Secondary cohomology and the Steenrod square

To Jan–Erik Roos on his sixty–fifth birthday

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Abstract

For a field k we choose the Eilenberg–MacLane space $Z^n = K(k,n)$ by the realization of the simplicial k–vector space generated by the non–basepoint singular simplices of the n–sphere $S^n = S^1 \wedge \ldots \wedge S^1$. The permutation of the smash product factors S^1 yields an action of the symmetric group σ_n on S^n and hence on Z^n. Moreover the quotient map $S^n \times S^m \to S^{n+m}$ induces a cup product map $\mu : Z^n \times Z^m \to Z^{m+n}$; see the Appendix below.

It is well known that the (reduced) cohomology $\widetilde{H}^n(X,k)$ of a pointed space X is the same as the set $[X, Z^n]$ of homotopy classes $\{x\}$ of pointed maps $x : X \to Z^n$. Moreover the cup product of the cohomology algebra $H^* = H^*(X,k) = \widetilde{H}^* \oplus k$ is induced by the map μ, that is $\{x\} \cup \{y\} = \{\mu(x,y)\}$. The cohomology algebra is *graded commutative* in the sense that

$$\{x\} \cup \{y\} = (-1)^{nm}\{y\} \cup \{x\}$$

In this paper we replace the homotopy set $[X, Z^n]$ by the groupoid $[\![X, Z^n]\!]$. The objects of this groupoid are the pointed maps $x : X \to Z^n$ and the morphisms $x \Rightarrow y$ in $[\![X, Z^n]\!]$ are the homotopy classes of homotopies $x \simeq y$ termed tracks. The set of path components of $[\![X, Z^n]\!]$ is

$$\pi_0[\![X, Z^n]\!] = [X, Z^n] = \widetilde{H}^n$$

and the group of tracks $0 \Rightarrow 0$ of the trivial map $0 : X \to * \to Z^n$ in $[\![X, Z^n]\!]$ is

$$\pi_1[\![X, Z^n]\!] = [X, \Omega Z^n] = \widetilde{H}^{n-1}$$

We associate with $[\![X, Z^n]\!]$ the exact sequence $\mathcal{H}^n(X)$:

$$0 \to \widetilde{H}^{n-1} \to \mathcal{H}^n(X)_1 \xrightarrow{\partial} \mathcal{H}^n(X)_0 \to \widetilde{H}^n \to 0$$

Here $\mathcal{H}^n(X)_0$ is the set of all pointed maps $X \to Z^n$ and $\mathcal{H}^n(X)_1$ is the set of pairs (x, H) with $H : x \Rightarrow 0$ and $\partial(x,H) = x$.

The Eilenberg–MacLane spaces Z^n have the following *basic properties*:

(a) Z^n is a k–vector space object in the category **Top*** of pointed spaces.

(b) The symmetric group σ_n acts on Z^n via linear automorphisms inducing the sign of a permutation on $H_n(Z^n)$.

(c) The cup product map $\mu : Z^n \times Z^m \to Z^{n+m}$ is k–bilinear and equivariant with respect to the inclusion $\sigma_n \times \sigma_m \subset \sigma_{n+m}$. Moreover μ is associative in the obvious sense and the following diagram commutes.

$$\begin{tikzcd} Z^n \times Z^m \arrow[r, "\mu"] \arrow[d, "T"'] & Z^{m+n} \arrow[d, "\tau_{n,m}"] \\ Z^m \times Z^n \arrow[r, "\mu"] & Z^{m+n} \end{tikzcd}$$

The map T is the interchange map $T(x,y) = (y,x)$ and $\tau_{n,m} \in \sigma_{n+m}$ is the element interchanging the first n–block with the second m–block.

Properties (b) and (c) imply that $\{Z^n, n \geq 0\}$ is a "symmetric spectrum" in the sense of Hovey–Shipley–Smith [HSS] 1.2.5. We use the poperties (a), (b) and (c) of Z^n to show that the graded object

$$\mathcal{H}^*(X) = \{\mathcal{H}^n(X), n \geq 1\}$$

has the structure of a "secondary algebra" which we call the *secondary cohomology* of the space X. Using secondary algebras we introduce the third *symmetric cohomology* SH^3 of a graded commutative algebra and we show that the secondary cohomology $\mathcal{H}^*(X)$ represents an element

$$< \mathcal{H}^*(X) > \in \mathsf{SH}^3(H^*, \widetilde{H}^*[1])$$

which is an invariant of the homotopy type of X. There is a natural transformation from the symmetric cohomology SH^3 to the Hochschild cohomology HH^3 which carries the class $< \mathcal{H}^*(X) >$ to the class

$$< C^*(X) > \in \mathsf{HH}^*(H^*, \widetilde{H}^*[1])$$

defined by the algebra of cochains $C^*(X)$ of the space X. It is known that the class $< C^*(X) >$ determines all triple Massey products in the cohomology $H^*(X,k)$, see for example Berrick Davydov [BD] or Baues–Minian [BM]. The new class $< \mathcal{H}^*(X) >$ in addition determines for $k = \mathbb{F}_2$ the Steenrod operations

$$\mathrm{Sq}^{n-1} : H^n \to H^{2n-1},\ n \geq 1.$$

The Hochschild cohomology HH^* is defined for algebras and graded algebras in general while the symmetric cohomology SH^3 is only defined for commutative graded algebras.

1 Secondary modules

Motivated by properties of Eilenberg–MacLane spaces in topology we introduce the algebraic concept of a secondary module. Later we will consider functors from the category of spaces to the categories of secondary modules and secondary algebras respectively.

Let k be a field an let R be a k*–algebra* with *unit* i and *augmentation* ε

$$k \xrightarrow{i} R \xrightarrow{\varepsilon} k. \tag{1.1}$$

Here i and ε are algebra maps with $\varepsilon i = 1$. For example let G be a group together with a homomorphism $\varepsilon : G \to k^*$ where k^* is the group of units in the field k. Then ε induces an augmentation

$$\varepsilon : k[G] \to k \tag{1}$$

where $k[G]$ is the *group algebra* of G. Here $k[G]$ is a vector space with basis G and ε carries the basis element $g \in G$ to $\varepsilon(g)$. In particular we have for the *symmetric group* σ_n (which is the group of bijections of the set $\{1, \dots, n\}$) the sign–homomorphism

$$\mathsf{sign} : \sigma_n \to \{1, -1\} \to k^* \tag{2}$$

which induces the *sign–augmentation*

$$\varepsilon = \varepsilon_{\mathsf{sign}} : k[\sigma_n] \to k \tag{3}$$

These examples play a special role in applications of topology below.

For k–vector spaces A, B we use the *tensor product*

$$A \otimes B = A \otimes_k B \tag{1.2}$$

A homomorphism $f : A \to B$ is termed a *k–linear* map. If A and B are R–modules then the map f is *R–linear* if in addition $f(r \cdot x) = r \cdot f(x)$ for $r \in R, x \in A$. If R and K are k–algebras then also $R \otimes K$ is a k–algebra with augmentation

$$\varepsilon : R \otimes K \xrightarrow{\varepsilon \otimes \varepsilon} k \otimes k = k$$

The multiplication in $R \otimes K$ is defined as usual by $(\alpha \otimes \beta) \cdot (\alpha' \otimes \beta') = (\alpha\alpha') \otimes (\beta\beta')$. Moreover if X is an R–module and Y is a K–module then $X \otimes Y$ is an $R \otimes K$–module by $(\alpha \otimes \beta) \cdot (x \otimes y) = (\alpha x) \otimes (\beta y)$.

(1.3) Definition. Let R be a k–algebra. A *secondary module $X = X_R$ over R* consists of a diagram

$$R \otimes X_0 \xrightarrow{\Gamma} X_1 \xrightarrow{\partial} X_0$$

where X_0 and X_1 are R–modules and ∂ is R–linear and Γ is k–linear such that for $r, r' \in R, a \in X_1, x \in X_0$ the following equations hold.

$$\partial\Gamma(r \otimes x) = (r - \varepsilon(r))x \tag{1}$$

$$\Gamma(r \otimes \partial a) = (r - \varepsilon(r))a \tag{2}$$

$$\Gamma((r \cdot r') \otimes x) = r\Gamma(r' \otimes x) + \varepsilon(r')\Gamma(r \otimes x) \tag{3}$$

$$\Gamma((r \cdot r') \otimes x) = \Gamma(r \otimes r'x) + \varepsilon(r)\Gamma(r' \otimes x) \tag{4}$$

Now let X_R and Y_K be secondary modules over R and over K respectively. A *map* between secondary modules

$$f = f_h : X_R \to Y_K \tag{5}$$

consists of an augmented algebra map $h : R \to K$ and a commutative diagram

$$\begin{tikzcd} R \otimes X_0 \arrow[r, "\Gamma"] \arrow[d, "h \otimes f_0"'] & X_1 \arrow[r, "\partial"] \arrow[d, "f_1"] & X_0 \arrow[d, "f_0"] \\ k \otimes Y_0 \arrow[r, "\Gamma"] & Y_1 \arrow[r, "\partial"] & Y_0 \end{tikzcd} \tag{6}$$

where f_1 and f_0 are k–linear and *h–equivariant*, i. e. $f_i(r \cdot b) = h(r) \cdot f_i(b)$ for $r \in R, b \in X_i$ and $i = 0, 1$. Let **secmod** be the category of secondary modules and let **secmod**(R) be the subcategory of secondary modules over R and *R–equivariant* maps $f = f_h$ for which h is the identity of R.

One readily checks that $\mathbf{secmod}(R)$ is an *additive* category (in fact an *abelian* category) with the direct sum $X_R \oplus Y_R$ given by

$$R \otimes (X_0 \oplus Y_0) \xrightarrow{\Gamma\oplus\Gamma} X_1 \oplus Y_1 \xrightarrow{\partial\oplus\partial} X_0 \oplus Y_0 \tag{1.4}$$

Moreover for a map $f : X_R \to Y_R$ in $\mathbf{secmod}(R)$ the secondary modules kernel(f) and cokernel(f) are defined in $\mathbf{secmod}(R)$ by using kernel(f_i) and cokernel(f_i) for $i = 0, 1$ in the obvious way.

(1.5) Remark. Let $R = k[G]$ be a group algebra augmented by $\varepsilon : G \to k^*$ as in (1.1). Then a secondary module X_R over R can be identified with a diagram

$$G \times X_0 \xrightarrow{\Gamma} X_1 \xrightarrow{\partial} X_0$$

where X_1, X_0 are k–vector spaces with an action of G via k–linear automorphisms and where ∂ is k–linear and G–equivariant. Moreover $G \times X_0$ is the product set and Γ is a function between sets which is k–linear in X_0 (i. e. for $g \in G$ the function $X_0 \to X_1, x \mapsto \Gamma(g, x)$ is k–linear). Moreover for $g, g' \in G$ the following equations hold.

$$\partial\Gamma(g, x) = (g - \varepsilon(g))x \tag{1}$$
$$\Gamma(g, \partial a) = (g - \varepsilon(g))a \tag{2}$$
$$\Gamma(gg', x) = g\Gamma(g', x) + \varepsilon(g')\Gamma(g, x) \tag{3}$$
$$\Gamma(gg', x) = \Gamma(g, g'x) + \varepsilon(g)\Gamma(g', x) \tag{4}$$

Let $I(R) = \text{kernel}(\varepsilon : R \to k)$ be the *augmentation ideal* considered as an R–bimodule. For an R–module M the tensor product $I(R) \otimes_R M$ over R is defined and this tensor product is an R–module by $r \cdot (\overline{r} \otimes m) = (r\overline{r}) \otimes m$ for $r \in R, \overline{r} \in I(R)$ and $m \in M$. We have the equation $(\overline{r} \cdot r) \otimes m = \overline{r} \otimes (r \cdot m)$ in $I(R) \otimes_R M$.

(1.6) Lemma. *A secondary R–module X can be equivalently described by a commutative diagram of R–linear maps:*

$$\begin{tikzcd} I(R) \otimes_R X_1 \arrow[r, "1\otimes\partial"] \arrow[d, "\mu"'] & I(R) \otimes_R X_0 \arrow[d, "\mu"] \arrow[dl, "\widetilde{\Gamma}"'] \\ X_1 \arrow[r, "\partial"] & X_0 \end{tikzcd}$$

Here μ is given by $\mu(\overline{r} \otimes m) = \overline{r} \cdot m$ for $m \in X_1$ or $m \in X_0$.

This characterization of a secondary R–module is somewhat more direct than definition (1.3) which is motivated by topological examples below.

Proof of (1.6). Given (1.3) we observe that $\Gamma(1 \otimes x) = 0$ for $x \in X_0$ by (3). Hence Γ in (1.3) is determined by the restriction

$$\Gamma' : I(R) \otimes X_0 \subset R \otimes X_0 \xrightarrow{\Gamma} X_1$$

Now (4) shows that Γ' induces a map

$$\widetilde{\Gamma} : I(R) \otimes_R X_0 \to X_1$$

which is R–linear by (3). By (1) and (2) we see that the diagram in (1.6) commutes. Conversely given such a diagram we define Γ in (1.3) by

$$\Gamma(r \otimes x) = \widetilde{\Gamma}((r - \varepsilon r) \otimes x)$$

Now it is easy to show that equations (1), … ,(4) are satisfied. q.e.d.

We use (1.6) for the following construction of free secondary R–modules.

(1.7) Definition. Let $d : V \to X_0$ be an R–linear map. Then the *free secondary R–module* X with basis (V, d) is obtained by the following push out in the category of R–modules and R–linear maps:

$$\begin{tikzcd}
I(R) \otimes_R V \arrow[r, "1 \otimes d"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\text{push}"] & I(R) \otimes_R X_0 \arrow[d, "\widetilde{\Gamma}"] \arrow[ddr, bend left, "\mu"] & \\
V \arrow[r, "i"] \arrow[drr, bend right, "d"'] & X_1 \arrow[dr, "\partial"] & \\
 & & X_0
\end{tikzcd}$$

We also write $X_1 = X_1(d)$ and $X = X(d)$. Since $\mu(1 \otimes d) = d\mu$ the R–linear map ∂ is well defined. Moreover we show that X is a well defined secondary R–module:

Proof. By (1.6) we have to show that $\widetilde{\Gamma}(1 \otimes \partial) = \mu$ on $I(R) \otimes_R X_1$. This holds if $\widetilde{\Gamma}(1 \otimes \partial)(1 \otimes i) = \mu(1 \otimes i)$ on $I(R) \otimes_R V$ and $\widetilde{\Gamma}(1 \otimes \partial)(1 \otimes \widetilde{\Gamma}) = \mu(1 \otimes \widetilde{\Gamma})$ on $I(R) \otimes_R (I(R) \otimes_R X_0)$. Now the first equation holds since

$$\widetilde{\Gamma}(1 \otimes \partial)(1 \otimes i) = \widetilde{\Gamma}(1 \otimes (\partial i)) = \widetilde{\Gamma}(1 \otimes d) = i\mu = \mu(1 \otimes i)$$

Here $i\mu = \mu(1 \otimes i)$ holds since i is R–linear. For the second equation we get

$$\widetilde{\Gamma}(1 \otimes \partial)(1 \otimes \widetilde{\Gamma}) = \widetilde{\Gamma}(1 \otimes (\partial\widetilde{\Gamma})) = \widetilde{\Gamma}(1 \otimes \mu) = \mu(1 \otimes \widetilde{\Gamma})$$

Here the last equation holds since for $r, \overline{r} \in I(R), x \in X_0$ we have

$$\begin{aligned}
\widetilde{\Gamma}(1 \otimes \mu)(r \otimes \overline{r} \otimes x) &= \widetilde{\Gamma}(r \otimes (\overline{r} \cdot x)) \\
&= \widetilde{\Gamma}((r \cdot \overline{r}) \otimes x) \\
&= r\widetilde{\Gamma}(\overline{r} \otimes x) \\
&= \mu(1 \otimes \widetilde{\Gamma})(r \otimes \overline{r} \otimes x)
\end{aligned}$$

Here we use the fact the $\widetilde{\Gamma}$ is R–linear. q.e.d.

