SIMULTANEOUS $\mathbb{Q}$-FACTORIALIZATION OF DIVISORS AND APPLICATIONS TO 3-FOLD DIVISORIAL CONTRACTIONS THAT CONTRACT A DIVISOR ONTO A CURVE THROUGH A cD_5 SINGULARITY

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ABSTRACT. Let $\Gamma \subset X$ be a smooth curve on a 3-fold. In this paper we classify terminal divisorial contractions $E \subset Y \longrightarrow \Gamma \subset X$ contracting an irreducible surface E onto Γ, in the case when the general section of X containing Γ has D_5 type singularities. In doing so we consider the problem of existence of MMP in families of canonical 3-folds, and we get results about existence of simultaneous $\mathbb{Q}$-factorialisation of a family of divisors.

0. INTRODUCTION

In the birational classification of algebraic varieties, one wants to identify in each birational class some distinguished “simple” members which are called minimal models and then study the structure of birational maps between them. In dimension two, satisfactory results were known for over one hundred years. To treat the problem in higher dimensions, the minimal model program (MMP) was developed. After contributions of Kawamata, Kollár, Reid, Shokurov and others, the program was completed in dimension 3 by Mori in 1988. According to Mori's theorem, every $\mathbb{Q}$-factorial terminal projective threefold X, is birational to a threefold X' with either $K_{X'}$ nef (in which case it is called a minimal model), or X' is a Fano fiber space. It is known that any birational map between minimal models is a composition of flops [Kaw88]. Terminal flops have been classified by the work of Kollár [Ko91]. However, the structure of birational maps between Fano fiber spaces is one of the most important open problems of threefold birational geometry. The Sarkisov program was developed by Corti, Reid and Sarkisov in order to factorize every birational map between Fano fiber spaces, as a composition of “elementary links”. Therefore, in order to better understand the geometry of the Fano fiber spaces,

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it is important to understand these links which consist of flips, flops and divisorial contractions. Flips have been classified by the work of Kollár and Mori [Ko-Mo92]. The structure of divisorial contractions is still an open problem.

let $E \subset Y \xrightarrow{f} \Gamma \subset X$ be a divisorial contraction. The most results concerning the case that Γ is a point are due to Corti [Cor-Rei00] and Kawakita [Ka02], [Ka01]. The case that Γ is a curve has been studied by Kawamata [Kaw94], and the author [Tzi02]. This paper can be considered as a continuation of [Tzi02]. In that paper, divisorial contractions when Γ was a smooth curve and X had a DuVal section of type D_{2n} through Γ, were classified. It was also explained that the case when the general section of X through Γ is D_{2n+1}, is very different than the D_{2n} one. In this paper we completely classify divisorial contractions $E \subset Y \xrightarrow{f} \Gamma \subset X$, in the case that the general section of X through Γ has D_5 type singularities and Y is terminal. This is done in theorem 2.3 and corollaries 2.3 and 2.5. In doing so we follow the method presented in [Tzi02]. However, we need to consider cases with respect to the form of the equation of X. We deform X to an X_0 whose equation is much simpler and then we show that in some cases the divisorial contractions exist in families. This is not true in general as shown by example 1.6. Then we obtain information about X by using results about deformations of terminal singularities [Nak98]. In order to construct divisorial contractions in families, we must consider the problem of existence of simultaneous $\mathbb{Q}$-factorialization of divisors. In particular, given a family of threefolds $X \longrightarrow T$, and a global divisor $Z \subset X$, does there exist a morphism $Y/T \longrightarrow X/T$, such that Y_t is a $\mathbb{Q}$-factorialization of Z_t? If T is a point and X is a canonical threefold this was done by Kawamata [Kaw88]. The problem is of interest by itself from the point of view of the MMP in higher dimensions. The case that X_0 is terminal is treated in theorem 1.1. The canonical case is more complicated. Theorem 1.3 gives conditions for a simultaneous $\mathbb{Q}$-factorialization to exist. Example 1.6 shows that these conditions are necessary. Finally in order to construct divisorial contractions in families, it is essential to use MMP in families of canonical threefolds. The terminal case was done by Kollár and Mori [Ko-Mo92]. The main difficulty in the canonical case is the existence of canonical flops in families. This is also an interesting problem by itself from the higher dimensional point of view. I should mention that this is not solved by Shokurov's latest work. Propositions 1.9 and 1.10 give cases when canonical flops exist in families. These results are sufficient for the purposes of this paper but are a very superficial approach to this problem.

I plan to return to it in a future paper. Finally I should say that I believe that the general D_{2n+1} case could be done by the methods of this paper. However my feeling is that there is a big amount of calculations needed to do and that would result into a disproportionally large to the results paper.

1. $\mathbb{Q}$-factorialization in families of canonical 3-folds

In this section we will consider the following problem.

Problem: Let $X \longrightarrow T$ be a family of 3-folds, and $Z \subset X$ a divisor in X flat over T. Assume that the scheme theoretic intersection $Z_t = Z \cap X_t$ is a divisor. When is there a simultaneous $\mathbb{Q}$-factorialization of Z_t? That is, when is there a morphism $Y \longrightarrow X$ such that $Y_t \longrightarrow X_t$ is a $\mathbb{Q}$-factorialization of Z_t for all t?

The case of a family of terminal threefolds is particularly nice and it is possible to give a complete answer.

Theorem 1.1. *Let $X \xrightarrow{f} T$ be a one dimensional family of terminal 3-folds. Let $Z \subset X$ be a divisor in X that dominates T. Assume that the scheme theoretic intersection $Z_t = X_t \cap Z$ is a divisor in X_t. Then there exist a crepant morphism $Y \longrightarrow X$ flat over T, such that $Y_t \xrightarrow{f_t} X_t$ is a $\mathbb{Q}$-factorialization of Z_t. In particular, the birational transform Z'_t of Z_t in Y_t is $\mathbb{Q}$-Cartier and f_t-ample.*

The situation for families of canonical 3-folds is much more complicated. In general it is not true that a common $\mathbb{Q}$-factorialization exists as shown by example 1.6. The number of crepant divisors of the members of the family is important. In this direction we have the following:

Theorem 1.2. *Let $X \longrightarrow T$ be a family of canonical threefolds. Let $e(X_t)$ be the number of crepant exceptional divisors of X_t. Then*

$$e(X_0) \geq e(X_t)$$

for t in a small neighborhood of $0 \in T$. Moreover, if equality holds then there exist a common $\mathbb{Q}$-terminalization. That is, there is a morphism $Y \xrightarrow{g} X$, such that $Y_t \xrightarrow{g_t} X_t$ is a $\mathbb{Q}$-factorial and terminal.

Question: Is it true that with assumptions as in the problem above, a simultaneous $\mathbb{Q}$-factorialization of Z_t exists iff $e(X_0) = e(X_t)$, or better if we only consider crepant divisors with center in Z_t?

In the canonical case we have the following.

Theorem 1.3. *Let $X \longrightarrow T$ be a family of index 1 canonical 3-folds, with T smooth. Let $Z \subset X$ be a divisor in X that dominates T such* *that the scheme theoretic intersection $Z_t = X_t \cap Z$ is a smooth integral divisor in X_t. Moreover, assume that*

(1) $X_t^{sing} \cap Z_t$ *is a smooth curve* L_t. *Let* $S_t = Spec\mathcal{O}_{X_t,L_t}$. *This is a Duval surface singularity defined over* $\mathbb{C}(t)$. $Z_t|_{S_t}$ *is a line through the origin such that the extended dual graph of* S_t *and* $Z_t|_{S_t}$ *is independent of* t. *This means that* S_t *are isomorphic singularities and that the position of* $Z_t|_{S_t}$ *in the fundamental cycle does not change with* t.

(2) X_0 *is either cDV along* Z_0, *or there is a smooth curve* $L \subset X$ *dominant over* T *such that* $L \cap X_t$ *is a reduced point* P_t *and* $m_L X = 2, 3$ *or* $k \geq 3$. *This means that* $P_t \in X_t$ *is elliptic with constant* k. *Moreover, if* $k \geq 3$ *then* $B_{P_0} X_0$ *is cDV and if* $k = 1$ *or* 2, *then the* $(3, 2, 1, 1,)$ *or* $(2, 1, 1, 1)$ *weighted blow ups are cDV along the birational transform* $\tilde{Z}_t$.

Then there is a birational morphism $Y \longrightarrow X$ *flat over* T, *such that* $Y_t \longrightarrow X_t$ *is a* $\mathbb{Q}$*-factorialization of* Z_t, *for all* t.

We can use this result to construct divisorial contractions in families.

