THE HOMOTOPY CLASSIFICATION OF $(n-1)$-CONNECTED $(n+4)$-DIMENSIONAL POLYHEDRA WITH TORSION FREE HOMOLOGY, $n \geq 5$

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The classification of homotopy types of finite polyhedra is a classical and fundamental task of topology which in particular is an inevitable step for the classification of manifolds; see [B2]. A best possible solution is the classification by an explicit list of indecomposable homotopy types. Such final solutions were obtained for $(n-1)$-connected $(n+k)$-dimensional polyhedra for $k=2$ by Whitehead and Chang [W1], [Ch] and for $k=3$ by Baues-Hennes [BH], [B1]. In this paper we consider the case $k=4$ for torsion free polyhedra and we prove the following surprising result.

Theorem. *Let $n \geq 5$. There is a list $X(\mathcal{L})$, see (1.3), consisting of exactly 67 polyhedra such that each $(n-1)$-connected $(n+4)$-dimensional polyhedron X with finitely generated torsion free homology admits a homotopy equivalence*

$$X \simeq X_1 \vee X_2 \vee \ldots \vee X_k$$

with $X_i \in X(\mathcal{L})$ for $1 \leq i \leq k$. Here $X_1 \vee \ldots \vee X_k$ denotes the one point union of the spaces X_i.

We describe the elementary polyhedra in the list $X(\mathcal{L})$ explicitly. They turn out to be CW-complexes with at most four non trivial cells.

Let $\mathbf{F}_n^4$ be the homotopy category of $(n-1)$-connected $(n+4)$-dimensional polyhedra with finitely generated torsion free homology. In the stable range $n \geq 6$ the category $\mathbf{F}^4 = \mathbf{F}_n^4$ is an additive category which does not depend on n. Freyd [F] showed that the <u>isomorphism class group</u> $K_0(\mathbf{F}^4)$ is a free abelian group. Using the result above we compute this group:

Corollary. $K_0(\mathbf{F}^4) = \mathbb{Z}^{29}$

The generators in $K_0(\mathbf{F}^4)$ are given by five spheres, 23 congruence classes of 2-primary elementary polyhedra in the list $X(\mathcal{L})$ and one 3-primary elementary polyhedron in $X(\mathcal{L})$; see (1.4).

§ 1 The list $X(\mathcal{L})$ of elementary polyhedra

We need the following elements in stable homotopy groups of spheres, compare Toda [T]. Let $\eta_n = \eta \in \pi_{n+1}(S^n) = \mathbb{Z}/2$ be the Hopf map, and let $\eta_n^2 = \eta\eta \in \pi_{n+2}(S^n) = \mathbb{Z}/2$ be the double Hopf map, $n \geq 3$. Moreover let $\nu = \nu_n$, $\alpha = \alpha_n \in \pi_{n+3}(S^n) = \mathbb{Z}/24$ be the generator of order 8 and 3 respectively.

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(1.1) Definition. We define a list $\mathcal{L}$ of 67 elements as follows. The spherical elements S^0, S^1, S^2, S^3, S^4 belong to $\mathcal{L}$ and the Hopf elements η_0, η_1, η_2 and $(\eta\eta)_0$ and $(\eta\eta)_1$ belong to $\mathcal{L}$. Moreover the following words consisting of letters η and v belong to $\mathcal{L}$ where v is a number:

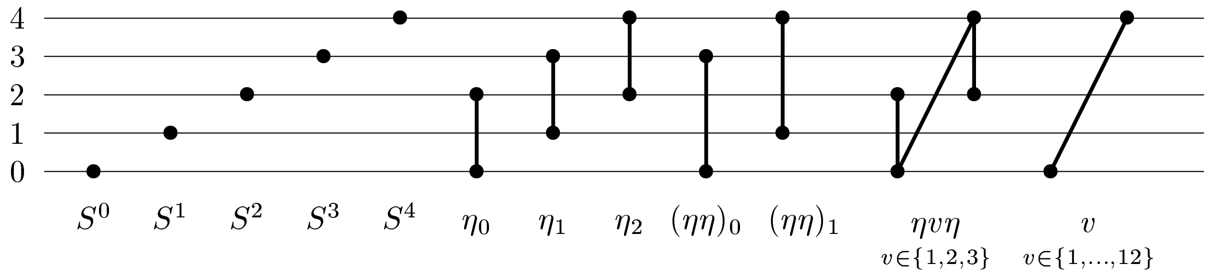
$$\left\{\begin{matrix} \eta\eta v\eta\eta, \eta\eta v\eta, \eta v\eta\eta \\ \eta\eta v, v\eta\eta, \eta v, v\eta \end{matrix}\right\} \quad \text{with} \quad v \in \{1, \dots, 6\},$$

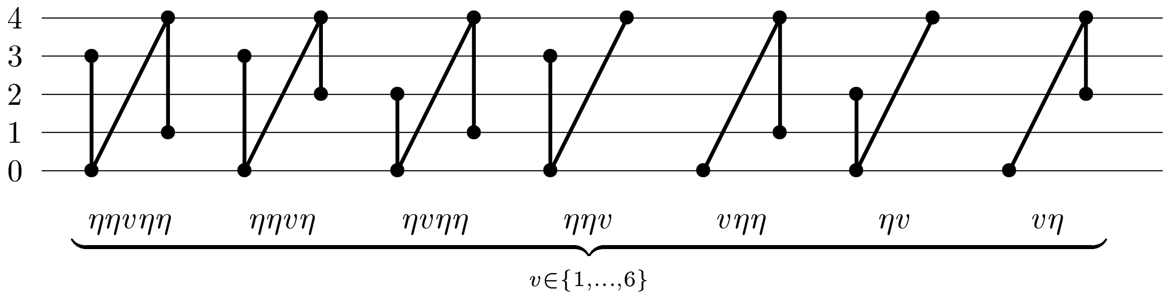
$$\eta v\eta \quad \text{with} \quad v \in \{1, 2, 3\},$$

$$v \quad \text{with} \quad v \in \{1, \dots, 12\}.$$

Here v is a one letter word consisting of the number v.

We point out that the words in $\mathcal{L}$ are subwords of $\eta\eta v\eta\eta$ containing v. We can visualize the elements of $\mathcal{L}$ by graphs as follows. First we describe spherical elements and Hopf elements by points and vertical edges. Then we describe the words in $\mathcal{L}$ by graphs consisting of such vertical edges and a diagonal edge which represents the letter v.





These graphs represent in the obvious way the words in $\mathcal{L}$. We define the duality operator D on $\mathcal{L}$, this is the function

$$D : \mathcal{L} \to \mathcal{L} \tag{1.2}$$

with $DD = 1$ defined by $D(S^i) = S^{4-i}$ for $i \in \{0, \dots, 4\}$ and $D(\eta_0) = \eta_2$, $D(\eta_1) = \eta_1$, $D(\eta\eta)_0 = (\eta\eta)_1$. Moreover for a subword w of $\eta\eta v\eta\eta$ let $D(w) = -w$ be the word obtained from w by reversing the order, that is $D(\eta\eta v\eta) = \eta v\eta\eta$. Hence if we look at the graph w then the graph $D(w)$ is obtained by turning w around.