One readily checks that the free secondary module $X(d)$ has the following *universal property*: Let X be an object in $\mathbf{secmod}(R)$ and let

$$\begin{tikzcd}
V \arrow[rr, "f"] \arrow[dr, "d"'] & & X_1 \arrow[dl, "\partial"] \\
 & X_0 &
\end{tikzcd} \tag{1.8}$$

be a commutative diagram of R–linear maps. Then there is a unique map $\overline{f} : X(d) \to X$ in **secmod**(R) of the form

$$\begin{tikzcd} I(R)\otimes_R X_0 \arrow[r, "\widetilde{\Gamma}"] \arrow[d, equal] & X_1(d) \arrow[r, "\partial"] \arrow[d, "\overline{f}_1"] & X_0 \arrow[d, equal, "\overline{f}_0=\text{identity}"] \\ I(R)\otimes_R X_0 \arrow[r, "\widetilde{\Gamma}"] & X_1 \arrow[r, "\partial"] & X_0 \end{tikzcd}$$

such that $\overline{f}_1 i = f$ for $i : V \to X_1(d)$ defined in (1.7).

2 Examples of secondary modules in topology

We describe examples of secondary modules which arise in topology. Let **Top*** be the category of pointed topological spaces with base point. This is a *groupoid enriched category* in the following sense. For objects X, Y in **Top*** the morphism object $[\![X;Y]\!]$ is the groupoid given as follows. Objects in $[\![X,Y]\!]$ are the pointed maps $X \to Y$ and for pointed maps $f, g : X \to Y$ the morphisms $H : f \Rightarrow g$ in $[\![X,Y]\!]$ are the *tracks* from f to g, that is H is a homotopy class of homotopies $f \simeq g$. The composite of tracks

$$h \overset{G}{\Longleftarrow} g \overset{H}{\Longleftarrow} f$$

is denoted by $G\widehat{+}H$ where $G\widehat{+}H$ is defined by adding homotopies in the usual way. The inverse of the track H is accordingly denoted by $\widehat{-}H : g \Rightarrow f$ with $(\widehat{-}H)\widehat{+}H = \widehat{0}_f$ and $H\widehat{+}(\widehat{-}H) = \widehat{0}_g$ where $\widehat{0}_f$ denotes the identity track of f.

If $Y \times Z$ is a product in **Top*** then

$$[\![X, Y \times Z]\!] = [\![X,Y]\!] \times [\![X,Z]\!] \tag{2.1}$$

is a product of groupoids. This shows that for an algebraic object Y in **Top*** the groupoid $[\![X,Y]\!]$ is a corresponding algebraic object in the category **Grd** of (small) groupoids. For example if Y is an abelian group object in **Top*** (i. e. an abelian topological group) then $[\![X,Y]\!]$ is a abelian group object in the category **Grd**. A map between abelian group objects which is a homomorphism of the group structure is termed a linear map.

Let **C** be a category. Then the category of pairs in **C** denoted by $\mathsf{pair}(\mathbf{C})$ is defined. Objects are morphisms $f : A \to B$ in **C** and morphisms $(\alpha, \beta) : f \to g$ in $\mathsf{pair}(\mathbf{C})$ are commutative diagrams in **C**

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & A' \arrow[d, "g"] \\ B \arrow[r, "\beta"] & B' \end{tikzcd}$$

Let **Ab** be the category of abelian groups.

(2.2) Proposition. *The category of abelian group objects in* **Grd** *and linear maps is isomorphic to the category* $\mathsf{pair}(\mathbf{Ab})$.

Proof. Given an abelian group object G in **Grd** we obtain the object

$$\partial : G_1 \to G_0$$

in $\mathsf{pair}(\mathbf{Ab})$ as follows. Here G_0 is the set of objects of G which is an abelian group since G is an abelian group object in **Grd**. Let $0 \in G_0$ be the neutral object in the abelian group G_0. Then G_1 is the set of all morphisms $f : a \Rightarrow 0$ in G with $a \in G_0$ and $\partial f = a$. The abelian group structure of G_1 is defined by

$$(f : a \Rightarrow 0) + (g : b \Rightarrow 0) = (f + g, a + b \to 0 + 0 = 0)$$

where the right hand side is defined since G is an abelian group object in **Grd**.

Conversely given an object $\partial : A_1 \to A_0$ in $\mathsf{pair}(\mathbf{Ab})$ we define the abelian group object $G(\partial)$ in **Grd** as follows. The set of objects of $G(\partial)$ is the set A_0. The set of morphisms of $G(\partial)$ is the product set $A_1 \times A_0$ where $(a_1, x) \in A_1 \times A_0$ is a morphism $(a_1, x) : \partial a_1 + x \to x$ in $G(\partial)$. The identity of x is $(0, x) : x = \partial 0 + x \to x$. Composition of

$$x \xleftarrow{(a_1,x)} \partial a_1 + x \xleftarrow{(b_1,\partial a_1+x)} \partial b_1 + \partial a_1 + x$$

is $(b_1 + a_1, x)$ for $a_1, b_1 \in A_1$ and $x \in A_0$. Now it is readily seen that this way one gets an isomorphism of categories. q.e.d.

Now given an abelian group object Y in $\mathbf{Top}^*$ the abelian group object $G = [\![X,Y]\!]$ in **Grd** is given via (2.2) by a homomorphism

$$G_1 = [\![X,Y]\!]_1 \xrightarrow{\partial} G_0 = [\![X,Y]\!]_0 \tag{2.3}$$

where G_0 is the set of all pointed maps $f : X \to Y$ and where G_1 is the set of all tracks $H : f \Rightarrow 0$ with $f \in G_0$ and $\partial H = f$. The group structure of Y induces the group structure on G_0 and G_1 in the obvious way.

(2.4) Definition. Let R be a k–algebra with augmentation $\varepsilon : R \to k$ as in (1.1). A *track module Y over R* is a R–module object Y in $\mathbf{Top}^*$ (i. e. a topological R–module) for which each map $r : Y \to Y$ given by $r \in R$ admits a *unique* track $\Gamma_r : r \Rightarrow \varepsilon r$ where $\varepsilon r : Y \to Y$ is defined by the k–vector space structure of Y.

(2.5) Example. Let $Z^n = K(k,n)$ be an Eilenberg–MacLane space of the underlying abelian group of the field k with the properties in the introduction, see Appendix A. This shows that for $R = k[\sigma_n]$ the space Z^n is a topological R–module, in fact, a track module over R since for $r \in R$ there is a unique track $r \Rightarrow \varepsilon r$ (with $r, \varepsilon r : Z^n \to Z^n$). Here ε is the sign–augmentation as in (1.1)(3).

(2.6) Proposition. *Let Y be a track module over R. Then for each X in $\mathbf{Top}^*$ one obtains canonically a secondary module over R*

$$R \otimes G_0 \xrightarrow{\Gamma} G_1 \xrightarrow{\partial} G_0$$

where ∂ is given by the groupoid $G = [\![X,Y]\!]$ as in (2.3) and where Γ is defined by the composite

$$\Gamma(r \otimes f) = \Gamma_{r-\varepsilon r} f$$

Here the track $\Gamma_{r-\varepsilon r} : r - \varepsilon r \Rightarrow 0$ is given for $r - \varepsilon r \in R$ by (2.4).

Using (2.6) we obtain for each track module Y over R a functor

$$\mathcal{H}(-;Y) : \mathbf{Top}^* \to \mathbf{secmod}(R) \tag{2.7}$$

which carries X to the track module $\mathcal{H}(X;Y) = (G_1, G_0, \partial, \Gamma)$ given by $[\![X,Y]\!]$ in (2.6). Of course we have

$$\pi_0\mathcal{H}(X,Y) = \mathrm{cokernel}(\partial) = [X,Y] \tag{1}$$
$$\pi_1\mathcal{H}(X,Y) = \mathrm{kernel}(\partial) = [X,\Omega Y] \tag{2}$$

Here $[X,Y]$ denotes the set of homotopy classes of pointed maps $X \to Y$ and ΩY is the loop space of Y. The elements $H \in [X,\Omega Y]$ are identified with the tracks $H : 0 \Rightarrow 0$ where $0 : X \to * \to Y$ is the *zero map*. By (1) and (2) we see that the functor (2.7) carries homotopy equivalences in $\mathbf{Top}^*$ to weak equivalences between secondary modules as defined in the next section.

We are mainly interested in the secondary module

$$\mathcal{H}^n(X) = \mathcal{H}(X, Z^n) \tag{2.8}$$

over $R_n = k[\sigma_n]$ given by (2.5) with $\pi_0\mathcal{H}^n(X) = \widetilde{H}^n(X,k)$ and $\pi_1\mathcal{H}^n(X) = \widetilde{H}^{n-1}(X,k)$. This corresponds to the boundary map ∂ in the introduction.

Proof of (2.6). By definition of ∂ we have $\partial(\Gamma_{r-\varepsilon r}f) = (r-\varepsilon r)f$ so that (1.3)(1) is satisfied. Now let $H : f \Rightarrow 0$ be an element in G_1 with $\partial H = f$. Then we get

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "0"'] & \bullet \arrow[l, "0"'] \\
\bullet & \bullet \arrow[l, "r-\varepsilon r"] \arrow[ul, phantom, "\Uparrow\Gamma_{r-\varepsilon r}"] & \bullet \arrow[l, "f"] \arrow[ul, phantom, "\Uparrow H"]
\end{tikzcd}$$

so that for $\Gamma = \Gamma_{r-\varepsilon r}$

$$\begin{aligned}
\Gamma * H &= 0H\hat{+}\Gamma f = \Gamma 0\hat{+}(r-\varepsilon r)H \\
&= \widehat{0}_0\hat{+}\Gamma f = \widehat{0}_0\hat{+}(r-\varepsilon r)H \\
&= \Gamma f = (r-\varepsilon r)H
\end{aligned}$$

and this implies (1.3)(2). Next we have by uniqueness of tracks in $[\![Y,Y]\!]$ the equations

$$\begin{aligned}
\Gamma_{rr'-\varepsilon(rr')} &= r\Gamma_{r-\varepsilon r'} + \varepsilon(r')\Gamma_{r-\varepsilon r} \\
&= \Gamma_{r-\varepsilon r}r' + \varepsilon(r)\Gamma_{r'-\varepsilon r'}
\end{aligned}$$

and these equations imply (1.3)(3),(4). q.e.d.

(2.9) Remark. Let Y be given as in (2.6) and let $\widehat{R} \subset [\![Y,Y]\!]$ be the full subgroupoid with objects given by maps $r : Y \to Y$ for $r \in R$. Then we obtain the action

$$\mu_R : \widehat{R} \times [\![X,Y]\!] \subset [\![Y,Y]\!] \times [\![X,Y]\!] \xrightarrow{\circ} [\![X,Y]\!]$$

where the second arrow is composition in the groupoid enriched category $\mathbf{Top}^*$. The action μ_R determines Γ in the secondary module $\mathcal{H}(X,Y)$ given by (2.6) and conversely $\mathcal{H}(X,Y)$ determines uniquely the action μ_R. In this sense a secondary module is a $\widehat{R}$–module in the category **Grd** of groupoids. Here $\widehat{R}$ is the groupoid with objects R, path components $\varepsilon^{-1}(x)$ with $x \in k$ and all automorphism groups in $\widehat{R}$ are trivial. The algebra structure of R yields a corresponding structure of $\widehat{R}$.

3 Weak equivalences

We can consider a secondary module X as a chain complex of k–vector spaces concentrated in degree 0 and 1. The homology of this chain complex is denoted by

$$\begin{aligned} \pi_0(X) &= \mathrm{cokernel}(\partial : X_1 \to X_0) \\ \pi_1(X) &= \mathrm{kernel}(\partial : X_1 \to X_0) \end{aligned} \tag{3.1}$$

A map $F = F_h : X_R \to Y_K$ between secondary modules is a *weak equivalence* if $h : R \to K$ is an isomorphism and f induces isomorphisms

$$\begin{aligned} f_* &: \pi_0(X_R) \cong \pi_0(Y_K) \\ f_* &: \pi_1(X_R) \cong \pi_1(Y_K) \end{aligned}$$

We point out that $\pi_0(X)$ and $\pi_1(X)$ are also R–modules for which, however, by (1.3)(1),(2) the R–module structure is induced by the augmentation ε, that is $r \cdot x = \varepsilon(r) \cdot x$ for $r \in R, x \in \pi_0(X), \pi_1(X)$. Hence $\pi_0(X)$ and $\pi_1(X)$ are just k–vector spaces with an action of R via ε. Such R–modules are termed *ε–modules.* If X_R is a secondary module with $\Gamma = 0$ then X_0 and X_1 are also ε–modules. Hence in this case X_R is given by a chain complex $\partial : X_1 \to X_0$ of k–vector spaces. We say that X_R is of *trivial type* if $\Gamma = 0$ and $\partial = 0$ so that in this case $X_0 = \pi_0$ and $X_1 = \pi_1$.

Two secondary modules X_R, Y_R are *weakly equivalent* if there exists a chain of R–equivariant weak equivalences

$$X_R \xleftarrow{\sim} X_1 \xrightarrow{\sim} X_2 \xleftarrow{\sim} \ldots X_n \xrightarrow{\sim} Y_R.$$

(3.2) Proposition. *Each secondary module is weakly equivalent to a secondary module of trivial type.*

Hence the only invariants of the weak equivalence class of a secondary module X are $\pi_0 X$ and $\pi_1 X$.