Corollary 1.4. *Let* $X \longrightarrow T$ *be a 1-dimensional family of terminal 3-fold singularities* $P_t \in X_t$. *Let* $\Gamma \subset X$ *be a surface such that the scheme theoretic intersection* $\Gamma_t = \Gamma \cap X_t$ *is a smooth curve through* P_t. *Let* H_t *be the general hyperplane section of* X_t *through* Γ_t *Assume it is* A_n *for all* t *and that* Γ_t *intersects an end of the fundamental cycle of* H_t *for all* t. *Then there exist a morphism* $g \colon W \longrightarrow X$ *flat over* T, *such that* $W_t \longrightarrow X_t$ *is a terminal extremal contraction of an irreducible divisor* E_t *onto* Γ_t. *Therefore divisorial contractions of a surface to a curve exist in families for this type of singularities.*

We can use the previous result to improve the result of [Tzi02, Theorem 5.1].

Corollary 1.5. *Let* $E \subset W \longrightarrow \Gamma \subset X$ *be a terminal 3-fold divisorial contraction contracting an irreducible divisor* E *onto a smooth curve* Γ. *Suppose that the general hyperplane* S *of* X *containing* Γ *is* A_n, *and that* Γ *intersects an end of the fundamental cycle of* S. *Then* W *has index* $n+1$.

Proof. By [Tzi02] it is possible to write the equation of X as

$$xy + z^{n+1} + t\phi(x, y, z, t) = 0,$$

where the curve Γ is given by $x = z = t = 0$. By a result of Hironaka-Rossi, the equation of X is equivalent to

$$xy + z^{n+1} + t\phi(x, y, z, t) + t^m = 0,$$

for $m >> 1$. Now we can deform X to X_0 given by $xy + z^{n+1} + t^m = 0$, and it can be explicitely seen with the same method as in the next example, that there is a terminal contraction $E_0 \subset W_0 \longrightarrow \Gamma_0 \subset X_0$, and W_0 has index $n+1$. Since the index is constant in families, the corollary follows. □

The conditions of theorem 1.3 are necessary. The next example gives a family of canonical 3-folds and a family of divisors that a simultaneous $\mathbb{Q}$-factorialization does not exist.

Example 1.6. *Let $X \subset \mathbb{C}^4_{x,y,z,t} \times \mathbb{C}^1_\epsilon$ be the family over $\mathbb{C}^1_\epsilon$ given by $\epsilon z^2 + xy + z^3 + t^3 = 0$. Let $\Gamma \subset X$ be the plane given by $x = z = t = 0$. Let $f\colon Y = B_\Gamma X \longrightarrow X$. Then $f^{-1}(\Gamma) = E + F$. E is not $\mathbb{Q}$-Cartier and there is no simultaneous $\mathbb{Q}$-factorialization of E in $Y \longrightarrow \mathbb{C}^1_\epsilon$.*

Proof. It is clear that $Y_\epsilon = B_{\Gamma_\epsilon} X_\epsilon$. Describe Y explicitely. By making the substitution $x = xt$, $z = zt$, we see that an affine open subset of Y is given by

$$\epsilon z^2 t + xy + z^3 t^2 + t^2 = 0.$$

$t = 0$ gives that $f^{-1}(\Gamma) = E + F$, where E is given by $x = t = 0$ and F by $y = t = 0$. The other charts are checked similarly. Suppose that a simultaneous $\mathbb{Q}$-factorialization $g\colon Z \longrightarrow Y$ of E_ϵ in Y_ϵ exists. According to [Ko-Mo92, theorem 12.3.1], (the theorem is only for terminal threefolds but the important is the existense of the versal deformation spaces Def(Y) which exists if Y is a modification of a terminal singularity which is true in our case), the proof of [Tzi02, theorem 1.8] runs for Z and X over T and we get a morphism $h\colon W \longrightarrow X$ flat over $\mathbb{C}^1_\epsilon$, and $W_\epsilon \longrightarrow X_\epsilon$ is an extremal contraction contracting an irreducible divisor E_ϵ onto the curve Γ_ϵ. The special fiber X_0 for $\epsilon = 0$ is given by $xy + z^3 + t^3 = 0$ and Γ_0 by $x = z = t = 0$. The general hyperplane section of X_0 through Γ_0 is A_2. For $\epsilon \neq 0$, X_ϵ is given by $\epsilon z^2 + xy + z^3 + t^3 = 0$. In this case, the general hyperplane section through Γ_ϵ is A_1. Therefore, W_ϵ has index 2 if $\epsilon \neq 0$ [Tzi02, theorem 5.1].

Claim: W_0 has index 3.

Since the index is constant in families, we get a contradiction and therefore a common $\mathbb{Q}$-factorialization does not exist.

Proof of claim. We will construct the contraction $W_0 \longrightarrow X_0$ explicitely. Y_0 is given by $xy + z^3t^2 + t^2 = 0$. E_0 is given by $x = t = 0$ and F_0 by $y = t = 0$. We want to construct a $\mathbb{Q}$-factorialization of E_0 and then contract F_0 to obtain W_0. The equation of Y_0 can be written as $xy + (z^3+1)t^2 = 0$. Therefore, $2E_0$ is Cartier at 0. To obtain a $\mathbb{Q}$-factorialization all we have to do is blow up the ideal $I = (x, t^2)$. So let

$g\colon Z_0 \longrightarrow Y_0$ be the blow up of I. g is an isomorphism in codimension 1. I claim that the g-exceptional curves C, are not contained in the birational transform F_0^Z and therefore $F_0^Z = F_0 = \mathbb{P}^2$. To see this we will describe Z_0 explicitely.

$Z \subset \mathbb{C}^4 \times \mathbb{P}^1_{u,v}$. It is given by

$$xu - t^2v = 0$$
$$xy + z^3t^2 + t^2 = 0$$

In the chart $v = 1$, Z_0 is given by

$$xu - t^2 = 0$$
$$y + z^3u + u = 0$$

It is easy to see that the g-exceptional curves are given by $x = y = t = z^3 + 1 = 0$, and that $F_0^{Z_0}$ by $y = t = u = 0$. Therefore no g-exceptional curve is contained in $F_0^{Z_0}$ and hence $F_0^{Z_0} = F_0 = \mathbb{P}^2$. Let $l \subset F_0^Z$ be a line. Then $l \cdot K_{Z_0} = l \cdot K_{Y_0} = -1 < 0$. Now let $p\colon Z_0 \longrightarrow W_0$ be the contraction of $F_0^{Z_0}$ over X_0. Hence there is also a commutative diagram:

$$\begin{tikzcd} & Z_0 \arrow[dl, "g_0"'] \arrow[dr, "p"] & \\ Y_0 \arrow[dr, "f_0"'] & & W_0 \arrow[dl, "q"] \\ & X_0 & \end{tikzcd}$$

We know that $2E_0^{Z_0}$ and $2F_0^{Z_0}$ are Cartier. Let $a > 0$ such that

$$K_{Z_0} = p^*K_{W_0} + aF_0^{Z_0}. \tag{1}$$

Let $l \subset F_0^{Z_0} = \mathbb{P}^2$ be a line. Then $l \cdot F_0^{Z_0} = -b/2$ with b a positive integer. We want to find b.

$$l \cdot (E_0^{Z_0} + F_0^{Z_0}) = l \cdot g_0^*(E_0 + F_0) = l \cdot (E_0 + F_0) = -1.$$

Now, $l \cdot E_0^{Z_0} = 1/2(2E_0^{Z_0}) \cdot l$. This can easily be computed. $2E_0^{Z_0}$ is given by $x = t^2 = y + z^3u + u = 0$ and $F_0^{Z_0}$ by $y = t = u = 0$. We can take any line in $F_0^{Z_0}$, so take the one given by $y = t = z = u = 0$. Clearly $l \cap 2E_0^{Z_0}$ is a reduced point. Therefore $l \cdot 2E_0^{Z_0} = 1$. Hence

$$l \cdot F_0^{Z_0} = -1 - l \cdot E_0^{Z_0} = -3/2.$$

Intersect (1) with l. We get that $a = 2/3$. Therefore the index of W_0 is 3. $\square$

$\square$

1.1. **Proof of theorem 1.1.** In order to prove the theorem we are going to use the MMP in dimesion 4 which is now known to work in this dimension [Sho02]. To do so we must show that X is terminal. This will follow from the next lemma.

