CROSSED INTERVAL GROUPS AND OPERATIONS ON THE HOCHSCHILD COHOMOLOGY

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Abstract. We introduce crossed interval groups and construct a crossed interval analog $\mathcal{I}S$ of the Fiedorowicz-Loday symmetric category ΔS. We prove that the functor $F_{\mathcal{S}}(-)$ of the free $\mathcal{I}S$-extension of an $\mathcal{I}$-object does not change homotopy type. We then observe that the operad $\mathcal{B}$ of natural operations on the Hochschild cohomology equals $F_{\mathcal{S}}(\mathcal{T})$, where $\mathcal{T}$ is an operad whose homotopy type is known. We conclude from these facts that $\mathcal{B}$ has the homotopy type of the operad of singular chains on the little disks operad.

1. Introduction and main results

It is well-known that, for any 'reasonable' type of algebras (where reasonable means algebras over a quadratic Koszul operad, see [12, II.3.3] or the original source [6] for the terminology), there exists the associated cohomology based on a 'standard construction.' For example, for associative algebras, the associated cohomology is the Hochschild cohomology, for Lie algebras the Chevalley-Eilenberg cohomology, for associative commutative algebras the Harrison cohomology, &c.

There arises a fundamental problem of describing all operations acting on this associated cohomology, by which one usually means the understanding of the homotopy type of the operad of natural operations on the standard construction [11, Problem 1.]. Our present paper aims at giving the answer for the Hochschild cohomology of associative algebras. To our best knowledge, it will be the first complete answer to the above question, compare also the introduction to [11].

So, let $\mathcal{B}$ be the operad of all natural operations on the Hochschild cochain complex of an associative algebra with coefficients in itself. In connection with the Deligne conjecture [3], various suboperads of $\mathcal{B}$ have been studied and several results about their homotopy types were claimed [8, 9, 15]. The only statement whose proof we were able to verify was, however, a theorem of [14] claiming that the brace suboperad $\mathcal{H} \subset \mathcal{B}$ has the homotopy type of the operad of singular chains of the little discs operad. There is also an intermediate operad $\mathcal{T}$ whose topological version was studied in [15], related to $\mathcal{H}$ and $\mathcal{B}$ via the inclusions $\mathcal{H} \hookrightarrow \mathcal{T} \hookrightarrow \mathcal{B}$. We will give, in [1], an 'elementary' proof that the inclusion $\mathcal{H} \hookrightarrow \mathcal{T}$ is a homotopy equivalence.

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The starting point of this work concerned the inclusion $\mathcal{T} \hookrightarrow \mathcal{B}$. It turns out that the operad $\mathcal{B}$ is, in a certain sense, freely generated by $\mathcal{T}$. More precisely, $\mathcal{B}$ is the free $\mathcal{I}S$-module generated by the $\mathcal{I}$-module $\mathcal{T}$, $\mathcal{B} = F_{\mathcal{S}}(\mathcal{T})$. Here $\mathcal{I}S$ is an analog of the symmetric category ΔS introduced in [5] (but see also [10]), with the simplicial category Δ replaced by Joyal's category $\mathcal{I}$ of intervals [7]. The fundamental result of this paper says that the functor $F_{\mathcal{S}}(-)$ preserves homotopy type. Combining this feature with the observations in the previous paragraph we conclude that the operad $\mathcal{B}$ has the homotopy type of the operad of singular chains on the little discs operad.

The category $\mathcal{I}\mathcal{S}$ is an example of a crossed interval group which we introduce and study in the first section. In Section 3 we use an acyclic-models-type argument to prove that the functor $F_{\mathcal{S}}(-)$ does not change the homotopy type. In the last section we apply our machinery to the operad $\mathcal{B}$.

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2. Crossed interval groups

Joyal's (skeletal) *category of intervals* $\mathcal{I}$ is the category whose objects are the linearly ordered sets $\langle n \rangle := \{-1, 0, \ldots, n, n+1\}$, $n \geq -1$, and morphisms non-decreasing maps $f : \langle m \rangle \to \langle n \rangle$ such that $f(-1) = -1$ and $f(m+1) = n+1$. The points $-1, n+1$ are called the *boundary points* of $\langle n \rangle$ and the ordinal $[n] := \{0, \ldots, n\}$ is the *interior* of $\langle n \rangle$. Intuitively, one may think of $\langle n \rangle$ as of the interval with endpoints -1 and $n+1$; morphisms in $\mathcal{I}$ are then non-decreasing, boundary preserving maps of intervals. By Joyal's duality [7], $\mathcal{I}$ is isomorphic to the opposite category Δ^{op} of the (skeletal) category Δ of finite ordered sets and their non-decreasing maps. The isomorphism $j : \Delta^{\mathrm{op}} \cong \mathcal{I}$ is constructed as follows.

Recall that Δ has object the finite ordinals $[n] = \{0, \ldots, n\}$, $n \geq 0$. Morphism of Δ are generated by the face maps $\delta_i : [n-1] \to [n]$ (misses i) and the degeneracy maps $\sigma_i : [n+1] \to [n]$ (hits i twice), $i = 0, \ldots, n$. Then $j([n]) := \langle n-1 \rangle$, for $n \geq 0$. Moreover, $d_i = j(\delta_i) : \langle n-1 \rangle \to \langle n-2 \rangle$ is the map that hits $i-1$ twice, and $s_i = j(\sigma_i) : \langle n-1 \rangle \to \langle n \rangle$ the map that misses i.

The category $\mathcal{I}$ contains a subcategory of open maps. The objects of this subcategory are intervals and the morphisms are interval morphisms $f : \langle m \rangle \to \langle n \rangle$ which preserve the interiors i.e. $f(\{0, \ldots, m\}) \subset \{0, \ldots, n\}$. This subcategory is isomorphic to the (skeletal) category of all finite ordinals called Δ_{alg} (algebraic Δ). The category Δ_{alg} has a well-known universal property – it is a monoidal category freely generated by a monoid. See [7] for details.

The following definition is motivated by [5, Definition 1.1].

2.1. **Definition.** A *crossed interval group* $\mathcal{H}$ is a sequence of groups $\{H^n\}_{n \geq 0}$ equipped with the following structure:

(a) a small category $\mathcal{I}H$ called the *associated category* such that the objects of $\mathcal{I}H$ are the ordinals $\langle n\rangle = \{-1, 0, \ldots, n, n+1\}$, $n \geq -1$,

(b) $\mathcal{I}H$ contains $\mathcal{I}$ as a subcategory,

(c) $\mathrm{Aut}_{\mathcal{I}H}(\langle n\rangle) = (H^{n+1})^{\mathrm{op}}$, $n \geq -1$, and

(d) any morphism $f : \langle m\rangle \to \langle n\rangle$ in $\mathcal{I}H$ can be uniquely written as $\phi \circ h$, where $\phi \in \mathrm{Hom}_{\mathcal{I}}(\langle m\rangle, \langle n\rangle)$ and $h \in (H^{m+1})^{\mathrm{op}}$.

We would like to extend the Joyal's duality to the case of crossed interval groups.

2.2. **Definition.** A *crossed cosimplicial group* $\mathcal{H}$ is a sequence of groups $\{H^n\}_{n\geq 0}$ equipped with the following structure:

(a) a small category $H\Delta$ such that the objects of $H\Delta$ are the ordinals $[n] = \{0,,,\ldots,n\}$, $n \geq 0$,

(b) $H\Delta$ contains Δ as a subcategory,

(c) $\mathrm{Aut}_{H\Delta}([n]) = H^n$, $n \geq 0$, and

(d) any morphism $f : [m] \to [n]$ in $H\Delta$ can be uniquely written as $h \circ \phi$, where $\phi \in \mathrm{Hom}_{\Delta}([m],[n])$ and $h \in H^n$.

Now, given a crossed interval group $\mathcal{H}$ we construct its *Joyal's dual* crossed cosimplicial group $J\mathcal{H}$ with the associated category $(JH)\Delta$ by taking the same sequence of groups $\{H^n\}_{n\geq 0}$ and putting $(JH)\Delta([n],[m]) := \mathcal{I}H(\langle m-1\rangle, \langle n-1\rangle)$. One can easily check using the ordinary Joyal's duality that we thus obtain a crossed cosimplicial group. Moreover, we also obtain an isomorphism of the associated categories $j_H : ((JH)\Delta)^{\mathrm{op}} \to \mathcal{I}H$. We, therefore, see that definitions 2.1 and 2.2 are equivalent.

One advantage of the crossed interval definition is the same variance of morphisms as in Loday-Fiedorowicz crossed simplicial groups. So, many basic results for crossed simplicial groups can be carried over to crossed interval groups without changing the formulas from [5]. For example, we have that for any $h \in H^n$ and any $\phi \in \mathcal{I}(\langle m\rangle, \langle n\rangle)$ there exist unique $\phi^*(h) \in H^m$ and $h^*(\phi) \in \mathcal{I}(\langle m\rangle, \langle n\rangle)$ such that $h \circ \phi = h^*(\phi) \circ \phi^*(h)$. Using these correspondences one can give an alternative characterization of crossed interval groups repeating [5, Proposition 1.6] verbatim.

Let us define an *automorphism of an interval* $\langle n\rangle$ as a bijection $\langle n\rangle$ to $\langle n\rangle$ which preserves the boundary points. Clearly, the group $\mathrm{Aut}\,\langle n\rangle$ of all automorphisms is isomorphic to $S_{n+1} \times \mathbb{Z}_2$, where S_{n+1} denotes the symmetric group on $n+1$ elements and $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

The following characterization of the crossed interval groups is slightly more complicated than its crossed simplicial analogue (see Proposition 1.7 from [5]).