(3.3) Remark. For the secondary module $\mathcal{H}^n(X)$ over $R_n = k[\sigma_n]$ in (2.7) we know by (3.2) that the weak equivalence type of $\mathcal{H}^n(X)$ is trivial. This can also be seen by the following topological argument. By Baues [B] there exists a sequence

$$Z^n \xleftarrow{f} Y_1 \xrightarrow{g} Y_2 \xleftarrow{h} Y_3$$

of topological R_n–modules and R_n–linear maps f, g, h with the following properties. The action of R_n an Y_3 satisfies $r \cdot y = \varepsilon(r) \cdot y$ for $r \in R_n$ and $y \in Y_3$ where ε is the sign augmentation of R_n. Moreover f, g and h are homotopy equivalences on $\mathbf{Top}^*$. Hence we obtain weak equivalences of secondary modules over R_n

$$\mathcal{H}^n(X) = \mathcal{H}(X, Z^n) \xleftarrow{\sim} \mathcal{H}(X; Y_1) \xrightarrow{\sim} \mathcal{H}(X, Y_2) \xleftarrow{\sim} \mathcal{H}(X, Y_3)$$

where $\mathcal{H}(X, Y_3)$ is easily seen to be weakly equivalent to a secondary module of trivial type.

For the proof of (3.2) in (3.8) below we need the following pull back construction for secondary modules. Let X_R be a secondary module and let Y_0 be an R–module and let

$f : Y_0 \to X_0$ be a R–linear map. Then we obtain the following commutative diagram in which the subdiagram 'pull' is a pull back in the category of vector spaces.

$$\begin{tikzcd} R\otimes Y_0 \arrow[r, "\overline{\Gamma}"'] \arrow[d, "R\otimes f"] \arrow[rr, bend left, "\gamma"] & f^*X_1 \arrow[r, "\overline{\partial}"'] \arrow[d, "\overline{f}"] \arrow[dr, phantom, "\text{pull}"] & Y_0 \arrow[d, "f"] \\ R\otimes X_0 \arrow[r, "\Gamma"] & X_1 \arrow[r, "\partial"] & X_0 \end{tikzcd}$$

Here $\overline{\Gamma}$ is defined by $\overline{\partial}\,\overline{\Gamma} = \gamma$ with $\gamma(r\otimes y) = (r - \varepsilon r)\cdot y$ and $\overline{f}\,\overline{\Gamma} = \Gamma(R\otimes f)$. Then f^*X_1 is an R–module and $\overline{\partial}$ is R–linear. Moreover we get the following fact.

(3.4) Lemma. *The top row $Y_R = (\overline{\partial}, \overline{\Gamma}) = f^*X_R$ of the diagram is a secondary module over R and $(\overline{f}, f) : Y_R \to X_R$ is a map in* **secmod**(R) *which is a weak equivalence if $(\partial, f) : X_1 \oplus Y_0 \to X_0$ is surjective.*

The map $f^*X_R \to X_R$ has the following property. Let $i : K \to R$ be an augmented map between k–algebras and let $g : Z_K \to X_R$ be an i–equivariant map between secondary modules for which a commutative diagram

$$\begin{tikzcd} & Y_0 \arrow[d, "f"] \\ Z_0 \arrow[ur, "h_0"] \arrow[r, "g_0"'] & X_0 \end{tikzcd} \tag{1}$$

is given. Here h_0 and g_0 are i–equivariant. Then there exists a unique i–equivariant map $h : Z_K \to f^*X_R$ for which the diagram

$$\begin{tikzcd} & f^*X_R \arrow[d, "(\overline{f}, f)"] \\ Z_K \arrow[ur, "h"] \arrow[r, "g"'] & X_R \end{tikzcd} \tag{2}$$

commutes in **secmod**.

Proof of (3.4). The elements of f^*X_1 are pairs (x_1, y) with $\partial x_1 = fy$. We define $r(x_1, y) = (rx_1, ry)$ so that $\overline{\partial}$ and $\overline{f}$ are R–linear with $\overline{\partial}(x_1, y) = y$ and $\overline{f}(x_1, y) = x_1$. Moreover

$$\overline{\Gamma}(r\otimes y) = (\Gamma(r\otimes fy), (r - \varepsilon r)y)$$

Hence (1.3)(1) holds for Y_R. Moreover

$$\begin{aligned} \overline{\Gamma}(r\otimes \overline{\partial}(x_1, y)) &= \overline{\Gamma}(r\otimes y) \\ &= (\Gamma(r\otimes fy), (r - \varepsilon r)y) \\ &= (\Gamma(r\otimes \partial x_1), (r - \varepsilon r)y) \\ &= ((r - \varepsilon r)x_1, (r - \varepsilon r)y) \\ &= (r - \varepsilon r)(x_1, y) \end{aligned}$$

This shows (1.3)(2) for Y_R. Next we consider (1.3)(3) for Y_R and we get

$$\begin{aligned}
\overline{\Gamma}(r\cdot r'\otimes y) &= (\Gamma(r\cdot r'\otimes fy),(r\cdot r'-\varepsilon(rr'))y)\\
r\overline{\Gamma}(r'\otimes y)+\varepsilon(r')\overline{\Gamma}(r\otimes y) &= r(\Gamma(r'\otimes fy),(r'-\varepsilon r')y)+\varepsilon(r')(\Gamma(r\otimes fy),(r-\varepsilon r)y)\\
&= (\Gamma(rr'\otimes fy), r(r'-\varepsilon r')y+\varepsilon(r')(r-\varepsilon r)y)\\
&= (\Gamma(rr'\otimes fg),(rr'-\varepsilon(rr'))y).
\end{aligned}$$

Similarly one checks (1.3)(4) for Y_R. Hence $(f,\overline{f}) : Y_R \to X_R$ is a well defined map between secondary modules. If (∂, f) is surjective then the pull back is also a push out and therefore $(\overline{f}, f)$ is a weak equivalence. q.e.d.

Now given a secondary module X_R we can choose a k–linear section $s:\pi_0\to X_0$ of the quotient map $q : X_0 \to X_0/\operatorname{im}(\partial)=\pi_0$. Hence the R–linear map

$$f : R\otimes\pi_0 \to X_0 \tag{3.5}$$

with $f(r\otimes x)=r\cdot sx$ is defined with $qf(r\otimes x)=\varepsilon(r)\cdot x$. Here $R\otimes\pi_0$ is a *free* R–module with the action of R given by $r\cdot(r'\otimes x)=(r\cdot r')\otimes x$ and qf coincides with the R–linear map

$$qf=\varepsilon\otimes 1 : R\otimes\pi_0\to k\otimes\pi_0=\pi_0 \tag{1}$$

For $I(R)=\ker(\varepsilon : R\to k)$ we have $\ker(\varepsilon\otimes 1)=I(R)\otimes\pi_0$. Using f in (3.5) we get as in (3.4) an R–linear map between secondary modules

$$(\overline{f},f) : Y_R = f^*X_R\to X_R \tag{2}$$

which is a weak equivalence since $(\partial, f) : X_1\oplus Y_0 = X_1\oplus R\otimes\pi_0\to X_0$ is surjective. Here Y_R is a secondary module which is special in the following sense. We say that a secondary module X_R is *special* if for $\pi_0=\pi_0X$ one has

$$\begin{cases} X_0 = R\otimes\pi_0 \text{ and}\\ \operatorname{im}(\partial : X_1\to X_0)=I(R)\otimes\pi_0.\end{cases}$$

(3.6) Proposition. *A special secondary R–module X_R admits an R–linear section $t : I(R)\otimes\pi_0\to X_1$ of $\partial : X_1\to I(R)\otimes\pi_0$.*

Proof. We define t by the map

$$\Gamma : R\otimes X_0 = R\otimes R\otimes\pi_0\to X_1,$$

namely for $r'\in I(R)$ and $x\in\pi_0$ let

$$t(r'\otimes x)=\Gamma(r'\otimes 1\otimes x).$$

Then we have $\partial t(r'\otimes x)=\partial\Gamma(r'\otimes 1\otimes x)=(r'-\varepsilon r')(1\otimes x)=r'\otimes x$ since $\varepsilon(r')=0$. Moreover t is R–linear since for $r\in R$

$$\begin{aligned}
t(r\cdot r'\otimes x) &= \Gamma(r\cdot r'\otimes 1\otimes x)\\
&= r\Gamma(r'\otimes 1\otimes x)+\varepsilon(r')\Gamma(r\otimes 1\otimes x)\\
&= r\Gamma(r'\otimes 1\otimes x)\\
&= r\cdot t(r'\otimes x).
\end{aligned}$$

q.e.d.

(3.7) Remark. A converse of (3.6) is also true. Let π_0 and π_1 be k–vector spaces and let

$$\partial : X_1 \to I(R) \otimes \pi_0$$

be a surjective R–linear map for which $\pi_1 = \ker(\partial)$ is an ε–module and let t be an R–linear section of ∂. Then a special secondary R–module X_R is defined in terms of ∂ and t as follows. Let $X_0 = R \otimes \pi_0$ and let

$$\Gamma : R \otimes X_0 = R \otimes R \otimes \pi_0 \to X_1$$

be given by $(r, r' \in R, x \in \pi_0)$

$$\Gamma(r \otimes r' \otimes x) = (r - \varepsilon r)t((r' - \varepsilon r') \otimes x) + \varepsilon(r')t((r - \varepsilon r) \otimes x)$$

Then one can check that Γ satisfies all the axioms in (1.3) so that X_R is a well defined special secondary module. Hence by (3.6) special secondary modules are up to isomorphism determined by π_0 and π_1 with $X_0 = R \otimes \pi_0$ and $X_1 = \pi_1 \oplus I(R) \otimes \pi_0$.

(3.8) Proof of (3.2). Let X_R be a secondary module. Then we obtain by (3.5) the special secondary module $Y_R = f^* X_R$ and the weak equivalence $Y_R \xrightarrow{\sim} X_R$. Moreover by (3.6) and (3.7) we have $Y_0 = R \otimes \pi_0$ and $Y_1 = \pi_1 \oplus I(R) \otimes \pi_0$ and

$$\partial : Y_1 = \pi_1 \oplus I(R) \otimes \pi_0 \to I(R) \otimes \pi_0 \subset R \otimes \pi_0 = Y_0$$

is given by the projection and the inclusion. Now we obtain a weak equivalence g with

$$\begin{tikzcd} Y_1 \arrow[r, "\partial"] \arrow[d, "g_1"'] & Y_0 \arrow[d, "g_0"] \\ \pi_1 \arrow[r, "0"] & \pi_0 \end{tikzcd}$$

where $g_0 = \varepsilon \otimes 1$ and g_1 is the projection. By the definition of Γ in (3.7) in terms of the section $t : I(R) \otimes \pi_0 \subset Y_1$ we see that $g_1 \Gamma = 0$ so that g is a well defined map between secondary modules where $0 : \pi_1 \to \pi_0$ is the secondary module of trivial type given by π_0 and π_1. q.e.d.

4 Tensor products of secondary modules

Here we introduce the tensor product of secondary modules which is needed for the definition of secondary algebras in the next section. For secondary modules X_R and Y_K the tensor product of the underlying chain complexes is given by the chain complex of k–vector spaces

$$X_1 \otimes Y_1 \xrightarrow{d_2} X_1 \otimes Y_0 \oplus X_0 \otimes Y_1 \xrightarrow{d_1} X_0 \otimes Y_0$$
$$d_2(a \otimes b) = (\partial a) \otimes b - a \otimes (\partial b)$$
$$d_1(a \otimes y) = (\partial a) \otimes y$$
$$d_1(x \otimes b) = x \otimes (\partial b)$$

with $x \in X_0, y \in Y_0, a \in X_1, b \in Y_1$. Hence d_1 induces the boundary map

$$\partial_\otimes : (X_1 \otimes Y_0 \oplus X_0 \otimes Y_1)/\operatorname{im}(d_2) \to X_0 \otimes Y_0 \qquad (4.1)$$

Since k is a field we get by the Künneth formula

$$\begin{aligned}\pi_0\partial_\otimes &= \operatorname{cok}(\partial_\otimes) = \pi_0(X)\otimes\pi_0(Y)\\ \pi_1\partial_\otimes &= \ker(\partial_\otimes) = \pi_1(X)\otimes\pi_0(Y)\oplus\pi_0(X)\otimes\pi_1(Y)\end{aligned}$$

One readily checks that $\partial_\otimes$ is an $R\otimes K$–equivariant k–linear map.

(4.2) Definition. We define the *tensor product* $X_R\otimes Y_K = (X\otimes Y)_{R\otimes K}$ of secondary modules X_R and Y_K by the diagram

$$R\otimes K\otimes X_0\otimes Y_0 \xrightarrow{\Gamma_\otimes} (X_1\otimes Y_0\oplus X_0\otimes Y_1)/\operatorname{im} d_2 \xrightarrow{\partial_\otimes} X_0\otimes Y_0$$

Here $\partial_\otimes$ is defined as in (4.1) and $\Gamma_\otimes$ is defined by the following formula

$$\begin{aligned}\Gamma_\otimes(\alpha\otimes\beta\otimes x\otimes y) &= \Gamma(\alpha\otimes x)\otimes(\beta y)+(\varepsilon(\alpha)x)\otimes\Gamma(\beta\otimes y)\\ &= (\alpha X)\otimes\Gamma(\beta\otimes y)+\Gamma(\alpha\otimes x)\otimes(\varepsilon(\beta)y)\end{aligned}$$

Here the second equation is a consequence of the first equation since $(\partial a)\otimes b = a\otimes(\partial b)$ by (3.1).