Lemma 1.7. *Let $f: X \longrightarrow T$ be a one dimensional family of canonical 3-folds over a smooth curve. Suppose that X has finite index. Then X is also canonical. If X_0 is terminal, then X is also terminal.*

Proof. First we reduce to the index 1 case. The index is invariant under deformation and therefore $\operatorname{index}(X) = \operatorname{index}(X_t)$, for all t. Let

$$p: \tilde{X} \longrightarrow X$$

be the index 1 cover of X. Then $\tilde{X}_t \longrightarrow X_t$ is the index 1 cover of X_t. Since $K_{\tilde{X}} = p^* K_X$, we only need to consider the case that X_t is index 1. So we assume that now. Since $f^*(t) = X_t$ which is Cartier in X, it follows by adjunction that

$$K_X|_{X_t} = K_{X_t}.$$

X_t has rational singularities. Therefore by [Ko-Mo98], X has also rational singularities. Since it is index 1, it must be canonical. Looking at the proof of [Ko-Mo98, theorem 5.35], we see that if X is canonical of any dimension and H a Cartier divisor that is also canonical, then the pair (X, H) is canonical too. Therefore (X, X_t) is canonical. This implies that there are no crepant exceptional divisors with center in X_t. Let $g: Y \longrightarrow X$ be a resolution of X. If an exceptional divisor E is crepant, then it's center is not in X_t. Therefore it does not appear in $g^* X_t$ and hence by adjunction, $E_t = E \cap \tilde{X}_t$ must be crepant and exceptional over X_t. But this is impossible since X_t is terminal near 0. Therefore X is also terminal. $\square$

Back to the proof of Theorem 1.1. Let

$$g: W \longrightarrow X$$

be a resolution of X and let E_i be the exceptional divisors. Run a $(W, 0)$ MMP over X. Since X is terminal, we arrive at a $Y' \xrightarrow{g'} X$ over T, such that g' is an isomorphism in codimension 1 and Y' is $\mathbb{Q}$-factorial. Therefore $Z' = {g'_*}^{-1} Z$ is $\mathbb{Q}$-Cartier, and hence so is Z'_t. To make it g'_t-ample, run a $(Y'_t, -\epsilon Z'_t)$ MMP over X_t. By [Ko-Mo92, Prop. 11.4, Theorem 11.10], and since X is terminal, this can be done in families. Therefore we obtain $Y \xrightarrow{g} X$ with the required properties.

Note: It would be of interest to get a proof of 1.1 without using the MMP in dimension 4.

1.2. **Proof of Theorem 1.2.** To see the first part, let $W \xrightarrow{g} X$ be a resolution of X. Let $\{F_i\}$ be the set of crepant exceptional divisors. Now Let $Y \xrightarrow{h} W$ be a log resolution of $(W, \sum_i F_i + g_*^{-1}X_0)$. In particular, $h_*^{-1}F_i$ is smooth, i.e., all crepant divisors are smooth and snc. Let $f = g \circ h$. By generic smoothness we can assume that Y_t is irreducible and smooth for $t \neq 0$. Let E be f-exceptional such that E_t is f_t-crepant. Write

$$K_Y = f^*K_X + aE + \{other\}.$$

Then by adjunction it follows that

$$K_{Y_t} = K_Y|_{Y_t} = f_t^*K_{Y_t} + aE_t + \{other\}|_{Y_t}.$$

Therefore $a = 0$ and hence E is crepant. It is possible that E_t has many components. Look at $E \longrightarrow T$. This is flat and surjective and E is smooth. Therefore by generic smoothness E_t is smooth and irreducible for general t. Since there are finitely many crepant divisors, we may assume that E_t is irreducible for $t \neq 0$. Let $X_0' = f_*^{-1}X_0$. Now I claim that $E_0 = E|_{X_0'}$ is also crepant. Let

$$f^*X_0 = X_0' + \sum_i b_i E_i.$$

Then by adjunction

$$K_{X_0'} = f^*K_{X_0} + \sum_i (a_i - b_i)E_i.$$

Since X_0 is canonical it follows that E_0 is also crepant. Therefore $e(X_0) \geq e(X_t)$.

Suppose that $e(X_0) = e(X_t)$. Let $W \xrightarrow{f} X$ be a resolution of X as before. By generic smoothness, for general $t \in T$, W_t is smooth and in particular irreducible. Therefore by removing a finite number of points of T, we can assume that W_t is smooth for all $t \neq 0$. Now run a $(W, 0)$ MMP over X. We arrive at a crepant morphism $Y \xrightarrow{g} X$ such that Y_t is terminal for $t \neq 0$. Moreover we claim that Y_0 is irreducible. To see this, observe that X is canonical since X_0 (we may only work with the index 1 cover of X, hence we may assume that X is index 1. Since X_0 is rational, so is X. Now rational of index 1 implies canonical). Therefore, as in the proof of lemma 1.6, the pair (X, X_0) is canonical. In particular this implies that X is terminal around X_0 and therefore there is no crepant divisor with center in X_0. Hence Y_0 is irreducible. Moreover, $Y_0 \longrightarrow X_0$ must be a small contraction since $e(X_0) = e(X_t)$. This concludes the proof of the theorem. $\square$

In order to prove theorem 1.3, we will use the minimal model program in families of canonical threefolds. For families of terminal threefolds it is known to work [Ko-Mo92]. To work the canonical case we need to prove existence of canonical flops in families. Since we will be applying it to non proper families, it is desirable to obtain some results about the behaviour of $\mathbb{Q}$-factoriality in families. The next result is similar to [Ko-Mo92, Theorem 12.1.10] but we have removed the properness condition and instead we have a stronger restriction for the singularities.

Proposition 1.8. *Let $X \xrightarrow{f} T$ be a 1-dimensional family of canonical threefolds such that Y_t^{sing} is proper, and let D be a divisor on X such that $D_0 = D|_{X_0}$ is $\mathbb{Q}$-Cartier. Then there is a neighborhood V of $0 \in T$ such that D is $\mathbb{Q}$-Cartier in $f^{-1}(V)$.*

Proof. Let $m > 0$ such that $m(D|_{X_0})$ is Cartier. Let $U = \{x \in X,$ s.t. mD is Cartier at $x\}$, and $Y = X - U$. Since X is normal, $codim(Y, X) \geq 2$. Let i, j denote the open immersions $U \longrightarrow X$ and $U \cap X_0 \longrightarrow X_0$. Then by assumption, the sheaf $F = \mathcal{O}_X(mD)|_U$ is invertible.

Claim:

$$(i_*F)|_{X_0} = j_*(F|U \cap X_0).$$

Since $i_*F = \mathcal{O}_X(mD)$, and $j_*(F|U \cap X_0) = \mathcal{O}_{X_0}(m(D|_{X_0}))$, which is invertible, the claim will imply that $\mathcal{O}_X(mD)$ is invertible in a neighborhood of X_0. Therefore by the properness of the singular part of X, Y is over a finite number of points of T and therefore there is a $O \in V \subset T$ such that $mD|_{f^{-1}(V)}$ is Cartier.

To prove the claim we follow the idea of [Ko-Mo92, Theorem 12.1.10]. There is an exact sequence

$$0 \longrightarrow F \longrightarrow F \longrightarrow F|U \cap X_0 \longrightarrow 0.$$

This gives

$$\begin{aligned} 0 \longrightarrow i_*F \longrightarrow i_*F \longrightarrow i_*(F|U \cap X_0) \longrightarrow \\ \longrightarrow R^1 i_*F \longrightarrow R^1 i_*F \longrightarrow R^1 i_*(F|U \cap X_0) \end{aligned} \tag{2}$$

Let $F_0 = F|U \cap X_0$. We will show that $R^1 i_* F_0 = R^1 i_* F = 0$. This is local, so we may take X to be affine. Then $R^1 i_* F_0 = H^1(X, i_* F_0) = H^1(X_0, j_* F_0)$. By using local cohomology we find that $H^1(U \cap X_0, F_0) = H^2_{X_0 \cap Y}(X_0, j_* F_0)$. But $j_* F_0 = \mathcal{O}_{X_0}(m(D|_{X_0}))$ and hence free and in particular S_3 and coherent. Hence by [Har67], $H^2_{X_0 \cap Y}(X_0, j_* F_0) = 0$. To show that $R^1 i_* F = 0$, we must show that it is coherent. Then from

(2) it will follow that it must be zero. By [Har67], it suffices to show that

$$\operatorname{depth}_z F + \operatorname{codim}(Z \cap Y, Z) \geq 3$$

for all $z \in U$. Since F is invertible, it is CM and hence we only need to check the above condition at the generic point of X. This will follow once we show that $codim(Y, X) \geq 3$. Since X_0 is canonical, X is also canonical. If $\dim Y = 2$, then at the generic point of Y, X is a rational surface singularity. Therefore $\mathbb{Q}$-factorial. So $\dim Y = 1$. Therefore $R^1 i_* F$ is coherent and hence it must be zero. This proves the claim and the proposition. □

Problem: Do canonical flops exist in families?