2.3. **Proposition.** *A crossed interval group $\mathcal{H}$ is a cosimplicial set $H^\bullet = \{H^n\}_{n\geq 0}$ such that each H^n is a group, together with a group homomorphism*

$$\rho_n : H^n \to \mathrm{Aut}\,\langle n-1\rangle \cong S_n \times \mathbb{Z}_2 \tag{1}$$

for each $n \geq 0$, such that, for $h, h' \in H^n$,

(1) $d_i(hh') = d_{\overline{h}(i)}(h')d_i(h)$, *where* $\overline{h}(i) := \rho(h)(i-1)+1$, $0 \leq i \leq n+1$,
(2) $s_i(hh') = s_{\underline{h}(i)}(h')s_i(h)$, *where* $\underline{h}(i) := \rho(h)(i)$, $0 \leq i \leq n-1$.
(3) *Moreover, the following set diagrams are commutative:*

$$\begin{tikzcd}
\langle n\rangle \arrow[r, "d_{\overline{h}(i)}"] \arrow[d, "d_i(h)"'] & \langle n-1\rangle \arrow[d, "h"] \\
\langle n\rangle \arrow[r, "d_i"] & \langle n-1\rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
\langle n-2\rangle \arrow[r, "s_{\underline{h}(i)}"] \arrow[d, "s_i(h)"'] & \langle n-1\rangle \arrow[d, "h"] \\
\langle n-2\rangle \arrow[r, "s_i"] & \langle n-1\rangle
\end{tikzcd}$$

where $0 \leq i \leq n+1$ in the first and $0 \leq i \leq n-1$ in the second diagram.

The proof is basically identical to the proof of Proposition 1.7 in [5] but we have to take into account that among the coface operators d_i in $\mathcal{I}$ the first and the last operators are not open maps of intervals. One can check that under the correspondence $h^* : \mathcal{I}(\langle m\rangle, \langle n\rangle) \to \mathcal{I}(\langle m\rangle, \langle n\rangle)$ the open maps go to the open maps, so the first and the last operators can only be stable or permuted by h^*. This explains the factor $\mathbb{Z}_2$ in (1).

We also observe that ρ_* is a homomorphism of crossed interval groups $\mathcal{H} \to \mathcal{S} \times \mathcal{Z}_2$ (see examples 2.4 and 2.5 below for the definitions of $\mathcal{S}$ and $\mathcal{Z}_2$), so the value of ρ_n is determined by its action on the set of open coface operators $\{d_i;\ 1 \leq i \leq n\}$ or equally on the set of codegenaracies (which are automatically open maps of intervals) $\{s_i;\ 0 \leq i \leq n-1\}$, and the homomorphism $\rho_0 : H^0 \to \mathbb{Z}_2$.

An important crossed interval group is introduced in the following example.

2.4. **Example.** Let $\mathcal{I}S$ be the category whose objects are the ordinals $\langle n\rangle = \{-1, 0, \ldots, n, n+1\}$, $n \geq -1$, and morphisms are maps $f : \langle m\rangle \to \langle n\rangle$ such that

(a) a linear order on each non-empty fiber $f^{-1}(i)$, $i \in \langle n\rangle$ is specified,
(b) f preserves the endpoints, $f(-1) = -1$ and $f(m+1) = n+1$, and
(c) -1 is the minimal element in $f^{-1}(-1)$ and $m+1$ is the maximal element in $f^{-1}(n+1)$.

The group $\mathrm{Aut}_{\mathcal{I}S}(\langle n\rangle)$ equals the symmetric group S_{n+1}, whence the notation. We leave as an exercise to verify that the $\mathcal{I}S$ defined above is an associated category of a crossed interval group. We denote this crossed interval group $\mathcal{S}$.

Let ΔS be the category whose objects are the ordinals $[n] := \{0, \dots, n\}$ and morphisms are maps $g : [m] \to [n]$ with a specified linear order on the fibres $g^{-1}(i)$, $i \in [n]$. Recall that ΔS, called the *symmetric category*, is an associated category of a *crossed simplicial group* S_* in the sense of [5, Definition 1.1]. There is also an 'algebraic' version of the symmetric category $(\Delta S)_{alg}$ introduced by Day and Street in [2], which has a universal property similar to Δ_{alg}: the category $(\Delta S)_{alg}$ is the symmetric monoidal category freely generated by a (noncommutative) monoid.

Similarly to the case of $\Delta_{alg} \hookrightarrow \mathcal{I}$ there is the inclusion

$$I : (\Delta S)_{alg} \hookrightarrow \mathcal{I}S$$

such that $I([n]) := \langle n \rangle$. If $g : [m] \to [n]$ is a morphism in $(\Delta S)_{alg}$, then $I(g)$ is the unique morphism $f : \langle m \rangle \to \langle n \rangle$ such that $f|_{\{0,\dots,m\}} = g$. The image of I is the subcategory of *open maps*, i.e. maps $f : \langle m \rangle \to \langle n \rangle$ such that $f(\{0, \dots, m\}) \subset \{0, \dots, n\}$. Likewise, there is the inclusion

$$\iota : \mathcal{I}S \hookrightarrow \Delta S \tag{2}$$

such that $\iota(\langle n \rangle) := [n + 2]$ and, for $f : \langle m \rangle \to \langle n \rangle$, $g = \iota(f)$ is the map with $g(i) := f(i - 1)$, for $i \in [n + 2]$.

2.5. **Example.** Another, rather trivial example of a crossed interval group is the 'constant' group $\mathcal{Z}_2$. The objects are the intervals $\langle n \rangle$, $n \geq -1$, and the morphisms are maps $f : \langle m \rangle \to \langle n \rangle$ such that f restricted to the interior of the interval is order preserving and f preserves the boundary, that is, $f(\{-1, m + 1\}) = \{-1, n + 1\}$. It is not hard to see that all the automorphisms groups in this category are $\mathbb{Z}_2$.

2.6. **Example.** The crossed interval group $\mathcal{F}lip$. The objects are again the intervals. The morphisms from $\langle n \rangle$ to $\langle m \rangle$ are maps $f : \langle m \rangle \to \langle n \rangle$ that preserve the boundary, which are either order preserving or order reversing. Again, it is not difficult to see that all automorphism groups are $\mathbb{Z}_2$. The unique nontrivial automorphism of an interval is the flip around its center. The crossed interval groups $\mathcal{Z}_2$ and $\mathcal{F}lip$ are, however, not isomorphic.

2.7. **Example.** One can construct an example of a crossed interval group $\mathcal{B}r$ if one replaces symmetric groups by braid groups in Example 2.4. The braid analogue of Example 2.5 will be the 'constant' crossed interval group $\mathcal{Z}$. We do not know if there is a braid analogue of the crossed interval group $\mathcal{F}lip$ of Example 2.6.

The examples of $\mathcal{Z}_2$ and $\mathcal{F}lip$ indicate that the classification of the crossed interval groups is, probably, more subtle than the classification of crossed simplicial groups given in [5].

2.8. **Definition.** Let $\mathcal{H}$ be a crossed interval group and let $\mathcal{C}$ be a category. An $\mathcal{H}$-*object* in $\mathcal{C}$ is a functor $X : (\mathcal{I}H)^{\mathrm{op}} \to \mathcal{C}$. A *morphism* between two $\mathcal{H}$-objects is a natural transformation of functors.

Similarly to the crossed simplicial case we have the following simple characterization of $\mathcal{H}$-objects (compare with Lemma 4.2 of [5]).

2.9. **Lemma.** *The category of $\mathcal{H}$-objects in a cocomplete category $\mathcal{C}$ is equivalent to the category whose objects are cosimplicial objects $X^\bullet$ in $\mathcal{C}$ equipped with the following additional structure:*

(1) *right group actions $X^n \times H^n \to X^n$, $n \geq 0$,*
(2) *coface relations $d_i(xh) = d_{\overline{h}(i)}(x)d_i(h)$, for $x \in X^n$, $h \in H^n$, $0 \leq i \leq n+1$,*
(3) *codegeneracy relations $s_i(xh) = s_{\underline{h}(i)}(x)s_i(h)$, for $x \in X^n$, $h \in H^n$, $0 \leq i \leq n-1$,*

and whose morphisms are cosimplicial morphisms which are degreewise equivariant.

In the above lemma, $\overline{h}(i)$ and $\underline{h}(i)$ have the same meaning as in Proposition 2.3. There is a natural restriction functor $\mathcal{U}$ from the category of $\mathcal{H}$-objects in $\mathcal{C}$ to the category of cosimplicial objects in $\mathcal{C}$. This functor has a left (right) adjoint provided $\mathcal{C}$ is cocomplete (complete). We will denote the left adjoint by $\mathcal{F}_{\mathcal{H}}$. It is easy to get an explicit formulas for $\mathcal{F}_{\mathcal{H}}$ similar to the formulas from [5, Definition 4.3].

2.10. **Lemma.** *Let $X = X^\bullet$ be a cosimplicial object in a cocomplete category $\mathcal{C}$. Then $\mathcal{F}_{\mathcal{H}}(X)$ is an $\mathcal{H}$-object with $\mathcal{F}_{\mathcal{H}}(X)^n := X^n \times H^n$, $n \geq 0$, and with cofaces and codegeneracies given by*

$$d_i(x,h) := (d_{\overline{h}(i)}(x), d_i(h)), \text{ for } x \in X^n,\ h \in H^n,\ 0 \leq i \leq n+1,$$
$$s_i(x,h) := (s_{\underline{h}(i)}(x), s_i(h)), \text{ for } x \in X^n,\ h \in H^n,\ 0 \leq i \leq n-1.$$

The unit $\iota : X \mapsto F_{\mathcal{H}}(X)$ of the monad $F_{\mathcal{H}}$ generated by the adjunction $\mathcal{F}_{\mathcal{H}} \dashv \mathcal{U}$ is given by the cosimplicial map $\iota(x) := (x, 1)$.

Let $\mathcal{C}$ be the category of abelian groups, chain complexes, (multi)simplicial abelian groups or topological spaces. In all these cases there is a notion of geometric realization (or better to say *totalization*) of cosimplicial objects in $\mathcal{C}$. Given such a cosimplicial object $X^\bullet$ we will denote by $\mathrm{Tot}(X^\bullet)$ its totalization. We want to stress that in a fixed $\mathcal{C}$ there may exist different notions of totalization so we should specify which Tot is used in which case.