(1.3) Definition. Let $n \geq 5$. We associate with each element $g \in \mathcal{L}$ a CW-complex $X(g)$. The vertices of the graph given by g correspond exactly to the non-trivial cells of $X(g)$. We call $X(\mathcal{L}) = \{(X(g), g \in \mathcal{L}\}$ the list of elementary polyhedra associated to $\mathcal{L}$. For the spherical elements we set

$$X(S^i) = S^{n+i} \quad \text{with} \quad i \in \{0, 1, 2, 3, 4\}$$

where S^{n+i} is the $(n+i)$-sphere. For the Hopf elements we obtain the following 2-cell CW-complexes where Σ denotes the suspension.

$$\begin{aligned}
X(\eta_0) &= S^n \cup_\eta e^{n+1} \\
X(\eta_1) &= S^{n+1} \cup_\eta e^{n+2} = \Sigma X(\eta_0) \\
X(\eta_2) &= S^{n+2} \cup_\eta e^{n+3} = \Sigma^2 X(\eta_0) \\
X(\eta\eta)_0 &= S^n \cup_{\eta\eta} e^{n+2} \\
X(\eta\eta)_1 &= S^{n+1} \cup_{\eta\eta} e^{n+3} = \Sigma X(\eta\eta).
\end{aligned}$$

Moreover for the words in $\mathcal{L}$ we get the following CW-complexes with attaching maps defined by the edges in the graphs above. For a one point union $A \vee B$ let $i_1 : A \to A \vee B$ and $i_2 : B \to A \vee B$ be the inclusions and we set $v = v \cdot (\nu + \alpha)$ for $v \in \mathbb{N}$.

$$v \in \{1,2,3\} \qquad X(\eta v \eta) = S^n \vee S^{n+2} \cup_{i_1\eta} e^{n+2} \cup_{i_1 v + i_2 \eta} e^{n+4}$$

$$v \in \{1,\dots,6\} \begin{cases}
X(\eta\eta v \eta\eta) = S^n \vee S^{n+1} \cup_{i_1\eta\eta} e^{n+3} \cup_{i_1 v + i_2\eta\eta} e^{n+4} \\
X(\eta\eta v \eta) = S^n \vee S^{n+2} \cup_{i_1\eta\eta} e^{n+3} \cup_{i_1 v + i_2\eta} e^{n+4} \\
X(\eta v \eta\eta) = S^n \vee S^{n+1} \cup_{i_1\eta} e^{n+2} \cup_{i_1 v + i_2\eta\eta} e^{n+4} \\
X(\eta\eta v) = S^n \cup_{\eta\eta} e^{n+3} \cup_v e^{n+4} \\
X(v\eta\eta) = S^n \vee S^{n+1} \cup_{i_1 v + i_2\eta\eta} e^{n+4} \\
X(\eta v) = S^n \cup_\eta e^{n+2} \cup_v e^{n+4} \\
X(v\eta) = S^n \vee S^{n+2} \cup_{i_1 v + i_2\eta} e^{n+4}
\end{cases}$$

$$v \in \{1,\dots,12\} \qquad X(v) = S^n \cup_v e^{n+4}$$

The cells of the CW-complexes $X(w)$ with $w \in \mathcal{L}$ correspond exactly to the vertices of the graph w above and the edges in the graph w show precisely how these cells are attached.

(1.4) Definition. We say that $X(w)$ with $w \in \mathcal{L}$ is 2-primary if w is a Hopf element or if w is a subword of $\eta\eta v \eta\eta$ where v is divisible by 3 so that $v(\nu+\alpha)$ is a multiple of ν. There are exactly 24 elements in $\mathcal{L}$ which are 2-primary with only one congruence given by (3.2) below, that is $v = 3$ is congruent to $v = 1$. The only 3-primary polyhedron in $X(\mathcal{L})$ is $X(v) \simeq S^n \cup_\alpha e^{n+4}$ where $v = 8$ so that $v(\nu+\alpha) = -\alpha$. This definition is compatible with (3.5) below.

(1.5) Examples. The stabilization of the complex projective plane $\mathbb{C}P_2$ yields the space

$$X(\eta_0) = \Sigma^{n-2}\mathbb{C}P_2.$$

Similarly the stabilization of the quaterionic projective plane $\mathbb{H}P_2$ gives us

$$X(v) = \Sigma^{n-4}\mathbb{H}P_2 \qquad \text{with} \qquad v = 1.$$

Moreover by (3.1) [B3] we see that the stabilization of $\mathbb{C}P_3$ yields

$$X(\eta v) = \Sigma^{n-2}\mathbb{C}P_3 \qquad \text{with} \qquad v = 1$$

in the list above.

§ 2 The decomposition problem

Let $\mathbf{C}$ be an additive category with zero object $*$ and biproducts $A \oplus B$. An object X in $\mathbf{C}$ is <u>decomposable</u> if there exists an isomorphism $X \cong A \oplus B$ where A and B are not isomorphic to $*$. A <u>decomposition</u> of X is an isomorphism

$$X = A_1 \oplus \ldots \oplus A_n, n < \infty, \tag{2.1}$$

where A_i is indecomposable for all $i \in \{1, \ldots, n\}$. The decomposition of X is <u>unique</u> if $B_1 \oplus \ldots \oplus B_m \cong X \cong A_1 \oplus \ldots A_n$ implies that $m = n$ and that there is a permutation σ with $B_{\sigma(i)} \cong A_i$. The <u>decomposition problem</u> in $\mathbf{C}$ can be described by the following task: find a complete list of indecomposable isomorphism types in $\mathbf{C}$ and describe the possible decompositions of objects in $\mathbf{C}$. This problem is considered by representation theory. We say that the decomposition problem in $\mathbf{C}$ is wild or equivalently that $\mathbf{C}$ has <u>wild representation type</u> if the solution of the decomposition problem would imply a solution of the following problem.

(2.2) Problem. Let k be a field and consider the following additive category $\mathbf{V}^{\alpha,\beta}$. Objects are finite dimensional k-vector spaces V together with two endomorphisms $\alpha_V, \beta_V : V \to V$. Morphisms are k-linear maps $f : V \to W$ satisfying $f\alpha_V = \alpha_W f$ and $f\beta_V = \beta_W f$. The decomposition problem in $\mathbf{V}^{\alpha,\beta}$ for any field k is termed a "wild problem of representation theory".

If the list of all indecomposable objects of $\mathbf{C}$ is finite then $\mathbf{C}$ has <u>finite</u> representation type. If the representation type of $\mathbf{C}$ is neither finite nor wild then $\mathbf{C}$ is of <u>tame representation type</u>. In representation theory there are in general means to compute an explicit list of all indecomposable objects in $\mathbf{C}$ if $\mathbf{C}$ has tame representation type; in the tame cases described below such explicit lists actually are computed in the literature. If the number of objects in this list which satisfy a given torsion restraint is finite then we say that $\mathbf{C}$ has <u>essentially finite</u> representation type.

For example consider the category of finitely generated (*f.g.*) abelian groups; this category has essentially finite representation type since the list of indecomposable objects is given by the indecomposable cyclic groups $\mathbb{Z}$ and $\mathbb{Z}/p^i$ where p is a prime and $i \geq 1$. Here the torsion restraint is given by a number N which bounds the order of the torsion subgroup.

