Twisted Legendre Equations
with Solutions of Height ≤ 3

William L. Hoyt

and

Charles F. Schwartz

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

William L. Hoyt
Dept. of Mathematics
Hill Center
Rutgers University
New Brunswick NJ 08904

USA

Charles F. Schwartz
Dept. of Mathematics
Rider College
Lawrenceville NJ

USA

MPI/90-55

DISCRETE GROUPS, EXPANDING GRAPHS and INVARIANT MEASURES

(preliminary version)

Chapters 1-3

by

ALEXANDER LUBOTZKY
Institute of Mathematics
Hebrew University
Jerusalem, 91904
Israel

The author would be grateful for any corrections, remarks and suggestions about these notes. ~~They will be especially valuable if received by 1/1/90~~.

Addresses: Sept. 88–Aug. 89: Department of Mathematics, Yale University, Box 2155 Yale Station, New Haven, CT 06520, USA
Sept. 89–Dec. 89: Department of Mathematics, Columbia University, New York, NY 10027
Jan. 90—: Institute of Mathematics, Hebrew University, Jerusalem, 91904, Israel.

1. Introduction

The purpose of this paper is to derive explicit algebraic conditions on points $(a_1,a_2,a_3) \in \mathbb{C}^3$ which imply that the equation

$$(1.1) \qquad y^2 = (t-a_1)(t-a_2)(t-a_3)x(x-1)(x-t)$$

has a non-trivial $\mathbb{C}(t)$-rational solution, to determine irreducible components and intersections for some of these conditions, to list maximal independent sets of $\mathbb{C}(t)$-rational solutions for certain values of (a_1,a_2,a_3), and to discuss some related problems.

In this paper the discussion is simplified by restricting attention to points (a_1,a_2,a_3) in the complement $V \subset \mathbb{C}^3$ of the planes defined by $a_i = 0$, $a_i = 1$, $a_i = a_j$ with $i \neq j$. A non-trivial $\mathbb{C}(t)$-rational solution of (1.1) is defined to be a pair $(x(t),y(t))$ of rational functions which satisfies (1.1) and is different from $(0,0)$, $(1,0)$ and $(t,0)$. The height of $(x(t),y(t))$ is defined to be the degree of the curve in $\mathbb{P}_2$ parametrized by $(1:t:x(t))$. For each $N > 0$, W_N denotes the set of points $(a_1,a_2,a_3) \in V$ such that (1.1) has a non-trivial $\mathbb{C}(t)$-rational solution of height $\leq N$. In an earlier paper [6] it is shown that each W_N is the union of a finite number of closed irreducible algebraic surfaces in V. The main results of the present paper concern the three algebraic sets $W_1 \subset W_2 \subset W_3$, and can be outlined as followed: In § 2 irreducible components of W_1, W_2 and W_3 are described in terms of suitable extra contact for configurations $\{\Sigma L_i, C\}$ defined by

$$\Sigma L_i : (t-a_1)(t-a_2)(t-a_3) \times (x-1)(x-t) = 0 \text{ and}$$
$$C : x = x(t) \ .$$

In § 3 equations and intersections are determined for irreducible components of W_1. Minimal models, intersection products, Neron–Severi lattices and Mordell–Weil groups for (1.1) are described in general in § 4 and are computed explicitly for the example $(a_1,a_2,a_3) = (-1,1/2,2)$ in § 5. A subsequent paper (in preparation) will determine parameterizations and intersections for other components of W_2 and W_3 and will compute intersection products, Neron–Severi lattices and Mordell–Weil groups for (1.1) for generic (a_1,a_2,a_3) as these components.

The work of one of the authors (Hoyt) on this paper has been supported by the Faculty Academic Study Program of Rutgers University and by Max–Planck–Institut für Mathematik in Bonn. The authors are very grateful to both Rutgers University and Max–Planck–Institut for helping to make this work possible. We are also very grateful to J. Stienstra for advice on this work and to Frau Sarlette for excellent typing.

2. Diagrams for irreducible components of W_1, W_2 and W_3

(2.1) Diagrams (2.4.1)–(2.4.10) in Table (2.4) below represent configurations $\{\Sigma L_i, C\}$ consisting of six lines $L_1,\dots,L_6$ and a rational curve C of degree ≤ 3 in $\mathbb{P}_2$ for which there exist a point $(a_1,a_2,a_3) \in V$ and a non–trivial $\mathbb{C}(t)$–rational solution $(x(t),y(t))$ of (1.1) such that ΣL_i and C are defined by

$$(2.1.1) \qquad \begin{cases} (t-a_1)(t-a_2)(t-a_3)x(x-1)(x-t) = 0 \text{ and} \\ x = x(t) \text{ , resp.,} \end{cases}$$

relative to some choice of inhomogeneous coordinates in $\mathbb{P}_2$. For each such configuration here are several different choices of (a_1,a_2,a_3) and $(x(t),y(t))$ which satisfy (2.1.1). One such choice is indicated for each diagram in the adjacent columns of Table (2.4), with $y(t)$ determined up to sign by the indicated values for $x(t)$, $x(t)-1$ and $x(t)-t$. Other choices can be obtained by replacing $y(t)$ by $-y(t)$, by permuting the labels $t = a_1$, $t = a_2$, $t = a_3$ on the three concurrent L_i, and/or by permuting the labels $x = 0$, $x = 1$, $x = t$ on the other three L_i in Table (2.4). The latter permutations are induced as indicated in Figure (2.5) by the six projective transformations of $\mathbb{P}_2$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&1&1\\-1&0&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}0&-1&0\\1&1&1\\0&0&-1\end{bmatrix}, \begin{bmatrix}1&1&1\\0&-1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}-1&0&0\\1&1&1\\0&0&-1\end{bmatrix}$$

which leave fixed the point $(0,0,1)$ at ∞ on vertical lines and which alter labels on vertical lines from $t = a$ to $t = b$ with

$$b = a \text{ , } 1/(1-a) \text{ , } 1-(1/a) \text{ , } 1-a \text{ , } 1/a \text{ , } a/(a-1) \text{ , resp.}$$

The effect of such changes on components of W_1, W_2 and W_3 will be discussed in §§ 3 and 6 below.

(2.2) Conversely, for each (a_1,a_2,a_3) in W_1, W_2 or W_3 and for each non–trivial $\mathbb{C}(t)$–rational solution $(x(t),y(t))$ of (1.1) of height ≤ 3, the configuration $\{\Sigma L_i,C\}$ defined by (2.1.1) is of one of the types (2.4.1)–(2.4.10). In particular ΣL_i has a single triple point (at ∞ on $t = a_i$) ; C is either a line, a conic, a nodal cubic, or a cuspidal cubic; and $x(t)$ must have one of the forms

$$b_0+b_1t\ ,\ \ c_0+\ldots+c_3t^3\ ,\ \ (d_0+\ldots+d_2t^2)/(t-a_2)\ ,$$

$$(e_0+\ldots+e_3t^3)/(t-a_i)(t-a_j)\ \text{ or }\ (f_0+\ldots+f_3t^3)/(t-h)^2$$

with $b_1 \neq 0$, $c_3 \neq 0$, $a_i \neq a_j \neq h$, since other forms violate either the condition that C has degree ≤ 3 or the condition that

$$(t-a_1)(t-a_2)(t-a_3)x(t)(x(t)-1)(x(t)-t) = (y(t))^2$$

is a perfect square in $\mathbb{C}(t)$. Furthermore points and branches of C at which two L_i meet or at which some L_i is tangent correspond to common factors or a square factor of $x(t)$, $x(t)-1$ and/or $x(t)-t$. Consideration of these conditions shows that $\{\Sigma L_i;C\}$ must be of one of the types (2.4.1)–(2.4.10).