The answer for terminal flops is yes [Ko-Mo92]. I will treat this problem in another paper. However for the purposes of this paper, we only need existence under some very strong assumptions.

Proposition 1.9. *Let $f_0\colon Y_0 \longrightarrow X_0$ be a flopping contraction in any dimension. Suppose that X_0 has only hypersurface double points. Let $Y \longrightarrow T$ be a deformation of Y_0. Then there is a projective morphism $f\colon Y \longrightarrow X$ over T, such that $f_0\colon Y_0 \longrightarrow X_0$, is the given contraction. Let $f'_0\colon Y'_0 \longrightarrow X_0$ be the flop of f_0. Then there is a projective morphism $f' : Y' \longrightarrow X$, such that $f'_0 : Y'_0 \longrightarrow X_0$ is the flop of f_0. In other words flops to double points exist in families.*

Proof. $0 \in X_0$ can be written as

$$x^2 + f(y_1, \ldots, y_n) = 0.$$

X exists by [Ma-Ro71]. Then X can be written as

$$x^2 + f(y_1, \ldots, y_n, t_1, \ldots, t_m) = 0.$$

Now we proceed as in the 3-fold terminal case. If $-D$ is f-ample, then to prove the existence of the flop for f it suffices to show that $R(X, D) = \oplus_d \mathcal{O}_X(df_* D)$ is finitely generated. X has an involution i. Then $D + i^* D = 0$. Since $R(X, -D)$ is finitely generated, then so is $R(X, D)$. Therefore there is an $Y' \longrightarrow X$ the flop of f. But $Y' = Y$ and the involution is also in every fiber. Therefore the flop exists in families. □

Proposition 1.10. *Let $f\colon Y/T \longrightarrow X/T$ be a morphism between two families of canonical 3-folds over T. Suppose that $f_0\colon Y_0 \longrightarrow X_0$ is a flopping contraction. Suppose that there is a morphism $g\colon Z \longrightarrow T$, such that $Z_0 \longrightarrow Y_0$ is a $\mathbb{Q}$-factorial terminalization of X_0. Then there is a morphism $f'\colon Y' \longrightarrow X$, such that $f'_0\colon Y'_0 \longrightarrow X_0$, is the flop of f_0. Therefore in this case flops exist in families.*

Note that the conditions of this proposition are not always satisfied as shown by example 1.6.

Proof. Let $f_0' \colon Y_0' \longrightarrow X_0$ be the flop of f_0. Let $g_0' \colon Z_0' \longrightarrow Y_0'$ be a $\mathbb{Q}$-factorial terminalization of Y_0'. Z_0, Z_0' are birational and nef over X_0. Therefore, $\phi_0 \colon Z_0 \dashrightarrow Z_0'$ is an isomorphism in codimension 1 and a composition of terminal flops [Kaw88]. Terminal flops exist in families [Ko-Mo92], therefore ϕ_0 extends to a map $\phi \colon Z \dashrightarrow Z'$ over T. Then by [Ma-Ro71], there is a morphism $g' \colon Z' \longrightarrow Y'$ over X, extending g_0'. Hence $f' \colon Y' \longrightarrow X$ is the required flop. $\square$

In order to prove theorem 1.3, we will need a few simple lemmas.

Lemma 1.11. *Let $Z \subset X$ be a smooth divisor in a normal variety X which has canonical hypersurface singularities only, and that every irreducible component of $X^{sing} \cap Z$ is smooth. Let $Y = B_Z X \xrightarrow{f} X$ be the blow up of X along Z. Then*

1. *Y is normal and has hypersurface singularities only,*
2. *$K_Y = f^* K_X$,*
3. *every irreducible component E_i of the f-exceptional divisor E is smooth.*

Proof. Let $n = \dim X$. Then we may assume that $X \subset \mathbb{C}^{n+1}$. Let $f \colon B_Z \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$ be the blow up of $\mathbb{C}^{n+1}$ along Z. Then $B_Z \mathbb{C}^{n+1}$ is smooth and $Y = B_Z X \subset B_Z \mathbb{C}^{n+1}$ is codimension 1 and hence Y has hypersurface singularities only. Let F be the f-exceptional divisor. Then by adjunction, $K_{B_Z \mathbb{C}^{n+1}} = f^* K_{\mathbb{C}^{n+1}} + F$, and $f^* X = Y + F$. Therefore

$$K_Y = (K_{B_Z \mathbb{C}^{n+1}} + Y)|_Y = f^* K_X.$$

Since $\mathbb{C}^{n+1}$ is smooth and X canonical, the proof of [Ko-Mo98, Theorem 5.34] show that the pair $(\mathbb{C}^{n+1}, X)$ is also canonical. By adjunction we see that

$$K_{B_Z \mathbb{C}^{n+1}} + Y = f^*(K_{\mathbb{C}^{n+1}} + X).$$

Therefore the pair $(\mathbb{C}^{n+1}, Y)$ is also canonical. Hence by [Ko-Mo98, Proposition 5.51], Y is normal. Finally, since every irreducible component of $X^{sing} \cap Z$ is smooth and $f^{-1}(z) = \mathbb{P}^1$, for all $z \in Z$, it follows that every irreducible component of $F \cap Y$ is also smooth. $\square$

Lemma 1.12. *Let $(0 \in X)$ be a canonical 3-fold singularity such that the general hyperplane section H of X through 0 is an elliptic surface singularity with invariant $k = 1$, 2 or 3. Let $0 \in Z \subset X$ be a smooth divisor. Let $Y \xrightarrow{f} X$ be either the blow up of X at 0 if $k = 3$, or the $(3,2,1,1)$ or $(2,1,1,1)$ weighted blow up of X at 0 in the case that*

$k=2$ or 1. Then the birational transform Z' of Z in Y is smooth and $Y^{sing}\cap Z$ is a union of rational curves.

Proof. Suppose that $k=3$. Then Y is just the blow up of X at 0 and therefore $Z'=B_0Z$ is smooth.

Now suppose that $k=2$ or 3. Then $m_0X=2$. Let Z be given by $x=y=0$. Then X is given by an equation of the form

$$xf(x,y,z,t)+yg(x,y,z,t)=0.$$

Since $t=0$ is elliptic, we can write the equation of X as

$$x^2+xf(x,y,z,t)+yg(x,y,z,t)=0.$$

By using the Weierstrass preparation theorem, the equation of X can be written as

$$x^2+2x\phi(y,z,t)+yh(y,z,t)=0. \tag{1}$$

Moreover, $\phi(y,z,t),\ h(y,z,t)\in(y,z,t)^2$, because otherwise $0\in X$ is cA_n. Eliminate x from (1). X becomes

$$F(x,y,z,t)=x^2+yh(y,z,t)-\phi^2(y,z,t)=0, \tag{2}$$

ans Z is given by $x-\phi(y,z,t)=y=0$. Moreover from the previous discussion, $\phi(y,z,t),\ h(y,z,t)\in(y,z,t)^2$. Now consider cases with respect to h.

Case 1: $mult_0h(y,z,t)\geq 3$.

In this case assign weights to x, y, z and t as follows. Let $w(x)=2$ and $w(y)=w(z)=w(t)=1$. Let $Y=B_0^wX\xrightarrow{f}X$ be the $(2,1,1,1)$ weighted blow up of X. We want to understand the birational transform Z' of Z in Y.

Claim: $Z'=B_0Z$ and therefore Z is smooth.

By the definition of the weighted blow up,

$$Z'=\text{Proj}\oplus_{n\geq 0}\frac{m^w(n)+I_Z}{I_Z},$$

and $I_Z=(x-\phi(y,z,t),y)$. We may look at Z as given by $x-\phi_1(z,t)=0$ in $\mathbb{C}^3_{x,z,t}$, where $\phi_1(z,t)=\phi(0,z,t)$. Note that $\phi_1(z,t)\neq 0$, since otherwise $\phi(y,z,t)=y\lambda(y,z,t)$ and hence $2Z$ given by $x^2=y=0$ will be Cartier. Then $mult_0\phi_1(z,t)\geq 2$. Now

$$\frac{m^w(1)+I_Z}{I_Z}=(\overline{x},\overline{z},\overline{t})=(\overline{z},\overline{t},\phi_1(\overline{z},\overline{t}))=(\overline{z},\overline{t})=\frac{m+I_Z}{I_Z}$$

In general,

$$\frac{m^w(n)+I_Z}{I_Z} = (\overline{x}^i\overline{t}^j\overline{z}^k, 2i+j+k \geq n) =$$

$$(\phi_1(\overline{z},\overline{t})^i\overline{t}^j\overline{z}^k, 2i+j+k \geq n) = (\overline{t}^j\overline{z}^k, j+k \geq n) = \frac{m^n+I_Z}{I_Z}$$

Therefore, $Z' = B_0Z$, and hence Z' is smooth. Note that this is the case that $k=2$.

