CELLS AND REPRESENTATIONS OF RIGHT-ANGLED COXETER GROUPS

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ABSTRACT. We study Kazhdan-Lusztig cells and the corresponding representations of right-angled Coxeter groups and Hecke algebras associated to them. In case of the infinite groups generated by reflections in the hyperbolic plane about the sides of right-angled polygons we obtain an explicit description of the left and two-sided cells. In particular, we prove that there are infinitely many left cells but they all form only three two-sided cells.

1. INTRODUCTION

A Coxeter group is said to be right-angled if for any two distinct simple reflections their product has order 2 or ∞. Due to their special properties and rich structure right-angled Coxeter groups often arise in different geometric and algebraic problems. In a sense, the most interesting right-angled Coxeter groups are those which can be presented as groups generated by reflections in hyperbolic spaces. Examples of geometric applications of such groups can be found in [D], [MV], [BGI]. These groups also occur as Weyl groups of certain Kac-Moody Lie algebras [KP]. We are interested in the representations of right-angled Coxeter groups W and the corresponding Hecke algebras $\mathcal{H}$.

An important difference between hyperbolic reflection groups and affine or finite Coxeter groups is that the former do not usually have a local system of generators in the sense of [OV]. In particular, the groups P_n (see 3.1), that are generated by reflections in the hyperbolic plane about the sides of right-angled n-gons can be called anti-local since only adjacent generators in the canonical system of simple reflections commute. This means that we can hardly hope to construct the representations of these groups inductively using the approach suggested in [OV]. At the same time, we can still use the global methods of [KL] and try to describe the Kazhdan-Lusztig cells in our groups. It appears that certain symmetries of the initial groups are reflected in the structure of the partitions into cells making the cells trackable. Our main results concern the groups P_n but the methods can be applied to the other right-angled Coxeter groups as well.

We show that the partition of the group P_n ($n \geq 5$) into left cells consists of infinitely many elements and give an explicit description of the cells. At the same time, all the left cells in our groups form precisely three two-sided cells (one of which is, of course, the trivial cell). It was first shown in [B] that the number of one-sided cells of a hyperbolic Coxeter group can be infinite, but there the author used an implicit argument and obtained only a conjectural structure of the corresponding cell-partitions (see also [C] for discussion). The fact that infinitely many left cells

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can still fall into finitely many two-sided equivalence classes and hence give rise to only finitely many $\mathcal{H}$-bimodules seems have been previously unnoticed.

The explicit description of the cells makes it possible to consider corresponding representations of the Coxeter groups and their Hecke algebras. Here we only start the related analysis leaving more detailed considerations for the future. We discuss the representations using W-graphs which were also introduced in [KL].

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2. Preliminaries

We recall some well-known facts about Coxeter groups and Hecke algebras associated with them. The basic reference is the fundamental paper [KL]. All the material cited here can be also found in the book [H].

2.1. Let W be a Coxeter group and let S be the corresponding set of simple reflections. With some ambiguity of language we shall also call by Coxeter group the Coxeter system (W, S). The *Hecke algebra* $\mathcal{H}$ over the ring $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of Laurent polynomials in $q^{1/2}$ is defined as follows. As an $\mathcal{A}$-module, $\mathcal{H}$ is free with basis T_w $(w \in W)$, the multiplication is defined by

$$\begin{aligned} &T_w T_{w'} = T_{ww'}, \text{ if } l(ww') = l(w) + l(w'),\\ &T_s^2 = q + (q-1)T_s, \text{ if } s \in S, \end{aligned}$$

where $l(w)$ is the length of w in (W, S).

It will be also convenient to define

$$\widetilde{T}_w = q^{-l(w)/2} T_w.$$

2.2. Let $a \to \overline{a}$ be the involution of the ring $\mathcal{A}$ defined by $\overline{q^{1/2}} = q^{-1/2}$. This extends to an involution $h \to \overline{h}$ of the ring $\mathcal{H}$ given by

$$\overline{\sum a_w T_w} = \sum \overline{a}_w T_w^{-1}.$$

Let $\leq$ be the Bruhat order on W (see [H], Ch. 5.9). Denote, as usual, $q_w = q^{l(w)}$, $\epsilon_w = (-1)^{l(w)}$ for all $w \in W$. In [KL] it was shown that for any $w \in W$ there exists a unique element element $C_w \in \mathcal{H}$ such that

$$\begin{aligned} &\overline{C}_w = C_w,\\ &C_w = \epsilon_w q_w^{1/2} \sum_{y \leq w} \epsilon_y q_y^{-1} \overline{P_{y,w}} T_y, \end{aligned}$$

where $P_{y,w} \in \mathcal{A}$ is a polynomial in q of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$, and $P_{w,w} = 1$.

Elements C_w form a basis (called C*-basis*) of $\mathcal{H}$ as an $\mathcal{A}$-module. This basis and the polynomials $P_{y,w}$ (called *Kazhdan-Lusztig polynomials*) turn out to be of fundamental interest in the representation theory of Coxeter groups and Hecke algebras.

2.3. Given $y, w \in W$, we write $y \prec w$ if $y < w$ and $P_{y,w}$ is a polynomial in q of degree exactly $\frac{1}{2}(l(w) - l(y) - 1)$ (which is, of course, possible only if $l(w) - l(y)$ is odd). In this case the coefficient of the highest power of q in $P_{y,w}$ is denoted by $\mu(y, w)$. If $w \prec y$ we set $\mu(w, y) = \mu(y, w)$, otherwise (if neither $y \prec w$ nor $w \prec y$) let $\mu(y, w) = \mu(w, y) = 0$. We write $y - w$ if $\mu(x, y) \neq 0$.

