Dedicated to the memory of Andrei Nikolaevich Tyurin

MODULI OF MATHEMATICAL INSTANTON VECTOR BUNDLES WITH ODD c_2 ON PROJECTIVE SPACE

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1. Introduction

By a *mathematical n-instanton vector bundle* (shortly, a *n-instanton*) on 3-dimensional projective space $\mathbb{P}^3$ we understand a rank-2 algebraic vector bundle E on $\mathbb{P}^3$ with Chern classes

$$c_1(E) = 0, \quad c_2(E) = n, \quad n \geq 1, \tag{1}$$

satisfying the vanishing conditions

$$h^0(E) = h^1(E(-2)) = 0. \tag{2}$$

Denote by I_n the set of isomorphism classes of n-instantons. This space is nonempty for any $n \geq 1$ - see, e.g., [BT], [NT]. The condition $h^0(E) = 0$ for a n-instanton E implies that E is stable in the sense of Gieseker-Maruyama. Hence I_n is a subset of the moduli scheme $M_{\mathbb{P}^3}(2; 0, 2, 0)$ of semistable rank-2 torsion-free sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = 0,\ c_2 = n,\ c_3 = 0$. The condition $h^1(E(-2)) = 0$ for $[E] \in I_n$ (called the *instanton condition*) by the semicontinuity implies that I_n is a Zariski open subset of $M_{\mathbb{P}^3}(2; 0, 2, 0)$, i.e. I_n is a quasiprojective scheme. It is called the *moduli scheme of mathematical n-instantons*.

In this paper we study the problem of the irreducibility of the scheme I_n. This problem has an affirmative solution for small values of n, up to $n = 5$. Namely, the cases $n = 1, 3, 3, 4$ and 5 were settled in papers [B1], [H], [ES], [B3] and [CTT], respectively. The aim of this paper is to prove the following result.

Theorem 1.1. *For each $n = 2m + 1, \quad m \geq 0$, the moduli scheme I_n of mathematical n-instantons is reduced and irreducible of dimension $8n - 3$.*

A guide to the paper is as follows. In section 3 we remind a well-known relation between mathematical n-instantons and nets of quadrics in arithmetic n-dimensional vector space $\mathbf{k}^n$. The nets of quadrics are considered as vectors of the space $\mathbf{S}_n = S^2(\mathbf{k}^n)^\vee \otimes \wedge^2 V^\vee$, where $V = H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$, and those nets which correspond to n-instantons (we call them n-instanton nets) satisfy the so-called Barth's conditions - see definition (13). Thus the description of the moduli space I_n of n-instantons reduces to that of the locally closed subset $MI_n \subset \mathbf{S}_n$ of n-instanton nets of quadrics which is crucial for our study.

In section 4 we prove one result of general position for the set of $(2m + 1)$-instanton nets of quadrics $MI_{2m+1},\ m \geq 1$. Essentially, this result means that the natural map $MI_{2m+1} \to \mathbf{S}_{m+1}$ induced by a generic embedding $\mathbf{k}^{m+1} \hookrightarrow \mathbf{k}^{2m+1}$ is dominating - see Remark 8.1.

Section 5 is a study of some linear algebra related to a direct sum decomposition $\xi : \mathbf{k}^{m+1} \oplus \mathbf{k}^m \xrightarrow{\sim} \mathbf{k}^{2m+1}$ giving the above embedding $\mathbf{k}^{m+1} \hookrightarrow \mathbf{k}^{2m+1}$. Using the result of section 4 we obtain here the relation (61) which is a key instrument for our further considerations. Also, the decomposition ξ enables us to relate $(2m+1)$-instantons E to rank-$(2m+2)$ symplectic vector bundles E_{2m+2} on $\mathbb{P}^3$ satisfying the vanishing conditions $h^0(E_{2m+2}) = h^2(E_{2m+2}(-2)) = 0$.

In section 6 we introduce a new scheme X_m as a locally closed subset of the vector space $\mathbf{S}_{m+1} \times \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k}^{m+1})^\vee \otimes \wedge^2 V^\vee$ which is defined by linear algebraic data somewhat similar to Barth's conditions. We prove that X_m as a reduced scheme is isomorphic to a certain dense open subset $MI_{2m+1}(\xi)$ of MI_{2m+1} determined by the choice of the direct sum decomposition ξ above. This reduces the problem of the irreducibility of I_{2m+1} to that of X_m.

The last ingredient in the proof of Theorem 1.1 is a scheme Z_m introduced in section 7 as a closed subscheme of the vector space $\mathbf{S}_m^\vee \times \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k}^m)^\vee) \otimes \wedge^2 V^\vee$ defined by explicit equations. We relate the scheme Z_m to the so-called t'Hooft instantons. Using the properties of t'Hooft instantons (see subsection 5.2) we show that the scheme Z_m is reduced and irreducible.