Case 2: $mult_0 h(y,z,t) = 2$, and $k=1$.

Let $h_2(y,z,t)$ be the degree 2 part of $h(y,z,t)$. Since the section $t=0$ is elliptic, the cubic term of $F(x,y,z,0)$ must be a cube because otherwise the section $t=0$ is DuVal. Therefore,

$$h_2(y,z,t) = ay^2 + tl(y,z,t),$$

where $l(y,z,t)$ is linear in y, z, t. Therefore the equation of X becomes

$$x^2 + ay^3 + ytl(y,z,t) + yh_{\geq 4}(y,z,t) - \phi^2(y,z,t) = 0. \tag{3}$$

If $l(y,z,t) \neq 0$, then one of y^2z, yzt, yt^2, appears in the above equation and the section $z=t$ is given by $x^2+\psi(y,t)=0$, and $\psi_3(y,t)$ is not a cube and hence it must be DuVal. Therefore $l(y,z,t)=0$. Therefore the equation of X becomes

$$x^2 + y^3 + yh_{\geq 4}(y,z,t) - \phi^2(y,z,t) = 0, \tag{4}$$

where $m_0\phi \geq 2$. Moreover, Z is given by $x-\phi(y,z,t)=y=0$, and as before, $\phi(0,z,t)\neq 0$. By using the Weierstrass preparation theorem it is possible to write

$$x^2 + y^3 + yh_{\geq 4}(y,z,t) - \phi^2(y,z,t) =$$

$$x^2 + u_1[y^3 + \alpha(z,t)y^2 + \beta(z,t)y + \gamma(z,t)] =$$

$$u_1[(\frac{x}{\sqrt{u_1}})^2 + y^3 + \alpha(z,t)y^2 + \beta(z,t)y + \gamma(z,t)]$$

From the above equations, it is clear that $\gamma(z,t) = unit\cdot\phi^2_{\geq 2}(0,z,t) = -\psi^2_{\geq 2}(z,t)$. Therefore Z is given by $x-\psi_{\geq 2}(z,t) = y = 0$. Now eliminate y^2. The equation of X can be written

$$x^2 + y^3 + 3\alpha(z,t)y^2 + \beta(z,t)y - \psi^2_{\geq 2}(z,t) = 0.$$

The change of variables $y \mapsto y - \alpha(z,t)$ makes the equation of X

$$x^2 + y^3 + (y-\alpha(z,t))\delta(z,t) - \alpha^3(z,t) - \psi^2(z,t) = 0, \tag{5}$$

where $\delta(z,t) = \beta(z,t) - 3\alpha^2(z,t)$. Z is given by $x - \psi_{\geq 2}(z,t) = y - \alpha(z,t) = 0$. Now it is easy to see that $m_0(\alpha^3(z,t)+\psi^2(z,t)) \geq 6$. Therefore, $m_0(\psi)\geq 3$ and $m_0(\alpha)\geq 2$. Now an argument as in case 1 shows that $Z' = B_0Z$ and therefore it is smooth. $\square$

Lemma 1.13. *Let $X \longrightarrow T$ be a 1-dimensional family such that X_0 is canonical. Assume that* $\dim X_0 \geq 3$ *and that X has finite index. Then the pair (X, X_0) is also canonical.*

Proof. By lemma 1.6 it follows that X is canonical. Since X has finite index it must have index 1. X_0 is canonical and therefore as in the proof of lemma 1.10, (X, X_0) is also canonical since X_0 is Cartier in X. □

Lemma 1.14. *Let T be a smooth curve and $Y/T \xrightarrow{f} X/T$ be a morphism over T. Assume that X_0 is a canonical 3-fold. Suppose that $K_Y = f^* K_X$. Then* $\dim f^{-1}(x) \leq 2$ *for any $x \in X_0$.*

Proof. As before we see that the pair (X, X_0) is canonical. Therefore, for any $x_0 \in X_0$, $x_0 \in X$ is a terminal singularity. If there is a divisor $E \subset f^{-1}(x_0)$, then E is crepant over x_0 which is impossible. Therefore $\dim f^{-1}(x_0) \leq 2$. □

Proof of Theorem 1.3.

Suppose that X_0 is given by $F(x, y, z, w) = 0$ in $\mathbb{C}^4$. Then X is given by $F(x, y, z, w) + t\Psi(x, y, z, w) = 0$ in $\mathbb{C}^5$. Let $X_1 \xrightarrow{f_1} X$ be the $(1, 1, 1, 1, 1)$, or $(2, 1, 1, 1, 1)$, or $(3, 2, 1, 1, 1)$ weighted blow up of X along L. Consider the $(1, 1, 1, 1, 1)$ blow up case first. Let $\tilde{\mathbb{C}} = B_0\mathbb{C}^5$. Then

$$K_{\tilde{\mathbb{C}}} = f_1^* K_{\mathbb{C}^5} + 3E$$
$$f_1^* X = Y + 3E.$$

Therefore

$$K_{X_1} = f_1^* K_X.$$

By lemma 1.13, it follows that $\dim f_1^{-1}(x) =\leq 2$ for any $x \in X_0$. Therefore $X_1 \longrightarrow T$ is also a family of canonical 3-folds. Moreover, $e(X_1) < e(X)$. By lemma 1.11, $Z_1 = (f_1)_*^{-1} Z$ is smooth and $X_1^{sing} \cap (Y_1)_0$ is a union of smooth rational curves. Let $X_2 \xrightarrow{f_2} X_1$ be the blow up of X_1 along Z_1. Then by lemma 1.10, $K_{X_2} = f_2^* K_{X_1}$, X_2 is normal and $e(X_2) < e(X_1)$. Moreover, the conditions of part 1 of the theorem guarantee that there is a smooth f_2-exceptional divisor (over the generic point of L_0 it is just the blow up of a line through a DuVal singularity). We continue this process and get a sequence

$$\cdots \longrightarrow X_k \xrightarrow{f_k} X_{k-1} \longrightarrow \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X,$$

such that

$$e(X_1) > e(X_2) > \cdots e(X_k) > \cdots$$

Since there are finitely many crepant divisors, there is a $n > 0$ such that $e(X_n) = 0$ and therefore X_n is terminal and $g_n\colon X_n \longrightarrow X$ is crepant. Moreover $X_n \longrightarrow T$ is a family of terminal 3-folds. Let E_i be the g_n-exceptional divisors. By lemma 1.10 they are all smooth. Since $(X_n)_0$ is terminal, it has only isolated hypersurface singularities. Therefore we can obtain a $\mathbb{Q}$-factorialization of it, W, by just blowing up Z_n and the exceptional E_i. Now consider the map $W/T \longrightarrow X/T$. A $\mathbb{Q}$-factorialization of Z_0 can be obtained by running a $(W_0, \epsilon \sum E_i^W)$ MMP over X_0. Since W_0 is terminal and by propositions 1.8 and 1.9, the operations of the MMP extend in the family. Therefore we obtain a morphism $Y/T \xrightarrow{f} X/T$, such that $Y_0 \xrightarrow{f_0} X_0$ is a $\mathbb{Q}$-factorialization of Z_0. Now by proposition 1.8, $Y_t \longrightarrow X_t$ is a $\mathbb{Q}$-factorialization of Z_t in a neighborhood of $0 \in T$. □

Proof of Corollary 1.4. Let $f\colon Y \longrightarrow X$ be the blow up of X along Γ. Then as in [Tzi02], there are two f-exceptional divisors E_1 and E_2, which are smooth. E_1 is over Γ. Then from [Tzi02] it follows that Y_0 is cDV. The conditions of theorem 1.3 are satisfied and therefore there is a morphism $g\colon Z \longrightarrow Y$, such that $Z_0 \longrightarrow Y_0$ is a $\mathbb{Q}$-factorialization of $(E_1)_0$. Now the construction in [Tzi02] can be done in families and hence there is a morphism $W \longrightarrow X$ which is a family of divisorial contractions. □

2. cD_5 TYPE DIVISORIAL CONTRACTIONS.

In this section we will get a complete classification of terminal 3-fold divisorial contractions $E \subset W \longrightarrow \Gamma \subset X$, in the case that Γ is smooth and the general section of X through Γ is of type D_5. At this moment I would like to point out that I believe that the same method that will treat the D_5 case, can also give the general D_{2n+1} case. However, my feeling is that the amount of calculations needed is disproportionate to the value of the result. I should also mention that the D_{2n} case was completely done in [Tzi02], but as it was shown there it is very different than the D_{2n+1} case.