Let us assume that we choose a functor of totalization for $\mathcal{C}$. It makes sense to ask when does the unit of the adjunction

$$\iota : X \to F_{\mathcal{H}}(X),$$

induce a weak equivalence after totalization? In the dual case of crossed simplicial groups and $\mathcal{C}$ the category of topological spaces, Theorem 5.3 from [5] gives answers this question by providing an equivariant homeomorphism

$$\mathrm{Tot}(F_G(X_\bullet)) \to \mathrm{Tot}(G_\bullet) \times \mathrm{Tot}(X_\bullet).$$

It is also not too difficult to see by providing an explicit homotopy that for G equal ΔS or ΔBr the space $\mathrm{Tot}(G_\bullet)$ is contractible and, hence, $\mathrm{Tot}(i)$ is a homotopy equivalence.

Unfortunately, the methods of [5] do not work for the crossed interval groups due to the fact that the behavior of cosimplicial totalization is much less understood than its simplicial counterpart. The next section is devoted to some partial result about cosimplicial totalization of $\mathcal{I}S$-chain complexes, see Proposition 3.6 and its Corollary 3.7. This result will suffice for our applications in Section 4.

3. A homotopy equivalence

The main result of this section, Theorem 3.10, states that the cochain complex associated to $F_S(C_*^\bullet)$, where $C_*^\bullet$ is a cosimplicial chain complex with finitely generated torsion-free components, has the homotopy type of the cochain complex associated to $C_*^\bullet$. Let us introduce a couple of categories that will become important later in this section.

We denote by $\mathcal{C}hain$ the category of $\mathbb{Z}$-graded chain complexes of abelian groups considered as a closed symmetric monoidal category with respect to the usual tensor product of chain complexes. Let $\mathcal{C}hain^\Delta$ be the category of cosimplicial objects in $\mathcal{C}hain$. Elements of $\mathcal{C}hain^\Delta$ are systems $C_*^\bullet = (C_*^n, \partial)$, $n \geq 0$, of chain complexes with face and degeneracy operators $d_i : C_*^{n-1} \to C_*^n$, $s_i : C_*^{n+1} \to C_*^n$, $0 \leq i \leq n$, which are chain maps and satisfy the standard cosimplicial identities.

Another important category will be the category $\mathcal{C}och\mathcal{C}hain$ of non-negatively graded cochain complexes in the category $\mathcal{C}hain$. Elements of $\mathcal{C}och\mathcal{C}hain$ are systems (C_*^*, d, ∂) consisting of abelian groups C_k^m, $m \geq 0$, $k \in \mathbb{Z}$, and linear maps $d : C_k^m \to C_k^{m+1}$ (*horizontal* differentials), $\partial : C_{k+1}^m \to C_k^m$ (*vertical* differentials) that square to zero and satisfy $\partial d + d\partial = 0$.

Each cosimplicial chain complex $C_*^\bullet \in \mathcal{C}hain^\Delta$ determines an element C_*^* of $\mathcal{C}och\mathcal{C}hain$, with the differential $d : C_k^m \to C_k^{m+1}$ given by the standard formula

$$d(c) := \sum_{0 \leq i \leq m+1} (-1)^i d_i(c), \ c \in C_k^m, \ m \geq 0, \ k \in \mathbb{Z}.$$

For $C_*^* \in \mathcal{C}och\mathcal{C}hain$ denote by $|C_*^*|^*$ the cochain complex with $|C_*^*|^m := \prod_{n-k=m} C_k^n$, with the differential given by

$$d(a^0, a^1, a^2, \ldots) := (-\partial a^0, da^1 - \partial a^0, da^2 - \partial a^1, \ldots),$$

for $(a^0, a^1, a^2, \ldots) \in C_*^0 \times C_*^1 \times C_*^2 \times \cdots$. Finally, define the *total complex* of a cosimplicial chain complex $C_*^\bullet$ to be the cochain complex $\mathrm{Tot}(C_*^\bullet)^* := |C_*^*|^*$.

Warning. Elements of $\mathcal{C}och\mathcal{C}hain$ are particular types of bicomplexes of abelian groups (with a slightly unusual degree convention), but our definition of $|C_*^*|^*$ differs from the traditional definition of the total complex of a bicomplex in that it involves the direct *product* instead of the direct *sum*.

3.1. **Example.** The components of the 'big' operad $\mathcal{B}$ of Section 4 form a cosimplicial, non-negatively graded chain complex $B^\bullet_*(n)$ with $B^q_m(n) = \bigoplus_{k_1+\cdots+k_n=m} B^q_{k_1,\ldots,k_n}$. The corresponding total space $\mathrm{Tot}(B^\bullet_*(n))$ equals, by (2.3), the arity n component $\mathcal{B}(n)$ of the dg operad $\mathcal{B}$. Another example is the cosimplicial chain complex determined, in the same manner, by the components of the suboperad $\mathcal{T} \subset \mathcal{B}$.

At this moment, we need to introduce a theory of models for functors with values in cochain complexes. Due to the lack of a suitable concept of cofree modules, this theory is not merely a dualization of the chain version.

Let $\mathcal{D}$ be a dg-category, that is, a category enriched over $\mathcal{C}hain$. Let $F : \mathcal{D} \to \mathcal{C}hain$ be a covariant dg-functor and $\mathfrak{M}$ a set of objects of $\mathcal{D}$. Let us denote, for $K \in \mathcal{D}$, by $\widetilde{F}(K)_*$ the chain complex

$$\widetilde{F}(K)_* := \underline{\mathcal{C}hain}\Big(\bigoplus_{M\in\mathfrak{M}} \underline{\mathcal{D}}(K,M)_*, F(M)_*\Big)_* = \prod_{M\in\mathfrak{M}} \underline{\mathcal{C}hain}(\underline{\mathcal{D}}(K,M)_*, F(M)_*)_*. \tag{3}$$

In the above display, as well as in the rest of this section, we used the notation $\underline{\mathcal{C}}(-,-)$ for the enriched Hom-functor of a dg-category $\mathcal{C}$.

It is obvious that (3) defines a dg-functor $\widetilde{F} : \mathcal{D} \to \mathcal{C}hain$ of dg-categories. There exists the canonical dg-transformation $\lambda : F \to \widetilde{F}$ such that, for $k \in F(K)_*$,

$$\lambda(k) = \prod_{M\in\mathfrak{M}} \lambda_M(k) \in \widetilde{F}(K)_*,$$

where $\lambda_M(k) \in \underline{\mathcal{C}hain}(\underline{\mathcal{D}}(K,M)_*, F(K)_*)_*$ sends $\mu \in \underline{\mathcal{D}}(K,M)_*$ into $F(\mu)(k) \in F(M)_*$.

3.2. **Definition.** We say that $F : \mathcal{D} \to \mathcal{C}hain$ as above is *corepresented with models* $\mathfrak{M}$ if there exists a (not necessarily dg) transformation $\chi : \widetilde{F} \to F$ such that $\chi\lambda = id_F$.

In the next two examples we will use the following elementary property of chain complexes of abelian groups. Let us denote, for $V_* \in \mathcal{C}hain$, by $V^\#_*$ the linear dual, $V^\#_* := \underline{\mathcal{C}hain}(V_*, \mathbb{Z})_*$, where $\mathbb{Z}$ are the integers considered as the chain complex concentrated in degree 0. We choose this notation to avoid possible conflicts with the star indicating the grading. Observe that the components $(V^\#_*)_n$ of the dual $V^\#_*$ are given by

$$(V^\#_*)_n = V^\#_{-n},\ n \in \mathbb{Z}. \tag{4}$$

Then, for chain complexes with *finitely generated torsion free components* U_* and V_*, one has the canonical isomorphisms of enriched Hom-functors

$$\underline{\mathcal{C}hain}(V_*, W_*)_* \cong \underline{\mathcal{C}hain}(W^\#_*, V^\#_*)_*. \tag{5}$$

Notation. For $q \geq 0$ we denote by $\Delta[q]$ the standard simplicial q-simplex and $M[q]$ the simplicial abelian group spanned by $\Delta[q]$. Let $D[q]$ be the cosimplicial abelian group given as the componentwise linear dual of $M[q]$ (an explicit description of $D[q]$ is given in Example 3.12 below). We will interpret $M[q]$ (resp. $D[q]$) as a simplicial (resp. cosimplicial) chain complex concentrated in degree zero, with trivial vertical differentials.

Let $\mathcal{C}h \subset \mathcal{C}hain$ be the self-dual subcategory consisting of chain complexes with finitely generated torsion-free components. Let $\mathcal{C}h^{\Delta}$ denote the category of cosimplicial objects in $\mathcal{C}h$ and $\mathcal{C}h^{\Delta^{op}}$ the category of simplicial objects in $\mathcal{C}h$. The componentwise linear dual $C_*^\bullet \mapsto C_*^{\bullet\#}$ defines a contravariant isomorphism $\mathcal{C}h^{\Delta} \cong \mathcal{C}h^{\Delta^{op}}$.