In the last section 8 we finish the proof of Theorem 1.1. The proof is based on a study of certain scheme $\overline{X}_m$ containing X_m and fibred over the vector space $\mathrm{Hom}(\mathbf{k}^\vee, \mathbf{k}^{m+1}) \otimes \wedge^2 V$. We show that the zero fibre of this projection is scheme-theoretically isomorphic to a direct product of Z_m and a certain vector space. This together with the irreducibility of Z_m and some other results stated earlier leads to the irreducibility of X_m.

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2. Notation and conventions

Our notations are mostly standard. The base field $\mathbf{k}$ is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or scheme X, then $n\mathcal{F}$ denotes a direct sum of n copies of the sheaf $\mathcal{F}$, $H^i(\mathcal{F})$ denotes the i^{th} cohomology group of $\mathcal{F}$, $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and $\mathcal{F}^\vee$ denotes the dual to $\mathcal{F}$ sheaf, i.e. the sheaf $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If Z is a subscheme of X, by $\mathcal{I}_{Z,X}$ we denote the ideal sheaf corresponding to a subscheme Z. If $X = \mathbb{P}^r$ and t is an integer, then by $\mathcal{F}(t)$ we denote the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. $[\mathcal{F}]$ will denote the isomorphism class of a sheaf $\mathcal{F}$. For any morphism of $\mathcal{O}_X$-sheaves $f : \mathcal{F} \to \mathcal{F}'$ and any $\mathbf{k}$-vector space U (respectively, for any homomorphism $f : U \to U'$ of $\mathbf{k}$-vector spaces) we will denote, for short, by the same letter f the induced morphism of sheaves $id \otimes f : U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$ (respectively, the induced morphism $f \otimes id : U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$).

Everywhere in the paper V will denote a fixed vector space of dimension 4 over $\mathbf{k}$ and we set $\mathbb{P}^3 := P(V)$. Also verywhere below we will reserve the letters u and v for denoting the two morphisms in the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \to 0$. For any $\mathbf{k}$-vector spaces U and W and any vector $\phi \in \mathrm{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \mathrm{Hom}(U \otimes V, W \otimes V^\vee)$ understood as a homomorphism $\phi : U \otimes V \to W \otimes V^\vee$ or, equivalently, as a homomorphism $^\sharp\phi : U \to W \otimes \wedge^2 V^\vee$, we will denote by $\widetilde{\phi}$ the composition $U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{^\sharp\phi} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} W \otimes \Omega_{\mathbb{P}^3}(2)$, where ϵ is the induced morphism in the exact triple $0 \to \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 v^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} \Omega_{\mathbb{P}^3}(2) \to 0$ obtained by passing to the second wedge power in the dual Euler exact sequence. Also, shortening the notation, we will omit sometimes the subscript $\mathbb{P}^3$ in the notation of sheaves on $\mathbb{P}^3$, e.g., write $\mathcal{O}$, Ω etc., instead of $\mathcal{O}_{\mathbb{P}^3}$, $\Omega_{\mathbb{P}^3}$ etc., respectively.

Everywhere in the paper for $m \geq 1$ we denote by $\mathbf{S}_m$ the vector space $S^2(\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee$. Following W.Barth [B2], [B3] and A.Tyurin [T1], [T2] we call this space *the space of nets of quadrics in the space* $\mathbf{k}^m$.

3. Some generalities on instantons. Set MI_n

In this section we recall some well known facts about mathematical instanton bundles - see, e.g., [CTT].

For a given n-instanton E, the conditions (1), (2), Riemann-Roch and Serre duality imply

$$h^1(E(-1)) = h^2(E(-3)) = n, \quad h^1(E\otimes\Omega^1_{\mathbb{P}^3}) = h^2(E\otimes\Omega^2_{\mathbb{P}^3}) = 2n+2, \tag{3}$$

$$h^1(E) = h^2(E(-4)) = 2n-2.$$

Furthermore, the condition $c_1(E)=0$ yields an isomorphism $\wedge^2 E \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^3}$, hence a symplectic isomorphism $j : E \xrightarrow{\simeq} E^\vee$. This symplectic structure j on E is unique up to a scalar, since E as a stable bundle is a simple bundle, i.e. $\mathrm{Hom}(E,E) = \mathbf{k}id$. Consider a triple (E,f,j) where E is an n-instanton, f is an isomorphism $\mathbf{k}^n \xrightarrow{\simeq} H^2(E(-3))$ and $j : E \xrightarrow{\simeq} E^\vee$ is a symplectic structure on E. We call two such triples (E,f,j) and $(E'f',j')$ equivalent if there is an isomorphism $g : E \xrightarrow{\simeq} E'$ such that $g_*\circ f = \lambda f'$ with $\lambda\in\{1,-1\}$ and $j = g^\vee\circ j'\circ g$, where $g_* : H^2(E(-3)) \xrightarrow{\simeq} H^2(E'(-3))$ is the induced isomorphism. We denote by $[E,f,j]$ the equivalence class of a triple (E,f,j). From this definition one easily deduces that the set $F_{[E]}$ of all equivalence classes $[E,f,j]$ with given $[E]$ is a homogeneous space of the group $GL(\mathbf{k}^n)/\{\pm id\}$.