We begin with a definition.

Definition 2.1. *A divisorial contraction is a morphism $f : E \subset W \longrightarrow \Gamma \subset X$, such that X and W are $\mathbb{Q}$-factorial, $W - E \cong X - \Gamma$, E is a prime divisor, $-K_W$ is f-ample, and $\rho(W/X) = 1$.*

The contraction is called terminal if X and W have terminal singularities. Also note that when working in the analytic category, the $\mathbb{Q}$-factoriality of X and the requirement that $\rho(W/X) = 1$ is lost.

We also need the following definition.

Definition 2.2 ([Tzi02]). *Let $0 \in S$ be a DuVal singularity of type D_n, and let $0 \in \Gamma \subset S$ be a smooth curve through the singularity. Let $U \xrightarrow{f} S$ be the minimal resolution and $\Gamma' = f_*^{-1}\Gamma$. Let*

$$\begin{array}{ccccccc} & & & & & & \overset{E_{n-1}}{\circ} \\ & & & & & & | \\ \overset{E_1}{\circ} & — & \overset{E_2}{\circ} & — & \cdots & — & \overset{E_{n-2}}{\circ} \\ & & & & & & | \\ & & & & & & \overset{E_n}{\circ} \end{array}.$$

be the dual graph. Then Γ' intersects either E_1, or E_{n-1} or E_n. In the first case we call Γ of type DF_l, and in the others of type DF_r.

In the D_5 case we have the following.

Theorem 2.3. *Let $0 \in \Gamma \subset S \subset X$. Suppose that $P \in S$ is a D_5 singularity for the general S through Γ. Then*

(1) *If $\Gamma \subset S$ is of type DF_l, then there is no terminal contraction.*
(2) *If $\Gamma \subset S$ is of type DF_r, then it is always possible to write the equation of $\Gamma \subset S \subset X$ as*

$$x^2 + y^2z + 2xz^2 + t[xz\psi(z,t) + axt + bxt^k + \phi_{\geq 2}(y,z,t)] = 0,$$

so that y^2, yz and z^ν do not appear in $\phi_{\geq 2}(y,z,t)$ for any ν. Moreover, $k \geq 1$, $ab = 0$ and $I_\Gamma = (x,y,t)$. Let $a_{i,j,k}$ denote the coefficient of $y^iz^jt^k$ in $\phi_{\geq 2}(y,z,t)$. Then a terminal contraction does not exist iff

$$\phi_2(y,z,t) = 0$$

and
(a)
$$a_{0,0,4} = a_{1,0,2} = 2a_{0,1,2} - a\psi(0,0) = 4a_{0,0,3} - a^2 = 0$$
or
(b)
$$a_{0,2,1}^2 - 2aa_{0,2,1} + 4a_{0,0,3} = 0$$
$$2a_{0,1,2} - a_{0,2,1}\psi(0,0) = 0.$$

If there is a terminal contraction $W \longrightarrow X$, then W has index 4.

Example 2.4. *Let $0 \in \Gamma \subset S \subset X$ be given by*

$$X : x^2 + y^2z + 2xz^2 + t\phi_{\geq 4}(y,z,t) = 0,$$

$I_\Gamma = (x,y,t)$. Then there is no terminal divisorial contraction of an irreducible suface to Γ.

Corollary 2.5. *Let $\Gamma \subset X$ as in the theorem. Let $E \subset W \xrightarrow{f} \Gamma \subset X$ the canonical divisorial contraction, which by* [Tzi02] *always exists. Let Σ be the general section of X through 0. Then W is terminal iff $\Sigma' = f_*^{-1}\Sigma$ is normal.*

Conjecture 2.6. *Let $0 \in \Gamma \subset X$. Assume that the general section S of X through Γ is DuVal. Let Σ be the general section of X through 0. Let $E \subset W \xrightarrow{f} \Gamma \subset X$ the canonical divisorial contraction. Then W is terminal iff $\Sigma' = f_*^{-1}\Sigma$ is normal.*

This is true for all the D_{2n} cases as well as the D_5.

Corollary 2.7. *Let $0 \in \Gamma \subset X$ as above. Let Σ be the general hyperplane section of X through 0. Write X as deformation of Σ, $X \longrightarrow \Delta$, using Γ as the parameter. There is a morphism $\Delta \xrightarrow{\phi} Def(\Sigma)$ inducing the above deformation. Then there is a subset Z of $Def(\Sigma)$, such that, a terminal divisorial contraction exists iff $Im(\phi) \subset Z$.*

Proof of Theorem 2.3. The method that we are going to use is based on the method that appears in [Tzi02] which I describe next. There is always a canonical contraction whose construction is described from the diagram

$$\begin{tikzcd}
Z \arrow[rr, dashed, "\phi"] \arrow[d, "g"'] & & Z' \arrow[d, "g'"] \\
Y \arrow[dr, "f"'] & & W \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

We start by taking the blow up Y of X along Γ. There are two f-exceptional divisors, E_1 and E_2. E_1 is a ruled surface over Γ and $E_2 \cong \mathbb{P}^2$ is over the singular point $0 \in X$. Then take a $\mathbb{Q}$-factorialisation Z of E_1. After running a suitable MMP over X, E_2 contracts giving the required contraction. The main difficulty is to calculate Z and especially it's singularities away from E_2^Z. In the D_{2n} case, Z was relatively straightforward to obtain. To treat the D_5 case we will do the following to understand Z. We will first put some restrictions on the equation of X. Then Z will be obtained from the following diagram:

$$\begin{tikzcd} & Y_2 \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ Y_1 \arrow[dr, "f_1"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

Y_1 is the blow up of Y along E_1. Let F be the f_1-exceptional divisor. Then we will get an explicit $\mathbb{Q}$-factorialisation Y_2 of $E_1^{Y_1}$. We will then show that no f_2-exceptional curve is contained in F^{Y_2} and hence we can contract F^{Y_2} to obtain Z. Then in order to decide whether W is terminal, all we must do is to study the singularities of Y_2 away from $F^{Y_2} \cup E_2^{Y_2}$. Finally in order to remove the restrictions that we imposed on the equation of X, we will use the deformation theory that was developed earlier to deform any contraction to one that can be treated as we just explained. Then we will use the fact that a deformation of a terminal singularity is again terminal [Nak98].

Part 1. of the theorem follows from [Tzi02, Theorem 6.1].

To see part 2. By [Tzi02, Proposition 4.8], $\Gamma \subset S \subset X$ is given by

$$x^2 + y^2 z + 2xz^2 + t[xz\psi(z,t) + axt + bxt^k + \phi_{\geq 2}(y,z,t)] = 0,$$

and $I_\Gamma = (x, y, t)$.

The technical condition that we need to impose at the time is that yt and t^2 do not appear in $\phi_{\geq 2}(y,z,t)$.

If $a = 0$ then we already have the claimed form. If $a \neq 0$ then write $axt + bxt^k = axt \cdot (unit)$. The change of variables $t \mapsto t/\sqrt[2]{(unit)}$ will give the required normal form.

Start by describing $Y = B_\Gamma X$. In the chart $x = xt$, $y = yt$ it is given by

$$x^2 t + y^2 tz + 2xz^2 + xtz\psi(z,t) + axt^2 + bxt^{k+1} + \phi_{\geq 2}(yt, z, t) = 0.$$

$t = 0$ gives that $E_1 : x = t = 0$, $E_2 : z = t = 0$ and $L = E_1 \cap E_2 : x = z = t = 0$. Now it is easy to see that Y is singular along L. This is what makes this case so different from the D_{2n} cases where Y had only one singularity along L, and much more difficult to work.

As we said before, we will try to show that none of the f_2-exceptional curves is contained in $F_1^{Y_2}$ and also study the singularities of Y_2 away from $E_2^{Y_2} \cup F_1^{Y_2}$. Moreover, it is clear that Z can only have isolated cDV point over a cDV point and therefore we restrict our attention to what happens over non cDV points. One can also check the other charts and see that all the non cDV point are contained in the first one given by $x = xt$, $y = yt$. So it suffices to do all our calculations in that chart.