(4.3) Lemma. *The tensor product $X_R\otimes Y_K$ of secondary modules X_R and Y_K is a well defined secondary module over $R\otimes K$.*

Proof. The map $\partial_\otimes$ is $R\otimes K$–linear and $\Gamma_\otimes$ is a well defined k–linear map. Hence we have to check the equations (1) ... (4) in (1.3): We first check (1).

$$\begin{aligned}\partial_\otimes\Gamma_\otimes(\alpha\otimes\beta\otimes x\otimes y) &= \partial\Gamma(\alpha\otimes x)\otimes\beta y+\varepsilon(\alpha)x\otimes\partial\Gamma(\beta\otimes y)\\ &= (\alpha-\varepsilon(\alpha))x\otimes\beta y+\varepsilon(\alpha)x\otimes(\beta-\varepsilon(\beta))y\\ &= \alpha x\otimes\beta y-\varepsilon(\alpha)\varepsilon(\beta)x\otimes y\\ &= (\alpha\otimes\beta-\varepsilon(\alpha\otimes\beta))(x\otimes y)\end{aligned}$$

Next we check (2) for $\Gamma_\otimes$.

$$\begin{aligned}\Gamma_\otimes(\alpha\otimes\beta\otimes\partial_\otimes(a\otimes y)) &= \Gamma_\otimes(\alpha\otimes\beta\otimes\partial a\otimes y)\\ &= \Gamma(\alpha\otimes\partial a)\otimes\beta y+\varepsilon(\alpha)\partial a\otimes\Gamma(\beta\otimes y)\\ &= (\alpha-\varepsilon(\alpha))a\otimes\beta y+\varepsilon(\alpha)a\otimes\partial\Gamma(\beta\otimes y) \quad \text{, see (4.1),}\\ &= (\alpha-\varepsilon(\alpha))a\otimes\beta y+\varepsilon(\alpha)a\otimes(\beta-\varepsilon(\beta))y\\ &= \alpha a\otimes\beta y-\varepsilon(\alpha)\varepsilon(\beta)a\otimes y\\ &= (\alpha\otimes\beta-\varepsilon(\alpha\otimes\beta))(a\otimes y)\end{aligned}$$

$$\begin{aligned}\Gamma_\otimes(\alpha\otimes\beta\otimes\partial_\otimes(x\otimes b)) &= \Gamma_\otimes(\alpha\otimes\beta\otimes x\otimes\partial b)\\ &= \Gamma(\alpha\otimes x)\otimes\beta\partial b+\varepsilon(\alpha)x\otimes\Gamma(\beta\otimes\partial b)\\ &= \Gamma(\alpha\otimes x)\otimes\partial(\beta b)+\varepsilon(\alpha)x\otimes(\beta-\varepsilon(\beta))b\\ &= \partial\Gamma(\alpha\otimes x)\otimes\beta b+\varepsilon(\alpha)x\otimes(\beta-\varepsilon(\beta))b \quad \text{, see (4.1),}\\ &= (\alpha-\varepsilon(\alpha))x\otimes\beta b+\varepsilon(\alpha)x\otimes(\beta-\varepsilon(\beta))b\\ &= \alpha x\otimes\beta b-\varepsilon(\alpha)\varepsilon(\beta)x\otimes b\\ &= (\alpha\otimes\beta-\varepsilon(\alpha\otimes\beta))(x\otimes b)\end{aligned}$$

Now we check (3) for $\Gamma_\otimes$.

$$\begin{aligned}\Gamma_\otimes((\alpha\otimes\beta)(\alpha'\otimes\beta')\otimes x\otimes y) &= \Gamma_\otimes(\alpha\alpha'\otimes\beta\beta'\otimes x\otimes y)\\ &= \Gamma(\alpha\alpha'\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes\Gamma(\beta\beta'\otimes y) = \text{(i)}\end{aligned}$$

Now we get by (3) that (i)=(ii) coincides with

$$\text{(ii)} = (\alpha\Gamma(\alpha'\otimes x)+\varepsilon(\alpha')\Gamma(\alpha\otimes x))\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes(\beta\Gamma(\beta'\otimes y)+\varepsilon(\beta')\Gamma(\beta\otimes y))$$

On the other hand we have

$$\begin{aligned}\text{(iii)} &= (\alpha\otimes\beta)\Gamma_\otimes(\alpha'\otimes\beta'\otimes x\otimes y)+\varepsilon(\alpha'\otimes\beta')\Gamma_\otimes(\alpha\otimes\beta\otimes x\otimes y)\\ &= \alpha\Gamma(\alpha'\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha')\alpha x\otimes\beta\Gamma(\beta'\otimes y)\\ &\quad +\varepsilon(\alpha'\otimes\beta')(\Gamma(\alpha\otimes x)\otimes\beta y+\varepsilon(\alpha)x\otimes\Gamma(\beta\otimes y)\end{aligned}$$

We have to check (ii)=(iii). But this is equivalent to

$$\varepsilon(\alpha')\Gamma(\alpha\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes\beta\Gamma(\beta'\otimes y) = \varepsilon(\alpha')\alpha x\otimes\beta\Gamma(\beta'\otimes y)+\varepsilon(\alpha'\otimes\beta')\Gamma(\alpha\otimes x)\otimes\beta y$$

This equation is equivalent to

$$\begin{tikzcd}\Gamma(\alpha\otimes x)\otimes\beta(\beta'-\varepsilon\beta')y \arrow[r, equal] \arrow[d, equal] & (\alpha-\varepsilon\alpha')x\otimes\beta\Gamma(\beta'\otimes y) \arrow[d, equal]\\ \Gamma(\alpha\otimes x)\otimes\beta\partial\Gamma(\beta'\otimes y) & \partial\Gamma(\alpha\otimes x)\otimes\beta\Gamma(\beta'\otimes y)\end{tikzcd}$$

By (4.1) we know that

$$\begin{aligned}\Gamma(\alpha\otimes x)\otimes\beta\partial\Gamma(\beta'\otimes y) &= \Gamma(\alpha\otimes x)\otimes\partial\beta\Gamma(\beta'\otimes y)\\ &= \partial\Gamma(\alpha\otimes x)\otimes\beta\Gamma(\beta'\otimes y)\end{aligned}$$

This completes the proof that (ii)=(iii) and hence (i)=(iii) and hence (3) holds for $X\otimes Y$. Finally we have to check (4). For this we apply (4) to (i) above and we get (i)=(iv) where

$$\text{(iv)} = (\Gamma(\alpha\otimes\alpha' x)+\varepsilon(\alpha)\Gamma(\alpha'\otimes x))\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes(\Gamma(\beta\otimes\beta' y)+\varepsilon(\beta)\Gamma(\beta'\otimes y))$$

On the other hand we have

$$\begin{aligned}\text{(v)} &= \Gamma_\otimes(\alpha\otimes\beta\otimes\alpha' x\otimes\beta' y)+\varepsilon(\alpha\otimes\beta)\Gamma_\otimes(\alpha'\otimes\beta'\otimes x\otimes y)\\ &= \Gamma(\alpha\otimes\alpha' x)\otimes\beta\beta' y+\varepsilon(\alpha)\alpha')x\otimes\Gamma(\beta\otimes\beta' y)\\ &\quad +\varepsilon(\alpha\otimes\beta)(\Gamma(\alpha'\otimes x)\otimes\beta' y+\varepsilon(\alpha')x\otimes\Gamma(\beta'\otimes y))\end{aligned}$$

We have to check (iv)=(v). This is the case if and only if the following equation holds.

$$\begin{aligned}&\varepsilon(\alpha)\Gamma(\alpha'\otimes x)\otimes\beta\beta' y+\varepsilon(\alpha\alpha')x\otimes\Gamma(\beta\otimes\beta' y)\\ &\qquad = \varepsilon(\alpha)\alpha' x\otimes\Gamma(\beta\otimes\beta' y)+\varepsilon(\alpha\otimes\beta)\Gamma(\alpha'\otimes x)\otimes\beta' y\end{aligned}$$

This equation holds if and only if the following equation is true

$$\begin{tikzcd}\Gamma(\alpha'\otimes x)\otimes(\beta-\varepsilon\beta)\beta' y \arrow[r, equal] \arrow[d, equal] & (\alpha'-\varepsilon(\alpha'))x\otimes\Gamma(\beta\otimes\beta' y) \arrow[d, equal]\\ \Gamma(\alpha'\otimes x)\otimes\partial\Gamma(\beta\otimes\beta' y) & \partial\Gamma(\alpha'\otimes x)\otimes\Gamma(\beta\otimes\beta' y)\end{tikzcd}$$

Now again (4.1) shows that this equation is true. Hence we have shown (i)=(v) and this corresponds to equation (4) for $X\otimes Y$. Now the proof of the lemma is complete. q.e.d.

(4.4) Lemma. *The tensor product of secondary modules is associative and bilinear, that is:*

$$\begin{aligned}(X_R \otimes Y_K) \otimes Z_L &= X_R \otimes (Y_K \otimes Z_L)\\ (X_R \oplus Y_R) \otimes Z_L &= X_R \otimes Z_L \oplus Y_R \otimes Z_L\\ Z_L \otimes (X_R \oplus Y_R) &= Z_L \otimes X_R \oplus Z_L \otimes Y_R\end{aligned}$$

We point out that the chain complex $k = (0 \to k)$ is a unit for the tensor product, that is

$$X_R \otimes k = X_R = k \otimes X_R \tag{4.5}$$

Here we use the obvious identification $V \otimes k = V = k \otimes V$ for a k–vector space V.

We shall use the tensor product of secondary modules mainly for the next result.

(4.6) Proposition. *The cup product map $\mu : Z^n \times Z^n \to Z^{n+m}$ induces an $i_{n,m}$–equivariant map between secondary modules*

$$\mu_* : \mathcal{H}^n(X) \otimes \mathcal{H}^m(X) \to \mathcal{H}^{n+m}(X)$$

where $i_{n,m} : k[\sigma_n] \otimes k[\sigma_m] \to k[\sigma_{n+m}]$ is induced by the inclusion $\sigma_n \times \sigma_m \subset \sigma_{n+m}$.

Proof. The map μ_* carries $f \otimes g \in \mathcal{H}^n_0 \otimes \mathcal{H}^m_0$ with $f : X \to Z^n, g : X \to Z^m$ to the composite $\mu(f,g) : X \to Z^n \times Z^m \to Z^{n+m}$. Since μ is k–bilinear and $i_{n,m}$–equivariant $\mu_* : \mathcal{H}^n_0 \otimes \mathcal{H}^m_0 \to \mathcal{H}^{n+m}_0$ is well defined. Moreover μ_* is induced on $(\mathcal{H}^n \otimes \mathcal{H}^m)_1$ by the map

$$\bar{\mu} : \mathcal{H}^n_0 \otimes \mathcal{H}^m_1 \oplus \mathcal{H}^n_1 \otimes \mathcal{H}^m_0 \to \mathcal{H}^{n+m}_1 \tag{1}$$

which carries $f \otimes G$ with $G : g \Rightarrow 0 \in \mathcal{H}^m_1$ to $\mu(f,G) : \mu(f,g) \Rightarrow 0$ and carries $H \otimes g$ with $H : f \Rightarrow 0 \in \mathcal{H}^n_1$ to $\mu(H,g) : \mu(f,g) \Rightarrow 0$. Here we use the fact that the bilinearity of μ implies that $\mu(f,0) = 0 = \mu(0,g)$. If $H : f \Rightarrow 0$ and $G : g \Rightarrow 0$ are given then in fact

$$\bar{\mu}(f \otimes G) = \bar{\mu}(H \otimes g) \tag{2}$$

so that μ_* is well defined and $i_{n,m}$–equivariant. In fact, we have for the track $(H,G) : (f,g) \Rightarrow (0,0)$ with $(f,g) : X \to Z^n \times Z^m$ given by the homotopy (H_t, G_t) the formula

$$\begin{aligned}\mu(H,G) &= \mu((0,G)\widehat{+}(H,g))\\ &= \mu((H,0)\widehat{+}(f,G))\end{aligned} \tag{3}$$

where $\widehat{+}$ denotes addition of tracks. Hence we get

$$\begin{aligned}\mu(H,g) &= 0\widehat{+}\mu(H,g) = \mu((0,G))\widehat{+}\mu(H,g)\\ &= \mu((0,G)\widehat{+}(H,g)) = \mu(H,G)\\ &= \mu((H,0)\widehat{+}(f,G))\\ &= \mu(H,0)\widehat{+}\mu(f,G)\\ &= 0\widehat{+}\mu(f,G) = \mu(f,G)\end{aligned}$$

and this proves (2). Finally we have to show that μ_* is compatible with the Γ–operator. For this let $r \in R_n$ and $s \in R_m$ and let

$$\begin{aligned}\Gamma_{r-\varepsilon r} &: r - \varepsilon r \Rightarrow 0 : Z^n \to Z^n\\ \Gamma_{s-\varepsilon s} &: s - \varepsilon s \Rightarrow 0 : Z^m \to Z^m\\ \Gamma_{r\odot s-\varepsilon(r)\cdot\varepsilon(s)} &: r \odot s - \varepsilon(r \odot s) \Rightarrow 0 : Z^{n+m} \to Z^{n+m}\end{aligned} \tag{4}$$

where $r \odot s = i_{n,m}(r,s) \in R_{n+m}$. We observe that in R_{n+m} we have the following equations

$$\begin{aligned}(r-\varepsilon r)\odot s+\varepsilon(r)(1_n\odot(s-\varepsilon s)) &= \\ (r\odot s)-\varepsilon(r)(1_n\odot s)+\varepsilon(r)(1_n\odot s)-\varepsilon(r)\varepsilon(s)(1_n\odot 1_m) & \\ = r\odot s-\varepsilon(r\odot s)\in R_{n+m} &\end{aligned}\tag{5}$$

Let $Z^n \wedge Z^m = Z^n \times Z^m / Z^n \times \{0\} \cup \{0\} \times Z^m$ be the smash product. Then the cup product map μ induces a map $\widetilde{\mu} : Z^n \wedge Z^m \to Z^{n+m}$ and we get the composites $a, b, c : Z^n \wedge Z^m \to Z^{n+m}$ by

$$\begin{aligned} a &= \widetilde{\mu}(r-\varepsilon r)\wedge s, \\ b &= \varepsilon(r)\widetilde{\mu}(1_n \wedge (s-\varepsilon s)), \\ c &= (r\odot s-\varepsilon(r\odot s))\widetilde{\mu}. \end{aligned}\tag{6}$$

Then (5) shows that $a + b = c$. Hence

$$\begin{aligned} A &= \widetilde{\mu}(\Gamma_{r-\varepsilon r}\wedge s)+\varepsilon(r)\widetilde{\mu}(1_n\wedge\Gamma_{s-\varepsilon s}) \text{ and} \\ B &= \Gamma_{r\odot s-\varepsilon(r\odot s)}\widetilde{\mu} \end{aligned}\tag{7}$$

are both tracks $c \Rightarrow 0$. Now obstruction theory shows that these tracks A and B coincide since the set of homotopy classes $[\Sigma Z^n \wedge Z^m, Z^{n+m}]$ is trivial. The equation $A = B$ implies that μ satisfies

$$\mu_*\Gamma(r\otimes s\otimes f\otimes g)=\Gamma(r\odot s\otimes\mu(f,g))\tag{8}$$

by definition on Γ in (2.6) and by formula (2.4), which exactly corresponds to $A = B$. q.e.d.