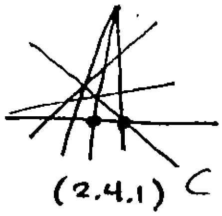
(2.3) In Diagram (2.4.1) the points labelled $(a_1,0)$, $(a_2,0)$ can be varied on the line $x=0$ in a manner which leads to another configuration of type (2.4.1). This leads to the explicit birational parametrization

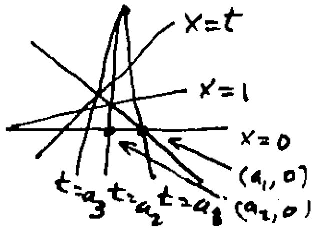
$$(a_1,a_2) \longrightarrow (a_1,a_2,a_1/(1+a_1-a_2))$$

of one of the irreducible components of W_1 . (See (3.2) below). Similarly in Diagram (2.4.9) the points labelled (α_i^3, α_i^2) can be varied on C in a manner which yields a rational parametrization of a component of W_3 by a quotient of $C \times C \times C$ modulo an action of the group of homotheties $(\alpha^3, \alpha^2) \longrightarrow ((\gamma\alpha)^3, (\gamma\alpha)^2)$. It is expected that similar rational parametrizations of each component of W_2 and W_3 can be defined by varying points marked in Diagrams (2.4.2)–(2.4.10).

Table (2.4)

Type of configuration	One choice of labels for ΣL_i	Corresponding values for $x(t)$, $x(t)-1$, $x(t)-t$

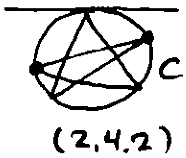


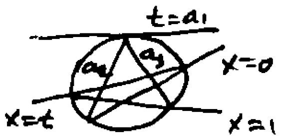


$x = p(t-a_1)$

$x-1 = q(t-a_2)$

$x-t = r(t-a_3)$

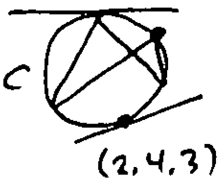


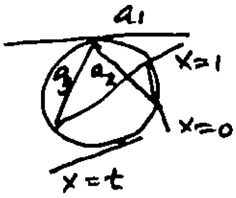


$x = p(t-a_2)t/(t-a_1)$

$x-1 = q(t-a_3)(t-1)/(t-a_1)$

$x-t = rt(t-1)/(t-a_1)$

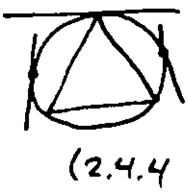


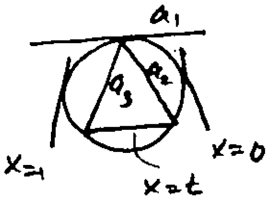


$x = p(t-a_2)/(t-a_1)$

$x-1 = q(t-a_3)/(t-a_1)$

$x-t = r(t-h)^2/(t-a_1)$





$x = p(t-h_1)^2/(t-a_1)$

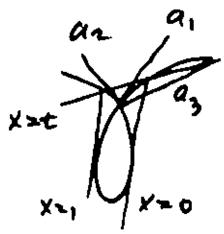
$x-1 = q(t-h_2)^2/(t-a_1)$

$x-t = r(t-a_2)(t-a_3)/(t-a_1)$

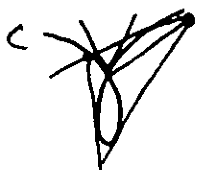
Table (2.4)

(continued)

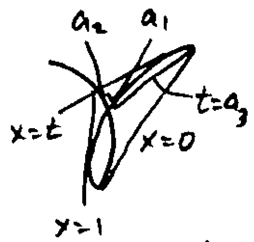
(2.4.5)



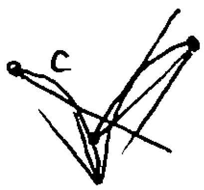
$$x = pt(t-h_1)^2/(t-a_1)(t-a_2)$$
$$x-1 = q(t-1)(t-h_2)/(t-a_1)(t-a_2)$$
$$x-t = rt(t-1)(t-a_3)/(t-a_1)(t-a_2)$$



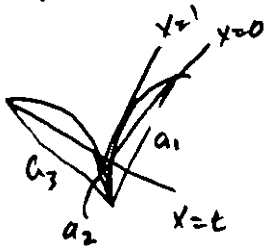
(2.4.6)



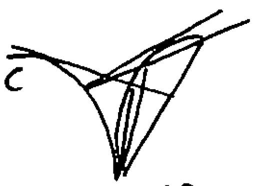
$$x = p(t-a_3)(t-h_1)^2/(t-a_1)(t-a_2)$$
$$x-1 = q(t-1)(t-h_2)^2(t-a_1)(t-a_2)$$
$$x-t = r(t-1)(t-h_3)^2/(t-a_1)(t-a_2)$$



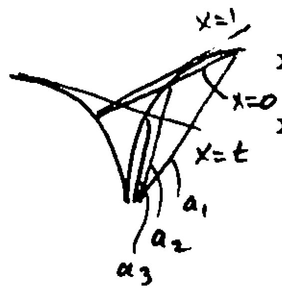
(2.4.7)



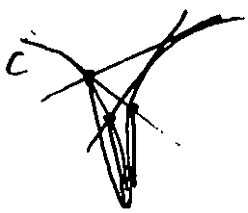
$$x = pt(t-a_1)/(t-h_1)^2$$
$$x-1 = q(t-h_2)^2/(t-h_1)^2$$
$$x-t = rt(t-a_2)(t-a_3)/(t-h_1)^2$$



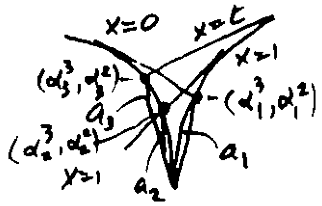
(2.4.8)



$$x = p(t-a_1)(t-a_2)/(t-h_1)^2$$
$$x-1 = q(t-h_2)^2/(t-h_1)^2$$
$$x-t = r(t-a_3)/(t-h_3)^2/(t-h_1)^2$$



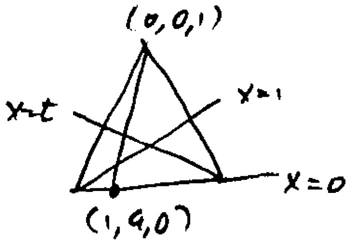
(2.4.9)



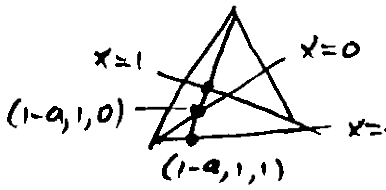
$$x = p(t-a_1)(t-h_2)^2/(t-h_1)^2$$
$$x-1 = q(t-a_3)(t-h_3)^2/(t-h_1)^2$$
$$x-t = r(t-a_3)(t-h_4)^2/(t-h_1)^2$$

(2.4.10)

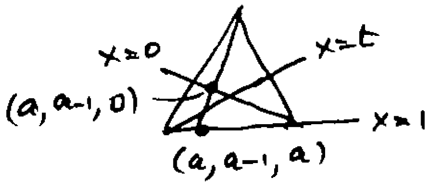
Figure (2.5)



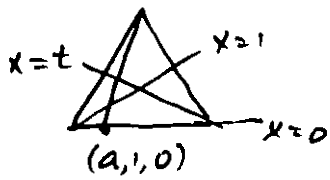
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



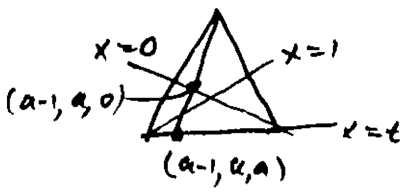
$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



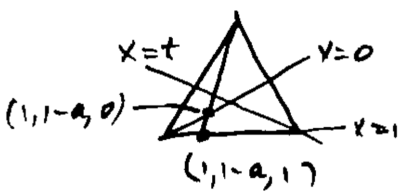
$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

3. Equations and intersections for components of W_1

(3.1) If $(a_1,a_2,a_3) \in V$ and if (1.1) has a non–trivial $\mathbb{C}(t)$–rational solution (x,y) of type (2.4.1), then there exist a permutation (ijk) of (123) and non–zero values $p,q,r \in \mathbb{C}$ such that

$$x = p(t-a_i)\ ,\ x-1 = q(t-a_j)\ ,\ x-t = r(t-a_k)\ ,$$
$$q = p\ ,\ r = p-1\ ,\ pa_i + \frac{1}{i} = pa_j\ ,\ pa_i = (p-1)a_k \text{ and}$$
$$a_i - (1 + a_i - a_j)a_k = 0\ .$$

Conversely if $(a_1,a_2,a_3) \in V$ satisfies this last relation for some (ijk), then (1.1) has a non–trivial $\mathbb{C}(t)$–rational solution (x,y) which satisfies the preceding relations with the same (ijk).