Next we describe the decomposition problem of homotopy theory. Let $\mathbf{Top}^*/\simeq$ be the homotopy category of pointed topological spaces. The set of morphisms $X \to Y$ in $\mathbf{Top}^*/\simeq$ is the set of homotopy classes $[X, Y]$. Isomorphisms in $\mathbf{Top}^*/\simeq$ are called homotopy equivalences and isomorphism types in $\mathbf{Top}^*/\simeq$ are homotopy types. Let $\mathbf{A}_n^k$ be the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of $(n-1)$-connected $(n+k)$-dimensional CW-complexes which are finite. The objects of $\mathbf{A}_n^k$ are also called A_n^k-polyhedra, see [W1]. The suspension Σ gives us sequences of functors

$$\mathbf{A}_1^k \xrightarrow{\Sigma} \mathbf{A}_2^k \to \ldots \to \mathbf{A}_n^k \xrightarrow{\Sigma} \mathbf{A}_{n+1}^k \to \ldots \tag{2.3}$$

which describe the k-stem of homotopy theory. The Freudenthal suspension theorem shows that these sequences stabelize in the sense that for $k+1<n$ the functor $\Sigma : \mathbf{A}_n^k \to A_{n+1}^k$ is an equivalence of additive categories so that

$$\mathbf{A}^k = \mathbf{A}_n^k \qquad \text{with} \qquad k+1<n \tag{1}$$

does not depend on n. This is the stable homotopy category of (-1)-connected finite type k-dimensional spectra. The biproduct in the additive category $\mathbf{A}^k$ is the one point union of spaces. We point out that for $k+1=n$ the functor Σ is full and a 1-1 correspondence of homotopy types. The Spanier-Whitehead duality is a contravariant additive functor

$$D : \mathbf{A}^k \to \mathbf{A}^k \tag{2}$$

satisfying $DD=1$ and $D(S^{n+i}) = S^{n+k-i}$ for $i \in \{0,\dots,k\}$; compare for example [Co].

The k-stem of homotopy groups of spheres, denoted by $\pi_{n+k}(S^n), n \geq 2$, is now known for fairly large k; for example one can find a complete list for $k \leq 19$ in Toda's book [T]. The k-stem of homotopy types, however, is still mysterious even for small k.

For $k=0$ it is well known that each A_n^0-polyhedron X is homotopy equivalent to a one point union $X \simeq S^n \vee \ldots \vee S^n$ of n-dimensional spheres. Hence $\mathbf{A}^0$ has finite representation type. Moreover the following results are known.

- $\mathbf{A}^1$ and $\mathbf{A}^2$ have essentially finite representation type; see [W1], [Ch]
- $\mathbf{A}^3$ has tame representation type; see [BH], [B1]
- $\mathbf{A}^k$ with $k \geq 6$ has wild representation type; see [BD].

Therefore only the representation types of $\mathbf{A}^4$ and $\mathbf{A}^5$ remain unknown. In [BD] we show:

- The full subcategory of $\mathbf{A}^4 (= \mathbf{A}_n^4$ with $n \geq 6)$ consisting of spaces X with $\pi_{n+1}X = \pi_{n+2}X = 0$ has tame representation type.

Clearly the main result of this paper (compare the theorem in the introduction) shows

- The full subcategory of $\mathbf{A}^4$ consisting of spaces X with torsion free homology has finite representation type.

§ 3 Decomposition and congruence of spaces with torsion free homology

In this paper we consider the full subcategory

$$\mathbf{F}_n^k \subset \mathbf{A}_n^k \tag{3.1}$$

consisting of $(n-1)$-connected $(n+k)$-dimensional CW-complexes which are finite and have torsion free homology. For example, each nontrivial element α in the $(k-1)$-stem, $\alpha \in \pi_{n+k-1}(S^n)$, yields the canonical 2-cell complex $S^n \cup_\alpha e^{n+k} \in \mathbf{F}_n^k, n \geq 2$, which is indecomposable. Hence elements in homotopy groups of spheres can essentially be identified with special indecomposable objects in $\mathbf{F}_n^k$, $k \geq 2$. The decomposition in $\mathbf{F}_n^4$ is not unique. For example Freyd [F] points out that for $n \geq 5$ there is a homotopy equivalence

$$S^n \vee (S^n \cup_\nu e^{n+4}) \simeq S^n \vee (S^n \cup_{3\nu} e^{n+4}) \tag{3.2}$$

in $\mathbf{F}_n^4$ where, however, the CW-complexes $S^n \cup_\nu e^{n+4}$ and $S^n \cup_{3\nu} e^{n+4}$ are not homotopy equivalent. Here $\nu \in \pi_{n+3}(S^n)$ is a generator of order 8 as in § 1.

Unsöld [U1], [U2] obtained an algebraic characterization of the homotopy types in $\mathbf{F}_n^4$ with $n \geq 3$. Using this result for $n \geq 5$ we show the following main result of this paper which implies the theorem in the introduction.

(3.3) Theorem. *The list $X(\mathcal{L})$ of 67 elementary polyhedra in (1.3) is a complete list of all indecomposable spaces in $\mathbf{F}^4$. Hence for $n \geq 5$ each $(n-1)$-connected $(n+4)$-dimensional finite polyhedron X with torsion free homology admits a decomposition*

$$X \simeq X_1 \vee \ldots \vee X_j$$

with $X_i \in X(\mathcal{L})$ for $1 \leq i \leq j$. Moreover the Spanier-Whitehead duality functor $D : \mathbf{F}^4 \to \mathbf{F}^4$ is completely understood on objects since $D(X(w)) = X(Dw)$ for $w \in \mathcal{L}$. Here we use the duality operator in (1.2).

Following Freyd [F] and Cohen [Co] 4.26 we use the following notation.

(3.4) Definition. We say that two spaces X, Y in $\mathbf{A}^k$ are congruent and we write $X \equiv Y$ if (a) or equivalently (b) is satisfied.

(a) There exists a space Z in $\mathbf{A}^k$ such that $X \vee Z \simeq Y \vee Z$ are homotopy equivalent

(b) There exists a homotopy equivalence $X \vee B_X \simeq Y \vee B_X$ where B_X is the unique one point union of spheres which has the same Betti numbers as X, that is $H_*(X)/\text{torsion} = H_*(B_X)$.

(3.5) Definition. Let p be a prime. A space X in $\mathbf{A}^k$ is a p-primary space if there exists a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{p^N \cdot 1_X} & X \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

where B is a one point union of spheres. Here p^N is a power of the prime p and $p^N \cdot 1_X$ is a multiple of the identity of X in the abelian group of homotopy classes $[X, X]$ in $\mathbf{A}^k$ and N can not be chosen to be $N = 0$. This implies that X is not a one point union of spheres.

Recall that for any small additive category $\mathbf{C}$ (for example $\mathbf{C} = \mathbf{A}^k$ or $\mathbf{C} = \mathbf{F}^k$, $k \geq 0$) we have the isomorphism class group $K_0(\mathbf{C})$. This is the abelian group with one generator [A] for each isomorphism class of objects $A \in \mathbf{C}$ with relations $[A]+[B] = [A \oplus B]$. This is just the Grothendieck group of $\mathbf{C}$ as defined by Bass [Ba]. A typical element of $K_0(A)$ is a formal difference $[A]-[B]$ with $[A]-[B] = [A']-[B']$ if and only if there exists an isomorphism in $\mathbf{C}$ of the form $A \oplus B' \oplus C \cong A' \oplus B \oplus C$ for some object C in $\mathbf{C}$. The following result is due to Freyd [F]; see also Cohen [Co] 4.44.

(3.6) Theorem of Freyd. *Let $k \geq 0$. Then $K_0(\mathbf{A}^k)$, resp. $K_0(\mathbf{F}^k)$, is a free abelian group generated by the spheres in $\mathbf{A}^k$ and by the congruence classes of indecomposable p-primary spaces in $\mathbf{A}^k$ (resp. $\mathbf{F}^k$) where p runs through all primes.*

Such a wonderful result yields the crucial task to compute the generators of $K_0(\mathbf{A}^k)$, resp. $K_0(\mathbf{F}^k)$, explicitly. According to the remarks in (2.3) we get:

- $K_0(\mathbf{A}^0) = \mathbb{Z}$ generated by S^n
- $K_0(\mathbf{A}^1) = \mathbb{Z}^\infty$ generated by S^n, S^{n+1} and all elementary Moore spaces $M(C, n)$ where $C = \mathbb{Z}/p^N$ with p a prime and $N \geq 1$.
- $K_0(\mathbf{A}^2) = \mathbb{Z}^\infty$ generated by spheres, elementary Moore spaces, and elementary Chang polyhedra, see [Ch], [B1], [Hi1], [Hi2].
- $K_0(\mathbf{A}^3) = \mathbb{Z}^\infty$ generated by the elementary polyhedra obtained in Baues-Hennes [BH]; see also [B1].
- The computation of generators in $K_0(\mathbf{A}^k), k \geq 6$, is a wild problem in the sense of representation theory; this follows from [BD].