Each class $[E,f,j]$ defines a point

$$A_n = A_n([E,f,j]) \in S^2(\mathbf{k}^n)^\vee\otimes\wedge^2V^\vee \tag{4}$$

in the following way. Consider the exact sequences

$$0\to\Omega^1_{\mathbb{P}^3}\xrightarrow{i_1} V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to\mathcal{O}_{\mathbb{P}^3}\to 0, \tag{5}$$

$$0\to\Omega^2_{\mathbb{P}^3}\to\wedge^2V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-2)\to\Omega^1_{\mathbb{P}^3}\to 0, 0\to\wedge^4V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-4)\to\wedge^3V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-3)\xrightarrow{i_2}\Omega^2_{\mathbb{P}^3}\to 0,$$

induced by the Koszul complex of $V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{ev}\mathcal{O}_{\mathbb{P}^3}$. Twisting these sequences by E and passing to cohomoligy in view of (2) gives the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & H^2(E(-4))\otimes\wedge^4V^\vee & \to & H^2(E(-3))\otimes\wedge^3V^\vee & \xrightarrow{i_2} & H^2(E\otimes\Omega^2_{\mathbb{P}^3}) & \to & 0 \\ & & & & \downarrow A' & & \cong\uparrow\partial & & \\ 0 & \leftarrow & H^1(E)) & \leftarrow & H^1(E(-1))\otimes V^\vee & \xleftarrow{i_1} & H^1(E\otimes\Omega_{\mathbb{P}^3}) & \leftarrow & 0, \end{array} \tag{6}$$

where $A' := i_1\circ\partial^{-1}\circ i_2$. The Euler exact sequence (5) yields the canonical isomorphism $\omega_{\mathbb{P}^3}\xrightarrow{\simeq}\wedge^4V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau : \mathbf{k}\xrightarrow{\simeq}\wedge^4V^\vee$ induces the isomorphisms $\tilde\tau : V\xrightarrow{\simeq}\wedge^3V^\vee$ and $\hat\tau : \omega_{\mathbb{P}^3}\xrightarrow{\simeq}\mathcal{O}_{\mathbb{P}^3}(-4)$. Now the point $A = A_n$ in (4) is defined as the composition

$$A : \mathbf{k}^n\otimes V \xrightarrow[\simeq]{\tilde\tau} \mathbf{k}^n\otimes\wedge^3V^\vee \xrightarrow[\simeq]{f} H^2(E(-3))\otimes\wedge^3V^\vee \xrightarrow{A'} H^1(E(-1))\otimes V^\vee \xrightarrow[\simeq]{j} \tag{7}$$

$$\xrightarrow[\simeq]{j} H^1(E^\vee(-1))\otimes V^\vee \xrightarrow[\simeq]{SD} H^2(E(1)\otimes\omega_{\mathbb{P}^3})^\vee\otimes V^\vee \xrightarrow[\simeq]{\hat\tau} H^2(E(-3))^\vee\otimes V^\vee \xrightarrow[\simeq]{f^\vee} (\mathbf{k}^n)^\vee\otimes V^\vee,$$

where SD is the Serre duality isomorphism. One checks that A_n is a skew symmetric map depending only on the class $[E,f,j]$ and not depending on the choice of τ, and that this point $A_n\in\wedge^2((\mathbf{k}^n)^\vee\otimes V^\vee)$ lies in the direct summand $\mathbf{S}_n = S^2(\mathbf{k}^n)^\vee\otimes\wedge^2V^\vee$ of the canonical decomposition

$$\wedge^2((\mathbf{k}^n)^\vee\otimes V^\vee) = S^2(\mathbf{k}^n)^\vee\otimes\wedge^2V^\vee\oplus\wedge^2(\mathbf{k}^n)^\vee\otimes S^2V^\vee. \tag{8}$$

Here $\mathbf{S}_n$ is the space of nets of quadrics in $\mathbf{k}^n$. Following [B3], [T1] and [T2] we call A the *n-instanton net of quadrics* corresponding to the data $[E,f,j]$.