5 Squaring operations

Let K, L, R be k–algebras. Then $K \otimes L$ and $L \otimes K$ are k–algebras and we consider homomorphisms of k–algebras

$$K\otimes L\xrightarrow{i}R\xleftarrow{j}L\otimes K\tag{5.1}$$

compatible with augmentations. For $\alpha \in K$ and $\beta \in L$ we write

$$\alpha\odot\beta=i(\alpha\otimes\beta) \text{ and } \beta\odot\alpha=j(\alpha\otimes\beta)\tag{1}$$

We also consider elements $\tau, \bar\tau \in R$ with $\tau\bar\tau = 1 = \bar\tau\tau$ and

$$\begin{aligned}\tau(\alpha\odot\beta)&=\beta\odot\alpha, \\ \bar\tau(\beta\odot\alpha)&=\alpha\odot\beta\end{aligned}\tag{2}$$

for $\alpha \in K, \beta \in L$. As a special case let $R_n = k[\sigma_n]$ be the group algebra of the symmetric group σ_n. Then we have the element $\tau = \tau_{n,m} \in \sigma_{n+m}$ interchanging the n–block $\{1, \dots, n\}$ and the m–block $\{n + 1, \dots, n + m\}$ such that the diagram

$$\begin{tikzcd} \sigma_n\times\sigma_m \arrow[r, "i_{n,m}"] \arrow[d, "T"] & \sigma_{n+m} \arrow[d, "\tau"] \\ \sigma_m\times\sigma_n \arrow[r, "i_{m,n}"] & \sigma_{n+m} \end{tikzcd}\tag{3}$$

commutes. Here T is the interchange map and $i_{n,m}$ is the inclusion. The sign of $\tau_{n,m}$ is $(-1)^{nm}$. We get algebra homomorphisms

$$R_n \times R_m \xrightarrow{i} R_{n+m} \xleftarrow{j} R_m \otimes R_n \tag{4}$$

induced by the inclusions $i_{n,m}$ and $i_{m,n}$ above and the element $\tau = \tau_{n,m} \in R_{n+m}$ satisfies (2).

The following definition of τ–pairs and τ–squares is part of the concept of a τ–commutative secondary algebra in the next section.

(5.2) Definition. A *τ–pair* consists of secondary modules Z_K, Y_L and X_R and maps

$$Z_K \otimes Y_L \xrightarrow{\mu_i} X_R \xleftarrow{\mu_j} Y_L \otimes Z_K \tag{1}$$

where i, j have the properties in (5.1) with respect to an element $\tau \in R$. We write for $z \in Z_0, y \in Y_0, z_1 \in Z_1, y_1 \in Y_1$,

$$\begin{aligned} &\mu_i(z \otimes y) = z \cdot y,\ \mu_i(z_1 \otimes y) = z_1 \cdot y,\ \mu_i(z \otimes y_1) = z \cdot y_1, \\ &\mu_j(y \otimes z) = y \cdot z,\ \mu_j(y \otimes z_1) = y \cdot z_1,\ \mu_j(y_1 \otimes z) = y_1 \cdot z. \end{aligned} \tag{2}$$

Now the element τ (termed the *interchange element*) satisfies

$$\begin{aligned} \tau(z \cdot y) &= y \cdot z, \\ \tau(z_1 \cdot y) &= y \cdot z_1, \\ \tau(z \cdot y_1) &= y_1 \cdot z. \end{aligned} \tag{3}$$

For $K = L$ and $Z_K = Y_L$ and $i = j$ and $\mu = \mu_i = \mu_j$ we call the τ–pair

$$Y_L \otimes Y_L \xrightarrow{\mu} X_R \tag{4}$$

a *τ–square.*

For a τ–pair μ_i, μ_j the induced k–linear maps $\pi_0\mu_i, \pi_0\mu_j, \pi_1\mu_i, \pi_1\mu_j$ yield

$$\begin{gathered} \pi_0 Z \otimes \pi_0 Y \to \pi_0 X \leftarrow \pi_0 Y \otimes \pi_0 Z \\ \pi_0 Z \otimes \pi_1 Y \oplus \pi_1 Z \otimes \pi_0 Y \to \pi_1 X \leftarrow \pi_0 Y \otimes \pi_1 Z \oplus \pi_1 Y \otimes \pi_0 Z \end{gathered} \tag{5}$$

These *product maps* carry $u \otimes v$ to the products $u \cdot v$ as in (2). Moreover (3) implies for $a \in \pi_0 Y, b \in \pi_0 Z, a_1 \in \pi_1 Y, b_1 \in \pi_1 Z$,

$$\begin{aligned} a \cdot b &= \varepsilon(\tau) b \cdot a, \\ a \cdot b_1 &= \varepsilon(\tau) b_1 \cdot a, \\ a_1 \cdot b &= \varepsilon(\tau) b \cdot a_1. \end{aligned} \tag{6}$$

A *map* between τ–pairs is a commutative diagram

$$\begin{tikzcd} Z_K \otimes Y_L \arrow[r, "\mu_i"] \arrow[d, "f \otimes g"'] & X_R \arrow[d, "d"] & Y_L \otimes Z_K \arrow[l, "\mu_j"'] \arrow[d, "g \otimes f"] \\ Z'_K \otimes Y'_L \arrow[r, "\mu'_i"] & X'_R & Y'_L \otimes Z'_K \arrow[l, "\mu'_j"'] \end{tikzcd} \tag{7}$$

where f, g resp. d are K, L, resp. R–equivariant. Such a map is a weak equivalence if f, g and d are weak equivalences. Of course the product maps (5) are invariants of the weak equivalence type of a τ–pair. A map between τ–squares is accordingly a map (f, g, d) with $f = g$. Below we describe a *squaring operation* Sq which is an invariant of the weak equivalence type of a τ–square. Hence though secondary modules X by (5.2) are up to weak equivalence only determined by $\pi_0 X$ and $\pi_1 X$ a τ–pair or a τ–square have nontrivial weak equivalence types.

(5.3) Lemma. *For a τ–pair as in* (5.2) *we have* $\Gamma(\tau \otimes (y \cdot z + \varepsilon(\tau) z \cdot y)) = 0$.

Proof. Since $\tau\tau = 1$ we have

$$\begin{aligned} 0 = \Gamma(1, z \cdot y) = \Gamma(\tau\tau, z \cdot y) &= \Gamma(\tau \otimes \tau(z \cdot y)) + \varepsilon(\tau)\Gamma(\tau \otimes z \cdot y) \\ &= \Gamma(\tau \otimes y \cdot z) + \varepsilon(\tau)\Gamma(\tau \otimes z \cdot y) \\ &= \Gamma(\tau \otimes (y \cdot z + \varepsilon(\tau) z \cdot y)). \end{aligned}$$

q.e.d.

(5.4) Lemma. *For a τ–square as in* (5.2) *we have* $\Gamma(\tau, \partial y_1 \cdot \partial y_1) = (1 - \varepsilon(\tau))(\partial y_1) \cdot y_1$.

Proof. We have $\partial_\otimes(y_1 \otimes \partial y_1) = \partial y_1 \otimes \partial y_1$ in $Y_L \otimes Y_L$ so that $\partial(y_1 \cdot \partial y_1) = \partial y_1 \cdot \partial y_1$ in X_R. Hence

$$\begin{aligned} \Gamma(\tau, \partial y_1 \cdot \partial y_1) &= \Gamma(\tau, \partial(y_1 \cdot \partial y_1)) \\ &= (\tau - \varepsilon(\tau)) y_1 \cdot (\partial y_1) \\ &= (\partial y_1) \cdot y_1 - \varepsilon(\tau) y_1 \cdot (\partial y_1) \end{aligned}$$

Using (4.1) we see that $y_1 \cdot (\partial y_1) = (\partial y_1) \cdot y_1$. q.e.d.

(5.5) Proposition. *Let $\mu : Y_L \to X_R$ be a τ–square as in* (5.2) *with $\varepsilon(\tau) = 1$. Then there is a well defined k–linear map*

$$\mathrm{Sq} : \pi_0(Y) \to \pi_1(X)$$

which carries the element $\{y\} \in \pi_0(Y)$ represented by $y \in Y_0$ to the element $\mathrm{Sq}\{y\} = \Gamma(\tau, y \cdot y)$. Moreover we have $2\,\mathrm{Sq} = 0$.

Proof. We have in X_R the equation

$$\partial\Gamma(\tau, y \cdot y) = (\tau - \varepsilon(\tau))(y \cdot y) = \tau(y \cdot y) - y \cdot y = 0$$

since $\varepsilon(\tau) = 1$ and $\tau(y \cdot y) = y \cdot y$ by (5.2). Hence $\Gamma(\tau, y \cdot y)$ is an element in $\pi_1(X) = \mathrm{kernel}(\partial : X_1 \to X_0)$. On the other hand we have $\pi_0(Y) = \mathrm{cokernel}(\partial : Y_1 \to Y_0)$. If we replace y by $y + \partial y_1 \in Y_0$ with $y_1 \in Y_1$ we get

$$\Gamma(\tau, (y + \partial y_1) \cdot (y + \partial y_1)) = \Gamma(\tau, y \cdot y) + \Delta$$

where

$$\Delta = \Gamma(\tau, y \cdot (\partial y_1) + (\partial y_1) \cdot y) + \Gamma(\tau, (\partial y_1) \cdot (\partial y_1)).$$

Since $\varepsilon(\tau) = 1$ we see by (5.3) and (5.4) that $\Delta = 0$. Hence the function Sq with $\mathrm{Sq}\{y\} = \Gamma(\tau, y \cdot y)$ is well defined. We obviously have $\Gamma(\tau, (r \cdot y)(r \cdot y)) = \Gamma(\tau, r^2 y \cdot y) = r^2\Gamma(\tau, y \cdot y)$ and we have $\Gamma(\tau, (y + z)(y + z)) = \Gamma(\tau, y \cdot y) + \Gamma(\tau, z \cdot z) + \Gamma(\tau, y \cdot z + z \cdot y)$. Here the last term vanishes by (5.3). Hence we get the equations $\mathrm{Sq}(ra) = r^2\,\mathrm{Sq}(a)$ and $\mathrm{Sq}(a + b) = \mathrm{Sq}(a) + \mathrm{Sq}(b)$ with $a, b \in \pi_0(Y)$. Hence $\mathrm{Sq}(0) = 0$ and then $0 = \mathrm{Sq}\,a + \mathrm{Sq}(-a)$ where $\mathrm{Sq}(-a) = (-1)^2\,\mathrm{Sq}(a) = \mathrm{Sq}(a)$. Hence $2\,\mathrm{Sq}(a) = 0$. q.e.d.

The main examples of τ–pairs and τ–squares arise by the following result.

(5.6) Proposition. *For a pointed space X we obtain by $\tau_{n,m} \in \sigma_{n+m}$ and (4.6) the τ–pair*

$$\mathcal{H}^n(X) \otimes \mathcal{H}^m(X) \xrightarrow{\mu_*} \mathcal{H}^{n+m}(X) \xleftarrow{\mu_*} \mathcal{H}^m(X) \otimes \mathcal{H}^n(X)$$

which is a τ–square if $n = m$.

Proof. This is an easy application of the commutative diagram in property (c) of the cup product map μ in the introduction. q.e.d.

6 Secondary cohomology

Using the tensor product of secondary modules we introduce the notion of a secondary algebra. We define a functor which associates with each space X a secondary cohomology algebra $\mathcal{H}^*(X)$.

We consider a sequence R_* of augmented k–algebras $R_n, n \geq 0$, together with augmented algebra maps

$$i_{n,m} = \odot : R_n \otimes R_m \to R_{n+m} \tag{6.1}$$

carrying $\alpha \otimes \beta$ to $\alpha \odot \beta$ such that for $\gamma \in R_k$ we have

$$(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma)$$

in R_{n+m+k}. Since $\odot$ is an algebra map we have

$$(\alpha \cdot \alpha') \odot (\beta \cdot \beta') = (\alpha \odot \beta) \cdot (\alpha' \odot \beta')$$

where $\alpha \cdot \alpha'$ denotes the product in R_n. Let $1_n \in R_n$ be the unit element of R_n with $1_n \odot 1_m = 1_{n+m}$. For $n = 0$ we have $R_0 = k$ and $1_0 \in R_0$ satisfies $1_0 \odot \alpha = \alpha \odot 1_0 = \alpha$. We call $R_* = (R_*, \odot)$ a *coefficient algebra.*

Of course we have the *trivial* coefficient algebra k with $R_n = k$ for $n \geq 0$. On the other hand we shall use the *symmetric coefficient algebra* $k[\sigma_*]$ given by the augmented group algebras $R_n = k[\sigma_n]$ where σ_n is the symmetric group and R_n has the sign augmentation (1.1)(3). Moreover $\odot = i_{m,n}$ is induced by the inclusion of groups $\sigma_n \times \sigma_m \subset \sigma_{n+m}$ as in (5.1)(4).

(6.2) Definition. An *algebra V over a coefficient algebra R_** is a sequence of R_n–modules $V^n, n \geq 0$, together with k–linear maps

$$V^n \otimes V^m \to V^{n+m}$$

carrying $x \otimes y$ to $x \cdot y$. For $z \in V^k$ we have in V^{n+m+k}

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

and for $\alpha \in R_n, \beta \in R_m$ we have

$$(\alpha x) \cdot (\beta y) = (\alpha \odot \beta)(x \cdot y).$$

We do not assume that the algebra V has a unit. Let V and W be such algebras over R_*. Then a *map* $f : V \to W$ over R_* is given by an R_n–linear map $f = f^n : V^n \to W^n, n \geq 0$, with $f(x \cdot y) = f(x) \cdot f(y)$. This defines the category of algebras over R_*.

If $R_* = k$ is the trivial coefficient algebras then V in (6.2) is just a *graded algebra* over k. A graded algebra V is *commutative* if for $x \in V^n, y \in V^m$ we have

$$y \cdot x = (-1)^{nm} x \cdot y \tag{6.3}$$

For example the reduced cohomology $\widetilde{H}^*(X, k)$ of a pointed space X with coefficients in k is a commutative graded algebra. We generalize this notion of commutative algebras as follows.

(6.4) Definition. Let R_* be a coefficient algebra and assume elements

$$\tau_{m,n} \in R_{n+m} \ (n, m \geq 0)$$

are given with the following properties $(m, n, k \geq 0)$.

$$\begin{aligned}
\tau_{m,n}\tau_{n,m} &= 1_{n+m} \\
\tau_{m,0} = \tau_{0,m} &= 1_m \\
\tau_{n,m}(\beta \odot \alpha) &= (\alpha \odot \beta)\tau_{n,m} \text{ for } \alpha \in R_n, \beta \in R_m \\
\tau_{m+n,k} &= (\tau_{m,k} \odot 1_n)(1_m \odot \tau_{n,k}) \\
\varepsilon(\tau_{m,n}) &= (-1)^{m,n} \in k
\end{aligned}$$

Then we say that an algebra V over R_* is *τ–commutative* if for $x \in V^n, y \in V^m$

$$y \cdot x = \tau_{n,m}(x \cdot y)$$

in V^{n+m}.

For example we have as in (5.1)(3) the interchange elements $\tau_{n,m} \in k[\sigma_{n+m}]$ in the symmetric coefficient algebra for which a τ–commutative algebra is the same as a "commutative twisted algebra" in the sense of Stover [St]. On the other hand we can define the interchange elements $\tau_{n,m} = (-1)^{nm} \in k$ in the trivial coefficient algebra so that in this case a τ–commutative algebra is the same as a commutative graded algebra in (6.3). We now are ready to introduce the notion of a secondary algebra.