For any $w \in W$ define subsets of S:

$$\mathcal{L}(w) = \{s \in S \mid sw < w\}, \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

Now define $y \leq_L w$ to mean that there is a chain $y = y_0, y_1, \ldots, y_n = w$ such that $y_i - y_{i+1}$ and $\mathcal{L}(y_i) \not\subset \mathcal{L}(y_{i+1})$ for $0 \leq i < n$. Similarly, say that $y \leq_R w$ if $y^{-1} \leq_L w^{-1}$. Finally, define $y \leq_{LR} w$ to mean that there exists a chain $y = y_0, y_1, \ldots, y_n = w$ such that for each $i < n$ either $y_i \leq_L y_{i+1}$ or $y_i \leq_R y_{i+1}$.

Let $\sim_L$, $\sim_R$, $\sim_{LR}$ be the equivalence relations associated to the preorders $\leq_L$, $\leq_R$, $\leq_{LR}$, respectively. The corresponding equivalence classes are called *left*, *right* and *two-sided cells* of W.

2.4. We shall often make use of the following properties of the defined relations.

LEMMA. Let $x, y \in W$ and $x < y$.

(i) If there exists $s \in S$ such that $x < sx$, $sy < y$, then $x \prec y$ if and only if $y = sx$.

(ii) If there exists $s \in S$ such that $x < xs$, $ys < y$, then $x \prec y$ if and only if $y = xs$.

Moreover, in each of the cases $\mu(x, y) = 1$.

(This statement can be found in the proof of Theorem 1.3 in [KL], it follows from the formula for the action of the elements T_s on the basis $\{C_w \mid w \in W\}$ of $\mathcal{H}$ and the definition of C_w.)

COROLLARY.

(i) If $x \leq_L y$, then $\mathcal{R}(x) \supset \mathcal{R}(y)$. Hence, $x \sim_L y$ implies $\mathcal{R}(x) = \mathcal{R}(y)$.

(ii) If $x \leq_R y$, then $\mathcal{L}(x) \supset \mathcal{L}(y)$. Hence, $x \sim_R y$ implies $\mathcal{L}(x) = \mathcal{L}(y)$.

(To prove the corollary it is enough to consider the case $x - y$ with $\mathcal{L}(x) \not\subset \mathcal{L}(y)$, details can be found in [KL].)

2.5. The main purpose of the partition of a Coxeter group W into cells is that it gives rise to the representations of group W and its Hecke algebra $\mathcal{H}$. It is convenient to describe these representations using W-graphs.

Let X be a set. Consider an oriented graph Γ whose set of vertices is X, for each $x \in X$ there is assigned a subset I_x of S, and if $I_x \not\subset I_y$ then there is an edge $(x, y) \in X \times X$ labeled with an integer $\mu(x, y)$. Graph Γ is called *W-graph* if map $s \to \tau_s$, such that

$$\tau_s x = \begin{cases} -x, & \text{if } x \in X,\ s \in I_x, \\ qx + q^{1/2} \sum\limits_{y \in X,\ s \in I_y} \mu(x, y) y, & \text{if } x \in X,\ s \notin I_x, \end{cases}$$

defines a representation of $\mathcal{H}$ on the free $\mathcal{A}$-module $\mathcal{A}(X)$.

In [KL] it was shown that $X = W$ with $I_x = \mathcal{L}(x)$ and $\mu(x, y)$ defined by the polynomial $P_{x,y}$ as in 2.3 gives a W-graph. Moreover, its full subgraphs corresponding to the left cells with the same sets I_x and the same function μ are W-graphs themselves. Finally, when W is a symmetric group S_n it was proved that all the irreducible representations of $\mathcal{H}$ are defined by the W-graphs associated to the left cells of W.

3. Right-angled Coxeter groups

3.1. Recall that Coxeter group (W, S) is called *right-angled Coxeter group* if for any $s \neq t$ in S the product st has order 2 or ∞. The Coxeter graph of W has only edges labeled via ∞ (see examples on Figure 1).

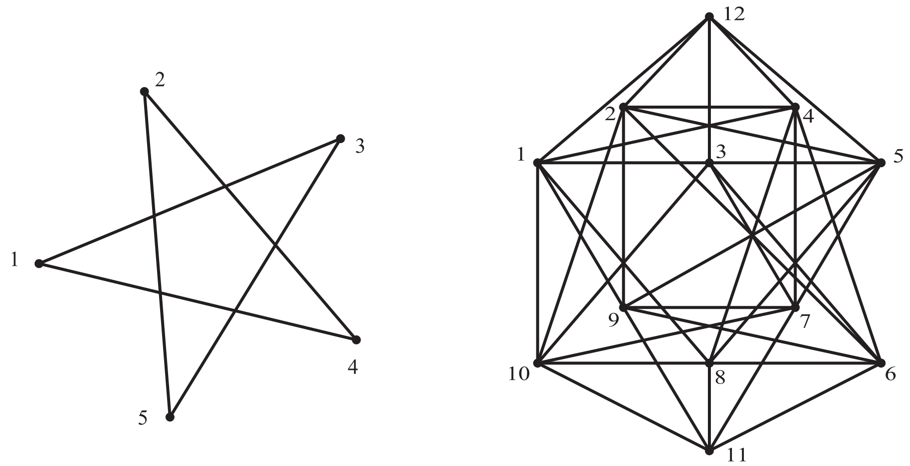

Figure 1. Coxeter graphs of the groups generated by reflections about the sides of a right-angled pentagon in the hyperbolic plane (the group P_5) and the regular right-angled dodecahedron in the hyperbolic 3-space.