Calculate $f_1 : Y_1 \longrightarrow Y$. In the chart $x = xt$, Y_1 is given by

$$x^2t^2 + y^2z + 2xz^2 + xzt\psi(z,t) + axt^2 + bxt^{k+1} + \frac{1}{t}\phi_{\geq 2}(yt, z, t) = 0. \tag{6}$$

Let F_1 be the f_1-exceptional divisor. $t = 0$ gives that

$$F_1 : z = t = 0 \tag{7}$$

$$E_1^{Y_1} : t = y^2 + 2xz + \frac{1}{z}[\frac{1}{t}\phi_{\geq 2}(yt, z, t)]_{t=0} = 0. \tag{8}$$

Observe that $E_2^{Y_2}$ does not appear in this chart. Now let

$$\phi_2(y, z, t) = a_3zt.$$

Then $\frac{1}{t}\phi_2(yt, z, t) = a_3z$. Hence $[\frac{1}{t}\phi_{\geq 2}(yt, z, t)]_{t=0} = a_3z$.

The singular locus of Y_1 is contained in $F_1 \cap E_1^{Y_1}$. This is the two lines, l_d, given by

$$z = t = 0$$
$$y = d$$

with d such that $d^2 + a_3 = 0$. We will now study the singularities of Y_1 along these lines. The change of variables $y \mapsto y + d$ brings the equations of l_d to $y = z = t = 0$, and

$$Y_1 : x^2t^2 + (y+d)^2z + 2xz^2 + xzt\psi(z,t) + axt^2 + bxt^{k+1} + \frac{1}{t}\phi_{\geq 2}((y+d)t, z, t) = 0.$$

Look what happens along l_d by making the change of variables $x \mapsto x - \delta$. Y_1 is given by

$$(x-\delta)^2t^2 + (y+d)^2z + 2(x-\delta)z^2 + (x-\delta)zt\psi(z,t) + a(x-\delta)t^2 + b(x-\delta)t^{k+1} + a_3z + \frac{1}{t}\phi_{\geq 3}((y+d)t, z, t) = 0,$$

and this gives

$$(x-\delta)^2t^2 + y^2z + 2dyz + 2(x-\delta)z^2 + (x-\delta)zt\psi(z,t) + a(x-\delta)t^2 + b(x-\delta)t^{k+1} + \frac{1}{t}\phi_{\geq 3}((y+d)t, z, t) = 0. \tag{9}$$

It is now clear that Y_1 is singular along these two lines, and smooth away from them. Therefore we want to calculate Y_2 in a neighborhood of these lines.

Now it is possible to write

$$\frac{1}{t}\phi_{\geq 3}(yt,z,t) = z^2 f(y,z) + t\phi(y,z,t)$$

and

$$\phi(y,z,t) = z\phi_1(y,z) + t\phi_2(y,z,t).$$

Then (2) becomes

$$\begin{aligned}(10)\quad x^2t^2 + y^2z + 2xz^2 + a_3z + xzt\psi(z,t) + axt^2 + bxt^{k+1} \\ + z^2f(y,z) + zt\phi_1(y,z) + t^2\phi_2(y,z,t) = 0.\end{aligned}$$

Moreover,

$$\begin{gathered} F_1 : z = t = 0 \\ E_1^{Y_1} : t = y^2 + 2xz + a_3 + zf(y,z) = 0 \\ \cup l_d : y^2 + a_3 = z = t = 0 \end{gathered}$$

Now we want to describe Y_2. It can be obtained by blowing up a suitable multiple of $E_1^{Y_1}$. To find it look what happens along l_d. The change of variables $y \mapsto y + d$ and $x \mapsto x - \delta$ shows that $2E_1^{Y_1}$ given by

$$I_{2E_1^{Y_1}} = (t^2, y^2 + 2xz + a_3 + xt\psi(z,t) + zf(y,z) + t\phi_1(y,z))$$

is Cartier at all but finitely many points of l_d. Therefore we can get Y_2 by blowing up $2E_1^{Y_1}$. At this point I want to mention that the reason that i have the condition that yt does not exist in the equation of X is that if it appears then I don't see that $2E_1^{Y_1}$ is Cartier at all but finitely many points and therefore i don't know what needs to be blown up to get Y_2.

Therefore $Y_2 \subset \mathbb{C}^4_{x,y,z,t} \times \mathbb{P}^1_{u,w}$ is given by the equations

$$wt^2 - u[y^2 + 2xz + zf(y,z) + xt\psi(z,t) + t\phi_1(y,z)] = 0 \tag{11}$$

$$\begin{aligned} x^2t^2 + y^2z + 2xz^2 + a_3z + xzt\psi(z,t) + axt^2 + bxt^{k+1} \\ +z^2f(y,z) + zt\phi_1(y,z) + t^2\phi_2(y,z,t) = 0. \end{aligned} \tag{12}$$

In the affine chart $u = 1$, Y_2 is given by

$$y^2 + 2xz + zf(y,z) + xt\psi(z,t) + t\phi_1(y,z) - wt^2 = 0 \tag{13}$$

$$x^2 + wz + ax + bxt^{k-1} + \phi_2(y,z,t) = 0. \tag{14}$$

Now we study what happens over l_d. It is easy to see that $C = f_2^{-1}(l_d)$ is given by

$$y^2 + a_3 = z = t = 0 \qquad (15)$$
$$x^2 + ax \mid_{t=0} + \phi_2(y,0,0) = 0.$$

It is easy to check that no component of C is contained in $F_1^{Y_2}$. Next we want to see what kind of singularities Y_2 has alog C. To do this we must calculate the jacobian, J, of Y_2 along C. An easy computation gives that

$$J = \begin{pmatrix} 0 & 2d & 2x_0 + f(0,0) & x_0\psi(0,0) + \phi_1(0,0) & 0 \\ 2x_0 + a & \frac{\partial \phi_2}{\partial y}(d,0,0) & w + \frac{\partial \phi_2}{\partial z}(d,0,0) & \frac{\partial \phi_2}{\partial t}(d,0,0) & 0 \end{pmatrix}$$

where $d^2 + a_3 = 0$ and $x_0^2 + ax_0 + \phi_2(d,0,0) = 0$. If $d \neq 0$ then from (9) it follows that Y_1 is cA_n along l_d and therefore Y_2 can have at worst isolated cDV points on C. Hence the case of interest is when $d = 0$ which implies that $a_3 = 0$ as well.

It is easy to see that $x_0\psi(0,0) + \phi_1(0,0)$ is the coefficient of zt in (9) at any point on l given by $x \mapsto x + x_0$. Moreover, $x_0^2 + ax_0 + \phi_2(0,0,0)$ is the coefficient of t^2. Hence if it is nonzero then Y_1 is cA_n under C, and therefore Y_2 has isolated cDV points along C. So we want to investigate what happens if $x_0\psi(0,0) + \phi_1(0,0) = 0$. Now by looking at the Jacobian matrix J it is clear that Y_2 is either singular along C or smooth. In the first case the resulting contraction will be only canonical and in the second we will have to see what happens in the other chart. Y_2 is singular along C iff

$$(2x_0 + a)(2x_0 + f(0,0)) = 0 \qquad (16)$$
$$[(b(k-1)x_0t^{k-2}) \mid_{t=0} + \frac{\partial \phi_2}{\partial t}(0,0,0)](2x_0 + f(0,0)) = 0$$
$$\frac{\partial \phi_2}{\partial y}(0,0,0)(2x_0 + f(0,0)) = 0$$
$$x_0\psi(0,0) + \phi_1(0,0) = 0$$
$$x_0^2 + ax_0 + \phi_2(0,0,0) = 0$$

A similar calculation in the other affine piece of Y_2 given by $w = 1$ leads to the same conclusion about the singularities of Y_2 and we again obtain the equations (16).

Now we will show that there are no other f_2-exceptional curves that appear over the other affine piece of Y_1 given by $t = tx$, and therefore the conditions given by (16) are necessary and sufficient.

Y_1 is given by

$$x^2t + y^2tz + 2z^2 + xzt\psi(z, xt) + ax^2t^2 + bx^{k+1}t^{k+1} + \frac{1}{x}\phi_{\geq 2}(xyt, z, xt) = 0. \tag{17}$$

Moreover, $x = 0$ gives that $F_1 : z = x = 0$, $E_2^{Y_1} : z = t = 0$ and

$$E_1^{Y_1} = (x, y^2t + 2z + \frac{1}{z}[\frac{1}{x}\phi_{\geq 2}(xyt, z, xt)]_{x=0}).$$

The singularities of Y_1 are along $L_1 \cup l_d = E_1^{Y_1} \cap F_1$. $L_1 : x = z = t = 0$ is over L and $l_d : x = y - d = t = 0$, with $d^2 + ad + \phi_2(0, 0, 0) = 0$ as before, is over a point and we already studied the part away from zero in the other chart. So we only need to see what happens along L_1.