Denote $W_A := \mathbf{k}^n \otimes V/\ker A$. Using the above chain of isomorphisms we can rewrite the diagram (6) as

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker A & \longrightarrow & \mathbf{k}^n \otimes V & \xrightarrow{c_A} & W_A & \longrightarrow & 0 \\
 & & & & \downarrow A & & \cong \downarrow q_A & & \\
0 & \longleftarrow & \ker A^\vee & \longleftarrow & (\mathbf{k}^n)^\vee \otimes V^\vee & \xleftarrow{c_A^\vee} & W_A^\vee & \longleftarrow & 0.
\end{array} \tag{9}
$$

Here $\dim W_A = 2n+2$ and $q_A : W_A \xrightarrow{\simeq} W_A^\vee$ is the induced skew-symmetric isomorphism. An important property of $A = A_n([E,f,j])$ is that the induced morphism of sheaves

$$
a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{c_A^\vee} (\mathbf{k}^n)^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{ev} (\mathbf{k}^n)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \tag{10}
$$

is an epimorphism such that the composition $\mathbf{k}^n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{q_A} W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee} (\mathbf{k}^n)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ is zero, and $E = \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A$. Thus A defines a monad

$$
\mathcal{M}_A : \quad 0 \to \mathbf{k}^n \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{a_A} W_A \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{a_A^\vee \circ q_A} (\mathbf{k}^n)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0 \tag{11}
$$

with the cohomology sheaf E,

$$
E = E(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A. \tag{12}
$$

Note that passing to cohomology in the monad $\mathcal{M}_A$ twisted by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and using (12) yields the isomorphism $f : \mathbf{k}^n \xrightarrow{\simeq} H^2(E(-3))$. Furthermore, the simplecticity of the form q_A in the monad $\mathcal{M}_A$ implies that there is a canonical isomorphism of $\mathcal{M}_A$ with its dual which induces the symplectic isomorphism $j : E \xrightarrow{\simeq} E^\vee$. Thus, the data $[E,f,j]$ are recovered from the net A. This leads to the following description of the moduli space I_n. Consider the *set of n-instanton nets of quadrics*

$$
MI_n := \left\{ A \in \mathbf{S}_n \;\middle|\; \begin{array}{l} (i)\ \operatorname{rk}(A : \mathbf{k}^n \otimes V \to (\mathbf{k}^n)^\vee \otimes V^\vee) = 2n+2, \\ (ii)\ \text{the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to (\mathbf{k}^n)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \\ \qquad \text{defined by } A \text{ in (10) is surjective,} \\ (iii)\ h^0(E_2(A)) = 0, \ \text{where } E_2(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A \\ \qquad \text{and } q_A : W_A \xrightarrow{\simeq} W_A^\vee \text{ is a symplectic isomorphism} \\ \qquad \text{defined by } A \text{ in (9)} \end{array} \right\} \tag{13}
$$

The conditions (i)-(iii) here are called *Barth's coditions.* These conditions show that MI_n is naturally supplied with a structure of a locally closed subscheme of the vector space $\mathbf{S}_n$. Moreover, the above description shows that there is defined a morphism $\pi_n : MI_n \to I_n : \ A \mapsto [E(A)]$, and it is known that this morphism is a principal $GL(\mathbf{k}^n)/\{\pm id\}$-bundle in the étale topology - cf. [CTT]. Here by construction the fibre $\pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_n$ coincides with the homogeneous space $F_{[E]}$ of the group $GL(\mathbf{k}^n)/\{\pm id\}$ described above. Hence the irreducibility of $(I_n)_{red}$ is equivalent to the irreducibility of the scheme $(MI_n)_{red}$.

The definition (13) yields the following.

Theorem 3.1. *For each $n \geq 1$, the space of n-instanton nets of quadrics MI_n is a locally closed subscheme of the vector space $\mathbf{S}_n$ given locally at any point $A_n \in MI_n$ by*

$$
\binom{2n-2}{2} = 2n^2 - 5n + 3 \tag{14}
$$

equations obtained as the rank condition (i) in (13).

Note that from (14) it follows that

$$
\dim_{[A]} MI_n \geq \dim \mathbf{S}_n - (2n^2 - 5n + 3) = n^2 + 8n - 3 \tag{15}
$$

at any point $A_n \in MI_n$. On the other hand, by deformation theory for any n-instanton E we have $\dim_{[E]} I_n \geq 8n-3$. This agrees with (15), since $MI_n \to I_n$ is a principal $GL(\mathbf{k}^n)/\{\pm id\}$-bundle in the étale topology.