The simplest example of a right-angled Coxeter group is the infinite dihedral group $D_\infty = < s_1, s_2 \mid s_1^2 = s_2^2 = 1 >$. We are mainly interested in right-angled Coxeter groups generated by reflections in the hyperbolic n-space. The important example for us are the groups P_n having representations

$$P_n = < s_1, s_2, \ldots, s_n \mid s_i^2 = 1,\ (s_j s_{j+1})^2 = 1,\ (s_n s_1)^2 = 1 >$$

where $i = 1, \ldots, n$; $j = 1, \ldots n-1$. If $n \geq 5$ the group P_n can be presented as a group of isometries of the hyperbolic plane generated by reflections about the sides of a right-angled hyperbolic n-gon (for more information about this and other geometric facts mentioned here we refer the reader to [VS]).

3.2. If a Coxeter group (W, S) is presented by isometries of the hyperbolic space, then it gives rise to a tessellation of the space by the fundamental chambers of the group. The dual graph G of this tessellation endowed with the standard graph metric reflects the structure of the initial group: fixing a vertex $e \in G$ which will correspond to the identity element of W and labelling all the edges with the corresponding simple reflections from S, we can associate to an element $w \in W$ represented by a word on S a geodesic path starting from e in G. Two paths define the same element if and only if they have the same ends. Reduced expressions in

W correspond to the shortest geodesics in G. The graph G is called Cayley graph of (W, S), this graph can be defined for an arbitrary group with a fixed system of generators.

3.3. Let (W, S) be an arbitrary Coxeter group. We call by *lines* elements $w \in W$ that have unique reduced expressions. By a subword of a word $s_1 s_2 \dots s_n$, $s_i \in S$ we mean any expression of the form $s_i s_{i+1} \dots s_j$, $1 \le i \le j \le n$. Subword u of word $w_0 u w_1$ is called *segment* of the word and of the corresponding element $w \in W$, if u represents a maximal line such that any reduced expression of w has the form $w_0' u w_1'$ with $w_0 = w_0'$, $w_1 = w_1'$ as the elements of the group W (here "maximal" means that u is not contained in any other subword with the same properties).

Lines are exactly the unique shortest geodesics of Cayley graph that start from e. The same way, segments of $w \in W$ correspond to the (maximal) geodesics of Cayley graph that are contained in any shortest geodesic corresponding to w. The only segment of a line is the line itself.

Let us give a characterization of the lines and segments in a right-angled Coxeter group.

Proposition. Let (W, S) be a right-angled Coxeter group. Then

(i) a word $s_1 \dots s_k$ represents a line in W if and only if for any $i = 1, \dots, k-1$ the product $s_i s_{i+1}$ has order ∞;

(ii) a subword u of $w_0 u w_1$ is a segment if and only if u gives a minimal line such that $\#\mathcal{R}(w_0) \ne 1$ and $\#\mathcal{L}(w_1) \ne 1$ (here "minimal" means that u does not contain any proper or empty subword with the same property).

(Proof easily follows from the definitions.)

4. Distinguished Involutions

We recall some definitions and results from [L2], [L3].

As in Section 2 we fix a Coxeter group (W, S) and denote by $\mathcal{H}$ the corresponding Hecke algebra over $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. For any $x, y, z \in W$ define elements $f_{x,y,z}$, $g_{x,y,z}$, $h_{x,y,z}$ in $\mathcal{A}$ so that:

$$\widetilde{T}_x \widetilde{T}_y = \sum_z f_{x,y,z} \widetilde{T}_z,$$

$$\widetilde{T}_x C_y = \sum_z g_{x,y,z} C_z,$$

$$C_x C_y = \sum_z h_{x,y,z} C_z.$$

4.1. Let $\mathcal{A}^+ = \mathbb{Z}[q^{1/2}]$. To go further we need the following assumptions about the Coxeter group:

- (W, S) is *crystallographic*, which means that for any $s \ne t$ in S the product st has order 2, 3, 4, 6 or ∞;
- (W, S) is *bounded*, which means that there exists an integer $N \ge 0$ such that $q^{N/2} f_{x,y,z} \in \mathcal{A}^+$ for all $x, y, z \in W$, or equivalently, $q^{N/2} h_{x,y,z} \in \mathcal{A}^+$ for all $x, y, z \in W$.

4.2. Right-angled Coxeter groups are crystallographic. We prove that they are bounded:

Lemma. If in right-angled Coxeter group (W,S) $ts_1 \dots s_n = \hat{t}s_1 \dots \hat{s}_j \dots s_n$ $(t, s_i \in S,\, i = 1, \dots, n)$ then $t = s_j$ and t commutes with $s_1, \dots, s_{j-1}$.