Claim: $E_1^{Y_1}$ is $\mathbb{Q}$-Cartier along L_1. Therefore there are no f_2-exceptional curves over L_1 and hence the conditions for nonexistence of a terminal contraction are precisely those given by (16).

We know that $2E_2^{Y_1}$ is Cartier. Consider the index 1 cover of $E_2^{Y_1}$

$$\pi : \tilde{Y_1} \longrightarrow Y_1.$$

We will show that $F_1^{\tilde{Y_1}}$ is $\mathbb{Q}$-Cartier over L_1 and therefore F_1, and hence $E_1^{Y_1}$ too, is $\mathbb{Q}$-Cartier along L_1.

So now we calculate the index 1 cover of $E_2^{Y_1}$. By definition [Ko-Mo98],

$$\tilde{Y_1} = \mathrm{Spec}(\mathcal{O}_{Y_2} \oplus I_{E_2^{Y_1}, Y_1})$$

It is easy to see now that $\tilde{Y_1}$ is given by

$$x^2 + y^2uw + 2u^2 + xuw\psi(uw, xw^2) + ax^2w^2 + bx^{k+1}w^{2k} + \frac{1}{xw^2}\phi_{\geq 2}(xyw^2, uw, xw^2) = 0 \tag{18}$$

and π_1 is given by $t = w^2$ and $uw = z$. Moreover,

$$F_1^{\tilde{Y_1}} : u = x = 0$$

and therefore, since t^2 does not appear in $\phi_{\geq 2}(y, z, t)$, it is clear that it is $\mathbb{Q}$-Cartier along $\pi_1^{-1}(L_1) : x = u = w = 0$. The claim now follows easily.

Therefore, a terminal contraction does not exist iff (16) are satisfied. Let $a_{i,j,k}$ be the coefficient of $y^iz^jt^k$ in $\phi_{\geq 2}(y, z, t)$. Then $f(0, 0) = a_{0,2,1}$, $\frac{\partial \phi_2}{\partial t}(0, 0, 0) = a_{0,0,4}$, $\frac{\partial \phi_2}{\partial y}(0, 0, 0) = a_{1,0,2}$, $\phi_2(0, 0, 0) = a_{0,0,1}$ and $\phi_1(0, 0) = a_{0,1,2}$. Moreover, $ab = 0$. Substitute these to (16) to get the conditions claimed by the statement of the theorem.

To find the index of the singularities of W in the case that it is terminal, we will do the same as in the proof of [Tzi02, Theorem 5.1] and example 1.6.

Let S be the general hyperplane section of X contaning Γ and S' its birational transform in Y. Then by [Tzi02] it follows that S' has exacty one singular point that is A_4. So Y has an A_4 point at the generic point of L and hence $5E_1$ and $(5E_2$ are Cartier at the generic point of L. Again as in the proof of [Tzi02, Theorem 5.1] we conclude that

$$\text{index}(E_1^Z) = \text{index}(E_2^Z) = 5.$$

Note that the above arguments show that $Z = Z'$ and $E_2^Z \cong \mathbb{P}^2$. Let $a > 0$ such that

$$K_Z = g'^* K_W + aE_2^Z. \tag{19}$$

Let $l \subset E_2^Z \cong \mathbb{P}^2$ be a line. Then $l \cdot E_2^Z = -b/5$, with b a positive integer. We want to find b.

$$l \cdot (E_1^Z + 2E_2^Z) = l \cdot g^*(E_1 + 2E_2) = -1. \tag{20}$$

Moreover, $l \cdot E_1^Z = 1/5(5E_1^Z) \cdot l$. This is easily computable. I claim that the scheme theoretic intersection of $5E_1^Z$ and E_2^Z is just $3L$. Then it is clear that for general l, $l \cdot E_1^Z = 3/5$. To see the claim it suffices to work at the generic point of L. Then as we explained earlier, Z is an A_4 surface singularity there and E_1^Z, E_2^Z are two lines through the singular point. Moreover, by [Tzi02, Theorem 4.1, Proposition 4.6], it is possible to write at the generic point of L, Z as $xy - z^5 = 0$, E_1^Z is given by $x - z = y - z^4 = 0$, and E_2^Z by $x - z^2 = y - z^3 = 0$. Now it is not difficult to see that $length(5E_1^Z \cap E_2^Z) = 3$ (at the generic point of L), and hence $5E_1^Z \cap E_2^Z = 3L$. Therefore from (20) it follows that

$$l \cdot E_2^Z = 1/2(-1 - 3/5) = -4/5.$$

Now from (19) it follows that $a = 5/4$, and therefore W has index 4.

Now we want to remove the technical condition that we imposed on the equation of X. So, suppose that the monomials yt or t^2 appear in $\phi_{\geq 2}(y, z, t)$. Then write the equation of X as

$$x^2 + y^2z + xz^2 + yt^2 + t\psi(x, y, z, t) = 0.$$

We can now deform X to X_0 given by

$$x^2 + y^2z + xz^2 + yt^2 = 0.$$

Let $\mathcal{X} \longrightarrow \Delta$ be the deformation. Let $\diamond \subset \mathcal{X}$ be the plane $x = y = t = 0$. Let $\mathcal{Y} \xrightarrow{\{} \mathcal{X}$ be the blow up of $\mathcal{X}$ along Π. Then it is easy to see that $\mathcal{Y} \longrightarrow \Delta$ is a deformation of $Y = BX_\Gamma$ with special fiber $Y_0 = BX_{0\Gamma_0}$.

Now it is easy to check that Y_0 satisfies the conditions of theorem 1.3. Therefore there exist a common $\mathbb{Q}$-factorialisation $\mathcal{Z} \longrightarrow \mathcal{Y}$ of E_1, E_2, and hence as in the proof of corollary 1.4, the divisorial contractions form a family, i.e., there is a morphism $\mathcal{E} \subset \mathcal{W} \xrightarrow{g} \Pi \subset \mathcal{X}$, such that $\mathcal{E}_0 \subset \mathcal{W}_0 \xrightarrow{g} \Pi_0 \subset \mathcal{X}$, is the divisorial contraction $E_0 \subset W_0 \longrightarrow \Gamma_0 \subset X_0$. Therefore W deforms to W_0. Hence if W_0 is terminal, then by [Nak98] W is also terminal. We will now show that indeed W_0 is terminal. Hence in order to study when there is no terminal contraction, it is no restriction to assume that there is no yt or t^2 in $\phi_{\geq 2}(y,z,t)$ as we did. At this point I would like to mention that all the arguments so far work for the general D_{2n+1} case. However I do not know how to show that W deforms to W_0 for $n \geq 4$. I believe that a more carefull look at theorem 1.3 will treat the general case but the amount of calculations involved exceed the value of the result.

To show that W_0 is terminal. Observe that the equation of X_0 is equivalent to

$$x^2 + y^2z + xz^2 + yt^2 + yt^3 = 0.$$

Therefore it is possible to deform X_0 to X_0' given by

$$x^2 + y^2z + xz^2 + yt^3 = 0.$$

Again theorem 1.3 applies and hence W_0 deforms to W_0'. But now X_0' satisfies the conditions imposed in the beginning of the proof of theorem 2.3 and the equations (16) show that W_0' is terminal. Hence W_0 is terminal too. A similar argument works for the case that t^2 appears.

Now the general section Σ of X through 0 is given by $z = ax + by + ct$. If one looks carefully through the calculations in the proof of theorem 2.3, it can be seen that the equations (16) are exactly the conditions for Σ not to be normal. Hence we get corollary 2.4. It is unfortunate that the only proof I know of this result is a computational one. □

Proof of corollary 2.6. Let $\Sigma' \xrightarrow{f} \Sigma$ be a birational morphism such that Σ' is normal, $-K_{\Sigma'}$ is f-ample, and Σ' admits a terminal 1-parameter deformation. Then there is a morphism $Def(\Sigma') \xrightarrow{\phi} Def(\Sigma)$ [Ko-Mo92, Prop. 11.4]. Let $Z \subset Def(\Sigma)$ be the union of the images of all such morphisms. We may consider X as a deformation of Σ along Γ, say over a base Δ. Let $W \xrightarrow{g} X$ be the canonical contraction that always exists [Tzi02]. Then W can be considered as a deformation of $\Sigma' = g^*\Sigma$. If W is terminal, then we showed that Σ' is normal. Hence $Im[\Delta \longrightarrow Def(\Sigma)] \subset Z$. Conversely, if $Im[\Delta \longrightarrow Def(\Sigma)] \subset Z$, then there is a morphism $\Sigma' \longrightarrow \Sigma$ that lifts to a morphism $W \longrightarrow X$, where W is a terminal deformation of Σ', and is the required contraction. $\square$

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