Let $\mathcal{S}_n = \{[E] \in I_n |$ there exists a line $l \in \mathbb{P}^3$ of maximal jump for E, i.e. such a line l that $h^0(E(-n)|_l) \neq 0\}$. It is known [S] that $\mathcal{S}_n$ is a closed subset of I_n of dimension $6n+2$. Thus, since $\dim_{[E]} I_n \geq 8n-3$ at any $[E] \in I_n$, it follows that

$$I'_n := I_n \smallsetminus \mathcal{S}_n \tag{16}$$

is an open subset of I_n and $(I'_n)_{red}$ is dense open in $(I_n)_{red}$; respectively,

$$MI'_n := \pi_n^{-1}(I'_n) \tag{17}$$

is an open subset of MI_n and we have a dense open embedding

$$(MI'_n)_{red} \xhookrightarrow{\text{dense open}} (MI_n)_{red} . \tag{18}$$

For technical reasons we will below restrict ourselves to MI'_n instead of MI_n.

4. A result of general position for $(2m+1)$-instanton nets

Definition 4.1. Let U and U' be two vector spaces of dimensions respectively m and n, where $m \geq n$. Consider the projective space $P(U \otimes U')$. We say that *a point $x \in P(U \otimes U')$ has rank r* (and denote this as $\mathrm{rk}(x) = r$), if

(i) there exist unique subspaces $U_r(x) \subset U$ and $U'_r(x) \subset U'$ of dimensions $\dim U_k(x) = \dim U'_k(x) = r$ such that $x \in P(U_r(x) \otimes U'_r(x))$, and

(ii) there do not exist subspaces $\tilde{U} \subset U$ and $\tilde{U}' \subset U'$ of dimension $\dim \tilde{U} = \dim \tilde{U}' < r$ such that $x \in P(\tilde{U} \otimes \tilde{U}')$.

It is well known that each point $x \in P(U \otimes U')$ has a uniquely defined rank $1 \leq \mathrm{rk}(x) \leq n$.

Fix a positive integer $m \geq 3$ and a $(2m+1)$-instanton vector bundle E such that $[E] \in I'_{2m+1}$ and denote $H_{2m+1} = H^2(E(-3))$ and $H_{4m} = H^2(E(-4))$. The Euler Exact sequence induces the exact triple $0 \to E \otimes \Omega_{\mathbb{P}^3} \to V^\vee \otimes E(-1) \to E \to 0$ which gives a natural multiplication map in the first cohomology:

$$H^\vee_{2m+1} \otimes V^\vee \stackrel{mult}{\to} H^\vee_{4m} \to H^2(E \otimes \Omega_{\mathbb{P}^3}). \tag{19}$$

Passing to cohomology of the exact triple $0 \to E \otimes \Omega^2_{\mathbb{P}^3} \to \wedge^2 V^\vee \otimes E(-2) \to E \otimes \Omega_{\mathbb{P}^3} \to 0$ and using standard equalities $0 = h^2(E(-2)),\ h^3(E \otimes \Omega^2_{\mathbb{P}^3}) = h^0(E \otimes \Omega_{\mathbb{P}^3}) \leq h^0(E(-1) \otimes V^\vee) = 0$ for the instanton bundle E, we obtain: $H^2(E \otimes \Omega_{\mathbb{P}^3}) = 0$. Hence (19) gives the exact triple

$$0 \to W^\vee_{4m+4} \to H^\vee_{2m+1} \otimes V^\vee \stackrel{mult}{\to} H^\vee_{4m} \to 0 \tag{20}$$

where

$$W^\vee_{4m+4} := H^1(E \otimes \Omega_{\mathbb{P}^3}). \tag{21}$$

We now prove the following main result of this section.

Theorem 4.2. *Let $m \geq 3$ and let E be a $(2m+1)$-instanton, $[E] \in I'_{2m+1}$. Consider the spaces $H_{2m+1} = H^2(E(-3))$ and $W_{4m+4} = H^1(E \otimes \Omega_{\mathbb{P}^3})^\vee$ together with the injection $W^\vee_{4m+4} \hookrightarrow H^\vee_{2m+1} \otimes V^\vee$ defined in (20). Then for a generic m-dimensional subspace V_m of $H^\vee_{2m+1}$ one has*

$$W^\vee_{4m+4} \cap V_m \otimes V^\vee = \{0\}.$$

Доказательство. According to Definition 4.1 in which we put $U = H^\vee_{2m+1}$, $U' = V^\vee$, each point $x \in P(H^\vee_{2m+1} \otimes V^\vee)$ has rank $1 \leq \mathrm{rk}(x) \leq \dim V^\vee = 4$. Thus

$$P(W^\vee_{4m+4}) = \overset{4}{\underset{r=1}{\cup}} Z_r, \tag{22}$$

where

$$Z_r := \{x \in P(W^\vee_{4m+4}) \mid rk(x) = r\}, \quad 1 \leq r \leq 4,$$

are locally closed subsets of $P(W^\vee_{4m+4})$. Consider the Grassmannian

$$G := G(m, H^\vee_{2m+1})$$

and its locally closed subsets

$$\Sigma_r = \{V_m \in G \mid V_m \supset U_r(x) \text{ for some point } x \in Z_r\}, \quad 1 \leq r \leq 4. \tag{23}$$