For the category $\mathbf{F}^k$ of torsion free polyhedra we get accordingly:

- $K_0(\mathbf{F}^0) = \mathbb{Z}$ generated by S^n
- $K_0(\mathbf{F}^1) = \mathbb{Z}^2$ generated by S^n, S^{n+1}
- $K_0(\mathbf{F}^2) = \mathbb{Z}^4$ generated by S^n, S^{n+1}, S^{n+2}, $X(\eta_0)$
- $K_0(\mathbf{F}^3) = \mathbb{Z}^7$ generated by S^n, S^{n+1}, S^{n+2}, S^{n+3}, and $X(\eta_0)$, $X(\eta_1)$, $X(\eta\eta)_0$.

As a next step we get:

(3.7) Theorem. $K_0(\mathbf{F}^4) = \mathbb{Z}^{29}$ *is generated by the 5 spheres $S^n, \ldots, S^{n+4}$ in $\mathbf{F}^4$, by the 23 congruence classes of 2-primary poylhedra in $X(\mathcal{L})$ and by the unique 3-primary polyhedron in $X(\mathcal{L})$; see (1.4).*

Proof. This is a consequence of (3.6), (3.3) and the description of p-primary indecomposable polyhedra in $X(\mathcal{L})$ in (1.4). The only congruence between 2-primary polyhedra in $X(\mathcal{L})$ is the one described in (3.2). This shows that there are exactly 23 congruence classes of 2-primary polyhedra in $X(\mathcal{L})$. q.e.d.

Moreover we see by the result (A.5) in the Appendix.

(3.8) Remark. Let p be a prime. The computation of the set of generators in $K_0(\mathbf{F}^k)$ given by congruence classes of p-primary polyhedra is a wild problem for $k \geq 10(p-1)$.

§ 4 The algebraic classification of spaces in $\mathbf{F}^4$

We first repeat the classification theorem of Unsöld [U1], [U2]. He defines the following algebraic category $\mathbf{SF}^4$.

(4.1) Definition. Objects in $\mathbf{SF^4}$ are tuple of abelian groups

$$\mathcal{H} = (H_0, H_1, H_2, H_3, H_4, \pi_1, \pi_2) \in \mathbf{Ab}^7$$

where H_i with $i \in \{0, \ldots, 4\}$ is finitely generated and free abelian together with the following diagrams (1), (2), (3) in $\mathbf{Ab}$:

(1) An exact sequence

$$H_3 \to \pi_1 \otimes \mathbb{Z}/2 \to \pi_2 \to H_2 \xrightarrow{b} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_1 \to H_1 \to 0$$

(2) Let $P = \ker(H_0 \xrightarrow{q} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_1)$ where q is the quotient map. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0 & \xrightarrow{2} & P & \longrightarrow & \ker(\eta^1) & \longrightarrow & 0 \\ & & {\scriptstyle q\eta^1 q}\downarrow & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \Omega} & & \\ & & \pi_1 \otimes \mathbb{Z}/2 & \longrightarrow & \pi_2 \otimes \mathbb{Z}/2 & \longrightarrow & \ker(b) \otimes \mathbb{Z}/2 & & \end{array}$$

commutes where $q\eta^1 q$ is given by (1) and where Ω is determined by the extension

$$0 \to \ker(b) \to H_2 \to \ker(\eta^1) \to 0$$

given by (1). The top row of the diagram is short exact.

(3) Moreover for the abelian group

$$\Gamma_3 = (H_0 \otimes \mathbb{Z}/24 \oplus \pi_2 \otimes \mathbb{Z}/2)/\{(\xi \otimes 6, T(\xi));\ \xi \in P \subset H_0\}$$

defined by T in (2) a homomorphism

$$b_4 : H_4 \to \Gamma_3$$

is given.

A morphism between such objects in $\mathbf{SF}^4$ is a tuple of homomorphisms $\mathcal{H} \to \mathcal{H}'$ in $\mathbf{Ab}^7$ which is compatible with all arrows in the diagrams (1), (2) and (3). Clearly $\mathbf{SF}^4$ is an additive category with the direct sum of objects given by the direct sum of abelian groups and morphisms.

In [U1], [U2] one finds the proof of the following result.

(4.2) Classification theorem. *There is an additive functor $\lambda : \mathbf{F}^4 \to \mathbf{SF}^4$ which is full and representative and which reflects isomorphisms.*

The functor carries a space X to the certain exact sequence of J.H.C. Whitehead [W2] of X together with the secondary homotopy operation T which was introduced by Unsöld.

(4.3) Corollary. *There is a 1-1 correspondence of homotopy types in $\mathbf{F}^4$ and isomorphism types in $\mathbf{SF}^4$. Moreover this is also a 1-1 correspondence between indecomposable objects.*

In order to classify indecomposable objects in $\mathbf{SF}^4$ we transform Unsölds classification theorem into a "matrix problem" which can be solved by methods of representation theory. For this we observe that each space X in $\mathbf{F}^4$ as given by an attaching map

(4.4) $$f_4 : \bigvee_h S^3 \to X^3$$

where we use stable notation. Moreover using the homology decomposition of X we may assume that f_4 factors through the 2-skeleton X^2. Then f^4 defines the homomorphism in (4.1) (3)

(1) $$b_4 : H_4 = \mathbb{Z}^h \to \Gamma_3 X^3 = \Gamma_3$$

where $\Gamma_3 X^3 = \text{image}(\pi_3 X^2 \to \pi_3 X^3)$. We know by the classification of objects in $\mathbf{A}^3$ in Baues-Hennes [BH] that X^3 is a one point union of spaces

$$(2) \qquad \begin{cases} X_1 = S^0, & X_6 = S^1 \cup_\eta e^6 \\ X_2 = S^0 \cup_\eta e^2, & X_7 = S^3 \\ X_3 = S^0 \cup_{\eta\eta} e^3 & \\ X_4 = S^1 & \\ X_5 = S^2 & \end{cases}$$

For $0 \leq d < \infty$ we write

$$(3) \qquad dZ = \underbrace{Z \vee \ldots \vee Z}_{d}$$

for the d-fold one point union of d copies of the space Z. Then we can assume that f_4 in (4.4) is given by

$$(4) \qquad f_4 : hS^3 \to X^3 = \bigvee_{i \leq 7} d_i X_i$$

where $0 \leq d_i$. Here, however, we have

$$(5) \qquad \begin{cases} \Gamma_3(S^1 \cup_\eta e^6) = \Gamma_3(X_6) = 0 \\ \Gamma_3(S^3) = \Gamma_3(X_7) = 0 \end{cases}$$

Therefore f_4 has a factorization f'_4 to $\bigvee\limits_{i \leq 5} d_i X_i$ and the mapping cone $C(f_4)$ of f_4 satisfies

$$(6) \qquad X = C(f_4) = C(f'_4) \vee d_6 X_6 \vee d_7 X_7$$

Thus it suffices to consider the decompositions of $C(f'_4)$. We therefore assume now that X^3 in (4.4) satisfies

$$X^3 = \bigvee_{i \leq 5} d_i \cdot X_i$$

and that $f_4 = f'_4$. The homotopy class of f_4 is determined by the associated homomorphism b_4 in (1). Moreover $\Gamma_3 X_i$ with $i \leq 5$ is computed by the following list where we write $\bar{A} = A \otimes \mathbb{Z}/2$; compare (4.1) (1), (2), (3).