Proof: The hypotheses in the first assertion imply that $x = mt+b \neq 0,1,t$ and that $(t-a_1)(t-a_2)/(t-a_3)x(x-1)(x-t)$ is a perfect square in $\mathbb{C}(t)$. This and the equality defining V imply that x, $x-1$, $x-t$ must be multiples of $t-a_1$, $t-a_2$, $t-a_3$ in some order, with multiples p,q,r determined by the order as indicated. The converse follows easily.

(3.2) W_1 is the union of six irreducible components

$$V \cap \mathbf{Q}_{ijk} = V \cap \bar{Q}_{ijk}$$

with $Q_{ijk} \subset \mathbb{C}^3$ the quadric surface defined by

$$a_i - (1+a_1-a_j)a_k = 0$$

and with $\overline{Q}_{ijk}$ the closure of Q_{ijk} in $\mathbb{P}_3$.

Proof: This is an immediate corollary of (3.1).

(3.3) If $(a_1,a_2,a_3) \in V$ and if (1.1) has two different solutions which satisfy the conditions in (3.1) for two different permutations (ijk) , (ℓnm) , then

$$(a_i,a_j,a_k) = \begin{cases} (1-(1/a_j),a_j,1/(1-a_j)) & \text{if } (\ell mn) = (jki) \text{ or } (kij) , \\ (-1,a_j,1/a_j) & \text{if } (\ell mn) = (ikj) , \\ (a_i,2,a_i/(a_i-1)) & \text{if } (\ell mn) = (kji) , \\ (a_i,1-a_i,1/2) & \text{if } (\ell mn) = (jik) . \end{cases}$$

Furthermore the existence of two such solutions for two permutations (ijk) , (jki) of the same parity implies the existence of a third solution for the third permutation (kij) of this parity. Conversely if $(a_1,a_2,a_3) \in V$ satisfies one of the displayed relations, then (1.1) has a pair of solutions which satisfy the conditions in (3.1) for the corresponding pair of permutations.

Proof: The hypotheses and (3.1) imply

$$a_i-(1+a_i-a_j)a_k = 0 \text{ and}$$
$$a_\ell-(1+a_\ell-a_m)a_n = 0 .$$

If (ℓmn) = (jki) , then elimination of a_i yields

$$0 = a_j(1-a_k)-(1+a_j-a_k)(1-a_j)a_k =$$
$$= (a_j-a_k)(1-a_k+a_ja_k) \;;$$

hence $a_k = 1/(1-a_j)$ since $a_j \neq a_k$; etc. If $(\ell mn) = (ikj)$, resp. (kji) , resp. (jik) , then subtraction yields

$$0 = a_i-(1+a_i-a_j)a_k-a_i+(1+a_i-a_k)a_j = (1+a_i)(a_j-a_k) \text{ , resp.}$$
$$0 = a_i-(1+a_i-a_j)a_k-a_k+(1+a_k-a_j)a_i = (a_i-a_k)(2-a_j) \text{ , resp.}$$
$$0 = a_i-(1+a_i-a_j)a_k-a_j+(1+a_j-a_i)a_k = (a_i-a_j)(1-2a_k) \text{ ; etc.}$$

The case $(\ell mn) = (kij)$ and also the second assertion follow easily from the identity

(3.3.1) $$[a_i-(1+a_i-a_j)a_k]+[a_j-(1+a_j-a_k)a_i]+[a_k-(1+a_k-a_i)a_j] = 0 \;.$$

Also see (3.7) below. The converse of the first assertion follows easily from (3.1).

(3.4) It follows from (3.3) that the intersection $V \cap Q_{ijk} \cap V \cap Q_{\ell mn}$ of each pair of distinct components of W_1 is an irreducible curve which is the intersection of V either with one of two twisted cubics or with one of six plane conics or with one of three lines in $\mathbb{C}^3$. These curves are listed in Table (3.8) and their real parts are sketched in Figure (3.10). The components of $\overline{Q}_{123} \cap \overline{Q}_{ijk}$ in $\mathbb{P}_3$ are listed in Table (3.9) for compansion.

(3.5) As indicated in Figure (3.10) intersections of triples or quadruples of distinct irreducible components of W_1 are either empty or one of the six special points

$$(-1,1/2,2) \text{ , } (-1,2,1/2) \text{ , } (1/2,2,-1) \text{ , } (2,1/2,-1) \text{ , } (2,-1,1/2) \text{ , } (1/2,-1,2)$$

or one of the two twisted cubics in Table (3.8). In Figure (3.10) hour hand positions I, III, V, VII, IX, XI are used as labels for the preceding six points and also for permutations (ijk) and irreducible components $V \cap Q_{ijk}$ which correspond as in Figure (3.10). In addition parenthetical labels such as $(I \cap III)$ or $(I \cap III \cap VII \cap XI)$ are used for intersections of components of W_1 .

(3.6) Figure (3.11) represents combinations of configurations $\{\Sigma L_p, L_{ijk}\}$ and $\{\Sigma L_p, L_{\ell mn}\}$ which correspond to pairs of solutions in (3.3). Note that permutations (ijk) , (ℓmn) of different parity correspond to lines L_{ijk} , $L_{\ell mn}$ which meet at double points of ΣL_p but that permutations of the same parity correspond to lines which do not meet on ΣL_p .

(3.7) The relation (3.3.1) in the proof of (3.3) corresponds to a special case of the theorem of Pappus as in Figure (3.12): If A,B,C are collinear (say on L_{123}) and if A′,B′,C′ are collinear (say on L_{312}) , then A",B",C" are also collinear (on L_{231}). In this special case, the lines AB", A"B′, A′B are concurrent.

Table (3.8)

$$V \cap Q_{ijk} \cap V \cap Q_{\ell mn}$$

	III $V\cap Q_{123}$	V $V\cap Q_{321}$	VII $V\cap Q_{312}$	IX $V\cap Q_{213}$	XI $V\cap Q_{231}$
I $V\cap Q_{132}$	$(-1,a,\frac{1}{a})$	$(\frac{1}{1-a},a,1-\frac{1}{a})$	$(a,\frac{1}{2},1-a)$	$(\frac{1}{1-a},a,1-\frac{1}{a})$	$(a,\frac{a}{a-1},2)$
III		$(a,2,\frac{a}{a-1})$	$(1-\frac{1}{a},a,\frac{1}{1-a})$	$(a,1-a,\frac{1}{2})$	$(1-\frac{1}{a},a,\frac{1}{1-a})$
V			$(a,\frac{1}{a},-1)$	$(\frac{1}{1-a},a,1-\frac{1}{a})$	$(\frac{1}{2},1-a,a)$
VII				$(2,\frac{a}{1-a},a)$	$(1-\frac{1}{a},a,\frac{1}{1-a})$
IX					$(\frac{1}{a},-1,a)$

Table (3.9)