3.3. **Example.** We claim that the functor $F : \mathcal{C}h^{\Delta} \to \mathcal{C}hain$ given by $F(C_*^\bullet) := C_*^q$ (the functor of the q-th component of the cosimplicial chain complex, $q \geq 1$) is corepresentable, with the model $D[q]$. To prove this statement, observe that, for $C_*^\bullet \in \mathcal{C}h^{\Delta}$ and $n \in \mathbb{Z}$, (5) implies

$$\underline{\mathcal{C}h}^{\Delta}(C_*^\bullet, D[q])_n \cong \underline{\mathcal{C}h}^{\Delta^{op}}(M[q], C_*^{\bullet\#})_n$$

and that we have the following isomorphism in the category of sets

$$\underline{\mathcal{C}h}^{\Delta^{op}}(M[q], C_*^{\bullet\#})_n \cong \mathcal{S}et^{\Delta^{op}}(\Delta[q], {C_{-n}^\bullet}^{\#}) \cong {C_{-n}^q}^{\#},$$

therefore

$$\underline{\mathcal{C}h}^{\Delta}(C_*^\bullet, D[q])_* \cong C_*^{q\#},$$

see (4). Using the above isomorphisms, one easily sees that (in the category of sets again)

$$\begin{aligned} \widetilde{F}(A)_n &= \underline{\mathcal{C}hain}(\underline{\mathcal{C}h}^{\Delta}(C_*^\bullet, D[q]), D[q]^q)_n \cong \underline{\mathcal{C}hain}(C_*^{q\#}, D[q]^q)_n \\ &\cong \underline{\mathcal{C}hain}(M[q]_q, C_*^q)_n \cong \mathcal{S}et(\Delta[q]_q, C_n^q). \end{aligned} \tag{6}$$

With this identification, we define $\chi : \mathcal{S}et(\Delta[q]_q, C_*^q) \to C_*^q$ as the evaluation at the (unique) nondegenerate q-simplex e_q in $\Delta[q]$, $\chi(\phi) := \phi(e_q)$. One easily verifies that indeed $\chi\lambda = id_F$ and that the morphisms χ are actually components of a dg-natural transformations (even though we do not need dg-naturality for our purpose).

3.4. **Example.** For a cosimplicial chain complex $C_*^\bullet$ and the crossed interval group $\mathcal{S}$ introduced in Example 2.4, we denoted $F_{\mathcal{S}}(C_*^\bullet)$ the $\mathcal{S}$-chain complex generated by $C_*^\bullet$, see Lemma 2.10. Its qth component equals the chain complex

$$F_{\mathcal{S}}(C_*^\bullet)^q := C_*^q \times S_q,\ q \geq 0.$$

Given the calculations of Example 3.3, it is easy to show that also the functor $F : \mathcal{C}h^{\Delta} \to \mathcal{C}hain$ defined by $F(C_*^\bullet) := F_{\mathcal{S}}(C_*^\bullet)^q$ is corepresentable, with the model $D[q]$. As in (6) we observe that

$$\widetilde{F}(C_*^\bullet)_n \cong \mathcal{S}et(\Delta[q]_q, C_n^q) \times S_q,\ n \in \mathbb{Z},$$

and use this isomorphism to define the transformation $\chi : \mathcal{S}et(\Delta[q]_q, C_*^q) \times S_q \to F_{\mathcal{S}}(C_*^\bullet)^q$ by $\chi(\phi, \sigma) := (\phi(e_q), \sigma)$, for $\phi \in \underline{\mathcal{C}hain}(M[q]_q, C_*^q)$ and $\sigma \in S_q$.

3.5. **Example.** Examples 3.3 and 3.4 can be generalized as follows. Let $\mathcal{C}$ be a small category enriched over a closed symmetric monoidal category V. Assume that, for any two $a, b \in \mathcal{C}$, the object $\mathcal{C}(a,b) \in V$ is strongly dualisable. Let $V^{\mathcal{C}}$ be the enriched category of enriched functors $\mathcal{C} \to V$ and their enriched natural transformations. Let $M_a, a \in \mathcal{C}$, be a representable presheaf i.e. the functor $M_a : \mathcal{C}^{\rm op} \to V$ given by $M_a(b) := \mathcal{C}(b,a)$, for $b \in \mathcal{C}$. Let, finally, $D_a : \mathcal{C} \to V$ be the functor defined as $D_a(b) := M_a^{\#}(b)$.

Denote by $Dual \subset V$ be the subcategory of strongly dualisable objects. Let q be an object of $\mathcal{C}$. We leave as an exercise to verify that the enriched functor $F : Dual^{\mathcal{C}} \to V$, $F(B) := B(q)$, is corepresentable (in the sense of an obvious generalization of Definition 3.2) with the model D_q.

Example 3.3 is then a particular case of the above generalization with $V := \mathcal{C}hain$ and $\mathcal{C}$ the category with the same objects as Δ but the enriched Hom-sets $\mathcal{C}(a,b)$ the linear span of $\Delta(a,b)$ considered as the chain complex concentrated in degree zero and Example 3.4 admits a similar treatment.

3.6. **Proposition.** *Let $\mathcal{D}$ be a dg-category and $\mathcal{C}och\mathcal{C}hain$ the category of cochain complexes of chain complexes. Let $A_*^*, B_*^* : \mathcal{D} \to \mathcal{C}och\mathcal{C}hain$ be dg-functors and $f_*^*, g_*^* : A_*^* \to B_*^*$ (not necessarily dg) transformations such that $f_*^0 = g_*^0$. Suppose moreover that*

(i) $B_*^q : \mathcal{D} \to \mathcal{C}hain$ is corepresented with models $\mathfrak{M}$, for each $q \geq 1$,

(ii) $A_*^q(M)$ is a finitely generated torsion-free chain complex concentrated in degree 0*, for each $q \geq 1$, $M \in \mathfrak{M}$, and*

(iii) $H^{\geq 1}(A_0^*(M), d) = 0$, for each $M \in \mathfrak{M}$.

Then there exists a natural cochain homotopy $H_*^* : f_*^* \sim g_*^*$ that commutes with the vertical (chain) differentials.

Proof. Let us start the proof by showing that, for each $M \in \mathfrak{M}$ one has, in the cochain complex

$$0 \longrightarrow A_0^0(M) \xrightarrow{d^0} A_0^1(M) \xrightarrow{d^1} A_0^2(M) \xrightarrow{d^2} \cdots \xrightarrow{d^{m-1}} A_0^m(M) \xrightarrow{d^m} \cdots,$$

a direct sum decomposition

$$A_0^m(M) \cong Im(d^{m-1}) \oplus D^m, \ m \geq 1, \tag{7}$$

such that $d^m|_{D^m}$ is a monomorphism. To verify this statement, consider the short exact sequence

$$0 \longrightarrow Ker(d^m) \longrightarrow A_0^m(M) \xrightarrow{d^m} Im(d^m) \longrightarrow 0.$$

Since $Im(d^m)$ is, by (ii), a subgroup of the finitely generated torsion-free abelian group $A_0^{m+1}(M)$, it is free and the above sequence splits, giving rise to an isomorphism

$$A_0^m(M) \cong Ker(d^m) \oplus D^m$$

such that $d^m|_{D^m}$ is a monomorphism. The acyclicity (iii) implies that $Ker(d^m) \cong Im(d^{m-1})$ which gives (7).

Let us return to the proof of the proposition. Assume we have already constructed transformations $H^i_* : A^i_* \to B^{i-1}_*$, $1 \le i \le n$, such that

$$H^{i+1}_* d^i + d^{i-1} H^i_* = f^i_* - g^i_*,$$

for each $0 \le i < n$, that moreover commute with the vertical differentials. We need to construct a transformation $H^{n+1}_* : A^{n+1}_* \to B^n_*$ such that

$$H^{n+1}_* d^n + d^{n-1} H^n_* = f^n_* - g^n_*. \tag{8}$$

Fix $M \in \mathfrak{M}$ and define a linear map $h^{n+1}_M : A^{n+1}_0(M) \to B^n_0(M)$ using decomposition (7) as follows. On $Im(d^n)$, h^{n+1}_M is defined by the equation

$$h^{n+1}_M d^n(x) = -d^{n-1} H^n_0(x) + (f^n_0 - g^n_0)(x),\ x \in D^n. \tag{9}$$

This definition is correct since d^n induces an isomorphism $d^n|_{D^n} : D^n \cong Im(d^n)$. On D^{n+1}, h^{n+1}_M can be chosen arbitrarily.

Let us verify that the map h^{n+1}_M defined above satisfies the following equality in $B^n_0(M)$:

$$(h^{n+1}_M d^n + d^{n-1} H^n_0)(a) = (f^n_0 - g^n_0)(a),\ \text{for } a \in A^n_0(M). \tag{10}$$

By (7), each $a \in A^n_0(M)$ decomposes as $a = d^{n-1}(u) + x$, with some $u \in A^{n-1}_0(M)$ and $x \in D^n$. The left hand side of (10) then equals

$$(h^{n+1}_M d^n + d^{n-1} H^n_0)(d^{n-1}(u) + x) = h^{n+1}_M d^n(x) + d^{n-1} H^n_0 d^{n-1}(u) + d^{n-1} H^n_0(x). \tag{11}$$

By induction,

$$H^n_0 d^{n-1}(u) = -d^{n-2} H^{n-1}_0(u) + (f^{n-1}_0 - g^{n-1}_0)(u),$$

therefore the middle term of the right hand side of (11) equals

$$d^{n-1}(f^{n-1}_0 - g^{n-1}_0)(u) = (f^n_0 - g^n_0)(d^{n-1}u).$$

By (9), the remaining two terms of the right hand side of (11) equal $(f^n_0 - g^n_0)(x)$. We conclude that the left hand side of (10) equals

$$(f^n_0 - g^n_0)(d^{n-1}(u) + x) = (f^n_0 - g^n_0)(a),$$

which is the right hand side of (10).