(6.5) Definition. Let R_* be a coefficient algebra. A *secondary algebra $\mathcal{H}^*$ over R_** consists of a sequence of secondary modules $\mathcal{H}^n$ over R_n, $n \geq 0$, together with $i_{n,m}$–equivariant maps

$$\mu = \mu_{n,m} : \mathcal{H}^n \otimes \mathcal{H}^m \to \mathcal{H}^{n+m} \tag{1}$$

for $n, m \geq 0$ which are associative in the sense that the diagram

$$\begin{tikzcd}
\mathcal{H}^n \otimes \mathcal{H}^m \otimes \mathcal{H}^r \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & \mathcal{H}^n \otimes \mathcal{H}^{m+r} \arrow[d, "\mu"] \\
\mathcal{H}^{n+m} \otimes \mathcal{H}^r \arrow[r, "\mu"] & \mathcal{H}^{n+m+r}
\end{tikzcd} \tag{2}$$

commutes. Here we use the tensor product of secondary modules. If elements $\tau_{n,m} \in R_{n+m}$ are given as in (6.4) we say that the secondary algebra $\mathcal{H}^*$ is *τ–commutative* if

$$\mathcal{H}^n \otimes \mathcal{H}^m \xrightarrow{\mu} \mathcal{H}^{n+m} \xleftarrow{\mu} \mathcal{H}^m \otimes \mathcal{H}^n \tag{3}$$

is a $\tau_{n,m}$–pair in the sense of (5.2). We study τ–commutative secondary algebras in more detail in section §9 below.

We say that the secondary algebra $\mathcal{H}^*$ is *w–closed* if one has k–linear isomorphisms

$$w = w^n : \pi_1\mathcal{H}^n \cong \pi_0\mathcal{H}^{n-1} \tag{4}$$

for $n \geq 1$ which satisfy

$$\begin{aligned} w^{n+m}(y_1 \cdot z) &= w^n(y_1) \cdot z \\ w^{n+m}(y \cdot z_1) &= (-1)^n y \cdot w^m(z_1) \end{aligned} \tag{5}$$

for $y \in \pi_0\mathcal{H}^n, z \in \pi_0\mathcal{H}^m, y_1 \in \pi_1\mathcal{H}^n, z_1 \in \pi_1\mathcal{H}^m$. In (5) the multiplication is defined by the maps (1) as in (5.2)(5). A *map* $f : \mathcal{H}^* \to \mathcal{G}^*$ between secondary algebras is given by a sequence $f^n : \mathcal{H}^n \to \mathcal{G}^n$ of R_n–equivariant maps between secondary modules such that f^n is compatible with μ in (1) and w in (4). Such a map f is a *weak equivalence* if f^n is a weak equivalence in **secmod** for $n \geq 0$.

Let **secalg** be the category of secondary algebras over the symmetric coefficient algebra $k[\sigma_*]$ which are τ–commutative and w–closed. For an object $\mathcal{H}^*$ in **secalg** we obtain a commutative graded algebra H^* by

$$H^n = \pi_0\mathcal{H}^n \text{ for } n \geq 0 \tag{6.6}$$

with the multiplication $H^n \otimes H^m \to H^{n+m}$ induced by μ in (6.5)(1). As in (5.2)(6) we see that H^* is commutative since $\mathcal{H}^*$ is τ–commutative. Moreover $\mu : \mathcal{H}^n \otimes \mathcal{H}^n \to \mathcal{H}^{2n}$ for $n \geq 1$ is a τ–square which yields the *squaring operation*

$$\mathrm{Sq}^{n-1} = w^{2n-1}\,\mathrm{Sq} : H^n \to H^{2n-1} \tag{6.7}$$

with $2\,\mathrm{Sq}^{n-1} = 0$ by (5.5). Here we use the assumption that $\mathcal{H}^*$ is w–closed.

The next result describes the *secondary cohomology algebra* $\mathcal{H}^*(X)$ of a pointed space X.

(6.8) Theorem. *There is a functor*

$$\mathcal{H}^* : \mathbf{Top}^* \to \mathbf{secalg}$$

which carries a pointed space X to a secondary algebra $\mathcal{H}^*(X)$ which is τ–commutative and w–closed. Moreover the algebra $\widetilde{H}^* = \pi_0\mathcal{H}^*(X)$ is (6.6) coincides with the reduced cohomology algebra $\widetilde{H}^*(X;k)$ and for $k = \mathbb{F}_2$ the squaring operation Sq^{n-1} in (6.7) coincides with the corresponding Steenrod operation.

Proof. This is an easy consequence of (3.7) and property (c) of the cup product maps in the introduction. The result on Steenrod squares is a consequence of a result of Kristensen, lemma 2.5 in [K].

7 Crossed modules and Hochschild cohomology

We introduce the concept of a crossed module in the context of algebras and we show that (in the graded case) a crossed module is the same as a secondary algebra over the trivial coefficient algebra $R_* = k$. A crossed module and equivalently a secondary algebra over

k represent an element in the third Hochschild cohomology. This leads to the notion of a characteristic class of a differential algebra.

We here consider the graded and the non–graded case at the same time. A graded vector space V is assumed to be positively graded, i. e. $V^i = 0$ for $i < 0$.

We use the following notation. An *algebra* $\widetilde{A}$ is given by a (graded) k–vector space $\widetilde{A}$ and a multiplication map $\widetilde{A} \otimes \widetilde{A} \to \widetilde{A}$ which is associative. On the other hand a *k–algebra* A is an algebra with unit $k \to A$ and augmentation $\varepsilon : A \to k$. Hence a k–algebra A is an algebra under and over k. Then the augmentation ideal

$$\widetilde{A} = \mathrm{kernel}(\varepsilon : A \to k)$$

is an algebra which determines the k–algebra $A = \widetilde{A} \oplus k$ completely. Moreover an $\widetilde{A}$–module is also an A–module and vice versa.

Let A be a (graded) k–algebra. An *A–bimodule* V is a (graded) k–vector space which is a left and a right A–module such that for $a, b \in A, x \in V$ we have $(a \cdot x) \cdot b = a \cdot (x \cdot b)$. For example A can be considered as an A–bimodule via the multiplication in A.

(7.1) Definition. A *crossed module* is a map of A–bimodules

$$\partial : V \to A$$

satisfying $\varepsilon\partial = 0$ and $(\partial v) \cdot w = v \cdot (\partial w)$ for $v, w \in V$.

Let $\pi_0(\partial) = \mathrm{cokernel}(\partial)$ and $\pi_1(\partial) = \mathrm{kernel}(\partial)$ in the category of (graded) vector spaces. Then the algebra structure of A induces an algebra structure of $\pi_0(\partial)$ and the A–bimodule structure of V induces a $\pi_0(\partial)$–bimodule structure of $\pi_1(\partial)$. In fact for $\{a\} \in \pi_0(\partial)$ the multiplication $\{a\} \cdot v = a \cdot v$ with $v \in \pi_1(\partial)$ is well defined since $(a + \partial w) \cdot v = a \cdot v + (\partial w) \cdot v = a \cdot v + w \cdot \partial v = a \cdot v$ where $\partial v = 0$. Hence a crossed module yields the exact sequence

$$0 \longrightarrow \pi_1(\partial) \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{q} \pi_0(\partial) \longrightarrow 0$$

in which all maps are A–bimodule morphisms. Here the A–bimodule structure of $\pi_0(\partial)$ and $\pi_1(\partial)$ is induced by the algebra map q.

(7.2) Lemma. *A secondary algebra $\mathcal{H}^*$ over the trivial coefficient algebra k as defined in* (6.5) *is the same as a crossed module ∂.*

Proof. Given $\mathcal{H}^*$ we obtain

$$\partial : \mathcal{H}^*_1 \to \mathcal{H}^*_0$$

where $\mathcal{H}^*_0$ is an algebra by the multiplication μ in (6.5)(1). Moreover $\mathcal{H}^*_1$ is a $\mathcal{H}^*_0$–bimodule by the multiplication (6.5)(1). Using (4.1) we see that ∂ yields the crossed module

$$\partial = (\partial, 0) : \mathcal{H}^*_1 \to \mathcal{H}^*_0 \oplus k$$

where $\mathcal{H}^*_0 \oplus k$ is the k–algebra given by $\mathcal{H}^*_0$. Conversely it is easy to see that a crossed module (7.1) defines a secondary algebra over k. q.e.d.

We now use crossed modules or equivalently secondary algebras over k for the definition of Hochschild cohomology.

(7.3) Definition. Let H be a (graded) k–algebra and let M be an H–bimodule. A *crossed n–fold extension* of H by M with $n \geq 2$ is an exact sequence in the category of (graded) k–vector spaces

$$\mathcal{E} : 0 \longrightarrow M \xrightarrow{\gamma} V_{n-1} \xrightarrow{\partial_{n-1}} \ldots \longrightarrow V_1 \xrightarrow{\partial_1} A \xrightarrow{q} H \longrightarrow 0$$

where $V_{n-1}, \dots, V_2$ are H–bimodules and ∂_1 is a crossed module. Moreover all maps are A–bimodule maps with the A–bimodule structure induced by the algebra map $q : A \to H$. A *weak equivalence* between two such extensions is a commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, equal] & V_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \arrow[r] & V_1 \arrow[r] \arrow[d, "f_1"] & A \arrow[r] \arrow[d, "f_0"] & H \arrow[d, equal] \\
M \arrow[r] & W_{n-1} \arrow[r] & \cdots \arrow[r] & W_1 \arrow[r] & B \arrow[r] & H
\end{tikzcd}$$

where f_0 is an algebra map and f_1 is a f_0–biequivariant homomorphism and f_i for $n-1 \geq i \geq 2$ is an H–bimodule map. For $n \geq 2$ let $\mathsf{HH}^{n+1}(H, M)$ be the set of weak equivalence classes of crossed n–fold extensions of H by M.

In Baues–Minian [BM] we show:

(7.4) Proposition. *$\mathsf{HH}^{n+1}(H, M)$ as defined above coincides naturally with the Hochschild cohomology of H with coefficients in $M, n \geq 2$.*

The proposition holds in the graded and in the non–graded case.

We have seen in (7.2) that each secondary algebra $\mathcal{H}^*$ over k yields a canonical crossed extension

$$0 \longrightarrow \pi_1(\mathcal{H}^*) \longrightarrow \mathcal{H}_1^* \xrightarrow{\partial} \mathcal{H}_0^* \oplus k \longrightarrow \pi_0(\mathcal{H}^*) \oplus k \longrightarrow 0$$

Here $H = \pi_0(\mathcal{H}^*) \oplus k$ is a k–algebra and $M = \pi_1(\mathcal{H}^*)$ is an H–bimodule. Hence the crossed extension represents an element

$$<\mathcal{H}^*> \in \mathsf{HH}^3(H, M) \tag{7.5}$$

which is termed the *characteristic class* of the secondary algebra $\mathcal{H}^*$. On the other hand a differential algebra C (like the cochain algebra of a space) as well yields a crossed extension representing a characteristic class $<C>$ as in the following example.

(7.6) Example. Let C be a differential graded k–algebra, that is, $C = \{C^i, i \geq 0\}$ with $C^iC^j \subseteq C^{i+j}$ and $d : C \to C$ of degree $+1$ satisfying $d(xy) = (dx)y + (-1)^{|x|}xd(y)$ and $dd = 0$ and $\varepsilon d = 0$. Then d induces the map of graded k–vector spaces

$$V = \operatorname{coker}(\widetilde{d})[1] \xrightarrow{\partial} \ker(d) = A \tag{1}$$

Here we define for a graded vector space W the *shifted* graded vector space $W[1]$ by

$$W^n = (W[1])^{n+1}, w \mapsto s(w), \tag{2}$$

Hence for the cokernel of the differential $\operatorname{coker}(d) = \widetilde{C}/\operatorname{im}(\widetilde{d})$ we obtain the shifted object $V = \operatorname{coker}(\widetilde{d})[1]$. Since d is of degree $+1$ the boundary induces ∂ by $\partial s\{v\} = d(v)$ for $\{v\} \in \operatorname{coker}(d), v \in C$. The algebra C induces an algebra structure of $A = \ker(d)$. Moreover it induces the structure of an A–bimodule on V by setting

$$\begin{aligned} a \cdot (s\{v\}) &= (-1)^{|a|}s\{a \cdot v\} \\ (s\{v\}) \cdot b &= s\{v \cdot b\} \end{aligned} \tag{3}$$

One can check that $\partial : V \to A$ is a crossed module in the sense of (7.1), see [BM]. This proves that $\partial : V \to A$ is crossed module and therefore we obtain by (7.2) a secondary algebra $\partial : V \to \widetilde{A}$ over k which is, in fact, w–closed (see (6.5)) by defining

$$w : \pi_1(\partial) = \widetilde{H}^*(C)[1] \cong \widetilde{H}^*(C) = \pi_0(\partial) \tag{4}$$

with $ws(x) = x$ for $x \in H^*(C) = \ker(d)/\operatorname{im}(d)$. The equations in (3) for the A–bimodule structure of V correspond exactly to the equations in (6.5)(5).

According to (3) we define for a graded algebra H^* the H^*–*bimodule* $\widetilde{H}^*[1]$ by setting

$$\begin{aligned} a \cdot (sx) &= (-1)^{|a|} s(a \cdot x) \\ (sx) \cdot b &= s(x \cdot b) \end{aligned} \tag{5}$$

Then we obtain by (1) and (4) the crossed 2–extension

$$0 \longrightarrow \widetilde{H}^*[1] \longrightarrow V \xrightarrow{\partial} A \longrightarrow H^* \longrightarrow 0$$

which by (7.4) represents an element

$$<C> \in \mathsf{HH}^*(H^*, \widetilde{H}^*[1]) \tag{6}$$

where $H^* = H^*C$ is the cohomology algebra of the differential algebra C. A cocycle θ representing $<C>$ is considered in Berrick–Davydov [BD].

As a special case we obtain for a pointed space X the augmented algebra of cochains on X denoted by C^*X for which

$$H^*(C^*X) = H^*(X)$$

is the cohomology algebra of X. Hence we get by (7.6)(6) the class

$$<C^*(X)> \in \mathsf{HH}^*(H^*(X), \widetilde{H}^*(X)[1]) \tag{7.7}$$

which is an invariant of the homotopy type of X in the sense that a pointed map $f : X \to Y$ satisfies

$$(f^*)^* <C^*(X)> = (f^*[1])_* <C^*(Y)>$$

in $\mathsf{HH}^3(H^*(Y), \widetilde{H}^*(X)[1])$ where $f^* : H^*(Y) \to H^*(X)$ yields the structure of an $H^*(Y)$–bimodule on $\widetilde{H}^*(X)[1]$.

We now compare the class (7.7) with the secondary cohomology algebra $\mathcal{H}^*(X)$ in (6.8).