(7)

X	H_3	$\to$	$\bar{\pi}_1$	$\to$	π_2	$\to$	H_2	$\to$	$\bar{H}_0$	$\to$	π_1	P	$\xrightarrow{T}$	$\bar{\pi}_2$	Γ_3
X_1	0		$\mathbb{Z}/2$	$=$	$\mathbb{Z}/2$		0		$\mathbb{Z}/2$	$=$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\twoheadrightarrow$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
X_2	0		0		$\mathbb{Z}$	$\xrightarrow{2}$	$\mathbb{Z}$	$\to$	$\mathbb{Z}/2$		0	$\mathbb{Z}$	$\twoheadrightarrow$	$\mathbb{Z}/2$	$\mathbb{Z}/12$
X_3	$\mathbb{Z}$	$\twoheadrightarrow$	$\mathbb{Z}/2$		0		0		$\mathbb{Z}/2$	$=$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\to$	0	$\mathbb{Z}/12$
X_4	0		$\mathbb{Z}/2$	$=$	$\mathbb{Z}/2$		0		0		$\mathbb{Z}$	0	$\to$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
X_5	0		0		$\mathbb{Z}$	$=$	$\mathbb{Z}$		0		0	0		0	$\mathbb{Z}/2$

Now let $V_i = \mathbb{Z}^{d_i}$ be the free abelian group of rank d_i with d_i given by (4). Then the computation of Γ_3 in this list shows that b_4 in (1) is given by a homomorphism

$$(8) \qquad b_4 : H_4 = \mathbb{Z}^h \to \Gamma_3 X^3 = V_1 \otimes \mathbb{Z}/24 \oplus (V_2 \oplus V_3) \otimes \mathbb{Z}/12 \oplus (V_4 \oplus V_5) \otimes \mathbb{Z}/2$$

This leads to the following matrix problem where we describe the action of automorphisms in $\mathbf{SF}^4$ on b_4.

(4.5) Matrix Problem. Let V_1, V_2, V_3, V_4, V_5 and H_4 be finitely generated free abelian groups. Moreover let M be an automorphism of $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$ given by a matrix M_{ij} of the form

$$\begin{array}{c|ccccc} & V_1 & V_2 & V_3 & V_4 & V_5 \\ \hline V_1 & \cdot & 2\backslash & 2\backslash & 12\backslash & 12\backslash \\ V_2 & \cdot & \cdot & \cdot & 12\backslash & 6\backslash \\ V_3 & \cdot & 2\backslash & \cdot & 12\backslash & 12\backslash \\ V_4 & 0 & 0 & 0 & \cdot & \cdot \\ V_5 & 0 & 0 & 0 & 0 & \cdot \end{array}$$

Here 0 is the zero matrix and $\cdot$ denotes matrices. Moreover $n\backslash$ with $n \in \{2, 6, 12\}$ denotes a matrix divisible by n. The automorphisms M of the form above describe a subgroup G of $\mathrm{Aut}\,(V)$. The group G acts on the abelian group

$$\Gamma_3 = V_1 \otimes \mathbb{Z}/24 \oplus (V_2 \oplus V_3) \otimes \mathbb{Z}/12 \oplus (V_4 \oplus V_5) \otimes \mathbb{Z}/2$$

in such a way that the canonical quotient map $V \twoheadrightarrow \Gamma_3$ is G-equivariant. We define an equivalence relation for homomorphisms

$$b_4, b_4' : H_4 \to \Gamma_3$$

by setting $b_4 \sim b_4'$ if there exists an automorphism N of H_4 and $M \in G$ with $M b_4 N^{-1} = b_4'$. Each b_4 is equivalent to a sum $b_4^1 \oplus \ldots \oplus b_4^r$ where the b_4^i are indecomposable.

(4.6) Theorem. *The indecomposable objects in $\mathbf{F}^4$, resp. $\mathbf{SF}^4$, are given by S^3, $S^1 \cup_\eta e^3$ and by the indecomposable homomorphisms b_4 in the matrix problem (4.5).*

Proof. Given $X^3 = \bigvee_{i \le 5} d_i X_i$ as in (4.4) we say that the homology H_* of $X = C(f_4)$ satisfies

$$(1) \qquad \begin{aligned} H_0 &= V_1 \oplus V_2 \oplus V_3 \\ H_1 &= V_4 \\ H_2 &= V_2 \oplus V_5 \\ H_3 &= V_3 \\ H_4 &= \mathbb{Z}^h \end{aligned}$$

Moreover according to the list (4.4) (7) we get the following homotopy groups π_* of X

$$(2) \qquad \begin{aligned} \pi_1 &= (V_1 \oplus V_3) \otimes \mathbb{Z}/2 \oplus V_4 \\ \pi_2 &= (V_1 \oplus V_4) \otimes \mathbb{Z}/2 \oplus V_2 \oplus V_5 \end{aligned}$$

This yields the tuple $\mathcal{H}$ of abelian groups in (4.1). The associated exact sequence (4.1) (1) is obtained as follows where the operators are determined by the list (4.4) (7); we again set

$$\bar{V} = V \otimes \mathbb{Z}/2.$$

In the following a group in the top row is given by the direct sum of groups in the corresponding column.

$$\begin{array}{ccccccccccccc}
H_3 & \to & \bar{\pi}_1 & \to & \pi_2 & \to & H_2 & \to & \bar{H}_0 & \to & \pi_1 & \to & H_1 \\
\\
0 & & \bar{V}_1 & \xrightarrow{1} & \bar{V}_1 & & 0 & & \bar{V}_1 & \xrightarrow{1} & \bar{V}_1 & & 0 \\
0 & & 0 & & V_2 & \xrightarrow{2} & V_2 & \xrightarrow{1} & \bar{V}_2 & & 0 & & 0 \\
V_3 & \xrightarrow{1} & \bar{V}_3 & & 0 & & 0 & & \bar{V}_3 & \xrightarrow{1} & \bar{V}_3 & & 0 \\
0 & & \bar{V}_4 & \xrightarrow{1} & \bar{V}_4 & & 0 & & 0 & & V_4 & \xrightarrow{1} & V_4 \\
0 & & 0 & & V_5 & \xrightarrow{1} & V_5 & & 0 & & 0 & & 0
\end{array} \tag{3}$$

Moreover we obtain the following natural maps from (4.1) (2), (3) and (4.4) (7).

$$\begin{array}{ccccccccc}
H_0 & \supset & P & \xrightarrow{T} & \bar{\pi}_2 & \to & \Gamma_3 & \leftarrow & H_0 \otimes \mathbb{Z}/24 \\
\\
V_1 & \supset & 2V_1 & \xrightarrow{\frac{1}{2}} & \bar{V}_1 & \xrightarrow{12} & V_1 \otimes \mathbb{Z}/24 & \xleftarrow{1} & V_1 \otimes \mathbb{Z}/24 \\
V_2 & = & V_2 & \xrightarrow{1} & \bar{V}_2 & \xrightarrow{6} & V_2 \otimes \mathbb{Z}/12 & \xleftarrow{1} & V_2 \otimes \mathbb{Z}/24 \\
V_3 & \supset & 2V_3 & & 0 & & V_3 \otimes \mathbb{Z}/12 & \xleftarrow{1} & V_3 \otimes \mathbb{Z}/24 \\
 & & 0 & & \bar{V}_4 & \xrightarrow{1} & V_4 \otimes \mathbb{Z}/2 & & 0 \\
 & & 0 & & \bar{V}_5 & \xrightarrow{1} & V_5 \otimes \mathbb{Z}/2 & & 0
\end{array} \tag{4}$$