Let us define the natural transformation $\widetilde{H}^{n+1}_* : A^{n+1}_* \to \widetilde{B}^n_*$ by requiring that $k \in A^{n+1}_*(K)$ is mapped into $\prod_{M \in \mathfrak{M}} \widetilde{H}^{n+1}_M(k) \in \widetilde{B}^n_*(K)$, where $\widetilde{H}^{n+1}_M(k) \in \underline{\mathfrak{Chain}}(\underline{\mathcal{D}}(K,M)_*, B^n_*(M))$ takes a morphism $\mu \in \underline{\mathcal{D}}(K,M)_*$ into $h^{n+1}_M A^{n+1}(\mu)(k) \in B^n_0(M)$. We claim that

$$\widetilde{H}^{n+1}_* d^n = \lambda(-d^{n-1} H^n_* + f^n_* - g^n_*) \tag{12}$$

(equality of transformations of functors $A^{n+1}_* \to \widetilde{B}^n_*$). We must prove that, for each $l \in A^n_*(K)$,

$$\widetilde{H}^{n+1}_* d^n(l) = \lambda(-d^{n-1} H^n_* + f^n_* - g^n_*)(l)$$

(equality of elements of $\widetilde{B}^n_*(K)$). By definition, the M-component of the left hand side evaluated at $\mu \in \underline{\mathcal{D}}(K,M)_*$ equals

$$\widetilde{H}^{n+1}_M(d^n l)(\mu) = h^{n+1}_M A^{n+1}(\mu)(d^n l) = h^{n+1}_M d^n A^n(\mu)(l).$$

Similarly, the M-component of the right hand side evaluated at $\mu \in \underline{\mathcal{D}}(K,M)_*$ equals

$$\lambda_M((-d^{n-1}H^n_* + f^n_* - g^n_*)(l))(\mu) =$$
$$= B^n(\mu)(-d^{n-1}H^n_* + f^n_* - g^n_*)(l) = (-d^{n-1}H^n_* + f^n_* - g^n_*)A^n_*(\mu)(l).$$

Equation (12) now follows from (10) and immediately implies that the natural transformation $H^{n+1}_* := \chi \widetilde{H}^n_*$ satisfies (8). It is not difficult to verify that H^{n+1}_* commutes with the vertical differentials. □

Proposition 3.6 can be easily generalized into the following statement, compare Example 3.5.

3.6b. **Proposition.** *Let V be an abelian symmetric monoidal closed category. Let $\mathcal{D}$ be a V-category and $\mathcal{C}ochV$ the category of cochain complexes in V. Let $A^*, B^* : \mathcal{D} \to \mathcal{C}ochV$ be V-functors and $f^*, g^* : A^* \to B^*$ natural transformations such that $f^0 = g^0$. Suppose that*

(i) $B^q : \mathcal{D} \to V$ is corepresented with models $\mathfrak{M}$, for each $q \geq 1$,

(ii) for $m \geq 1$ the short exact sequence $0 \to Ker(d^m) \to A^m(M) \stackrel{d^m}{\to} Im(d^m) \to 0$ splits, and

(iii) $H^{\geq 1}(A^*(M), d) = 0$, for each $M \in \mathfrak{M}$.

Then there exists a natural cochain homotopy $H^* : f^* \sim g^*$.

Proposition 3.6 has the following important corollary.

3.7. **Corollary.** *In the situation of Proposition 3.6, the homotopy $H^*_* : f^*_* \sim g^*_*$ determines, for each $K \in \mathcal{D}$, a natural cochain homotopy between the induced maps* $\mathrm{Tot}(f^*_*)^*, \mathrm{Tot}(g^*_*)^* : \mathrm{Tot}(A^*_*(K))^* \to \mathrm{Tot}(B^*_*(K))^*$.

Proof. A direct verification. □

In Lemma 2.10 we mentioned that, for a cosimplicial chain complex $C^\bullet_*$, there exists the canonical inclusion of cosimplicial chain complexes

$$\iota : C^\bullet_* \to F_{\mathcal{S}}(C^\bullet_*),$$

given by $\iota(a) := (a, 1)$. It induces the cochain map (which we denote by the same symbol) $\iota : \mathrm{Tot}(C^\bullet_*)^* \to \mathrm{Tot}(F_{\mathcal{S}}(C^\bullet_*))^*$.

3.8. **Definition.** The *miraculous map* $m : \mathrm{Tot}(F_{\mathcal{S}}(C^\bullet_*))^* \to \mathrm{Tot}(C^\bullet_*)^*$ is defined by the formula

$$m(a,\sigma) := \mathrm{sgn}(\sigma)a,\ a \in C^q_*,\ \sigma \in S_q,\ q \geq 0.$$

We call m miraculous because it is not induced by a cosimplicial map. Obviously $m \circ \iota = id$.

3.9. **Proposition.** *The miraculous map m is a cochain map.*

Proof. Recall (see Lemma 2.10) that the differential acts on $(a, \sigma) \in C^q_* \times S_q$ by

$$d^q(a,\sigma) = \sum_{0 \le i \le q+1} (-1)^i (d_{\overline{\sigma}(i)}(a), d_i(\sigma)),$$

where $\overline{\sigma} : \{0, \dots, q+1\}$ fixes $0, q+1$ and equals σ on $\{1, \dots, q\}$. The coboundary operators $d_i : S_q \to S_{q+1}$ are given by $d_0(\sigma) := \mathbb{1} \times \sigma$, $d_{q+1}(\sigma) := \sigma \times \mathbb{1}$ and $d_i(\sigma)$ is the permutation obtained by doubling the ith input of σ. These coboundary operators were introduced in [11, Example 12]. So, for $(a, \sigma) \in C^q_* \times S_q = F_{\mathcal{S}}(C^\bullet_*)^q_*$,

$$md^q(a,\sigma) = \sum_{0 \le i \le q+1} (-1)^i \mathrm{sgn}(d_i(\sigma)) d_{\overline{\sigma}(i)}(a)$$

while

$$d^q m(a,\sigma) = \sum_{0 \le i \le q+1} (-1)^i \mathrm{sgn}(\sigma) d_i(a) = \sum_{0 \le i \le q+1} (-1)^{\overline{\sigma}(i)} \mathrm{sgn}(\sigma) d_{\overline{\sigma}(i)}(a).$$

Comparing the terms at $d_{\overline{\sigma}(i)}(a)$ one sees that m will be a cochain map if we prove that

$$(-1)^{\overline{\sigma}(i)} \mathrm{sgn}(\sigma) = (-1)^i \mathrm{sgn}(d_i(\sigma)), \quad \text{for } 0 \le i \le q+1.$$

The last equality can be established by a direct verification. □

The rest of this section is devoted to the proof of:

3.10. **Theorem.** *Let $C^\bullet_*$ be a cosimplicial chain complex with finitely generated torsion-free components. Then there is a natural cochain homotopy equivalence between* $\mathrm{Tot}(F_{\mathcal{S}}(C^\bullet_*))^*$ *and* $\mathrm{Tot}(C^\bullet_*)^*$.

We already constructed natural cochain maps

$$\iota : \mathrm{Tot}(C^\bullet_*)^* \to \mathrm{Tot}(F_{\mathcal{S}}(C^\bullet_*))^* \text{ and } m : \mathrm{Tot}(F_{\mathcal{S}}(C^\bullet_*))^* \to \mathrm{Tot}(C^\bullet_*)^*$$

such that $m \circ \iota = id$. Therefore it remains to prove the existence of a natural cochain homotopy between $\iota \circ m$ and id. This will be done by applying Proposition 3.6 and its Corollary 3.7 to the situation $\mathcal{D} = \mathcal{C}h^\Delta$, $A^*_* = B^*_* = |F_{\mathcal{S}}(-)|^*_*$, $f^*_* = \iota \circ m$, $g^*_* = id$ and $\mathfrak{M} = \{D[0], D[1], D[2], \dots\}$. We already established, in Example 3.4, that $|F_{\mathcal{S}}(-)|^q_* = F_{\mathcal{S}}(-)^q_*$ is corepresented, with models $\mathfrak{M}$. It is also obvious that $(\iota \circ m)^0_* = id^0_*$ and that $F_{\mathcal{S}}(D[q])^\bullet_*$ is concentrated in chain degree 0. It remains to verify the acyclicity of models. Since all objects below will be concentrated in chain degree 0, we will omit the lower star indicating the chain grading.

3.11. **Proposition.** *For each $q \ge 0$, $H^{\ge 1}(|F_{\mathcal{S}}(D[q])|^*) = 0$.*

Proposition 3.11 will follow from Lemma 3.14 below. The following example explicitly describes $F_{\mathcal{S}}(D[q])$.

3.12. **Example.** To fix the notation, recall that $\Delta[q]$ is a simplicial set with

$$\Delta[q]_n = \{(a_0, \dots, a_n);\ 0 \le a_0 \le \cdots \le a_n \le q\},\ n \ge 0,$$

the cosimplicial structure being given by

$$\begin{aligned} \partial_i(a_0, \dots, a_n) &:= (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n), \text{ and} \\ s_i(a_0, \dots, a_n) &:= (a_0, \dots, a_j, a_j, \dots, a_n), \text{ for } 0 \le i \le n. \end{aligned}$$

The simplicial abelian group $M[q]$ generated by $\Delta[q]$ has the basis

$$\{(a_0, \dots, a_n);\ 0 \le a_0 \le \cdots \le a_n \le q\}.$$

Let me denote by

$$\{\langle a_0, \dots, a_n\rangle;\ 0 \le a_0 \le \cdots \le a_n \le q\}.$$

the dual basis of $D[q]_n$. The coboundary operators are then given by

$$d_i\langle a_0, \dots, a_n\rangle = \sum_{a_{i-1} \le s \le a_i} \langle a_0, \dots, a_{i-1}, s, a_{i+1}, \dots, a_n)\rangle$$

where we assume $a_{-1} := 0$ and $a_{n+1} := q$ so the formula makes sense also for $i = 0$ or $i = n+1$. For instance, for the generator $\langle 1, 1\rangle \in D[3]_2$ we have

$$d_0\langle 1, 1\rangle = \langle 0, 1, 1\rangle + \langle 1, 1, 1\rangle,\ d_1\langle 1, 1\rangle = \langle 1, 1, 1\rangle,\ d_2\langle 1, 1\rangle = \langle 1, 1, 1\rangle + \langle 1, 1, 2\rangle + \langle 1, 1, 3\rangle.$$

The above formulas show that $D[q]$ is not spanned by a cosimplicial set.