Proof. By [T] there is a sequence of elementary M-operations taking word $ts_1 \dots s_n$ to $\hat{t}s_1 \dots \hat{s}_j \dots s_n$ $(t, s_i \in S)$. In case of right-angled groups the operations are:

(i) $ss \to 1$;
(ii) $st \to ts$

$(s, t \in S)$. Both operations preserve parity of the number of a simple reflection occurrences in the word, which follows $t = s_j$. We shall prove that t commutes with $s_1, \dots, s_{j-1}$ by induction on j. If $j - 1 = 1$ then we have $ts_1t = s_1$; $ts_1 = s_1t$. Now, if $ts_1 \dots s_n$ is equivalent to $\hat{t}s_1 \dots s_{j-1}\hat{t}s_{j+1} \dots s_n$ and $j > 2$ then there is an M-operation which decreases the number of simple reflections between two t's. It can be only an operation of type (ii), so t commutes with some s_i with $i \in \{1, \dots, j-1\}$. It follows by the induction hypothesis that t does also commute with the remaining s_i. □

Theorem. Let (W,S) be a right-angled Coxeter group. Then it is bounded with $N = max\ l(w_0^{S'})$, where S' runs through all subsets of S such that corresponding subgroup $(W^{S'}, S')$ is finite and $w_0^{S'}$ denotes the longest element of $W^{S'}$.

Proof. Take arbitrary $x, y \in W$ and let $s_k \dots s_1$, $t_1 \dots t_n$ be corresponding reduced expressions $(s_i, t_j \in S,\, i = 1, \dots, k,\, j = 1, \dots, n)$. We have:

$$\widetilde{T}_x \widetilde{T}_y = \widetilde{T}_{s_k} \dots \widetilde{T}_{s_1} \widetilde{T}_y.$$

If $l(s_1y) = 1 + l(y)$ then $\widetilde{T}_{s_1}\widetilde{T}_y = \widetilde{T}_{s_1y}$; otherwise, $l(s_1y) = l(y) - 1$ and $\widetilde{T}_{s_1}\widetilde{T}_y = (q^{1/2} - q^{-1/2})\widetilde{T}_y + \widetilde{T}_{s_1y}$.

Proceeding inductively we obtain (similarly to [L2]):

$$\widetilde{T}_x \widetilde{T}_y = \sum_I (q^{1/2} - q^{-1/2})^{p_I} \widetilde{T}_I$$

where I ranges over all subsets $i_1 < \dots < i_{p_I}$ of $\{1, \dots, k\}$ such that

$$s_{i_l} \dots \hat{s}_{i_{l-1}} \dots \hat{s}_{i_1} \dots s_1 y < \hat{s}_{i_l} \dots \hat{s}_{i_{l-1}} \dots \hat{s}_{i_1} \dots s_1 y$$

for $l = 1, \dots, p_I$, and $\widetilde{T}_I = s_k \dots \hat{s}_{i_p} \dots \hat{s}_{i_1} \dots s_1 y$, $p = p_I = \#I$.

From the lemma it follows that for any such I there exists a reduced expression of x in which $i_j = j$ and $s_{i_j} \in \mathcal{L}(y)$ for $j = 1, \dots, l$. Denoting by Γ_y the subgroup of W generated by simple reflections from $\mathcal{L}(y)$, we have $s_{i_p} \dots s_{i_1}$ is a reduced expression of an element from Γ_y. To complete the proof it remains to show that for any $y \in W$ subgroup $\Gamma_y \le W$ is finite. Really, let $s \ne t \in \mathcal{L}(y)$. This means that y has a reduced expression $su_2 \dots u_n$ $(s, u_j \in S)$ and $tsu_2 \dots u_n = \hat{t}su_2 \dots \hat{u}_i \dots u_n$, so again by the lemma $st = ts$, which follows $\Gamma_y \cong \mathbb{Z}_2^p$ $(p = \#\mathcal{L}(y))$ is a finite group. □

4.3. Let $a(z)$ be the smallest integer such that $q^{a(z)/2}h_{x,y,z} \in \mathcal{A}^+$ for any $x, y \in W$. It follows that $0 \leq a(z) \leq N$ for any $z \in W$. Here are some important properties of the function a obtained in [L2], [L3]:

PROPERTIES OF $a(z)$:

(i) $a(w) = a(w^{-1})$ for all $w \in W$;
(ii) $a(w) = 0$ if and only if $w = e$;
(iii) the function a is constant on the two-sided cells of W;
(iv) $a(w) \leq l(w)$ for all $w \in W$.

4.4. Define a subset $\mathcal{D} \subset W$ as follows:

$$\mathcal{D} = \{z \in W \mid a(z) = l(z) - 2\delta(z)\},$$

where $\delta(z)$ is the degree of $P_{e,z}$ as a polynomial in q.

It can be shown that $d^2 = e$ for any $d \in \mathcal{D}$. The elements of $\mathcal{D}$ are called *distinguished involutions* of W. The following theorem was proved in [L3]:

THEOREM. Any left cell contains a unique $d \in \mathcal{D}$.

In the next section we shall use this powerful result to distinguish left cells of the groups P_n.

5. Cells and representations

We mainly consider left cells and corresponding left $\mathcal{H}$-modules. The results concerning right cells are entirely similar. Two-sided cells and $\mathcal{H}$-bimodules are obtained via combination of the left and right-sided ones.

Let us first suppose that (W, S) is an arbitrary Coxeter group. We shall always use s, t (possibly with subscribed indexes which will be not connected with the initial ordering of generators in the case of the group P_n) to denote the elements from S.