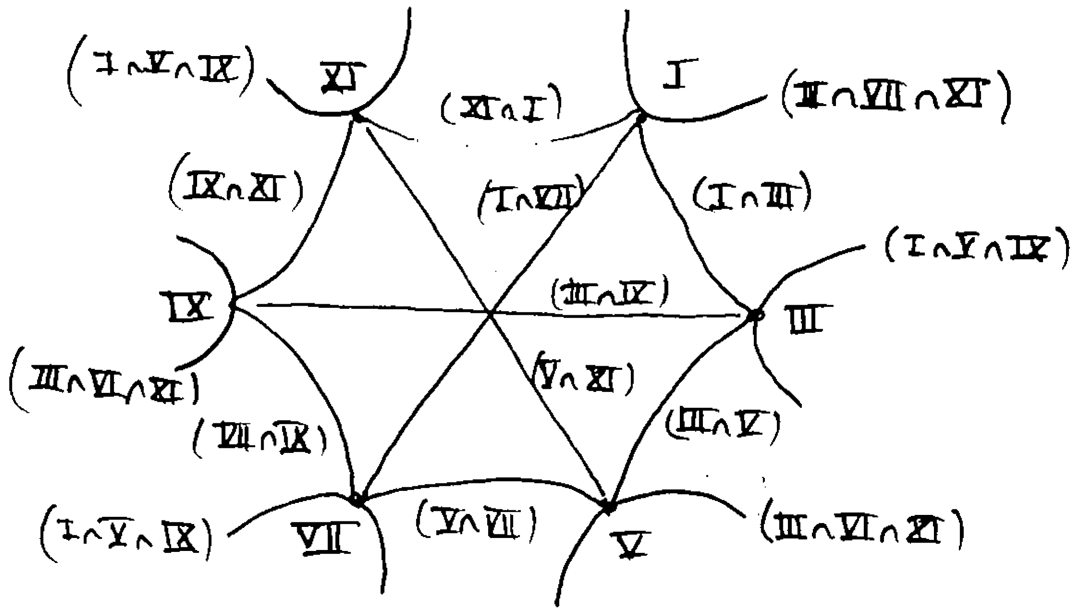
Components of

$\overline{Q}_{132} \cap \overline{Q}_{ijk} \subset \mathbb{P}_3$

	III $\overline{Q}_{123}$	V $\overline{Q}_{321}$	VII $\overline{Q}_{312}$	IX $\overline{Q}_{213}$	XI $\overline{Q}_{231}$
I $\overline{Q}_{132}$	$(1,-1,a,\frac{1}{a})$ $(1,a,a,a)$ $(1,a,1,1)$	$(1,\frac{1}{1-a},a,1-\frac{1}{a})$ $(1,a,a,a)$	$(1,a,\frac{1}{2},1-a)$ $(1,a,a,a)$ $(0,1,a,1)$ $(0,1,0,a)$	$(1,\frac{1}{1-a},a,1-\frac{1}{a})$ $(1,a,a,a)$	$(1,a,\frac{a}{a-1},2)$ $(1,a,a,a)$ $(0,1,0,a)$

Figure (3.10)

Real parts of
$V \cap Q_{ijk} \cap V \cap Q_{\ell mn}$



Labels in terms of hour hand positions

Hour	I	III	V	VII	IX	XI
Second point	$(-1,\frac{1}{2},2)$	$(-1,2,\frac{1}{2})$	$(\frac{1}{2},2,-1)$	$(\frac{1}{2},-1,2)$	$(2,-1,\frac{1}{2})$	$(2,\frac{1}{2},-1)$
permutat.	(132)	(123)	(321)	(312)	(213)	(231)
irreduc. component	$V \cap Q_{132}$	$V \cap Q_{123}$	$V \cap Q_{321}$	$V \cap Q_{312}$	$V \cap Q_{213}$	$V \cap Q_{231}$

Figure (3.11)

Configurations $\{\Sigma L_p, L_{ijk}, L_{\ell mn}\}$ which
correspond to pairs of solutions of type (2.4.1)

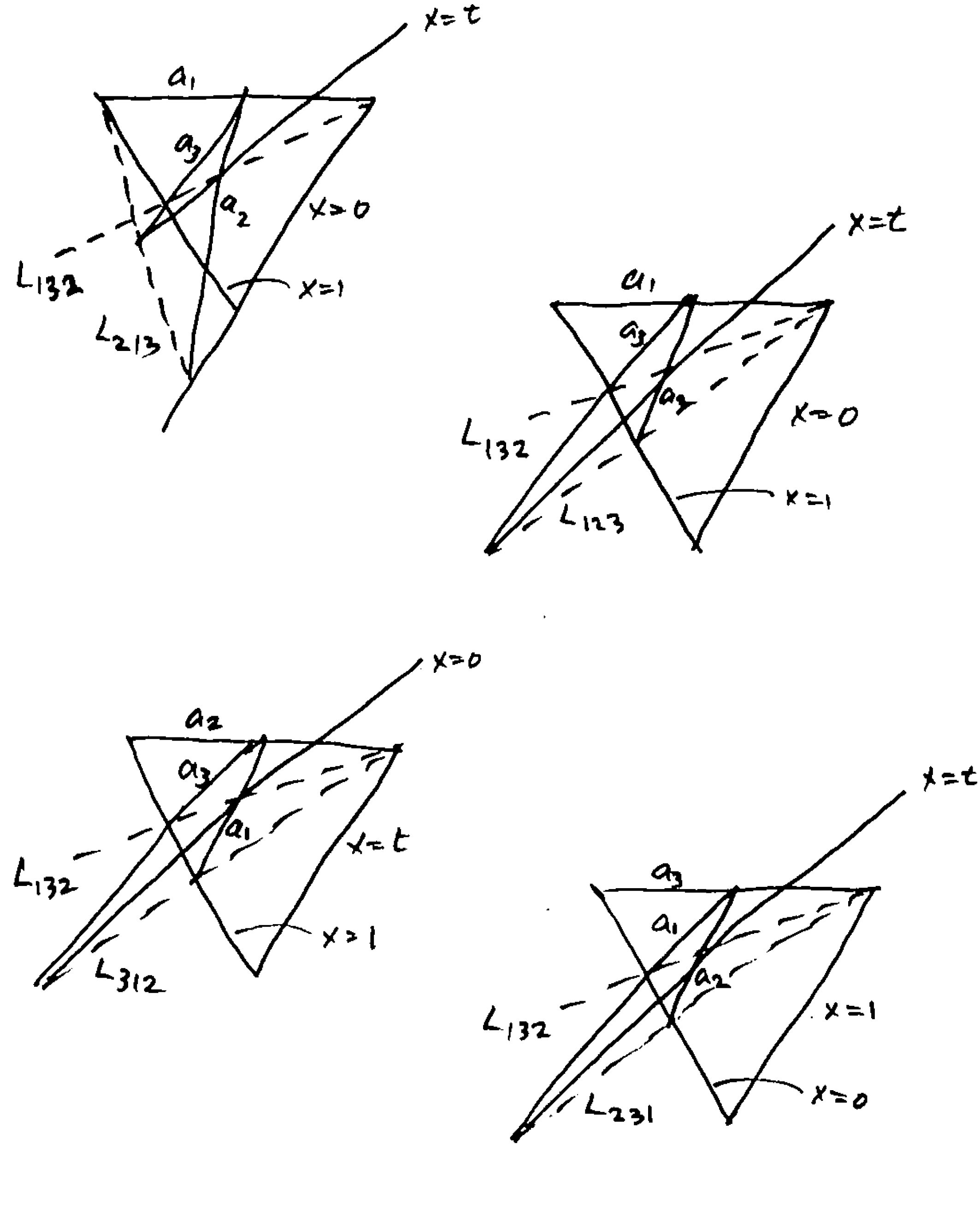
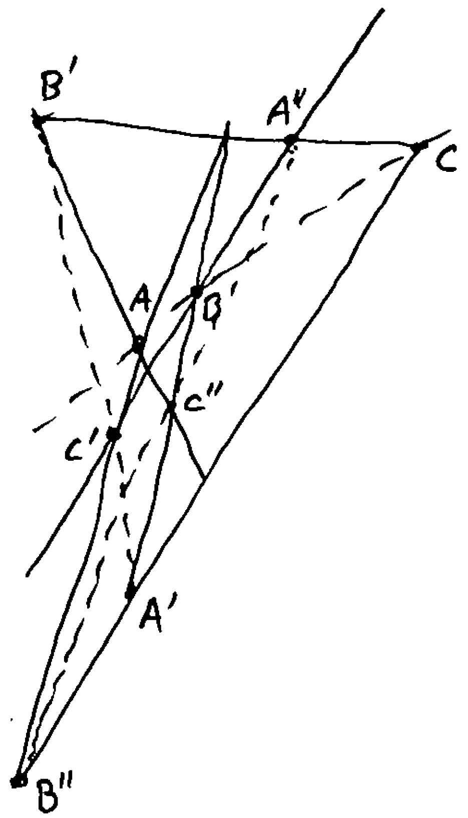