The condition that $Z_r \cap P(V_m \otimes V^\vee) \neq \emptyset$ means that there exists a point $x \in P(U_r) \cap Z_r$ for some r-dimensional subspace $U_r \subset V_m$. This together with (22) implies that

$$\{V_m \in G \mid P(V_m \otimes V^\vee) \cap P(W^\vee_{4m+4}) \neq \emptyset\} = \overset{4}{\underset{r=1}{\cup}} \Sigma_r.$$

Thus, to prove the Theorem, it is enough to show that

$$\dim \Sigma_r < \dim G, \quad 1 \leq r \leq 4. \tag{24}$$

We are starting now the proof of (24) for $r = 4, 3, 2, 1$.

(i) $r = 4$. Set $\Gamma_4 := \{(x, U) \in P(W^\vee_{4m+4}) \times G(4, H^\vee_{2m+1}) \mid \mathrm{rk}(x) = 4 \text{ and } U = U_4(x)\}$ and let $P(W^\vee_{4m+4}) \overset{p_4}{\leftarrow} \Gamma_4 \overset{q_4}{\rightarrow} G(4, H^\vee_{2m+1})$ be the projections. By construction, $p_4(\Gamma_4)) = Z_4$ and the morphism $p_4 : \Gamma_4 \to Z_4$ is an isomorphism. Hence

$$\dim q_4(\Gamma_4) \leq \dim \Gamma_4 = \dim Z_4 \leq \dim P(W^\vee_{4m+4}) = 4m + 3.$$

By construction we have the graph of incidence

$$\Pi_4 = \{(U, V_m) \in q_4(\Gamma_4) \times \Sigma_4 \mid U \subset V_m\}$$

with surjective projections $q_4(\Gamma_4) \overset{pr_1}{\leftarrow} \Pi_4 \overset{pr_2}{\rightarrow} \Sigma_4$ and a fibre

$$pr_1^{-1}(U) = G(m - 4, H^\vee_{2m+1}/U)$$

over an arbitrary point $U \in q_4(\Gamma_4)$. Hence

$$\dim \Sigma_4 \leq \dim \Pi_4 = \dim q_4(\Gamma_4) + \dim G(m-4, H^\vee_{2m+1}/U) \leq 4m+3+(m-4)(m+1) = m(m+1)-1 =$$

$= \dim G - 1 < \dim G$, i.e. (24) is true for $r = 4$.

(ii) $r = 3$. Consider a morphism $f_3 : Z_3 \to P(V^\vee)^\vee = \mathbb{P}^3 : x \mapsto V_3(x)$, where the pair of spaces $(U_3(x), V_3(x))$, $U_3(x) \subset H^\vee_{2m+1}$ and $V_3(x) \subset V^\vee$, is determined uniquely by the point x via the condition $x \in P(U_3(x) \otimes V_3(x))$, since $\mathrm{rk}(x) = 3$ (see Definition 4.1). Now for a given subspace $V_3 \subset V^\vee$ set

$$\Sigma_3(V_3) = \{V_m \in G \mid V_m \supset U_3(x) \text{ for some point } x \in f_3^{-1}(V_3)\}. \tag{25}$$

Comparing this with (23) for $r = 3$ yields

$$\Sigma_3 = \underset{V_3 \subset V^\vee}{\cup} \Sigma_3(V_3). \tag{26}$$

Hence,

$$\dim \Sigma_3 \leq \dim \Sigma_3(V_3) + 3. \tag{27}$$

We are going to obtain an estimate for the dimension of $\Sigma_3(V_3)$ for an arbitrary 3-dimensional subspace V_3 in $V^\vee$. This subspace defines a commutative diagram

(28)
$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F & \to & \Omega_{\mathbb{P}^3} & \to & \mathcal{I}_x(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \to & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \to & \mathcal{O}_{\mathbb{P}^3}(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_z & \to & \mathcal{O}_{\mathbb{P}^3} & \to & \mathbf{k}_z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}
$$

where $z = P(\ker : V \twoheadrightarrow V_3^\vee)$ is a point in $\mathbb{P}^3$ and the sheaf F has an $\mathcal{O}_{\mathbb{P}^3}$-resolution $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to 3\mathcal{O}_{\mathbb{P}^3}(-1) \to F \to 0$. Twisting this resolution by the vector bundle E and passing to cohomology we obtain the equalities $H^1(F \otimes E) \simeq H^2(E(-3)) = H_{2m+1},\ H^2(F \otimes E) = 0$. Respectively, passing to cohomology in diagram (28) twisted by E and using the above equalities and evident relations $H^0(E \otimes \mathbf{k}_z) \simeq \mathbf{k}^2,\ \ H^1(E \otimes \mathbf{k}_z) = 0$ implies the diagram