5.1. Let $w_1, w_2 \in W$. We say that w_1 and w_2 belong to the same *left precell* and write $w_1 \sim_{lp} w_2$ if there exists $w_L \neq 1$ such that w_1, w_2 have reduced expressions $w_1'w_L$, $w_2'w_L$, respectively, and

(a) if $l(w_L) > 1$ then w_1', w_2' are either trivial or segments in w_1, w_2, resp.;
(b) if $l(w_L) = 1$ then w_1 and w_2 are lines (with $\mathcal{R}(w_1) = \mathcal{R}(w_2) = w_L$).

Let us also suppose $1 \sim_{lp} 1$.

PROPOSITION. Relation $\sim_{lp}$ has the following properties:

(i) it is an equivalence relation on W;
(ii) each left precell Γ contains a unique shortest element $w_L = w_L(\Gamma)$ such that $\Gamma = \{w \mid w = w'w_L\}$ where w' is either trivial, or a segment in w, or $w'w_L$ is a line (as in the definition of $\sim_{lp}$).

Proof. To prove (i) we need only to check the transitivity of $\sim_{lp}$. Suppose

$$w_1 \sim_{lp} w_2, \; w_2 \sim_{lp} w_3,$$

then we have

$$w_1 = w_1'w_{L1}, \; w_2 = w_2'w_{L1} = w_2''w_{L2}, \; w_3 = w_3'w_{L2},$$

with w_1', w_2', w_2'', w_3' as in the definition of $\sim_{lp}$. We want to show $w_1 \sim_L w_3$ and we can suppose that w_1, w_2, w_3 are all unequal since otherwise it is trivial. We now need to do some routine case by case considerations:

1) $l(w_{L1}) = 1$. Then w_2 is a line, so w_2'' can not be its proper segment, consequently we have only two possibilities:

a) $l(w_{L2}) = 1$; $w_{L2} = w_{L1}$; $w_3 \sim_L w_1$.
b) $w_2'' = 1$; $w_3 = w_3' w_{L2} = w_3' w_2$, $w_3' \neq 1$ is a segment in w_3 but this is impossible by the definition of segment since w_2 is a line.

2) The case $l(w_{L2}) = 1$ is entirely similar.
3) $l(w_{L1}) > 1$, $l(w_{L2}) > 1$. There are again two possibilities:

a) $w_2' = 1$; $w_1 = w_1' w_{L1} = w_1' w_2 = w_1' w_2'' w_{L2}$, $w_1' \neq 1$ is a segment in w_1. We have either $w_2'' = 1$ which follows $w_3 \sim_L w_1$, or w_2'' is a segment in $w_2'' w_{L2}$ which leads to a contradiction with the definition of segment.
b) w_2' is a segment in w_2 then either $w_2' = w_2''$ and $w_3 \sim_L w_1$ or $w_2'' = 1$; $w_3 = w_3' w_{L2} = w_3' w_2 = w_3' w_2' w_{L1}$ which is impossible by the definition of segment.

So in all the possible cases we obtain $w_1 \sim_L w_3$.

To prove (ii) we can take for w_L (any) shortest element of the equivalence class Γ, then any w in Γ will have the required form by the definition and uniqueness of w_L follows. It is also possible to deduce the existence and uniqueness of w_L from the proof of (i). □

The existence of the canonical representatives w_L of the left precells implies that the following definitions make sense:

Dimension of a non-unit left precell Γ $\dim(\Gamma) = \#\mathcal{L}(w_L)$, since $l(w_L) \geq 1$ we have $\dim(\Gamma) \geq 1$. Given two left precells Γ_1, Γ_2 we say that $\Gamma_1 \leq_L \Gamma_2$, $\Gamma_1 \prec \Gamma_2$, $\Gamma_1 - \Gamma_2$ and $\Gamma_1 \sim_L \Gamma_2$ if $w_L(\Gamma_1) \leq_L w_L(\Gamma_2)$, $w_L(\Gamma_1) \prec w_L(\Gamma_2)$, $w_L(\Gamma_1) - w_L(\Gamma_2)$ and $w_L(\Gamma_1) \sim_L w_L(\Gamma_2)$, respectively. We also define $\mathcal{L}(\Gamma) = \mathcal{L}(w_L(\Gamma))$ and $\mathcal{R}(\Gamma) = \mathcal{R}(w_L(\Gamma))$.

5.2. Lemma. A left cell in W is a union of the left precells which are $\sim_L$-equivalent to each other.

Proof. By Lemma 2.4 if $w_1 \sim_{lp} w_2$ then $w_1 \sim_L w_2$. The corresponding chains $x_0 - x_1 - \cdots - x_k$, joining w_1 (w_2) with w_L and having property $\mathcal{L}(x_i) \cap \mathcal{L}(x_{i+1}) = \emptyset$ for any i (which is actually stronger than it is required for $\sim_L$), are obtained from the lines w_1' (w_2') defined in Proposition 5.1 by the rule $x_0 = w_1' w_L = t_1 \ldots t_k w_L$, $x_i = t_{i+1} \ldots t_k w_L$. Since both $\sim_L$ and $\sim_{lp}$ are the equivalence relations the remaining part of the statement follows easily. □

The language of precells seems to be very appropriate for the description of the cells of right-angled Coxeter groups. We shall consider in details the case $W = P_n$. Similar methods can be applied to the other right-angled Coxeter groups as well, we are going to study these cases elsewhere.