(7.8) Proposition. *By forgetting structure we obtain from the secondary cohomology $\mathcal{H}^*(X)$ a secondary algebra over k denoted by $\mathcal{H}^*(X)_{(k)}$. Then the classes*

$$<C^*X> = <\mathcal{H}^*(X)_{(k)}> \in \mathsf{HH}^3(H^*(X), \widetilde{H}^*(X)[1])$$

given by (7.5) *and* (7.7) *coincide.*

Proof. Using (3.3) and the definition of Y_3 in Baues [B] we see that $<\mathcal{H}(X, Y_3)> = <C^*X>$ for a simplicial set X. Here we use the universal property of Y_3 which says that a simplicial map $X \to Y_3$ can be identified with a cocycle in C^*X. q.e.d.

8 τ–crossed modules for commutative graded algebras

We have seen in §7 that a secondary algebra over the trivial coefficient algebra k is the same as a crossed module. We here show that in a similar way a τ–commutative secondary algebra over R_* is the same as a τ–crossed module. Weak equivalence classes of τ–crossed modules yield an abelian group which we call the third symmetric cohomology.

Let R_* be a coefficient algebra with interchange elements $\tau_{m,n} \in R_{m+n}$, for example let $R_* = k[\sigma_*]$ be the symmetric coefficient algebra. An *R_*–module* V is a sequence of (left) R_n–modules $V^n, n \geq 0$. A *map* or an *R_*–linear map* $f : V \to W$ between R_*–modules is given by a sequence of R_n–linear maps $f^n : V^n \to W^n$ for $n \geq 0$. The field k (concentrated in degree 0) is an R_*–module. Moreover using the augmentation ε of $R_n, n \geq 0$, we see that each graded k–vector space M is an R_*–module which we call an *ε–module.* For $x \in M^m$ we write $|x| = m$ where $|x|$ is the *degree* of x.

Given R_*–modules $V_1, \dots, V_k$ we define the *R_*–tensor product* $V_1 \overline{\otimes} \dots \overline{\otimes} V_k$ by

$$(V_1 \overline{\otimes} \dots \overline{\otimes} V_k)_n = \bigoplus_{n_1 + \dots + n_k = n} R_n \otimes_{R_{n_1} \otimes \dots \otimes R_{n_k}} V_1^{n_1} \otimes \dots \otimes V_k^{n_k} \tag{8.1}$$

where we use the algebra map $\odot : R_{n_1} \otimes \dots \otimes R_{n_k} \to R_n$ given by the structure of the coefficient algebra R_* in (6.1). One readily checks associativity

$$(V_{1,1} \overline{\otimes} \dots \overline{\otimes} V_{1,k_1}) \overline{\otimes} \dots \overline{\otimes} (V_{s,1} \overline{\otimes} \dots \overline{\otimes} V_{s,k_s}) = V_{1,1} \overline{\otimes} \dots \overline{\otimes} V_{1,k_1} \overline{\otimes} \dots \overline{\otimes} V_{s,1} \overline{\otimes} \dots \overline{\otimes} V_{s,k_s} \tag{1}$$

Compare Stover [St] 2.9. Moreover the interchange element τ in R_* yields the isomorphism

$$T : V \overline{\otimes} W \cong W \overline{\otimes} V \tag{2}$$

which carries $v \otimes w$ to $\tau_{w,v} w \otimes v$ where

$$\tau_{w,v} = \tau_{m,n} \in R_{m+n}$$

for $w \in W^m, v \in V^n$. Of course we have $k \overline{\otimes} V = V = V \overline{\otimes} k$.

(8.2) Definition. An *algebra A over R_** is given by an R_*–linear map $\mu : A \overline{\otimes} A \to A, \mu(a \otimes b) = a \cdot b$, which is associative in the sense that the diagram

$$\begin{tikzcd}
A \overline{\otimes} A \overline{\otimes} A \arrow[r, "1 \overline{\otimes} \mu"] \arrow[d, "\mu \overline{\otimes} 1"'] & A \overline{\otimes} A \arrow[d, "\mu"] \\
A \overline{\otimes} A \arrow[r, "\mu"] & A
\end{tikzcd}$$

commutes. Moreover A is *τ–commutative* if

$$\begin{tikzcd}
A \overline{\otimes} A \arrow[r, "\mu"] \arrow[d, "T"'] & A \arrow[d, equal] \\
A \overline{\otimes} A \arrow[r, "\mu"] & A
\end{tikzcd}$$

commutes. One readily checks that this coincides with the notation in (6.2) and (6.3). We say that A is a *k–algebra over R_** if algebra maps $k \xrightarrow{i} A \xrightarrow{\varepsilon} k$ are given with $\varepsilon i = 1$. Such a k–algebra A over R_* is completely determined by the algebra $\widetilde{A}$ over R_* with $\widetilde{A} = \text{kernel}(\varepsilon : A \to k)$ and $A = k \oplus \widetilde{A}$.

For an R_*–module V let $V_{(k)}$ be the underlying graded k–vectorspace. If A is a k–algebra over R_* then $A_{(k)}$ is a k–algebra (over k) in the sense of §7 above.

(8.3) Definition. Given an algebra A over R_* we say that an R_*–module V is an A–*module* if a map $m : A\overline{\otimes}V \to V$ is given such that

$$\begin{array}{ccc} A\overline{\otimes}A\overline{\otimes}V & \xrightarrow{1\overline{\otimes}\mu} & A\overline{\otimes}V \\ {\scriptstyle \mu\overline{\otimes}1}\downarrow & & \downarrow{\scriptstyle \mu} \\ A\overline{\otimes}V & \xrightarrow{\mu} & V \end{array}$$

commutes. Hence for $a \cdot x = \mu(a \otimes x)$ with $a \in A, x \in V$ we have $(\alpha a) \cdot (\beta x) = (\alpha \odot \beta)(a \cdot x)$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$. If A is a k–algebra over R_* we also assume that $1 \cdot x = x$ for $1 \in k, x \in V$. Then the A–module V is an $\widetilde{A}$–module and vice versa. In particular the algebra A is also an A–module in the obvious way.

(8.4) Lemma. *Let A be a τ–commutative k–algebra over R_* and let V be an A–module. Then $V_{(k)}$ is a $A_{(k)}$–bimodule by defining*

$$a \cdot x \cdot b = a \cdot \tau_{b,x}(b \cdot x)$$

for $a, b \in A, x \in V$.

Proof. We write $1_x = 1_n \in R_n$ for $x \in V^n$. Now we have for $a, b \in A$

$$\begin{aligned} (a \cdot x) \cdot b &= \tau_{b,a\cdot x} b \cdot (a \cdot x) = \tau_{b,a\cdot x}(b \cdot a) \cdot x \\ &= \tau_{b,a\cdot x}(\tau_{a,b} a \cdot b) \cdot x \\ &= \tau_{b,a\cdot x}(\tau_{a,b} \odot 1_x)(a \cdot b \cdot x) \\ a \cdot (x \cdot b) &= a \cdot \tau_{b,x}(b \cdot x) = (1_a \odot \tau_{b,x})(a \cdot b \cdot x) \end{aligned}$$

Here we have $\tau_{b,a\cdot x}(\tau_{a,b} \odot 1_x) = 1_a \odot \tau_{b,x}$ by one of the equations in (7.4). q.e.d.

If A and V in (8.4) are ε–modules then (8.4) corresponds to the following special case.

(8.5) Lemma. *Let H be a commutative graded k–algebra and let M be an H–module. Then M is an H–bimodule by defining*

$$a \cdot x \cdot b = a \cdot (-1)^{|b||x|} b \cdot x$$

for $a, b \in H$ and $x \in M$.

For an R_*–module V we obtain as in (1.6) the R_*–linear map

$$I(R_*) \odot_{R_*} V \xrightarrow{\mu} V$$

Here the left hand side is the R_*–module given in degree n by $I(R_n) \otimes_{R_n} V^n$ ond μ carries $a \otimes x$ to $a \cdot x$.

(8.6) Definition. Let A be a τ–commutative k–algebra over R_*. A τ–*crossed module* ∂ is given by a commutative diagram of R_*–linear maps

$$\begin{tikzcd} I(R_*) \odot_{R_*} V \arrow[r, "1\odot\partial"] \arrow[d, "\mu"'] & I(R_*) \odot_{R_*} A \arrow[d, "\mu"] \arrow[dl, "\widetilde{\Gamma}"'] \\ V \arrow[r, "\partial"] & A \end{tikzcd}$$

with the following properties (1) and (2). The R_*–module V is an A–module and ∂ is an A–module morphism, that is $\partial(a \cdot x) = a \cdot (\partial x)$ for $a \in A, x \in V$. Moreover $\varepsilon\partial = 0$ and for $x, y \in V$

$$(\partial x) \cdot y = \tau_{\partial y, x}(\partial y) \cdot x. \tag{1}$$

The R_*–linear map $\widetilde{\Gamma}$ satisfies for $\beta \in I(R_*)$ and $a, b \in A$ the equation

$$a \cdot \widetilde{\Gamma}(\beta \otimes b) = \widetilde{\Gamma}(1 \odot \beta \otimes a \cdot b) \tag{2}$$

Equation (2) shows that $\widetilde{\Gamma}$ is a map of left A–modules. Moreover (1) and (2) imply that for $\alpha \in I(R_*)$ the following equation holds.

$$\widetilde{\Gamma}(\alpha \otimes a) \cdot b = \widetilde{\Gamma}(\alpha \odot 1 \otimes a \cdot b) \tag{3}$$

Here the right hand action of b on $x = \widetilde{\Gamma}(\alpha \otimes a) \in V$ is defined as in (8.4) by $x \cdot b = \tau_{b,x} b \cdot x$. Using this notation (1) is equivalent to $(\partial x) \cdot y = x \cdot (\partial y)$, compare (7.1).

Proof of (3). For $x \in \widetilde{\Gamma}(\alpha \otimes a)$ we have $|x| = |a|$ and hence $\tau_{b,x} = \tau_{b,a}$. Therefore we get

$$\begin{aligned}
\widetilde{\Gamma}(\alpha \otimes a) \cdot b &= \tau_{b,x} b \cdot \widetilde{\Gamma}(\alpha \otimes a) \\
&= \tau_{b,x} \widetilde{\Gamma}(1 \odot \alpha \otimes b \cdot a) \\
&= \widetilde{\Gamma}(\tau_{b,a}(1 \odot \alpha) \otimes b \cdot a) \\
&= \widetilde{\Gamma}((\alpha \odot 1)\tau_{b,a} \otimes b \cdot a) \\
&= \widetilde{\Gamma}((\alpha \odot 1) \otimes \tau_{b,a} b \cdot a) \\
&= \widetilde{\Gamma}((\alpha \odot 1 \otimes a \cdot b).
\end{aligned}$$

(8.7) Lemma. *A τ–crossed module ∂ as in* (8.6) *yields for the underlying k–vector spaces a crossed module in the sense of* (7.1)

$$\partial : V_{(k)} \to A_{(k)}$$

where we use (8.4). *Moreover $\pi_0(\partial) = H$ is a commutative graded k–algebra and $\pi_1(\partial)$ is an H–module with the H–bimodule structure in* (8.5).

The lemma is based on the crucial property of a τ–crossed module, namely that $\pi_0(\partial) =$ cokernel(∂) and $\pi_1(\partial) =$ kernel(∂) are only ε–modules though V and A are R_*–modules. A τ–crossed module yields the exact sequence

$$0 \longrightarrow \pi_1(\partial) \xrightarrow{i} V \xrightarrow{\partial} A \xrightarrow{q} \pi_0(\partial) \longrightarrow 0 \tag{8.8}$$

in which all maps are A–module morphisms. Here the A–module structures of $\pi_0(\partial)$ and $\pi_1(\partial)$ are induced by the algebra map q. Moreover for the underlying k–vector spaces this is by (8.7) a crossed extension as in (7.1), (7.3).

The next result generalizes lemma (7.2) on crossed modules.

(8.9) Lemma. *A τ–commutative secondary algebra $\mathcal{H}^*$ over the coefficient algebra R_* as defined in* (6.5) *is the same as a τ–crossed module ∂ in* (8.6).

Proof. Given $\mathcal{H}^*$ we obtain

$$\partial : \mathcal{H}_1^* \to \mathcal{H}_0^* \oplus k = A$$

where A is a τ–commutative k–algebra by the multiplication μ in (6.5)(1). Moreover $\mathcal{H}_1^* = V$ is an A–module and one now readily checks by (4.2) and (1.6) that ∂ satisfies the properties of a τ–crossed module. Conversely given $(\partial, \widetilde{\Gamma})$ as in (8.6) we obtain the secondary module $\mathcal{H}^n$ over R_n by the commutative diagram (see (1.6))

$$\begin{tikzcd}
I(R_n) \otimes_{R_n} V^n \arrow[r, "1\otimes\partial"] \arrow[d, "\mu"'] & I(R_n) \otimes_{R_n} \widetilde{A}^n \arrow[d, "\mu"] \arrow[dl, "\widetilde{\Gamma}"'] \\
V^n \arrow[r, "\partial"] & \widetilde{A}^n
\end{tikzcd}$$

which we deduce from the diagram in (8.6). Moreover we define

$$\mu : \mathcal{H}^n \otimes \mathcal{H}^m \to \mathcal{H}^{n+m}$$

by the multiplication

$$\widetilde{A}^n \otimes \widetilde{A}^m = \mathcal{H}_0^n \otimes \mathcal{H}_0^m \xrightarrow{\mu} \widetilde{A}^{n+m} = \mathcal{H}_0^{n+m}$$

of the algebra $\widetilde{A}$ over R_* and by the map

$$\begin{tikzcd}
(\mathcal{H}^n \otimes \mathcal{H}^m)_1 \arrow[r, "\mu"] \arrow[d, equal] & \mathcal{H}_1^{n+m} \arrow[d, equal] \\
(V^n \otimes \widetilde{A}^m \oplus \widetilde{A}^n \otimes V^m)/\operatorname{im} d_2 \arrow[r, "\mu"] & V^{n+m}
\end{tikzcd}$$

with $\mu(x \otimes a) = x \cdot a$ and $\mu(b \otimes y) = b \cdot y$ where $x \cdot a = \tau_{a,x} a \cdot x$. By (8.6)(1) this map is trivial on $\operatorname{im}(d_2)$. Now it is easy to show that the multiplication μ on $\mathcal{H}^*$ is associative (compare the proof on (8.4)) and τ–commutative. Finally we have to check that μ on $\mathcal{H}^*$ is compatible with the equation in (4.2). This follows from (8.7)(2),(3) since for $\alpha' = \alpha + \alpha_k \in I(R^n) \oplus k = R^n$ and $\beta' = \beta + \beta_k \in I(R^m) \oplus k = R^m$ we have

$$\mu\Gamma_\otimes(\alpha' \otimes \beta' \otimes x \otimes y) = \widetilde{\Gamma}(\alpha \odot \beta \otimes x \cdot y) + \beta_k \widetilde{\Gamma}(\alpha \odot 1 \otimes x \cdot y) + \alpha_k \widetilde{\Gamma}(1 \odot \beta \otimes x \cdot y)$$

Here we have $\alpha \odot \beta = (\alpha \odot 1)(1 \odot \beta)$ and hence in $I(R^{n+m}) \otimes_{R^{n+m}} V^{n+m}$ we have

$$\begin{aligned}
\alpha \odot \beta \otimes x \cdot y &= (\alpha \odot 1)(1 \odot \beta) \otimes x \cdot y \\
&= \alpha \odot 1 \otimes (1 \odot \beta) x \cdot y \\
&= \alpha \odot 1 \otimes x \cdot \beta y
\end{aligned}$$

Therefore we get by (8.7)(2),(3)

$$\begin{aligned}
\mu\Gamma_\otimes(\alpha' \otimes \beta' \otimes x \otimes y) &= \widetilde{\Gamma}(\alpha \otimes x) \cdot (\beta y + \beta_k y) + \alpha_k x \cdot \widetilde{\Gamma}(\beta \otimes y) \\
&= \mu(\Gamma(\alpha' \otimes x) \otimes \beta' y + (\varepsilon(\alpha') x) \otimes \Gamma(\beta' \otimes y))
\end{aligned}$$

Hence μ is compatible with the equation in (4.2). q.e.d.