Figure (3.12)

A special case of the

theorem of Pappus



4. Neron–Severi Lattices and Mordell–Weil groups

(4.1) For fixed $(a_1,a_2,a_3) \in V$, the Neron model $X \longrightarrow \mathbb{P}_1$ for (1.1) relative to $\mathbb{C}(t)$ is an elliptic K3 surface with six singular fibers

$$C^s = \begin{cases} C_0^s + C_1^s & \text{at} \quad s = 0, 1, \infty \text{ or} \\ C_0^s + C_1^s + 2C_2^s + C_3^s + C_4^s & \text{at} \quad s = a_1, a_2, a_3 \end{cases}$$

of Kodaira types I_2 or I_0^*, with Picard number $\rho = 17, 18, 19$ or 20, and with Neron–Severi group $NS(X) \cong \mathbb{Z}^\rho$ generated by homology classes of sections and components of fibers. Furthermore there is a sublattice $L \subset NS(X)$ of rank 17 generated by the 17 independent classes of the curves

(4.1.1)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
σ_0	C^t	C_1^0	C_1^1	C_1^∞	$C_1^{a_1}$	$C_2^{a_1}$	$C_3^{a_1}$	$C_4^{a_1}$	$C_1^{a_2}$	$C_2^{a_2}$	$C_3^{a_2}$	$C_4^{a_2}$	$C_1^{a_3}$	$C_2^{a_3}$	$C_3^{a_3}$	$C_4^{a_4}$

with σ_0 the section at ∞, C^t a good fiber, and C_i^1, $i > 0$, the 15 components of singular fibers which do not meet σ_0.

Proof: These properties are special cases or easy consequences of results in Shioda [10] and Barth, Peters, and van de Ven [1]. Cf. [6]. In particular by [1, pp. 87, 183,189] X can be identified with the minimal non–singular resolution of a double sectic with A–D–E singularities which is defined by (1.1), and hence X is a K3 surface with $\rho \leq 20$. See (4.6) below.

(4.2) If $(a_1,a_2,a_3) \in V$ does not belong to any of the W_N in § 1, then $\rho = 17$. In this case the submodule L in (4.1) has index 4 in NS(X) with cosets represented by the classes of σ_0 and the sections τ_1, τ_3, τ_4 defined by the trivial solutions (0,0) , (1,0) , (t,0) , resp. of (1.1). However if $(a_1,a_2,a_3) \in V$ belongs to one of the W_N and if (x(t),y(t)) is an associated non–trivial solution of (1.1) as in § 2, then $\rho \geq 18$ and (x(t),y(t)) determines a section σ of $X \longrightarrow \mathbb{P}_1$ and a class $[\sigma] \in NS(X)$ which is independent modulo L .

Proof: Cf. [6].

(4.3) Intersection products $D \cdot E$ for curves on X determine a nondegenerate bilinear form on NS(X) with signature $(1+,(\rho-1)-)$ and with discriminant $d = \det(D_i \cdot D_j)$ where $D_1,\ldots,D_p$ represent a basis for NS(X) . Cf. [ ]. Intersection products for the 17 curves in (4.1.1) can be computed as in (4.5) and (4.6) below and form the 17×17 matrix in Table (4.7) with determinant 2^9 . In case $\rho = 17$ it follows that NS(X) has discriminant $d = 2^9/16 = 2^5$. There are partial results on discriminants for examples with $\rho > 17$ in §§5 and 6 below.

(4.4) As usual the set $\mathfrak{S}$ of holomorphic sections σ of $X \longrightarrow \mathbb{P}_1$ can be identified with the Mordell–Weil group of $\mathbb{C}(t)$–rational solutions of (1.1), with σ_0 as zero element and with $\sigma_1+\sigma_2+\sigma_3 = \sigma_0$ if and only if the corresponding $\mathbb{C}(t)$–rational solutions (x_i,y_i) are collinear on (1.1). In this case $\mathfrak{S} \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^2$ with $r = \rho-17$ and with torsion subgroup consisting of σ_0 and the sections τ_1, τ_3, τ_4 corresponding to the trivial solutions (0,0) , (1,0) , (t,0) of (1.1) . The numbering of the τ_i corresponds to non–empty intersections with components $C_i^{a_j}$ of the I_0^* fibers in Figure (4.8) below.

Each $\sigma \in \mathfrak{S}$ determines a class $[\sigma] \in NS(X)$, but difficulty with translation and torsion prevents the map $\sigma \longrightarrow [\sigma]$ from being a group homomorphism. However there is a modified map

$$\delta : \mathfrak{S} \longrightarrow NS(X) \otimes \mathbb{Q}$$

which is a homomorphism and which is characterized by the properties (4.5.1) that $[\sigma] - \delta(\sigma)$ is a $\mathbb{Q}$–linear combination of the curves D in (4.1.1) and (4.5.2) that $\delta(\sigma) \cdot D = 0$ for each D in (4.1.1). Furthermore the modified intersection product

$$\langle \sigma, \sigma' \rangle = -\delta(\sigma) \cdot \delta(\sigma')$$

is positive definite on $\mathfrak{S}$ modulo torsion.

Proof: See [10] or [6] for the first two assertions. See Cox and Zucker [2] for definition and properties of δ and $\langle \, , \rangle$. See (5.4) below for an example.

(4.5) Each of the 16 curves $\neq C_t$ in (4.1.1) is a nonsingular rational curve on X. For such curves D the adjunction formula yields

$$D \cdot D = D \cdot (D + K_X) = 2p_a(D) - 2 = -2$$

since X is a K3 surface with canonical class $K_X = 0$. This and the relation

$$C_t \cdot C_t = C_t \cdot C_{t'} = 0$$

yield the self–intersection numbers on the diagonal in Table (4.8). See [1].

(4.6) Let $Y \longrightarrow \mathbb{P}_2$ be the singular double cover defined by (1.1) with ramification locus $B = \Sigma L_i$; let $\mathbb{P}_2' \longrightarrow \mathbb{P}_2$ be the blow–up of the triple point of B ; and let $\mathbb{P}_2'' \longrightarrow \mathbb{P}_2'$ be the multiple blow–up of the double points of the total transform B' of B . Then X in (4.1) can be identified with the non–singular double cover of $\mathbb{P}_2''$ with disconnected, non–singular ramification locus consisting of the components of odd order in the total transform B'' of B' . This follows from definitions and results in [1, pp. 87, 183, 189] which show in particular that this double cover of $\mathbb{P}_2''$ is the minimal non–singular model for Y since Y is a double sextic with A–D–E singularities. It follows that the 16 curves $\neq C_t$ in (4.1.1) lie over the components of B'' with the configuration shown in Figure (4.8) below. It follows from Figure (4.8) together with (4.5) that the intersection products for the curves in (4.1.1) are those listed in Table (4.7). In Figure (4.8) the lines and ovals represent non–singular rational curves which intersect each other transversely at indicated points. The map $X \longrightarrow \mathbb{P}_2''$ is ramified on the curves σ_0 , τ_4 , τ_3 , τ_1 , $C_2^{a_1}$, $C_2^{a_2}$, $C_2^{a_3}$.