5.3. Theorem. The non-unit left cells of the group P_n ($n \geq 5$) are:

(i) n cells corresponding to the 1-dimensional left precells of P_n defined by the generators of P_n;
(ii) infinitely many cells which are equivalence classes of the left precells with the canonical representatives $\Gamma(w_L)$, such that $w_L = t_1 t_2 w_L'$ with $t_1 t_2 = t_2 t_1$ ($t_1, t_2 \in S$) and w_L' is a segment in w_L.

Proof. 1) We first show that each left precell of P_n belongs to an at least one left cell defined in the statement. If $\dim(\Gamma) = 1$ then $l(w_L(\Gamma)) = 1$ (as it is easily follows from the definitions of the precell and w_L), so Γ is one of the 1-dimensional precells from (i). By the definition of the group P_n the dimensions of its left precells are not greater then 2 (this also follows from the representation of P_n as a group of isometries of the hyperbolic plane), thus it remains to consider a left precell Γ with $\dim(\Gamma) = 2$.

Let $w_L = w_L(\Gamma)$, we have $w_L = t_{1,1} \dots t_{1,n_1} u_1 t_{2,1} \dots t_{2,n_2} u_2 \dots t_{k,1} \dots t_{k,n_k} u_k$, where for any admissible i, j the subwords u_i are segments in w_L, $n_i \geq 2$ and $t_{i,j} t_{i,j+1} = t_{i,j+1} t_{i,j}$. Define two elementary moves between reduced words:

A: $t_1 t_2 t_3 x \to t_2 t_3 x$ for $t_1 t_2 = t_2 t_1$, $t_2 t_3 = t_3 t_2$ and arbitrary subword x;

B: $s_1 s_2 u t_1 t_2 x \to t_1 t_2 x$ for $t_1 t_2 = t_2 t_1$, $s_1 s_2 = s_2 s_1$, u is a segment or $u = 1$ and arbitrary x.

By applying this elementary moves to w_L one can obtain the word $t_{k,n_k-1} t_{k,n_k} u_k$ (u_k is a segment) which defines a canonical precell in (ii). We shall show that the moves produce $\sim_L$-equivalent words.

The equivalence $w = t_1 t_2 t_3 x \sim_L w_0 = t_2 t_3 x$ is easy: we have $w = t_1 w_0$, $\mathcal{L}(w_0) = \{t_1, t_2\}$, $\mathcal{L}(w) = \{t_2, t_3\}$ and $t_3 \neq t_1$ (because all the expressions are reduced), so $w \succ w_0$, $\mathcal{L}(w_0) \not\subset \mathcal{L}(w)$ and $\mathcal{L}(w) \not\subset \mathcal{L}(w_0)$.

To prove that move B is a left equivalence we shall use a supplementary construction. Having $w = s_1 s_2 u t_1 t_2 x$, $w_0 = t_1 t_2 x$, define $w^* = t_1 u^{-1} s_1 s_2 u t_1 t_2 x$. Note, that u is a segment in w implies u^{-1} and u are segments in w^*. We are going to show the following relations:

$$
\begin{tikzcd}
w_0 \arrow[rrr, dash, "\leq_L"] \arrow[d, dash] & & & w^* \arrow[r, dash, "\sim_L"] \arrow[d, dash] & w_1^* \arrow[r, dash, "\sim_L"] & \dots \arrow[r, dash, "\sim_L"] & w_l^* \arrow[r, dash, "\sim_L"] & w \\
w_1 \arrow[r, dash, "\geq_L"] & w_2 \arrow[r, dash, "\geq_L"] & \dots \arrow[r, dash, "\geq_L"] & w_n & & & &
\end{tikzcd}
$$

where w_{i+1} is obtained from w_i by adding at the left the next letter of w^* and w^*_{j+1} is obtained from w^*_j by deleting a letter at the left.

The difficult part is to prove $w_0 \leq_L w^*$ since all the other chains are just of the form $x - y$ with $|l(x) - l(y)| = 1$ and so satisfy the definitions (one can also note that $w^* \sim_{lp} w$).

Let $u = u_1 \dots u_k$ with $u_i \in S$ is a (the) reduced expression of u. We need to show that the coefficient $\mu(w_0, w^*)$ of the $q^{(l(w^*) - l(w_0) - 1)/2} = q^{k+1}$ in P_{w_0, w^*} is not 0. Let us make use of the following formula for the polynomials $P_{y,w}$ obtained in the proof of existence of the C_w-basis in [KL]:

$$
P_{y,w} = q^{1-c} P_{sy,v} + q^c P_{y,v} - \sum_{\substack{y \leq z \prec v \\ sz < z}} \mu(z,v) q_z^{-1/2} q_v^{1/2} q^{1/2} P_{y,z} \qquad (y \leq w),
$$

where $w = sv$ with $l(w) = 1 + l(v)$, $c = 1$ if $sy < y$, $c = 0$ if $sy > y$ and $P_{x,v} = 0$ unless $x \leq v$.

We have:

$$
P_{w_0, w^*} = P_{t_1 t_2 x, t_1 u^{-1} s_1 s_2 u t_1 t_2 x} = P_{t_2 x, v} + q P_{t_1 t_2 x, v} - \Sigma_0
$$

with $v = u^{-1} s_1 s_2 u t_1 t_2 x$ and $\Sigma_0 = \sum \mu(z,v) q_z^{-1/2} q_v^{1/2} q^{1/2} P_{y,z}$ where the summation is over z such that $t_1 t_2 x \leq z \prec v$, $t_1 z < z$.