We now use τ-crossed modules (or by (8.9) equivalently τ–commutative secondary algebras) for the following definition of symmetric cohomology which is a symmetric analogue of Hochschild cohomology in (7.3).

(8.10) Definition. Let R_* be a coefficient algebra with interchange elements τ for example let $R_* = k[\sigma_*]$ be the symmetric coefficient algebra. Let H be a commutative graded k–algebra and let M be an H–module. A *τ–crossed extension* $\mathcal{E}$ of H by M is an exact sequence of graded k–vector spaces

$$\mathcal{E} : 0 \longrightarrow M \longrightarrow V \xrightarrow{\partial} A \xrightarrow{q} H \longrightarrow 0$$

Here ∂ is a τ–crossed module as in (8.6). Moreover all maps are A–module morphisms with the A–module structure induced by the algebra map $q : A \to H$. A *weak equivalence* between two such τ–crossed extensions is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d, "f_1"] & A \arrow[r] \arrow[d, "f_0"] & H \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & W \arrow[r] & B \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

Here f_0 is a morphism of k–algebras over R_* and f_1 is a f_0–equivariant homomorphism such that (f_0, f_1) is compatible with $\widetilde{\Gamma}$. Let $\mathsf{SH}^3(H, M)$ be the set of weak equivalence classes of τ–crossed extensions of H by M. Below we show that $\mathsf{SH}^3(H, M)$ is a well defined set with the structure of a k–vector space. We call $\mathsf{SH}^3(H, M)$ the *third symmetric cohomology* of H with coefficients in M.

At this moment we do not know a "cohomology theory" for commutative graded algebras H which yields the symmetric cohomology $\mathsf{SH}^3(H, M)$ above.

We have the canonical natural homomorphism

$$\mathsf{SH}^3(H, M) \to \mathsf{HH}^3(H, M) \tag{8.11}$$

which carries the weak equivalence class of the τ–crossed extension $\mathcal{E}$ to the weak equivalence class of the underlying crossed extension $\mathcal{E}_{(k)}$ given by (8.7), (8.4) and (8.5). We need the following "free" objects.

(8.12) Definition. Let V be a graded k–vector space. Then the free R_*–module $R_* \odot V$ generated by V is given by

$$(R_* \odot V)^n = R_n \otimes V^n.$$

For an R_*–module W we obtain the *tensor algebra over R_** by

$$\overline{T}(W) = \bigoplus_{n \geq 0} W^{\overline{\otimes} n}$$

where $W^{\overline{\otimes} 0} = k$ and $W^{\overline{\otimes} n}$ is the n–fold $\overline{\otimes}$–product $W \overline{\otimes} \ldots \overline{\otimes} W$ defined in (8.1). For the tensor algebra $T(V)$ over k we get

$$\overline{T}(R_* \odot V) = R_* \odot T(V)$$

so that $R_* \odot T(V)$ is the free k–algebra over R_* generated by V with the multiplication

$$(\alpha \otimes x) \cdot (\beta \otimes y) = \alpha \odot \beta \otimes x \cdot y$$

for $\alpha, \beta \in R_*$ and $x, y \in T(V)$. Let

$$K_\tau \subset R_* \odot T(V) = A$$

be the R_*–submodule generated by elements $1 \otimes y \cdot x - \tau_{x,y} \otimes x \cdot y$ for $x, y \in T(V)$. Then K_τ generates the ideal $A \cdot K_\tau \cdot A$ and the R_*–quotient module

$$\Lambda = A / A \cdot K_\tau \cdot A$$

is the *free τ–commutative k–algebra over R_*.*

Given a τ–commutative k–algebra A over R_* and a k–linear map $d : V \to A$ with $\varepsilon d = 0$ we obtain the following push out diagram in the category of R_*–modules

$$\begin{tikzcd}
I(R_*) \odot V \arrow[r, "1\odot d"] \arrow[d, "i\odot 1"'] \arrow[dr, phantom, "\text{push}"] & I(R_*) \odot_{R_*} A \arrow[d, "i'"] \arrow[ddr, bend left, "\mu"] & \\
R_* \odot V \arrow[r] \arrow[drr, bend right, "d''"'] & Y \arrow[dr, "d'"] & \\
& & A
\end{tikzcd}$$

Here d'' is defined by $d''(\alpha \otimes x) = \alpha \cdot d(x)$. The pair (μ, d'') induces the R_*–linear map d' which thus determines the map of A–modules ∂' and ∂ in the following commutative diagram

$$\begin{tikzcd}
& I(R_*) \odot_{R_*} A \arrow[r, "\mu"] \arrow[d, "i'"] \arrow[dddl, "\widetilde{\Gamma}"'] & A \arrow[d, equal] \\
& Y \arrow[r, "d'"] \arrow[d, "i''"] & A \arrow[d, equal] \\
& A \overline{\otimes} Y \arrow[r, "\partial'"] \arrow[d] & A \arrow[d, equal] \\
X_1(d) \arrow[r, equal] & A \overline{\otimes} Y / U \arrow[r, "\partial"] & A
\end{tikzcd}$$

with $\overset{\approx}{\Gamma} = i''i'$ and $\widetilde{\Gamma} = p\overset{\approx}{\Gamma} = pi''i'$. Here i'' is defined by $i''(y) = 1 \otimes y$ and ∂' is defined by $\partial'(a \otimes y) = a \cdot d'(y)$. Let U be the A–submodule of $A \overline{\otimes} Y$ generated by the elements

$$\begin{gathered}
(\partial' x) \cdot y - \tau_{\partial' y, x} (\partial' y) \cdot x, \\
a \cdot \overset{\approx}{\Gamma}(\beta \otimes b) - \overset{\approx}{\Gamma}(1 \odot \beta \otimes a \cdot b),
\end{gathered}$$

with $x, y \in A \overline{\otimes} Y$ and $a, b \in A, \beta \in I(R_*)$. One readily checks that U is in the kernel of ∂' so that ∂' induces the A–module map ∂ on the quotient $X_1(d) = A \overline{\otimes} Y / U$. We claim that $(\partial, \widetilde{\Gamma})$ is a well define τ–crossed module which is the *free τ–crossed module* with basis (V, d).

Finally we define for a commutative graded k–algebra the *free H–module* with basis V by $H \otimes V$.

(8.13) Definition. Let H be a commutative graded k–algebra and consider a long exact sequence

$$\mathcal{R} : \ldots \longrightarrow C_2 \longrightarrow C_1 \xrightarrow{\partial} \Lambda \xrightarrow{q} H \xrightarrow{0}$$

Here Λ is a free τ–commutative k–algebra over R_* and $(\partial, \widetilde{\Gamma})$ is a free τ–crossed module and $C_i, i \geq 2$, is a free H–module. All maps in the sequence are Λ–module morphisms where C_i and H are Λ–modules via the algebra map q. Then we call $\mathcal{R}$ a *free τ–crossed resolution* of H.

It is easy to see that free τ–crossed resolutions exist. Moreover given a τ–crossed extension $\mathcal{E}$ as in (8.10) we can choose a commutative diagram ($n \geq 2$)

$$\begin{tikzcd}
 & C_3 \arrow[r, "\partial"] \arrow[d] & C_2 \arrow[r] \arrow[d, "f_2"] & C_1 \arrow[r] \arrow[d, "f_1"] & \Lambda \arrow[r] \arrow[d, "f_0"] & H \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{E} : & 0 \arrow[r] & M \arrow[r] & V \arrow[r] & A \arrow[r] & H \arrow[r] & 0
\end{tikzcd}$$

so that one gets a weak equivalence

$$(f_2)_* \mathcal{R}_2 \to \mathcal{E}$$

where $\mathcal{R}_2$ is the τ–crossed extension

$$0 \to C_2/\partial C_3 \to C_1 \to \Lambda \to H \to 0$$

given by $\mathcal{R}$. This implies that $\mathsf{SH}^3(H, M)$ is actually a set. In fact, the function

$$\psi : \mathsf{Hom}_H(C_2/\partial C_3, M) \to \mathsf{SH}^3(H, M) \tag{8.14}$$

which carries $f_2 : C_2/\partial C_3 \to M$ to the weak equivalence class of $(f_2)_* \mathcal{R}_2$ is surjective. Using this surjection it is possible to define the k–vector space structure of $\mathsf{SH}^3(H, M)$.

Appendix: Eilenberg–MacLane spaces

Let k be a commutative ring, for example a field. For the definition of $Z^n = K(k, n)$ in (2.5) we shall need the following categories and functors; compare Goerss–Jardine [GJ]. Let **Set** and **Mod** be the category of sets and k–modules respectively and let Δ**Set** and Δ**Mod** be the corresponding categories of simplicial objects in **Set** and **Mod** respectively. We have functors

$$\mathbf{Top}^* \xrightarrow{\mathsf{Sing}} (\Delta\mathbf{Set})^* \xrightarrow{|\ |} \mathbf{Top}^*$$

given by the *singular set* functor Sing and the *realization* functor $|\ |$. Moreover we have

$$\Delta\mathbf{Set} \xrightarrow{R} \Delta\mathbf{Mod} \xrightarrow{\Phi} (\Delta\mathbf{Set})^*$$

where k carries the simplicial set X to the *free k–module* generated by X and where Φ is the *forgetful* functor which carries the simplicial module A to the underlying simplicial set. Moreover we need the Dold–Kan functors

$$\mathbf{Ch}_+ \xrightarrow{\Gamma} \Delta\mathbf{Mod} \xrightarrow{N} \mathbf{Ch}_+$$

where $\mathbf{Ch}_+$ is the category of chain complexes in $\mathbf{Mod}$ concentrated in degree ≥ 0. Here N is the *normalization* functor which by the Dold–Kan theorem is an equivalence of categories with inverse Γ. For a pointed space V let

$$K(V) = |\Phi S(V)| \text{ with } S(V) = \frac{k\,\mathsf{Sing}(V)}{k\,\mathsf{Sing}(*)}$$

Hence $K : \mathbf{Top}^* \to \mathbf{Top}^*$ carries a pointed space to a topological k–module. We define the binatural map

$$\bar{\otimes} : K(V) \times K(W) \to K(V \wedge W) \tag{*}$$

as follows. We have

$$\mathsf{Sing}(V \times W) = \mathsf{Sing}(V) \times \mathsf{Sing}(W)$$

and this bijection induces a commutative diagram in $\Delta\mathbf{Mod}$

$$\begin{tikzcd} R\,\mathsf{Sing}(V) \otimes R\,\mathsf{Sing}(W) \arrow[r, equal] \arrow[d] & R\,\mathsf{Sing}(V \times W) \arrow[d] \\ S(V) \otimes S(W) \arrow[r, "\Lambda"] & S(V \wedge W) \end{tikzcd}$$

The vertical arrows are induced by quotient maps. For k–modules A, B let $\otimes : A \times B \to A \otimes_k B$ be the map in $\mathbf{Set}$ which carries (a, b) to the tensor product $a \otimes b$. Of course this map $\otimes$ is bilinear. Moreover for A, B in $\Delta\mathbf{Mod}$ the map $\otimes$ induces the map $\otimes : \Phi(A \times B) \to \Phi(A \otimes B)$ in $\mathbf{Set}$ and the realization functor yields

$$|\otimes| : |\Phi A| \times |\Phi B| = |\Phi(A \times B)| \to |\Phi(A \otimes B)|$$

Hence for $A = S(V)$ and $B = S(W)$ we get the composite

$$|\Phi S(V)| \times |\Phi S(W)| \xrightarrow{|\otimes|} |\Phi(S(V) \otimes S(W))| \xrightarrow{|\Phi\Lambda|} |\Phi S(V \wedge W)|$$

and this is the map $\bar{\otimes}$ above. One readily checks that $\bar{\otimes}$ is bilinear with respect to the topological k–module structure of $K(V), K(W)$ and $K(V \wedge W)$ respectively. Moreover the following diagram commutes

$$\begin{tikzcd} K(V) \times K(W) \arrow[rr, "\bar{\otimes}"] & & K(V \wedge W) \\ |\mathsf{Sing} V| \times |\mathsf{Sing} W| \arrow[r, equal] \arrow[u, "h \times h"] & |\mathsf{Sing}(V \times W)| \arrow[r] & |\mathsf{Sing}(V \wedge W)| \arrow[u, "h"] \end{tikzcd} \tag{**}$$

Here the *Hurewicz map* h is the realization of the map in $\Delta\mathbf{Set}$

$$\mathsf{Sing}(V) \to \Phi k\,\mathsf{Sing}(V) \to \Phi S(V)$$

which carries an element x in $\mathsf{Sing}(V)$ to the corresponding generator in $k\,\mathsf{Sing}(V)$.

Let $S^n = S^1 \wedge \ldots \wedge S^1$ be the n–fold smash product of the 1–sphere S^1. Then the symmetric group σ_n acts on S^n by permuting the factors S^1. It is well known that this action of σ_n on S^n induces the sign–action of σ_n on the homology $H_n(S^n) = k$. We define the Eilenberg–MacLane space Z^n by

$$Z^n = K(S^n) = \left| \Phi \frac{k\mathsf{Sing}(S^n)}{k\mathsf{Sing}(*)} \right|$$

Since K is a functor we see that σ_n also acts on $K(S^n)$ via k–linear automorphisms. We define the multiplication map $\mu_{m,n}$ by

$$\mu : Z^m \times Z^n = K(S^m) \times K(S^n) \xrightarrow{\bar{\otimes}} K(S^m \wedge S^n) = Z^{m+n}$$

where $S^m \wedge S^n = S^{m+n}$ and where we use (*). Diagram (**) implies that μ induces the cup product in cohomology. The Eilenberg–MacLane spaces Z^n together with the multiplication map μ satisfy the axioms of Karoubi [Ka].

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