It is clear from the above explanation that elements of $F_{\mathcal{S}}(D[q])^n$ are linear combinations of the expressions of the form

$$(\langle a_0, \dots, a_n\rangle, \sigma), \quad \text{for } 0 \le a_0 \le \cdots \le a_n \le q,\ \sigma \in S_n.$$

The differential in $|F_{\mathcal{S}}(D[q])|^*$ is given by the formula

$$d^n(\langle a_0, \dots, a_n\rangle, \sigma) = \sum_{0 \le i \le n+1} (-1)^i (d_{\overline{\sigma}(i)}\langle a_0, \dots, a_n\rangle, d_i(\sigma)),$$

where the definition of $\overline{\sigma}(i)$ and $d_i(\sigma)$ was recalled in the proof of Proposition 3.9.

In [11, page 481] we introduced simple permutations and showed how each permutation σ determines a simple permutation $\kappa(\sigma)$. Since, for $\sigma \in S_n$ and $1 \le i \le n+1$, $\kappa(\sigma) = \kappa(d_i(\sigma))$, for each cosimplicial chain complex $C_*^\bullet$ one has the decomposition

$$F_{\mathcal{S}}(C_*^\bullet) = \bigoplus_{\chi \text{ simple}} F_{\mathcal{S}}^\chi(C_*^\bullet) \tag{13}$$

in which the subspaces

$$F_{\mathcal{S}}^\chi(C_*^\bullet) := \{(a, \sigma) \in F_{\mathcal{S}}(C_*^\bullet);\ \kappa(\sigma) = \chi\}$$

are invariant under coboundary operators. Let us see what happens if $C_*^\bullet = D[q]$.

[April 7, 2008]

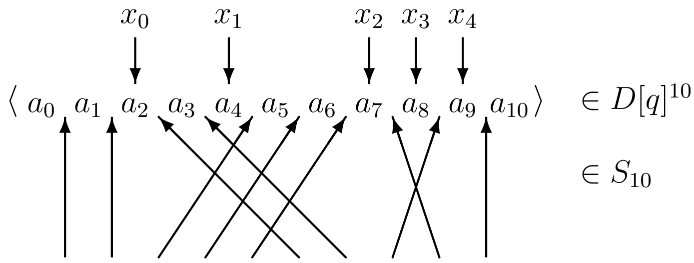


FIGURE 1. An element $(a, \sigma) \in F_{\mathcal{S}}(D[q])^{10}$.

Consider a simple permutation $\chi \in S_m$ different from $\mathbb{1} \in S_1$. I claim that there is a direct sum decomposition

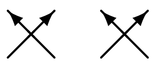
$$F^{\chi}_{\mathcal{S}}(D[q]) = \bigoplus_{0 \leq x_0 \leq \cdots \leq x_m \leq q} F^{\chi}_{\mathcal{S}}(D[q])_{x_0,\ldots,x_m}, \tag{14}$$

where $F^{\chi}_{S}(D[q])_{x_0,\ldots,x_m}$ is the smallest subspace of $F_{\mathcal{S}}(D[q])$ invariant under the coboundary operators and containing

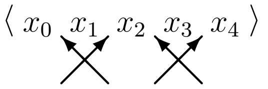
$$(\langle x_0, \ldots, x_m\rangle, \chi) \in F^{\chi}_{S}(D[q])^m.$$

Before I formulate a formal argument that $F^{\chi}_{S}(D[q])$ indeed decomposes as in (14), I give the following example.

3.13. **Example.** Figure 1 symbolizes an element $(a, \sigma) \in F_{\mathcal{S}}(D[q])^{10}$, with $a = \langle a_0, \ldots, a_{10}\rangle$ and $\sigma \in S_{10}$ the permutation determined by the arrows at the bottom of the picture. The corresponding simple permutation $\chi := \kappa(\sigma)$ is

 $\in S_4$

and (a, σ) belongs to the subspace $F^{\chi}_{S}(D[q])_{x_0,\ldots,x_4}$ generated by



$\in F_{\mathcal{S}}(D[q])^4$,

where $(x_0, x_1, x_2, x_3, x_4) := (a_2, a_4, a_7, a_8, a_9)$.

Let us describe how to decide to which subspace $F^{\chi}_{S}(D[q])_{x_0,\ldots,x_m}$ a general $(a, \sigma) \in F_{\mathcal{S}}(D[q])$ belongs. The permutation χ is determined easily, it is the simple permutation $\chi := \kappa(\sigma) \in S_m$ associated to σ. If $a = \langle a_0, \ldots, a_n\rangle$, then $x_0, \ldots, x_m$ is the sequence obtained from $a_0, \ldots, a_n$ by deleting $a_i's$ located between doubled arrows of σ, between $\langle$ and a vertical arrow $\uparrow$, or between $\uparrow$ and $\rangle$. I believe everything is clear from Figure 1. Let finally $a_\sigma := \langle x_0, \ldots, x_m\rangle$. It is easy to verify that then

$$F^{\chi}_{S}(D[q])_{x_0,\ldots,x_m} = \{(a, \sigma) \in F_{\mathcal{S}}(D[q]);\ \kappa(\sigma) = \chi,\ a_\sigma = \langle x_0, \ldots, x_m\rangle\}$$

which makes (14) obvious.

Let $\widehat{D}[q]$ by the 'extended' cosimplicial group obtained by adding to $D[q]$ the piece $\widehat{D}[q]^{-1}$ spanned by the empty bracket $\langle\ \rangle$ and $d_0 : \widehat{D}[q]^{-1} \to \widehat{D}[q]^0$ defined by

$$d_0\langle\ \rangle := \sum_{0\leq s\leq q} \langle s\rangle.$$

We claim that, for each simple permutation $\chi \in S_m$ different from $1\!\!1 \in S_1$, there is an isomorphisms of cochain complexes

$$|F^{\chi}_{S}(D[q])_{x_0,\ldots,x_m}|^* \cong \uparrow^{2m+2} |\widehat{D}[x_0]|^* \otimes |\widehat{D}[x_1-x_0]|^* \otimes \cdots \otimes |\widehat{D}[x_m-x_{m-1}]|^* \otimes |\widehat{D}[q-x_m]|^*, \tag{15}$$

where $\uparrow^{2m+2}$ in the right hand side means that the degrees are shifted up by $2m+2$. For instance, to $(a,\sigma) \in F_{\mathcal{S}}(D[q])^{10}$ in Figure 1 it corresponds

$$\begin{aligned}&\langle a_0, a_1\rangle \otimes \langle a_3\rangle \otimes \langle a_5, a_6\rangle \otimes \langle\ \rangle \otimes \langle\ \rangle \otimes \langle a_{10}\rangle \in\\ &\qquad \widehat{D}[a_2]^1 \otimes \widehat{D}[a_4-a_2]^0 \otimes \widehat{D}[a_7-a_4]^1 \otimes \widehat{D}[a_8-a_7]^{-1} \otimes \widehat{D}[a_9-a_8]^{-1} \otimes \widehat{D}[q-a_9]^0.\end{aligned}$$

Isomorphism (15) is defined as follows: to each $(\langle a_0,\ldots,a_n\rangle,\sigma) \in F^{\chi}_{S}(D[q])^n_{x_0,\ldots,x_m}$ one associates the element of the tensor product in the right hand side obtained by replacing, in $\langle a_0,\ldots,a_n\rangle$, each $x_j \in \{x_0,\ldots,x_m\}$ by $\rangle \otimes \langle$.

Combining isomorphisms (13), (14) and (15) with the obvious equality $F^{1\!\!1}_{\mathcal{S}}(A) \cong D[q]$, we finally get:

3.14. **Lemma.** *For each $q \geq 1$, one has the isomorphism of cochain complexes:*

$$\begin{aligned}&|F_{\mathcal{S}}(D[q])|^* \cong\\ &\quad \cong |D[q]|^* \oplus \bigoplus \uparrow^{2m+2} |\widehat{D}[x_0]|^* \otimes |\widehat{D}[x_1-x_0]|^* \otimes \cdots \otimes |\widehat{D}[x_m-x_{m-1}]|^* \otimes |\widehat{D}[q-x_m]|^*,\end{aligned}$$

where the summation in the right hand side is taken over all simple permutations $\chi \in S_m$ different from $1\!\!1 \in S_1$ and all $0 \leq x_0 \leq \cdots \leq x_m \leq q$.

Proof of Proposition 3.11. It is a standard fact that $H^i(|D[q]|^*) = 0$ for all $i \geq 1$ and that $H^i(|\widehat{D}[q]|^*) = 0$ for all i. The proposition then follows from Lemma 3.14 and the Künneth theorem. □

A particular case. Proposition 3.6 remains true if one assumes that all objects are concentrated in chain degree zero, that is, when $\mathcal{D}$ is enriched over abelian groups and the category $\mathcal{C}och\mathcal{C}hain$ is replaced by the category $\mathcal{C}och$ of cochain complexes of abelian groups. Proposition 3.6 then becomes a dual version of the classical Acyclic Models Theorem (see, for instance, [13, Theorem 28.3]).

[April 7, 2008]

4. Natural operations on the Hochschild cohomology

Let $B_*^{A\text{-}A}(A) = B_*(A, A, A)$ be the two-sided bar construction of a unital associative algebra $A = (A, \mu, 1)$ so that $B_n^{A\text{-}A}(A) = A \otimes A^{\otimes n} \otimes A$ is the free A-bimodule on $A^{\otimes n}$, $n \geq 0$. It extends into a functor $B_*^{A\text{-}A}(A) : \mathcal{I}S \to A\text{-}\mathtt{biMod}$ from the crossed interval group $\mathcal{I}S$ introduced in Definition 2.4 to the category A-$\mathtt{biMod}$ of A-bimodules as follows.