$\mu(t_2x, v) = 0$ because $\mathcal{L}(v) = u_k \notin \mathcal{L}(t_2x)$ since u is a segment in w, and so by Lemma 2.4 $P_{t_2x,v}$ has the maximal possible degree ($= k+1$) if and only if $u^{-1}s_1s_2ut_1t_2x = u_kt_2x$ which is impossible. Consider the second term:

$$qP_{t_1t_2x,v} = q^2P_{u_kt_1t_2x,u_{k-1}\ldots u_1s_1s_2u_1\ldots u_kt_1t_2x} + qP_1 - q\Sigma_1,$$

defining for $i = 1, \ldots, k$:

$$\begin{aligned} P_i &= P_{y_i,v_i}, \\ \Sigma_i &= \sum_{\substack{y_i \le z \prec v_i \\ u_{k-i+1}z<z}} \mu(z,v)q_z^{-1/2}q_{v_i}^{1/2}q^{1/2}P_{y_i,z}, \\ y_i &= u_{k-i+2}\ldots u_kt_1t_2x, \\ v_i &= u_{k-i}\ldots u_1s_1s_2u_1\ldots u_kt_1t_2x \end{aligned}$$

(with the conventions that $u_{k-i+2}\ldots u_k = 1$ for $i = 1$ and $u_{k-i}\ldots u_1 = 1$ for $i = k$).

The remarkable point is that the coefficients of q^{k-i+2} in qP_i and Σ_{i-1} for $i = 1, \ldots, k$ are equal! Really, consider the sum Σ_i. We have $\mathcal{L}(v_i) = u_{k-i} \notin \mathcal{L}(z)$ (since $u_{k-i+1} \in \mathcal{L}(z)$ and it does not commute with u_{k-i}) so by Lemma 2.4 $z \prec v_i$ implies $v_i = u_{k-i}z$, $\mu(z, v_i) = 1$, $z = u_{k-i-1}\ldots u_1s_1s_2u_1\ldots u_kt_1t_2x = v_{i+1}$ and this is possible only if

$$u_{k-i-1} = u_{k-i+1}. \qquad (*)$$

In this case we have $P_{y_i,z} = P_{y_i,v_{i+1}} = P_{y_{i+1},v_{i+1}}$ (the last equality is the consequence of $\mathcal{L}(y_i) = u_{k-i+2} \neq u_{k-i+1} = \mathcal{L}(v_{i+1})$) and so $\Sigma_i = qP_{i+1}$. It remains to check that if $\mu(y_{i+1}, v_{i+1}) \neq 0$ then the equality $(*)$ holds, which can be easily done by supposing in the contrary that $u_{k-i-1} \neq u_{k-i+1}$ and applying Lemma 2.4. The case of Σ_0 should be considered separately, of course, but appears to be very similar.

So the leading terms of qP_i and Σ_{i-1} do repetitively cancel each other and we finally obtain that the coefficient of q^{k+1} in P_{w_0,w^*} is equal to the leading coefficient of $q^{k+1}P_{u_1y_k,v_k}$, which is equal to 1 because $u_1y_k < v_k$ and $l(v_k) - l(u_1y_k) = 2$.

This proves $w_0 \prec w^*$ with $\mu(w_0, w^*) = 1$.

2) It remains to show that the cells defined in the statement do not intersect. We shall use the distinguished involutions.

For the non-unit elements $z \in P_n$ we have $a(z) \in \{1, 2\}$. If z is in a cell of type (i) then there exists $s \sim_L z$ and by the Properties 4.3 $a(z) = a(s) = 1$. Now let z is in a cell of type (ii). Using Moves A, B from the first part of this proof and their right-side analogs we see that z belongs to the same two-sided cell as st ($s, t \in S$, $(st)^2 = 1$). It is easy to show that $a(st) = 2$ (take $x = y = st$ in the definition of the function a), so again by 4.3 $a(z) = 2$.

It immediately follows from the definitions that $s_i \in \mathcal{D}$, $i = 1, \ldots, n$ so we have the distinguished involutions for each of the cells of type (i). Now consider a cell of type (ii) with the represenative $\Gamma(w_L)$ as in the statement of the theorem. An element $z = {w'_L}^{-1}t_1t_2w'_L \in \Gamma(w_L)$ is an involution, we shall see that $z \in \mathcal{D}$. Supposing $w'_L = u_1\ldots u_k$, all $u_i \in S$ we have

$$P_{e,z} = P_{e,u_k\ldots u_1t_1t_2u_1\ldots u_k} = P_{u_k,u_k\ldots u_1t_1t_2u_1\ldots u_k}.$$

After denoting u_k by t_1 we see that the argument, which was used to show that $w_0 \le_L w*$ in part (1) of this proof, works without any changes in this case either, giving $deg(P_{e,z}) = k$. So

$$l(z) - 2\delta(z) = 2k + 2 - 2k = 2 = a(z)$$

and $z \in \mathcal{D}$ by the definition.

It remains to apply Theorem 4.4 to distinguish all the left cells. $\square$

5.4. It was pointed out to me by V. Ostrik that the cells of type (i) were previously considered in [L1]. There the cells and corresponding representations were constructed for an arbitrary Coxeter group and then thoroughly studied in the finite and affine cases.