(29)
$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \mathbf{k}^2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_{2m+1} & \to & W_{4m+4}^\vee & \to & H^1(E \otimes \mathcal{I}_z(-1)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_{2m+1}^\vee \otimes V_3 & \xrightarrow{\lambda} & H_{2m+1}^\vee \otimes V^\vee & \to & H_{2m+1}^\vee \\
 & & \downarrow & & \downarrow{\scriptstyle mult} & & \downarrow \\
\mathbf{k}^2 & \rightarrowtail & H^1(E \otimes \mathcal{I}_z) & \to & H_{4m}^\vee & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0. & &
\end{array}
$$

In this diagram the composition $\epsilon := mult \circ \lambda$ is surjective. Hence, setting $W_{2m+3}(V_3) := \ker \epsilon$, where $\dim W_{2m+3}(V_3) = 2m+3$, we obtain a commutative diagram

(30)
$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & W_{2m+3}(V_3) & \xrightarrow{j} & W^\vee_{4m+4} & \to & H^\vee_{2m+1} \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & H^\vee_{2m+1}\otimes V_3 & \xrightarrow{\lambda} & H^\vee_{2m+1}\otimes V^\vee & \to & H^\vee_{2m+1}\\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle mult} & & \\
 & & H^\vee_{4m} & = & H^\vee_{4m} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0. & &
\end{array}
$$

Set

$$Z_3(V_3) := \{x \in P(W_{2m+3}(V_3)) \mid \mathrm{rk}(x) = 3\}.$$

The inclusion j in diagram (30) yields the bijection

$$Z_3(V_3) \overset{\simeq}{\to} f_3^{-1}(V_3). \tag{31}$$

Consider the graph of incidence $\Gamma_3(V_3) := \{(x, U) \in Z_3(V_3) \times G(3, H^\vee_{2m+1}) \mid U = U_3(x)\}$ with projections $Z_3(V_3) \overset{p_3}{\leftarrow} \Gamma_3(V_3) \overset{q_3}{\to} G(3, H^\vee_{2m+1})$. By construction, $p_3(\Gamma_3(V_3)) = Z_3(V_3)$ and the morphism $p_4 : \Gamma_3(V_3) \to Z_3(V_3)$ is an isomorphism. Hence

$$\dim q_3(\Gamma_3(V_3)) \le \dim \Gamma_3(V_3) = \dim Z_3(V_3) \le \dim P(W_{2m+3}(V_3)) = 2m+2. \tag{32}$$

Consider the graph of incidence

$$\Pi_3(V_3) = \{(U, V_m) \in q_3(\Gamma_3(V_3)) \times \Sigma_3(V_3) \mid U \subset V_m\}$$

with projections $q_3(\Gamma_3(V_3)) \overset{pr_1}{\leftarrow} \Pi_3(V_3) \overset{pr_2}{\to} \Sigma_3(V_3)$ and a fibre

$$pr_1^{-1}(U) = G(m-3, H^\vee_{2m+1}/U)$$

over an arbitrary point $U \in q_3(\Gamma_3(V_3))$. The projection $\Pi_3(V_3) \overset{pr_2}{\to} \Sigma_3(V_3)$ is surjective in view of (31). Hence, using (32), we obtain

$\dim \Sigma_3(V_3) \le \dim \Pi_3(V_3) = \dim q_3(\Gamma_3(V_3)) + \dim G(m-3, H^\vee_{2m+1}/U) \le 2m+2+(m-3)(m+1) =$

$= m^2 - 1$. This together with (27) and the assumption $m \ge 3$ yields $\dim \Sigma_3 \le m^2 + 2 = \dim G + 2 - m < \dim G$, i.e. (24) holds for $r = 3$.

Before proceeding to the case $r = 2$ we need to make a small digression on jumping lines of E. Introduce some more notation. For a given line $l \subset \mathbb{P}^3$ we have $E|l \simeq \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(-d)$ for a well-defined nonnegative integer d called the *jump of* $E|l$ and is denoted $d_E(l)$; respectively, the line l is called a *jumping line of jump* d *of* E. Set $G_{2,4} := G(2, V^\vee)$ and $J_k(E) := \{l \in G_{2,4} \mid d_E(l) \le k\}$, $J^*_k(E) := J_k(E) \smallsetminus J_{k+1}(E),\ 0 \le k$. From the semicontinuity of $E|l,\ l \in G_{2,4}$, it follows that $J_k(E)$ (resp., $J^*_k(E)$) is a closed (resp., locally closed) subset of $G_{2,4},\ k \ge 0$. Moreover, by Theorem of Grauert-Mülich, $J^*_0(E)$ is a dense open subset of $G_{2,4}$. Next, since $E \in I'_{2m+1}$, it follows that $J_{2m+1}(E) = \emptyset$, so that $J_{2m-1}(E) = J^*_{2m-1}(E) \sqcup J^*_{2m}(E)$. We will use below the following lemma.