Table (4.7)

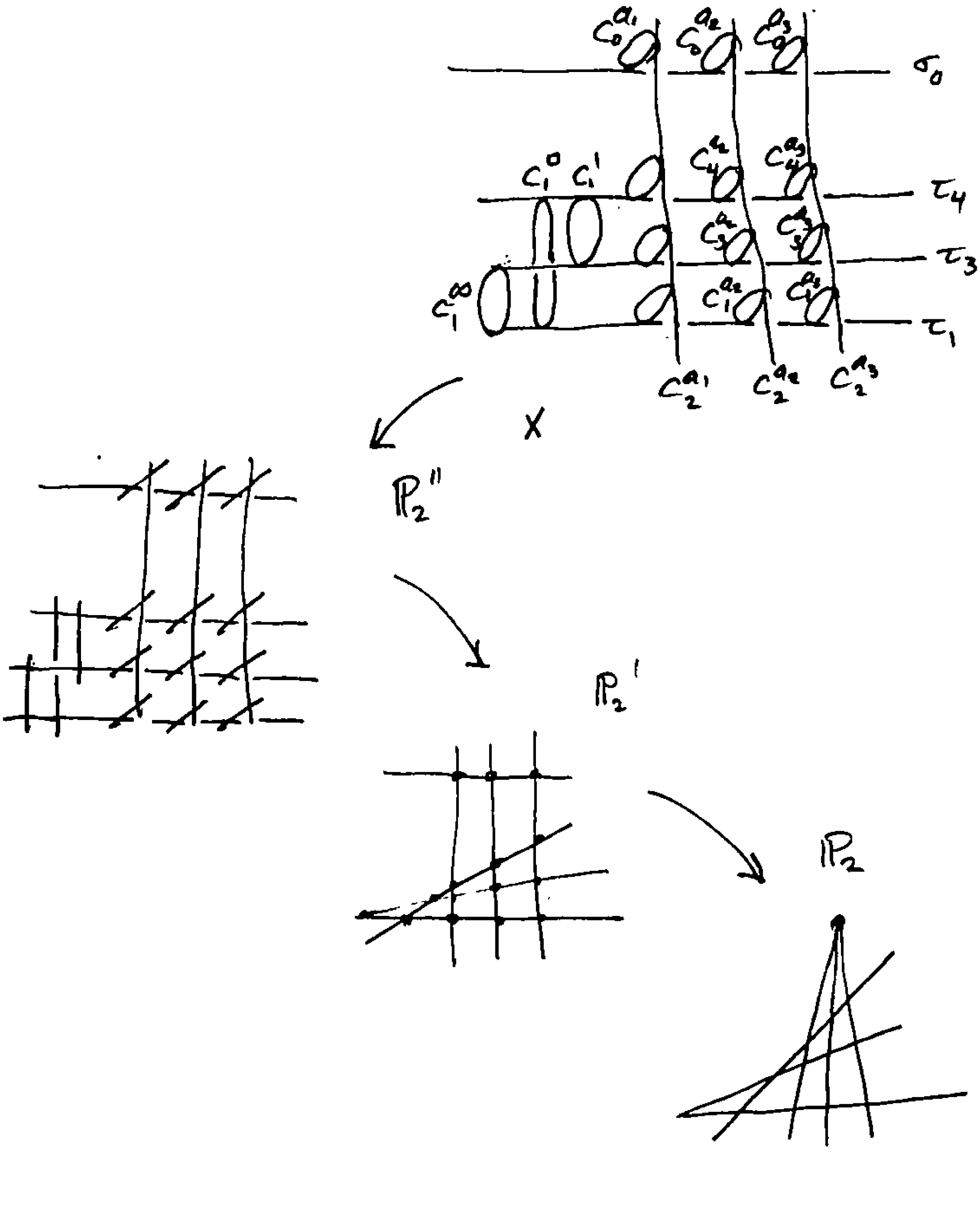
Intersection products for 17 curves in (4.2)

	1	2	3	4	5	6	9	8	7	10	13	12	11	14	17	16	15
1	–2	1															
2	1	0															
3			–2														
4				–2													
5					–2												
6						–2			1								
9							–2		1								
8								–2	1								
7						1	1	1	–2								
10										–2			1				
13											–2		1				
12												–2	1				
11										1	1	1	–2				
14														–2			1
17															–2		1
16																–2	1
15														1	1	1	–2

(all other entries are 0)

Figure (4.8)

Configurations for $X \longrightarrow \mathbb{P}_2'' \longrightarrow \mathbb{P}_2' \longrightarrow \mathbb{P}_2$



5. <u>The special case</u> $(a_1,a_2,a_3) = (-1,1/2,2)$ <u>with</u> $\rho = 20$

(5.1) As illustrated in Figure (3.10) there are four different permutations

$$(ijk) = (132), (123), (312), (231)$$

such that $(a_1,a_2,a_3) = (-1,1/2,2) \in V$ satisfies $a_i-(1+a_i-a_j)a_k = 0$ and lies in $V \cap Q_{ijk}$. As illustrated by Figure (5.6) there are four corresponding configurations $\{\Sigma L_p, L_{ijk}\}$ of type (2.4.1) with ΣL_p defined by

$$(t+1)(t-1/2)(t-2)x(x-1)(x-t) = 0$$

and with lines L_{ijk} which pass through double points

$$(a_i,0), (a_j,1), (a_k,a_k)$$

of ΣL_p and which satisfy the conditions in (3.1)

$$L_{132} : x = \tfrac{1}{3}(t+1),\ x-1 = \tfrac{1}{3}(t-2),\ x-t = -\tfrac{2}{3}(t-\tfrac{1}{2})\ ,$$

$$L_{123} : x = \tfrac{2}{3}(t+1),\ x-1 = \tfrac{2}{3}(t-\tfrac{1}{2}),\ x-t = -\tfrac{1}{3}(t-2)\ ,$$

$$L_{312} : x = -\tfrac{1}{3}(t-2),\ x-1 = -\tfrac{1}{3}(t+1),\ x-t = -\tfrac{4}{3}(t-\tfrac{1}{2})\ ,$$

$$L_{231} : x = \tfrac{2}{3}(t-\tfrac{1}{2}),\ x-1-\tfrac{2}{3}(t-2),\ x-t = -\tfrac{1}{3}(t+1)\ .$$

Therefore as in (3.1) the equation

(5.1.1) $$y^2 = (t+1)(t-1/2)(t-2)x(x-1)(x-t)$$

has four pairs of non–trivial $\mathbb{C}(t)$–rational solutions of type (2.4.1)

$$(x_{132}, \pm y_{132}) = (\tfrac{1}{3}(t+1), \pm\sqrt{\tfrac{2^2}{3^3}(t+1)(t-\tfrac{1}{2})(t-2)}) ,$$

$$(x_{123}, \pm y_{123}) = (\tfrac{2}{3}(t+1), \pm\sqrt{\tfrac{2^2}{3^3}(t+1)(t-\tfrac{1}{2})(t-2)}) ,$$

$$(x_{312} \pm y_{312}) = (-\tfrac{1}{3}(t-2), \pm\sqrt{\tfrac{2^2}{3^3}(t+1)(t-\tfrac{1}{2})(t-2)}) ,$$

$$(x_{231} \pm y_{231}) = (\tfrac{2}{3}(t-\tfrac{1}{2}), \pm\sqrt{\tfrac{2^2}{3^3}(t+1)(t-\tfrac{1}{2})(t-2)}) ,$$

which are determined up to the sign of $\pm y_{ijk}$ by $\{\Sigma L_p, L_{ijk}\}$.

(5.2) Let $X \longrightarrow \mathbb{P}_1$ be the Neron model for (5.1.1) relative to $\mathbb{C}(t)$, let $X \longrightarrow \mathbb{P}_2'' \longrightarrow \mathbb{P}_2' \longrightarrow \mathbb{P}_2$ be the maps specified in (4.6), and for each (ijk) let σ_{ijk} be the section determined by $(x_{ijk}, +y_{ijk})$ in (5.1). Each σ_{ijk} lies over L_{ijk} in $\mathbb{P}_2$ and also over the proper transform L''_{ijk} of L_{ijk} in $\mathbb{P}_2''$. The points (1:–1:0), (1:1/2:1/2), (1:2:1) $\in \mathbb{P}_2$ are blown up in $\mathbb{P}_2''$; they also are points of intersection of two distinct L_{ijk} which have different directions through these points; and consequently these points do <u>not</u> lift to points of intersection of distinct L''_{ijk} or σ_{ijk} . The other points of intersection of two distinct L_{ijk} are (1:0:2/3), (1:1:1/3), (0:1:1/3) ; these three points lift to transversal points of intersection of the corresponding σ_{ijk} ; they can be represented in Table (5.1) by solutions

$$(u,z,w) = \left(\frac{1}{t+c}, \frac{x}{t+c}, \frac{y}{(t+c)^3}\right)$$

of an equation

$$w^2 = (1-(a_1+c)u)(1-(a_2+c)u)(1-(a_3+c)u)z(z-u)(z-1+cu)$$

for a birationally equivalent model of X which is biholomorphically equivalent to X at each of these points. There are no other points of intersection of distinct σ_{ijk} .