The tuple $\mathcal{H}$ given by (1) and (2) together with (3) and (4) determine together with

$$b_4 : H_4 \to \Gamma_3 \tag{5}$$

an object (also denoted by $\mathcal{H}$) in $\mathbf{SF}_4$. We now consider an automorphism of this object $\mathcal{H}$ in $\mathbf{SF}_4$. Such an automorphism is given by automorphisms of the abelian groups in (1) and (2) which are matrices of the following form. For this we use the compatibility of the automorphism with the operators in (3) and (4). The automorphism of $H_0 = V_1 \oplus V_2 \oplus V_3$ is given by

$$\begin{pmatrix} y_0 & 2a & 2b \\ a' & w_0 & c \\ b' & 2c' & u_0 \end{pmatrix} \tag{6}$$

where for example $c \in \mathrm{Hom}(V_3, V_2)$ and $y_0 \in \mathrm{Hom}(V_1, V_1)$. The automorphism of $H_1 = V_4$ is given by

$$y_1 \in \mathrm{Aut}(V_4) \tag{7}$$

The automorphism of $H_2 = V_5 \oplus V_2$ is given by

$$\begin{pmatrix} y_2 & d \\ 2d' & w_2 \end{pmatrix} \tag{8}$$

Next the automorphism of $H_3 = V_3$ is given by

$$u_3 \in \mathrm{Aut}(V_3) \tag{9}$$

Moreover the automorphism of $\pi_1 = \bar{V}_1 \oplus \bar{V}_3 \oplus V_4$ is given by

$$\begin{pmatrix} y_0 & 0 & \alpha \\ b' & u_0 & \beta \\ 0 & 0 & y_1 \end{pmatrix} \qquad \text{where} \quad u_0 \otimes \mathbb{Z}/2 = u_3 \otimes \mathbb{Z}/2 \tag{10}$$

Finally the automorphism of $\pi_2 = \bar{V}_1 \oplus \bar{V}_4 \oplus V_5 \oplus V_2$ is given by

$$\begin{pmatrix} y_0 & \alpha & \gamma & \delta \\ 0 & y_1 & \rho & \lambda \\ 0 & 0 & y_2 & 2d \\ 0 & 0 & d' & w_2 \end{pmatrix} \qquad \text{where} \quad \begin{cases} \delta = a \otimes \mathbb{Z}/2, \\ \lambda = 0, \\ w_2 \otimes \mathbb{Z}/2 = w_0 \otimes \mathbb{Z}/2 \end{cases} \tag{11}$$

In fact, using $b_2 : H_2 \to \bar{H}_0$ in (3) we see that the entries $2a$, $2c'$ and $2d'$ have to be divisible by 2 and that $w_2 \otimes \mathbb{Z}/2 = w_0 \otimes \mathbb{Z}/2$. Moreover using $b_3 : H_3 \to \bar{\pi}_1$ in (3) we see that the entry $2b$ has to be divisible by 2 and that $u_3 \otimes \mathbb{Z}/2 = u_0 \otimes \mathbb{Z}/2$. Moreover T in (4) shows that $\delta = a \otimes \mathbb{Z}/2$, $\lambda = 0$ and $w_2 \otimes \mathbb{Z}/2 = w_0 \otimes \mathbb{Z}/2$.

The action of the automorphism on Γ_3 is given by $\bar{\pi}_2 \to \Gamma_3 \leftarrow H_0 \otimes \mathbb{Z}/24$ in (4). Hence only $(11) \otimes \mathbb{Z}/2$ is used for the action on $\bar{\pi}_2$ and hence on Γ_3. Here $y_2 \otimes \mathbb{Z}/2$ and $w_2 \otimes \mathbb{Z}/2$ are automorphisms. Since $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/2)$ is surjective for all n we can find $y_2^\sharp \in \mathrm{Aut}(V_5)$ with

$$y_2^\sharp \otimes \mathbb{Z}/2 = y_2 \otimes \mathbb{Z}/2 \tag{12}$$

We now construct a new automorphism of $\mathcal{H}$ which has the same effect on Γ_3 as the original automorphism chosen above. The new automorphism of $\mathcal{H}$ acts via (4.5) on Γ_3 by the automorphism of $V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5$ given by the matrix, compare (4.5):

$$\begin{pmatrix} y_0 & 2a & 2b & \alpha & \gamma \\ a' & w_0 & c & 0 & d^\sharp \\ b' & 2c' & u_0 & 0 & 0 \\ 0 & 0 & 0 & y_1 & \delta \\ 0 & 0 & 0 & 0 & y_2^\sharp \end{pmatrix} \qquad \text{with} \quad d^\sharp \otimes \mathbb{Z}/2 = d' \otimes Z/2 \tag{13}$$

In fact, given (13) we see that $y_2^\sharp \otimes \mathbb{Z}/2 \in \mathrm{Aut}\,(\bar{V}_5)$ and $w_0 \otimes \mathbb{Z}/2 \in \mathrm{Aut}\,(\bar{V}_2)$. Hence we can choose $y_2 \in \mathrm{Aut}\,(V_5), w_2 \in \mathrm{Aut}\,(V_2)$ with

$$\begin{cases} y_2 \otimes \mathbb{Z}/2 = y_2^\sharp \otimes \mathbb{Z}/2, \\ w_2 \otimes \mathbb{Z}/2 = w_0 \otimes \mathbb{Z}/2. \end{cases} \tag{14}$$

Then we get

(15) $$\begin{pmatrix} y_2, & 0 \\ 2d', & w_2 \end{pmatrix} \in \operatorname{Aut} V_5 \oplus V_2)$$

where we set $d = 0$. We are allowed to choose $d = 0$ since the action of $d \in \operatorname{Hom}(V_2, V_5)$ is trivial on Γ_3 (in fact, any map $S^0 \cup_\eta e^2 \to S^2$ induces the trivial homomorphism $0 : \pi_3(S^0 \cup_\eta e^2) \to \pi_3 S^2$). This yields (8) and therefore (6) … (11) are defined by (13) and (15) and by choosing an automorphism u_3 in (9) with $u_3 \otimes \mathbb{Z}/2 = u_0 \otimes \mathbb{Z}/2$ and some β in (10). This is the new automorphism of $\mathcal{H}$ with the property that the new automorphism induces the same automorphism of Γ_3 as the original automorphism. q.e.d.

§ 5 Computation of the indecomposable homomorphisms b_4 in the matrix problem (4.5)

Denote by Λ the ring of all 5×5 matrices of the form

(1) $$M = \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} & 12a_{14} & 12a_{15} \\ a_{21} & a_{22} & a_{23} & 12a_{24} & 6a_{25} \\ a_{31} & 2a_{32} & a_{33} & 12a_{34} & 12a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

where $a_{ij} \in \mathbb{Z}$. Let $U = U_1 \oplus U_2 \oplus U_3 \oplus U_4 \oplus U_5$ where $U_1 = \mathbb{Z}/24, U_2 = U_3 = \mathbb{Z}/12, U_4 = U_5 = \mathbb{Z}/2$. Consider U as Λ-$\mathbb{Z}$-bimodule ${}_\Lambda U_\mathbb{Z}$ in an obvious way. Remind that a *matrix over* U is, by definition (cf. [D]), an element of $P \otimes_\Lambda U \otimes H^*$, where P and H are, respectively, (finitely generated right) projective modules over Λ and $\mathbb{Z}$. It is more convenient to identify this tensor product with $U(H, P) = \operatorname{Hom}(H, P \otimes_\Lambda U)$. Two matrices $u \in U(H, P)$ and $u' \in U(H', P')$ are *isomorphic* if there are isomorphisms $\alpha : H \to H'$ and $\beta : P \to P'$ such that $\beta u = u' \alpha$. Put $e_i = e_{ii}$ (matrix unit) and $P_i = e_i \Lambda$. Then P can be uniquely decomposed as $P = \bigoplus_{i=1}^5 V_i \otimes P_i$ for some free abelian groups V_i. In this case $P \otimes_\Lambda U$ is just $\bigoplus_{i=1}^5 V_i \otimes U_i$. Thus we come to the same "matrix problem" as has been formulated above in (4.5).