On objects, $B_*^{A\text{-}A}(A)(\langle n \rangle) := B_{n+1}^{A\text{-}A}(A)$. For $a_{-1} \otimes a_0 \otimes \cdots \otimes a_m \otimes a_{m+1} \in B_{m+1}^{A\text{-}A}(A)$ and morphism $f : \langle m \rangle \to \langle n \rangle$ in $\mathcal{I}S$ put

$$B_*^{A\text{-}A}(A)(f)(a_{-1} \otimes a_0 \otimes \cdots \otimes a_m \otimes a_{m+1}) := \overline{a}_{-1} \otimes \overline{a}_0 \otimes \cdots \otimes \overline{a}_n \otimes \overline{a}_{n+1} \in B_{n+1}^{A\text{-}A}(A),$$

where $\overline{a}_i$ is, for $-1 \leq i \leq n+1$, the product of a_j's, $j \in f^{-1}(i)$, in the order specified by the linear order on the fiber $f^{-1}(i)$. If $f^{-1}(i) = \emptyset$ we put $\overline{a}_i := 1$.

It is easy to see that the composition $\Delta^{\mathrm{op}} \overset{joy}{\cong} \mathcal{I} \hookrightarrow \mathcal{I}S \xrightarrow{B_*^{A\text{-}A}(A)} A\text{-}\mathtt{biMod}$ is the standard simplicial A-A-bimodular structure of the bar construction. The functor $B_*^{A\text{-}A}(A) : \mathcal{I}S \to A\text{-}\mathtt{biMod}$ is in fact induced from the *symmetric bar construction* $B_*^{sym}(A) : \Delta S \to \mathbf{k}\text{-}\mathtt{Mod}$ introduced in [4], by the diagram

$$\begin{tikzcd} \mathcal{I}S \arrow[r, "B_*^{A\text{-}A}(A)"] \arrow[d, "\iota"'] & A\text{-}\mathtt{biMod} \arrow[d, "forget"] \\ \Delta S \arrow[r, "B_*^{sym}(A)"] & \mathbf{k}\text{-}\mathtt{Mod} \end{tikzcd}$$

in which ι is the inclusion (2).

Consequently, for any A-bimodule M, the space $C^*(A; M) := \mathrm{Hom}_{A\text{-}A}(B_*^{A\text{-}A}(A), M) \cong \mathrm{Lin}_{\mathbf{k}}(A^{\otimes *}, M)$ of Hochschild cochains of A with coefficients in M is an $(\mathcal{I}S)^{\mathrm{op}}$-$\mathbf{k}$-module, with $\mathbf{k}$ denoting the ground field. This structure will be crucial for us.

The big operad. Let us recall the dg-operad $\mathcal{B} = \{\mathcal{B}(n)\}_{n \geq 0}$ ($\mathcal{B}$ abbreviating the 'big') of *all natural multilinear operations* on the Hochschild cochain complex $C^*(A; A)$ of a 'generic' associative algebra A with coefficients in itself. A moment's reflection shows that any thinkable natural operation must be a linear combination of compositions of the following 'elementary' operations:

(a) The insertion $\circ_i : C^k(A; A) \otimes C^l(A; A) \to C^{k+l-1}(A; A)$ given, for $k, l \geq 0$ and $1 \leq i \leq k$, by the formula

$$\circ_i(f, g)(a_1, \ldots, a_{k+l-1}) := f(a_0, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+l-1}), a_{i+l}, \ldots, a_{k+l-1}).$$

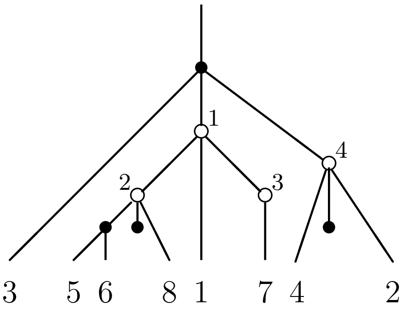


FIGURE 2. An $(8;3,3,1,3)$-tree representing an operation in $\mathcal{B}^8_{3,3,1,3}$. It has 4 white vertices, 2 black vertices and 2 stubs. We use the convention that directed edges point upwards.

(b) Let $\mu : A \otimes A \to A$ be the associative product, $id : A \to A$ the identity and $1 \in A$ the unit. Then elementary operations are also the 'constants' $\mu \in C^2(A;A)$, $id \in C^1(A;A)$ and $1 \in C^0(A;A)$.

(c) The assignment $f \mapsto f\sigma$ permuting the inputs of a cochain $f \in C^k(A;A)$ according to a permutation Σ_k is an elementary operation.

Let $B^l_{k_1,\ldots,k_n}$ denote, for $l, k_1, \ldots, k_n \geq 0$, the vector space of all natural operations

$$O : C^{k_1}(A;A) \otimes \cdots \otimes C^{k_n}(A;A) \to C^l(A;A).$$

Elements of $B^l_{k_1,\ldots,k_n}$ can be represented by linear combinations of $(l;k_1,\ldots,k_n)$-trees in the sense of the following definition in which, as usual, the *arity* of a vertex of a rooted tree is the number of its input edges and the *legs* are the input edges of a tree, see [12, II.1.5] for the terminology.

4.1. **Definition.** Let $l, k_1, \ldots, k_n$ be non-negative integers. An $(l;k_1,\ldots,k_n)$-tree is a planar tree with legs labeled by $1,\ldots,l$ and three types of vertices:

(a) 'white' vertices of arities $k_1,\ldots,k_n$ labeled by $1,\ldots,n$,
(b) 'black' vertices of arities ≥ 2 and
(c) 'special' vertices of arity 0 (no input edges).

We moreover require that there are no edges connecting two black vertices or a black vertex with a special vertex. For $n = 0$ we allow also the exceptional tree $|$ with no vertices.

We call an edge whose initial vertex is special a *stub* (also called, in [9], a *tail*). It follows from definition that the terminal vertex of a stub is white. An example of an $(l;k_1,\ldots,k_n)$-tree is given in Figure 2.

An $(l;k_1,\ldots,k_n)$-tree T as in Definition 4.1 determines the natural operation $O_T \in B^l_{k_1,\ldots,k_n}$ given by decorating, for each $1 \leq i \leq n$, the ith white vertex by $f_i \in C^{k_i}(A;A)$, the black vertices by the iterated multiplications, the special vertices by the unit 1, and preforming the composition along the tree. For instance, the tree in Figure 2 represents the operation

$$O(f_1, f_2, f_3, f_4)(a_1, \ldots, a_8) := a_3 f_1(f_2(a_5 a_6, 1, a_8), a_1, f_3(a_7)), f_4(a_4, 1, a_2))$$

where, as usual, we omit the symbol for the iteration of the associative multiplication μ. The exceptional $(1;)$-tree $|$ represents the identity $id \in C^1(A; A)$.

The spaces $B^l_{k_1,\ldots,k_n}$ form a colored operad B, with the set of colors equaling the set of natural numbers. The spaces B^l_k (i.e. $B^l_{k_1,\ldots,k_n}$ with $n = 1$ and $k = k_1$) are the Hom-sets of the underlying category of this colored operad. It turns out that this category is in fact a linearization of $(\mathcal{I}S)^{\mathrm{op}}$, by which we mean that

$$Span(\mathcal{I}S(\langle l-1\rangle, \langle k-1\rangle) \xrightarrow{\cong} B^l_k$$

for each $k, l \geq 0$, where $Span(-)$ denotes the $\mathbf{k}$-linear span. The above isomorphism is induced by the set-map that assigns to each morphism $g : \langle l-1\rangle \to \langle k-1\rangle$ in $\mathcal{I}S$ the natural operation $O_g : C^k(A; A) \to C^l(A; A) \in B^l_k$ constructed as follows.

Let $\overline{g} : \{0, \ldots, l-1\} \to \{-1, \ldots, k\} = \langle k-1\rangle$ be the restriction $\overline{g} := g|_{\{0,\ldots,l-1\}}$ and denote, for each $-1 \leq i \leq k$ for which $\overline{g}^{-1}(i)$ is non-empty

$$\overline{g}^{-1}(i) = (\sigma^i_1, \ldots, \sigma^i_{s(i)}),$$

where the numbers $\sigma^i_1, \ldots, \sigma^i_{s(i)} \in \{0, \ldots, l-1\}$ are ordered according to the restriction of the order of the fiber $g^{-1}(i)$ to its subset $\overline{g}^{-1}(i) \subset g^{-1}(i)$.

For $f : A^{\otimes k} \to A \in C^k(A; A)$ the cochain $O_g(f) : A^{\otimes l} \to A \in C^l(A; A)$ is given by the formula

$$O_g(f)(a_0, \ldots, a_{l-1}) = \overline{a}_{-1} \cdot f(\overline{a}_0, \ldots, \overline{a}_{k-1}) \cdot \overline{a}_k,$$

where, for $a_0, \ldots, a_{l-1} \in A$ and $i \in \{-1, \ldots, k\}$,

$$\overline{a}_i := \begin{cases} a_{\sigma^i_1} \cdot \ldots \cdot a_{\sigma^i_{s(i)}}, & \text{if } \overline{g}^{-1}(i) \neq \emptyset, \text{ and} \\ 1, & \text{otherwise.} \end{cases}$$

The $(l; k)$-tree encoding the operation O_g is depicted in Figure 3.

It follows from general properties of colored operads that B^l_k acts on $B^l_{k_1,\ldots,k_n}$ covariantly on the upper index and contravariantly on each lower index. Therefore, the spaces $B^l_{k_1,\ldots,k_n}$ assemble into a functor

$$\mathcal{B}^{\bullet}_{\bullet_1,\ldots,\bullet_n} : (\mathcal{I}S)^{\mathrm{op}} \times (\mathcal{I}S)^n \to \mathbf{k}\text{-}\mathtt{Mod}.$$

Precomposing this functor with the canonical functor

$$\Delta \times (\Delta^{\mathrm{op}})^n \to (\mathcal{I}S)^{\mathrm{op}} \times (\mathcal{I}S)^n$$

we get the functor

$$B^{\bullet}_{\bullet_1,\ldots,\bullet_n} : \Delta \times (\Delta^{\mathrm{op}})^n \to \mathbf{k}\text{-}\mathtt{Mod}.$$

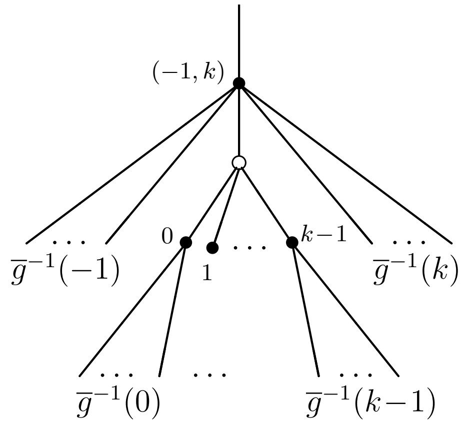


FIGURE 3. The (l,k)-tree representing the operation $O_g \in B^l_k$. It has one white vertex and $k+1$ black vertices. To each black vertex one attaches legs as indicated in the picture, labelled by the elements of the fibres of $\overline{g}$. If the corresponding fiber is empty, the black vertex becomes a special one – in the above picture this is the vertex labelled 1.