(5.3) Intersection products for the σ_{ijk} with each other and with the 17 curves in (4.1.1) are listed in Table (5.8). These values can be computed as follows: ${\sigma_{ijk}}^2 = -2$ by (4.5) since σ_{ijk} is a non–singular rational curve; three $\sigma_{ijk} \cdot \sigma_{\ell mn} = 1$ and the others $= 0$ by (5.2); $\sigma_{ijk} \cdot D = 0$ or 1 for D in (4.2) by inspection of Figure (5.9) for σ_{132} and similar diagrams for other σ_{ijk} . Intersection products are also listed in Table (5.8) for the preceding curves and the torsion sections τ_1, τ_3, τ_4 in (4.3).

(5.4) Intersection products for the 17 curves in (4.1.1) together with the three sections $\sigma_{132}, \sigma_{123}, \sigma_{312}$ form a 20×20 matrix A with determinant $\det A = -3 \cdot 2^7$. In fact by ordering these 20 curves as in the first column of Table (5.10) and by adding suitable fractional multiples of early entries to later entries as in the second column of Table (5.10) one obtaines the equivalent matrix ${}^t\!EAE$ in Table (5.11) which obviously has $\det({}^t\!EAE) = -3 \cdot 2^7$ and $\det E = 1$. Furthermore the classes in $NS(X) \otimes \mathbb{Q}$ of the last three entries in the second column of Table (5.10) must coincide with $\delta(\sigma_{132})$, $\delta(\sigma_{123})$, $\delta(\sigma_{312})$, resp., since the former classes obviously have the properties (4.5.1) and (4.5.2) which characterize the latter classes. Consequently the negative

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix}$$

of the lower right hand corner of ${}^{t}EAE$ is the matrix of corresponding modified intersection numbers

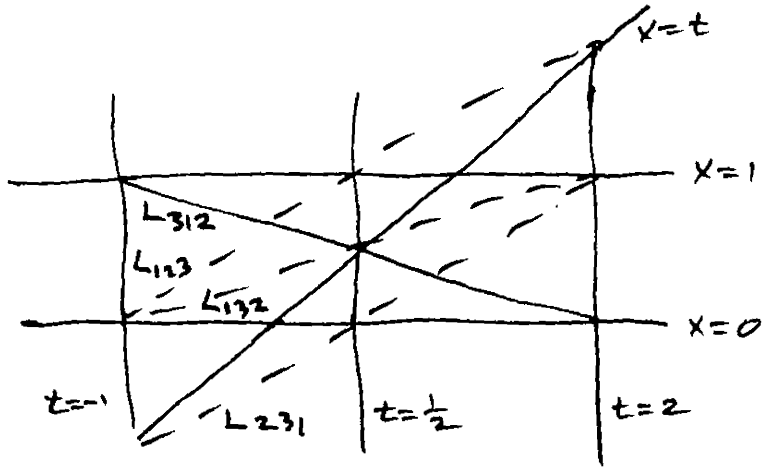
$$\langle \sigma_{ijk}, \sigma_{\ell mn} \rangle = -\delta(\sigma_{ijk}) \cdot \delta(\sigma_{\ell mn}) \ ,$$

$$(ijk), (\ell mn) = (132), (123), (312) \ .$$

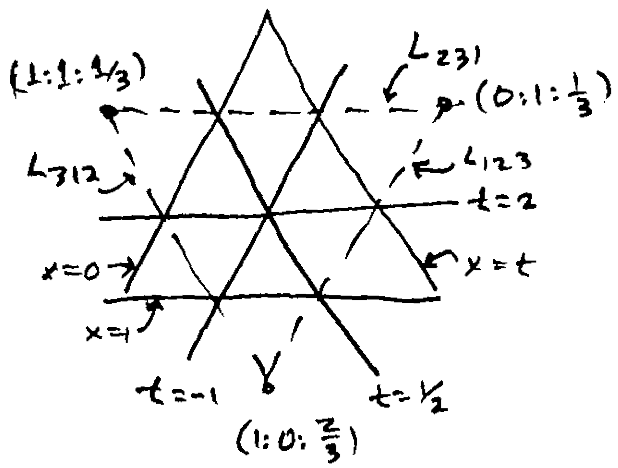
(5.5) It follows from (5.4) that X has Picard number $\rho = 20$, that the classes of the 20 curves in the first column of Table (5.10) form a basis for $NS(X) \otimes \mathbb{Q}$, that these classes generate a sublattics L' with finite index in NS(X) and with discriminant $-3 \cdot 2^7$, and that L' and the classes of τ_1, τ_3, τ_4 generate an intermediate lattice L" with discriminant $-3 \cdot 2^7/16 = -24$. Consequently either NS(X) = L" with discriminant $d = -24$, in which case $\mathfrak{S}$ is generated by $\sigma_{132}, \sigma_{123}, \sigma_{312}, \tau_1, \tau_3$; or NS(X) has discriminant $d = -6$ and contains L" as a sublattice of index 2, in which case σ_{132}, $\sigma_{123}, \sigma_{312}, \tau_1, \tau_3$ generate a subgroup of index 2 in $\mathfrak{S}$. J. Stienstra has pointed out that in fact $d = -24$ since otherwise X would have a transcendental lattice T_X with rank = 2 and discriminant = 6 , contrary to a result of Shioda and Inose [11].

Figure (5.6)

A configuration for (–1,1/2,2) which corresponds to four solutions of type (2.5.1)



(with (0:1:1/3) at ∞)



(with L_{132} at ∞)

Table (5.7)

Values of $(u,z,w) = (\frac{1}{t+c}, \frac{x}{t+c}, \frac{y}{(t+c)^3})$ at $t = 0,1,\infty$ on σ_{ijk}

ijk \ t	0	1	∞
132	$(\frac{1}{c}, \frac{1}{3c}, +\frac{1}{c^3}\sqrt{\frac{2}{3^3}})$	$(\frac{1}{1+c}, \frac{2}{3(1+c)}, \frac{1}{(1+c)^3}\sqrt{\frac{2}{3^3}})$	$(0, \frac{1}{3}, +\sqrt{\frac{2}{3^3}})$
123	$(\frac{1}{c}, \frac{2}{3c}, +\frac{1}{c^3}\sqrt{\frac{2^2}{3^3}})$	$(\frac{1}{1+c}, \frac{4}{3(1+c)}, \frac{1}{(1+c)^3}\sqrt{\frac{2^2}{3^3}})$	$(0, \frac{2}{3}, +\sqrt{\frac{2^2}{3^3}})$
312	$(\frac{1}{c}, \frac{2}{3c}, +\frac{1}{c^3}\sqrt{\frac{2^2}{3^3}})$	$(\frac{1}{1+c}, \frac{1}{3(1+c)}, \frac{1}{(1+c)^3}\sqrt{\frac{2^2}{3^3}})$	$(0, -\frac{1}{3}, +\sqrt{\frac{2^2}{3^3}})$
231	$(\frac{1}{c}, \frac{1}{3c}, +\frac{1}{c^3}\sqrt{\frac{2^2}{3^3}})$	$(\frac{1}{1+c}, \frac{1}{3(1+c)}, \frac{1}{(1+c)^3}\sqrt{\frac{2^2}{3^3}})$	$(0, \frac{2}{3}, +\sqrt{\frac{2^2}{3^3}})$

Table (5.8)