We shall write the elements of $U(H, P)$ as quintuples of matrices $(u_1, u_2, u_3, u_4, u_5)$, u_i being of size $m_i \times n$ if $H = n\mathbb{Z}, V_i = m_i \mathbb{Z}_i$, with the entries from $\mathbb{Z}/24$ for $i = 1$, from $\mathbb{Z}/12$ for $i = 2, 3$ and from $\mathbb{Z}/2$ for $i = 4, 5$. The matrices M of the form (1) define the "admissible transformations" of rows of the matrices u_i. For instance, as we have a_{21} in M we can add any multiple of a row of the matrix u_1 to any row of the matrix u_2. On the other hand, as we have $2a_{12}$ in M, we can add only even multiples of rows of u_2 to the rows of u_1, etc.

Certainly U is also a $\bar{\Lambda}$-$\mathbb{Z}$-bimodule, where $\bar{}$ always denotes the reduction modulo 24. For convenience we denote it by $\bar{U}$ though the elements from U and from $\bar{U}$ are the same. Moreover the matrices from $U(P, H)$ coincide with the matrices from $\bar{U}(\bar{P}, \bar{H})$ though nonisomorphic U-matrices can become isomorphic as $\bar{U}$-matrices. Consider first the 2-primary part $\tilde{U}$ of $\bar{U}$. It is a $\tilde{\Lambda}$-$\mathbb{Z}/8$-bimodule where $\tilde{\Lambda}$ is the ring of matrices of the form:

$$(2)\qquad \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} & 4a_{14} & 4a_{15} \\ a_{21} & a_{22} & a_{23} & 4a_{24} & 2a_{25} \\ a_{31} & 2a_{32} & a_{33} & 4a_{34} & 4a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

with $a_{ij} \in \mathbb{Z}/8$. Namely $\tilde{U} = \bigoplus_{i=1}^{5} \tilde{U}_i$ where $\tilde{U}_1 = \mathbb{Z}/8, \tilde{U}_2 = \tilde{U}_3 = \mathbb{Z}/4$ and $\tilde{U}_4 = \tilde{U}_5 = \mathbb{Z}/2$. Denote by z_i the image of $z \in \mathbb{Z}$ in $\tilde{U}_i$ and write $u < v$ for two elements $u, v \in \tilde{U}$ if there is an element $a \in \tilde{\Lambda}$ such that $au = v$ but not vice versa. Then

$$(3)\qquad 1_1 < 1_3 < 1_2 < 2_1 < 2_3 < 2_2 < 4_1 \quad \text{and} \quad 1_5 < 1_4 < 4_1 \quad \text{and} \quad 1_5 < 2_2$$

and there are no other "$<$" relations between these elements. We also put $a < 0$ for each nonzero a.

Let u be a $\tilde{U}$-matrix. Choose in the matrices u_1, u_2, u_3 the elements divisible by the least possible power of 2. If such an element exists in u_1 then using admissible transformations we can obtain zeroes for all other elements in the same row as well as in the same column in all three matrices u_1, u_2, u_3. The same is possible if there are no such elements in u_1 but there is one in u_3 just as in the case when such elements only exist in u_2. Therefore we may only consider elements u such that each row of u_1, u_2, u_3 contains at most one nonzero element, as well as each column in all these three matrices. For each column number i denote by a_i the nonzero element in this column in the matrices u_1, u_2, u_3 or 0 if there is no such element at all. For two column numbers i, j put $i \leq j$ if $a_i \leq a_j$ with respect to the order defined above.

Among the numbers of columns of the matrices u_4, u_5 containing a nonzero element b find the greatest possible with respect to this order. If b is the matrix u_5 then we can make all other elements of its row zeroes as well as all elements of its column in both matrices u_4, u_5. The same is true if there are no such elements in u_5 at all but there is one in u_4. After that the element u evidently splits into a direct sum of matrices of the following forms:

$$(4)\qquad \begin{cases} (1_4), (1_5), (a_1) \quad \text{with} \quad a \in \{1,2,4\}, \\ (a_2), (a_3) \quad \text{with} \quad a \in \{1,2\}, \\ \begin{pmatrix} 1_2 \\ 1_5 \end{pmatrix}, \quad \text{and} \\ \begin{pmatrix} a_1 \\ 1_4 \end{pmatrix}, \begin{pmatrix} a_1 \\ 1_5 \end{pmatrix}, \begin{pmatrix} a_2 \\ 1_4 \end{pmatrix}, \begin{pmatrix} a_3 \\ 1_4 \end{pmatrix}, \begin{pmatrix} a_3 \\ 1_5 \end{pmatrix}, \quad \text{with} \quad a \in \{1,2\}. \end{cases}$$

Hence the lower index refers to the index in (4.4) (2) and the index of V_i in (4.5) respectively. With $v = 3a$ these matrices correspond to the following 20 elements in the list $\mathcal{L}$.

$$
(5) \qquad \begin{cases} (\eta\eta)_1, \eta_2, v \quad \text{with} \quad v \in \{3,6,12\}, \\ \eta v, \eta\eta v \quad \text{with} \quad v \in \{3,6\}, \\ \eta v \eta \quad \text{with} \quad v = 3, \quad \text{and} \\ v\eta\eta, v\eta, \eta v\eta\eta, \eta\eta v\eta\eta, \eta\eta v\eta \quad \text{with} \quad v \in \{3,6\}. \end{cases}
$$

These describe a complete list of all congruence classes of 2-primary polyhedra with non trivial homology H_4. Hence together with η_1 in (4.5), η_0 and $(\eta\eta)_0$ the list (5) yields a complete list of 23 congruence classes of 2-primary polyhedra.

If we consider the 3-part of U-matrices the answer is quite evident: there is only one non-trivial indecomposable matrix 1_1 (isomorphic to 1_2 and to 1_3). At last gluing together 2-part and 3-part we get the following list of indecomposable U-matrices:

$$
(6) \qquad \begin{cases} (1_4), (1_5), (v_1) \quad \text{with} \quad v \in \{1, \dots, 12\}, \\ (v_2), (v_3) \quad \text{with} \quad v \in \{1, \dots, 6\}, \\ \\ \begin{pmatrix} v_2 \\ 1_5 \end{pmatrix} \quad \text{with} \quad v \in \{1,2,3\}, \quad \text{and} \\ \\ \begin{pmatrix} v_1 \\ 1_4 \end{pmatrix}, \begin{pmatrix} v_1 \\ 1_5 \end{pmatrix}, \begin{pmatrix} v_2 \\ 1_4 \end{pmatrix}, \begin{pmatrix} v_3 \\ 1_4 \end{pmatrix}, \begin{pmatrix} v_3 \\ 1_5 \end{pmatrix} \quad \text{with} \quad v \in \{1, \dots, 6\}. \end{cases}
$$

It is easy to check that all these $\mathcal{U}$-matrices are indeed non-isomorphic and that they form a complete list of all indecomposable solutions of the matrix problem (4.5). The matrices (6) correspond to the following 59 elements in the list $\mathcal{L}$.

$$
(7) \qquad \begin{cases} (\eta\eta)_1, \eta_2, v \quad \text{with} \quad v \in \{1, \dots, 12\}, \\ \eta v, \eta\eta v \quad \text{with} \quad v \in \{1, \dots, 6\}, \\ \eta v \eta \quad \text{with} \quad v \in \{1,2,3\}, \quad \text{and} \\ v\eta\eta, v\eta, \eta v\eta\eta, \eta\eta v\eta\eta, \eta\eta v\eta \quad \text{with} \quad v \in \{1, \dots, 6\}. \end{cases}
$$

The list (7) together with the sphere S^4 describes all indecomposable polyhedra with non trivial homology H_4. Together with $S^0, S^1, S^2, S^3, \eta_1, \eta_0$ and $(\eta\eta)_0$ this yields the complete list of 67 indecomposable polyhedra. q.e.d.