We can totalise the functor $B^{\bullet}_{\bullet_1,\ldots,\bullet_n}$ (i.e. we apply the cosimplicial $\overline{\mathrm{Tot}}$ with respect to the upper index and multisimplicial $\underline{\mathrm{Tot}}$ with respect to the lower indices). That is, for any $n \geq 0$ we put

$$\mathcal{B}^*(n) := \overline{\mathrm{Tot}}(\underline{\mathrm{Tot}} B^{\bullet}_{\bullet_1,\ldots,\bullet_n}) = \prod_{l-(k_1+\cdots+k_n)=*} B^l_{k_1,\ldots,k_n}.$$

So, the degree $+1$ differential $d : \mathcal{B}^* \to \mathcal{B}^{*+1}$ is given by $d := \delta - (\partial_1 + \cdots + \partial_n)$, where δ (resp. each ∂_i, $1 \leq i \leq n$) is induced from the corresponding cosimplicial (resp. simplicial) structure. It is evident that the collection $\mathcal{B} = \{\mathcal{B}^*(n)\}_{n\geq 0}$ with the operadic composition given by the composition of multilinear operations, is a dg operad.

The structure of the operad $\mathcal{B}$ is visualized in Figure 4. We need to emphasize that the structure there **is not** a bicomplex; the degree m-piece of $\mathcal{B}(n)$ is the direct *product*, not the direct sum, of elements on the diagonal $p+q=m$. Therefore the usual spectral sequence arguments do not apply. For instance, it can be shown that all rows in Figure 4 are acyclic, but $\mathcal{B}(n)$ is not acyclic!

Variants. An important suboperad of $\mathcal{B}$ is the suboperad $\overline{\mathcal{B}}$ generated by trees *without* stubs. The operad $\overline{\mathcal{B}}$ is the operad of all natural multilinear operations on the Hochschild complex of a *non-unital* associative algebra. It is generated by natural operations (a)–(c) above but without the unit $1 \in C^1(A;A)$ in (b).

There is also a suboperad $\mathcal{T}$ of $\mathcal{B}$ generated by elementary operations of types (a) and (b) only, without the use of permutations in (c). Its arity-n piece equals

$$\mathcal{T}^*(n) := \overline{\mathrm{Tot}}(\underline{\mathrm{Tot}} T^{\bullet}_{\bullet_1,\ldots,\bullet_n})^* = \prod_{l-(k_1+\cdots+k_n)=*} T^l_{k_1,\ldots,k_n},$$

$$
\begin{tikzcd}
 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 \arrow[r] & B_0^0(n) \arrow[r] \arrow[u] & B_0^1(n) \arrow[r] \arrow[u] & B_0^2(n) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & B_1^0(n) \arrow[r] \arrow[u] & B_1^1(n) \arrow[r] \arrow[u] & B_1^2(n) \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
0 \arrow[r] & B_{n-1}^0(n) \arrow[r] \arrow[u] & B_{n-1}^1(n) \arrow[r] \arrow[u] & B_{n-1}^2(n) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[r] \arrow[u] & B_n^1(n) \arrow[r] \arrow[u] & B_n^2(n) \arrow[r] \arrow[u] & \cdots \\
 & & 0 \arrow[r] \arrow[u] & B_{n+1}^2(n) \arrow[r] \arrow[u] & \cdots \\
 & & & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}
$$

FIGURE 4. diagram, $B_k^m(n) := \prod_{k_1+\cdots+k_n=k} B^m_{k_1,\ldots,k_n}$. The vertical arrows are the simplicial differentials ∂ and the horizontal arrows are the cosimplicial differentials δ.

where operations in $T^l_{k_1,\ldots,k_n}$ are represented by linear combinations of *unlabeled* $(l; k_1, \ldots, k_n)$-trees, that is, planar trees as in Definition 4.1 but without the labels of the legs. The inclusion $T^l_{k_1,\ldots,k_n} \hookrightarrow B^l_{k_1,\ldots,k_n}$ is realized by labeling the legs of an unlabeled tree counterclockwise in the orientation given by the planar embedding. The operad $\mathcal{T}$ is a chain version of the operad considered in [15, Section 3].

There is finally the operad $\overline{\mathcal{T}} := \overline{\mathcal{B}} \cap \mathcal{T}$ generated by unlabeled trees without stubs. It turns out that all these operads have the same homotopy type, see Theorem 4.3 below.

4.2. **Proposition.** *The functor $B^{\bullet}_{\bullet_1,\ldots,\bullet_n}$ is the result of application of the functor $F_{\mathbb{S}}$ described in Lemma 2.10 to the functor $T^{\bullet}_{\bullet_1,\ldots,\bullet_n}$. Therefore,*

$$\mathcal{B} = \overline{\mathrm{Tot}}(\underline{\mathrm{Tot}}(F_{\mathbb{S}} T^{\bullet}_{\bullet_1,\ldots,\bullet_n})) = \overline{\mathrm{Tot}}(F_{\mathbb{S}} \underline{\mathrm{Tot}}(T^{\bullet}_{\bullet_1,\ldots,\bullet_n})),$$

and the inclusion $\mathcal{T} \to \mathcal{B}$ is induced by the unit of the monad $F_{\mathbb{S}}$. The same relationship holds between $\overline{\mathcal{T}}$ and $\overline{\mathcal{B}}$.

Proof. The proposition easily follows from the explicit description of $\mathcal{B}$ and $\mathcal{T}$ given above. $\square$

Operads of braces. There is another very important suboperad $\mathcal{B}r$ of $\mathcal{B}$ generated by braces, cup-products and the unit introduced, under the notation $\mathcal{H}$, in [14, Section 1]. Let us recall its definition. The operad $\mathcal{B}r$ is the suboperad of the big operad $\mathcal{B}$ generated by the following operations.

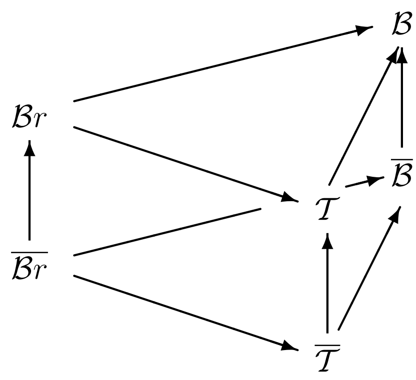


FIGURE 5. Operads of natural operations and their maps.

(a) The *cup product* $C^*(A;A) \otimes C^*(A;A) \to C^*(A;A)$ given, for $f \in C^k(A;A)$ and $g \in C^l(A;A)$, by

$$(f \cup g)(a_1, \ldots, a_{k+l}) := f(a_1, \ldots, a_k)g(a_{k+1}, \ldots, a_{k+l}).$$

(b) The constant $1 \in C^0(A;A)$.

(c) The *braces* $-\{-, \ldots, -\} : C^*(A;A)^{\otimes n} \to C^*(A;A)$, $n \geq 2$, given by

$$f\{g_2, \ldots, g_n\} := \sum (-1)^\epsilon f(id, \ldots, id, g_2, id, \ldots, id, g_n, id, \ldots, id),$$

where id is the identity map of A, the summation runs over all possible substitutions of $g_2, \ldots, g_n$ (in that order) into f and

$$\epsilon := \sum_{j=2}^{n} (\deg(g_i) - 1)t_j,$$

in which t_j is the total number of inputs in front of g_j.

The brace operad has also its non-unital version $\overline{\mathcal{B}r} := \overline{\mathcal{B}} \cap \mathcal{B}r$ generated by elementary operations (a) and (c). One can verify that both $\mathcal{B}r$ and $\overline{\mathcal{B}r}$ are indeed dg-suboperads of $\mathcal{B}$, see [14].

The operads introduced above can be organized into the diagram of inclusions shown in Figure 5. The main result of this section claims:

4.3. **Theorem.** *All operads in Figure 5 have the chain homotopy type of the operad $C_{-*}(\mathcal{D})$ of singular chains on the little disks operad $\mathcal{D}$ with the inverted grading. In particular, the big operad $\mathcal{B}$ of all natural operations has the homotopy type of $C_{-*}(\mathcal{D})$.*

Proof. It is easy to see that the operad $\overline{\mathcal{B}}$ is the simplicial normalization of the big operad $\mathcal{B}$, therefore the inclusion $\overline{\mathcal{B}} \hookrightarrow \mathcal{B}$ is a homotopy equivalence. The same relationship holds also between the operads $\overline{\mathcal{T}}$ and $\mathcal{T}$. The inclusion $\mathcal{T} \hookrightarrow \mathcal{B}$ is a weak equivalence by Proposition 4.2 and Theorem 3.10. The fact that also the inclusions $\mathcal{B}r \hookrightarrow \mathcal{T}$ and $\overline{\mathcal{B}r} \hookrightarrow \overline{\mathcal{T}}$ are weak equivalences will be proved in [1]. Our theorem now follows from the above observations and the fact, proved in [14], that the operad $\mathcal{B}r$ (denoted by $\mathcal{H}$ in [14]), has the homotopy type of $C_{-*}(\mathcal{D})$. $\square$

[April 7, 2008]

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