Left cells of the group P_n can be visualized on the corresponding tessellation of the hyperbolic plane. Figure 2 presents the cells for the group P_5: pentagon in the center is the unit cell, five shaded regions represent the cells of type (i) all giving a one two-sided cell (see Corollary 5.7), each white region represents a cell of type (ii) and altogether they form the third two-sided cell.

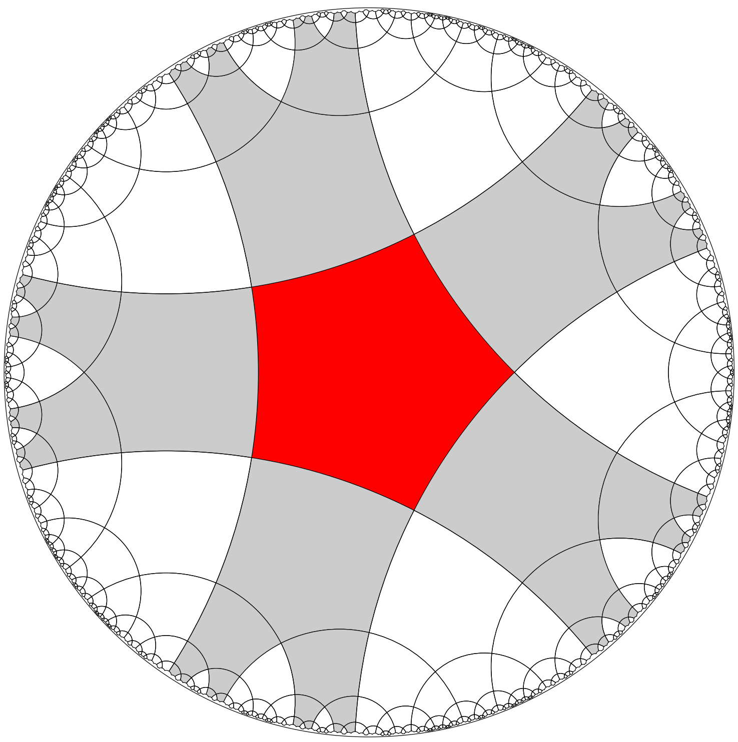

FIGURE 2. Cells of the group P_5.[1]

Below we give several corollaries that follow from Theorem 5.3 and its proof.

[1] This picture uses W. Casselman's PostScript library for hyperbolic geometry.

5.5. Corollary. The distinguished involutions of the group P_n are

$$\mathcal{D} = \{1\} \cup \{s_1, \ldots, s_n\} \cup \{ustu^{-1} \mid (st)^2 = 1,\ u \text{ is a segment in } ustu^{-1}\}.$$

The distinguished involutions are related with algebra $\mathcal{J}$ defined in [L3] which may be regarded as an asymptotic version of Hecke algebra $\mathcal{H}$. Using the methods from [L3] this corollary can be applied to retrieve a partial structure information about algebra $\mathcal{J}$ of the group P_n.

5.6. Corollary. W-graphs associated to the left cells of type (i) are infinite rooted trees (binary trees for the group P_5, see Figure 3), while the W-graphs associated to the type (ii) cells admit infinetely many different cycles.

One can see, that all the W-graphs corresponding to the cells of type (i) of the group P_n define equivalent representations of $\mathcal{H}$, with the equivalences induced by the cyclic permutations of the simple reflections $s_i \in S$. We suppose that the representations corresponding to the cells of type (ii) are also all equivalent but this does not readily follow from the above arguments.

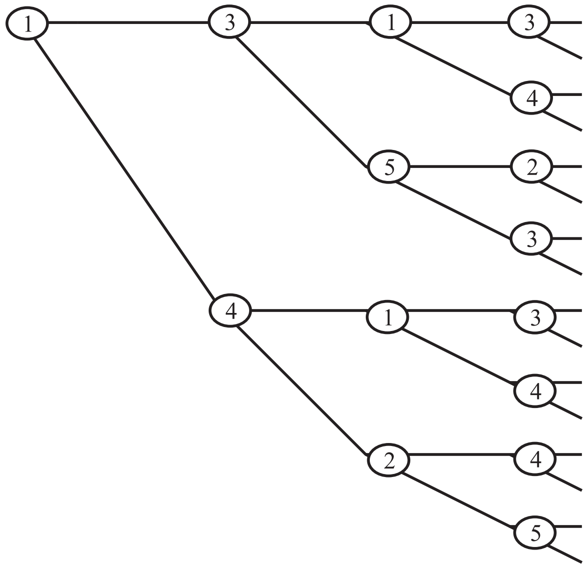

Figure 3. Example of a W-graph corresponding to a left cell of type (i) of the group P_5 (all $\mu(x, y) = 1$, the vertices are represented by circles with the corresponding subsets of S inside).

5.7. Corollary. The partition of the group P_n ($n \geq 5$) onto two-sided cells consits of 3 elements:

- the unit cell corresponding to the trivial representation of H;
- the union of the left cells of type (i) (a 1-dimensional cell), the corresponding W-graph is an infinite tree;
- the union of the left cells of type (ii) (a 2-dimensional cell), the corresponding W-graph admits infinitely many different cycles.

The remarkable point about this corollary is that to establish it we actually need only the Moves A, B from the proof 5.3 with their right-side analogs, and so we do not use part (2) of the argument which relies on certain very strong results about distinguished involutions.

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