Appendix: The p-local decomposition problem

Hans-Joachim Baues and Hans-Werner Henn

Let p be a prime and let $\mathbb{Z}_p$ be the smallest subring of $\mathbb{Q}$ containing $1/q$ for all primes q with $q \neq p$. A simply connected CW-complex X is p-<u>local finite type</u> if all homotopy groups or equivalently all homology groups of X are finitely generated $\mathbb{Z}_p$-modules. Moreover X has p-local dimension $\dim_p(X) \leq n$ if $H_iX = 0$ for $i > n$ and if H_nX is a free $\mathbb{Z}_p$-module. Let $n \geq 2$ and let $\mathbf{A}_n^k(p)$ be the full subcategory of the homotopy category $\mathbf{Top}^*/\simeq$ consisting of p-local finite type CW-complexes X which are $(n-1)$-connected and which satisfy $\dim_p(X) \leq n+k$. The suspension Σ gives us a sequence of functors

$$\mathbf{A}_2^k(p) \to \mathbf{A}_3^k(p) \to \ldots \to \mathbf{A}_n^k(p) \overset{\Sigma}{\to} \mathbf{A}_{n+1}^k(p) \to \ldots \tag{A.1}$$

which describes the p-local k-stem of homotopy theory. The Freudenthal suspension theorem shows that the sequence (A.1) stabilizes in the sense that for $k+1 < n$ the functor $\Sigma : \mathbf{A}_n^k(p) \to \mathbf{A}_{n+1}^k(p)$ is an equivalence of categories. Hence

$$\mathbf{A}^k(p) = \mathbf{A}_n^k(p) \quad \text{with} \quad k+1 < n \tag{A.2}$$

does not depend on n. This is the stable homotopy category of (-1)-connected p-local finite type k-dimensional spectra. The biproduct in the additive category $\mathbf{A}^k(p)$ is the one point union of spaces. The following result is well known; compare for example Wilkerson [Wi] and Freyd [F].

(A.3) Proposition. *Each space X in $\mathbf{A}^k(p)$ has a unique decomposition $X = X_1 \vee \ldots \vee X_j$ where all X_i with $i = 1, \ldots, j$ are indecomposable objects in $\mathbf{A}^k(p)$. Moreover the congruence classes of indecomposable p-primary spaces in $\mathbf{A}^k$ are in 1-1 correspondence with the indecomposable spaces in $\mathbf{A}^k(p)$ which are not p-local spheres.*

This leads to the problem to compute a complete list of indecomposable objects in $\mathbf{A}^k(p)$ and to determine the representation type of $\mathbf{A}^k(p)$. For example Baues-Hennes [BH] show that

- $\mathbf{A}^3(2)$ has tame representation type.

Moreover Henn [H] proved that for all odd primes p

- $\mathbf{A}^{4p-5}(p)$ has tame representation type.

Now let

$$\mathbf{F}^k(p) \subset \mathbf{A}^k(p) \tag{A.4}$$

be the full subcategory consisting of CW-complexes X for which all homology groups H_iX are free $\mathbb{Z}_p$-modules. The purpose of this Appendix is the proof of the following observation.

(A.5) Proposition. *For each prime p there exists $k < \infty$ such that the category $\mathbf{F}^k(p)$ has wild representation type.*

For example we show that $\mathbf{F}^{10}(2)$ has wild representation type. On the other hand by the theorem in the introduction above $\mathbf{F}^4(2)$ has finite representation type. The representation types of $\mathbf{F}^5(2)$, $\mathbf{F}^6(2)$, ... , $\mathbf{F}^9(2)$ are unknown. It is an interesting problem to determine for each prime p the smallest number $k = k(p)$ for which $\mathbf{F}^k(p)$ has wild representation type. For example we have $5 \leq k(2) \leq 10$ so that it is a nice finite task to compute $k(2)$. The decomposition problem in $\mathbf{F}^4(2)$ is solved in (3.5) above. Using the proof below we get for all p:

(A.6) Addendum. $4(p-1) \leq k(p) \leq 10(p-1)$.

Proof of (A.5). For $p = 2$ we know the q-stem $\pi_q^s(S^0) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$; compare Toda [T]. Hence we obtain spaces in $\mathbf{F}^{10}(2)$ by the stable attaching map

$$f : \bigvee_A S^q \to \bigvee_B S^0 \tag{1}$$

where A and B are finite index sets. Such attaching maps are in $1-1$ correspondence with homomorphisms

$$\varphi : V_A \to V_B \otimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2) \tag{2}$$

where V_A and V_B are $\mathbb{Z}/2$ -vector spaces with basis A and B respectively. Now φ is a quiver of the form

$$V_A \overset{\rightarrow}{\rightrightarrows} V_B \tag{3}$$

which is easily seen to be wild since we can choose $A = B$ and the top arrow an isomorphism. Then we get the wild problem in (2.2).

Next let p be a prime > 5. We look at p-local spectra X of finite type with p-local cells concentrated in dimension $2k(p-1)$ where $k \in \{0, \ldots, 5\}$ such that X/X^0 is a one point union of spheres. Then the homotopy types of such X are in $1-1$ correspondence with the isomorphism classes of 5 linear maps

$$g_k : V_k \to V_0 \quad \text{with} \quad k \in \{1, \ldots, 5\} \tag{4}$$

where $V_k = H_{2k(p-1)}(X) \otimes \mathbb{Z}/p$ for $k \geq 0$. Now (4) corresponds to the quiver with 5 arrows

$$\begin{array}{ccccc} & & \bullet & & \\ & & \uparrow & & \\ \bullet \nwarrow & & & & \nearrow \bullet \\ & & \bullet & & \\ \bullet \swarrow & & & & \searrow \bullet \end{array} \tag{5}$$

which is known to be wild; see Ringel [R]. In order to obtain (4) we only have to know that the p-torsion of stable homotopy groups of spheres satisfies for $0 < n < 2p(p-1) - 2$ the equation

$$\pi_n(S^0) = \begin{cases} \mathbb{Z}/p & \text{if} \quad n = 2k(p-1) - 1 \quad \text{and} \quad k \in \{1, \ldots, p-1\} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

See Ravenel [Ra] 1.1.13 and 1.1.14.

Now X is the mapping cone of

$$f : A = A_1 \vee A_2 \vee A_3 \vee A_4 \vee A_5 \to A_0 = X^0 \tag{7}$$

where $A_k, k \geq 1$, is a one point union of p-local spheres with

$$V_k = H_{2k(p-1)-1}(A_k) \otimes \mathbb{Z}/p.$$

Moreover the group of homotopy equivalences of A surjects to the product group $Aut\,(V_1) \times \ldots \times Aut\,(V_5)$ and for $i, j \geq 1$ and $i \neq j$ there are only trivial maps $A_i \to A_j$ as follows from (6). This shows the statement in (4) and hence for $p > 5$ $\mathbf{F}_n(p)$ is wild for $n = \dim(X) = 10(p-1)$.

For $p = 3$ and $p = 5$ we slightly modify the argument above. We use the same type of space as for $p > 5$ but we assume that the attaching map f in (7) is an element of order p. Then the same argument as above shows that $\mathbf{F}_n(p)$ is wild for $n = 10(p-1)$. q.e.d.

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