted (1990).

17. Vuillermot, P.A.: Almost–Periodic Attractors for a Class of Nonautonomous Reaction-Diffusion Equations on $\mathbb{R}^N$, III. Center Curves and Reduced Dynamics, Adv. Appl. Math. to be submitted (1991).

18. von Wahl, W.: Private communication.