	σ_{132}	σ_{123}	σ_{312}	σ_{231}	τ_1	τ_3	τ_4
1	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1
3	0	0	0	0	1	1	0
4	0	0	0	0	1	0	1
5	0	0	0	0	0	1	1
6	1	1	0	0	1	0	0
8	0	0	0	1	0	1	0
9	0	0	1	0	0	0	1
7	0	0	0	0	0	0	0
10	0	0	1	0	1	0	0
12	0	1	0	0	0	1	0
13	1	0	0	1	0	0	1
11	0	0	0	0	0	0	0
14	0	0	0	1	1	0	0
16	1	0	1	0	0	1	0
17	0	1	0	0	0	0	1
15	0	0	0	0	0	0	0
σ_{132}	–2	0	0	0	0	0	0
σ_{123}	0	–2	1	1	0	0	0
σ_{312}	0	1	–2	1	0	0	0
σ_{231}	0	1	1	–2	0	0	0
τ_1	0	0	0	0	–2	0	0
τ_3	0	0	0	0	0	–2	0
τ_4	0	0	0	0	0	0	–2

Figure (5.9)

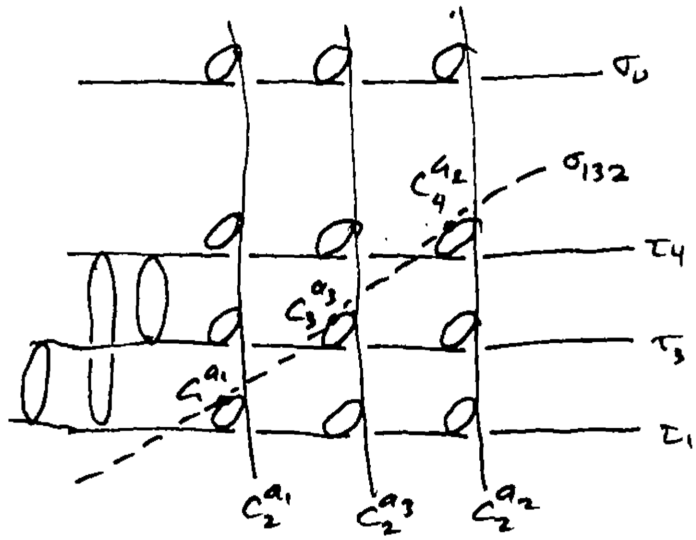


Table (5.10)

1	σ_0	σ_0
2	C^t	$C^t+\frac{1}{2}\sigma_0$
3	C_1^0	C_1^0
4	C_1^1	C_1^1
5	C_1^∞	C_1^∞
6	$C_1^{a_1}$	$C_1^{a_1}$
8	$C_3^{a_1}$	$C_3^{a_1}$
9	$C_4^{a_1}$	$C_4^{a_1}$
7	$C_2^{a_1}$	$C_2^{a_1}+\frac{1}{2}(C_1^{a_1}+C_3^{a_1}+C_4^{a_1}$
10	$C_1^{a_2}$	$C_1^{a_2}$
12	$C_3^{a_2}$	$C_3^{a_3}$
13	$C_4^{a_2}$	$C_4^{a_2}$
11	$C_2^{a_2}$	$C_2^{a_2}+\frac{1}{2}(C_1^{a_1}+C_3^{a_1}+C_4^{a_1})$
14	$C_1^{a_3}$	$C_1^{a_3}$
16	$C_3^{a_3}$	$C_3^{a_3}$
17	$C_4^{a_3}$	$C_4^{a_3}$
15	$C_2^{a_3}$	$C_2^{a_3}+\frac{1}{2}(C_1^{a_3}+C_3^{a_3}+C_4^{a_3})$
18	σ_{132}	$\sigma_{132}-2(C_2+\frac{1}{2}\sigma_0)+\frac{1}{2}C_1^{a_1}+C_2^{a_1}+\frac{1}{2}(C_1^{a_1}+C_3^{a_1}+C_4^{a_1})$
		$+\frac{1}{2}(C_4^{a_2}+C_2^{a_2}+\frac{1}{2}(C_1^{a_2}+C_3^{a_2}+C_4^{a_2})+\frac{1}{2}C_3^{a_3}+C_2^{a_3}+\frac{1}{2}(C_1^{a_3}+C_3^{a_3}+C_4^{a_3})$
		$=\delta(\sigma_{132})$

19	σ_{123}	$\sigma_{123}-2(C_2+\frac{1}{2}\sigma_0)+\frac{1}{2}C_1^{a_1}+C_2^{a_1}+\frac{1}{2}(C_1^{a_1}+C_3^{a_1}+C_4^{a_1})$ $+\frac{1}{2}C_3^{a_2}+C_2^{a_2}+\frac{1}{2}(C_1^{a_2}+C_3^{a_2}+C_4^{a_2})+\frac{1}{2}C_4^{a_3}+C_2^{a_3}+\frac{1}{2}(C_1^{a_3}+C_3^{a_3}+C_4^{a_3})$ $=\delta(\sigma_{123})$
20	σ_{312}	$\sigma_{312}-2(C_2+\frac{1}{2}\sigma_0)+\frac{1}{2}C_4^{a_1}+C_2^{a_1}+\frac{1}{2}(C_1^{a_1}+C_3^{a_1}+C_4^{a_1})$ $+\frac{1}{2}C_1^{a_2}+C_2^{a_2}+\frac{1}{2}(C_1^{a_2}+C_3^{a_2}+C_4^{a_2})+\frac{1}{2}C_3^{a_3}+C_2^{a_3}+\frac{1}{2}(C_1^{a_3}+C_3^{a_3}+C_4^{a_3})$ $=\delta(\sigma_{312})$

Table (5.11)

A =

–2	1																0	0	0
1	0																1	1	1
		–2																	
			–2																
				–2															
					–2			1									1	1	0
						–2		1									0	0	0
							–2	1									0	0	1
					1	1	1	–2									0	0	0
									–2			1					0	0	1
										–2		1					0	1	0
											–2	1					1	0	0
									1	1	1	–2					0	0	0
													–2			1	0	0	0
														–2		1	1	0	1
															–2	1	0	1	0
													1	1	1	–2	0	0	0
0	1				1	0	0	0	0	0	0	0	0	1	0	0	–2	0	0
0	1				1	0	0	0	0	1	0	0	0	0	1	0	0	–2	1
0	1				0	0	1	0	1	0	0	0	0	1	0	0	0	1	–2

$$
{}^{t}EAE = \begin{pmatrix}
\begin{matrix}-2 & 1\\ 0 & \frac{1}{2}\end{matrix} & & & & & \\
 & \begin{matrix}-2 & & \\ & -2 & \\ & & -2\end{matrix} & & & & \\
 & & \begin{matrix}-2 & & & \\ & -2 & & \\ & & -2 & \\ & & & -\frac{1}{2}\end{matrix} & & & \\
 & & & \begin{matrix}-2 & & & \\ & -2 & & \\ & & -2 & \\ & & & -\frac{1}{2}\end{matrix} & & \\
 & & & & \begin{matrix}-2 & & & \\ & -2 & & \\ & & -2 & \\ & & & -\frac{1}{2}\end{matrix} & \\
 & & & & & \begin{matrix}-1 & 0 & 0\\ 0 & -1 & \frac{1}{2}\\ 0 & \frac{1}{2} & -1\end{matrix}
\end{pmatrix}
$$

References

[1] Barth, Peters and Van de Ven, Complex Surfaces, Springer Verlag.

[2] D. Cox and S. Zucker, Intersection numbers of sections of elliptic surfaces, Inv. Math. 53 (1979), 1–44.

[5] W. Hoyt, Elliptic fiberings of Kummer surfaces, in Springer LNM No. 1383.

[6] ———, On twisted Legendre equations and Kummer surfaces, in Theta Functions, Bowdoin 1987, AMS Proc. Symp. Pure Math. vol. 49, Part I, 695–707.

[10] T. Shioda, On elliptic modular surface, J. Math. Soc. Japan 24 (1972), 20–59.

[11] T. Shioda and H. Inose, On singular K3 surfaces, in Complex Analysis and Algebraic Geometry, Iwanami Shoten 1977, 119–136.