Lemma 4.3. *(1)* $\dim J_{2m-1}(E) \le 1$.

(ii) $\dim J^*_k(E) \le 3$ *for* $1 \le k \le 2m-2$.

Proof of Lemma.

(1) Suppose the contrary, i.e. $\dim J_{2m}(E) \geq 2$. Take any irreducible surface $S \subset J_{2m}(E)$ and let D be the degree of S with respect to the sheaf $\mathcal{O}_{G_{2,4}}(1)$. Fix an integer $r \geq 5$ and take any irreducible curve C belonging to the linear series $\left|\mathcal{O}_{G_{2,4}}(r)|_S\right|$. Then the degree $\deg C$ w.r.t. $\mathcal{O}_{G_{2,4}}(1)$ equals to Dr, hence $\deg C \geq 5$. Hence by [C, Lemma 6] there exist two distinct lines, say, $l_1, l_2 \in C$, which intersect in $\mathbb{P}^3$. Let the plane $\mathbb{P}^2$ be the span of l_1 and l_2 in $\mathbb{P}^3$. Now the exact triple $0 \to E(-2)|_{\mathbb{P}^2} \to E|_{\mathbb{P}^2} \to E|_{l_1 \cup l_2} \to 0$ implies

$$H^0(E|_{\mathbb{P}^2}) \to H^0(E|_{l_1\cup l_2}) \to H^1(E(-2)|_{\mathbb{P}^2}). \tag{33}$$

Next, as $[E] \in I_{2m+1}$, we have $h^0(E(-1)) = h^1(E(-2)) = 0$, hence the exact triple $0 \to E(-2) \to E(-1) \to E(-1)|_{\mathbb{P}^2} \to 0$ implies

$$H^0(E(-1)|_{\mathbb{P}^2}) = 0. \tag{34}$$

Now assume $h^0(E|_{\mathbb{P}^2}) > 0$. Then a section $0 \neq s \in H^0(E|_{\mathbb{P}^2})$ defines an injection $\mathcal{O}_{\mathbb{P}^2} \overset{s}{\hookrightarrow} E|_{\mathbb{P}^2}$. This injection and (34) show that the zero-set Z of section s is 0-dimensional and the injection s extends to a triple $0 \to \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} E|_{\mathbb{P}^2} \to \mathcal{I}_{Z,\mathbb{P}^2} \to 0$. Whence

$$h^0(E|_{\mathbb{P}^2}) \leq 1. \tag{35}$$

Furthermore, equality together with Riemann-Roch and Serre duality for the vector bundle $E(-1)|_{\mathbb{P}^2}$ shows that $h^1(E(-2)|_{\mathbb{P}^2}) = 2m+1$. Whence in view of (33) and (34) we obtain

$$h^0(E|_{l_1\cup l_2}) \leq 2m+2. \tag{36}$$

On the other hand, let $x := l_1 \cap l_2$. Since by construction $l_1, l_2 \in J_{2m-1}(E)$, it follows that either $E|_{l_i} \simeq \mathcal{O}_{\mathbb{P}^2}(2m-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1-2m)$, or $E|_{l_i} \simeq \mathcal{O}_{\mathbb{P}^2}(2m) \oplus \mathcal{O}_{\mathbb{P}^2}(-2m)$, hence $h^0(E \otimes \mathcal{I}_{x,l_i}) \geq 2m-1$, $i = 1, 2$. This clearly implies $h^0(E|_{l_1\cup l_2}) \geq h^0(E \otimes \mathcal{I}_{x,l_1\cup l_2}) \geq h^0(E\otimes\mathcal{I}_{x,l_1}) + h^0(E\otimes\mathcal{I}_{x,l_2}) = 4m-2$. Comparing this with (36) we obtain the inequality $2m+2 \geq 4m-2$, i.e. $m \leq 2$. This contradicts to the assumption $m \geq 3$. Hence, the assertion (1) follows.

(2) This is an immediate corollary of Theorem of Grauert-Mülich. Lemma is proved. □

(iii) $r = 2$. Our notation and argument is completely parallel to that in the case $r = 3$. Consider a morphism $f_2 : Z_2 \to G_{2,4} : x \mapsto V_2(x)$, where the pair of spaces $(U_2(x), V_2(x))$, $U_2(x) \subset H^{\vee}_{2m+1}$ and $V_2(x) \subset V^{\vee}$, is determined uniquely by the point x via the condition $x \in P(U_2(x) \otimes V_2(x))$, since $\mathrm{rk}(x) = 2$ (see Definition 4.1).

According to the above remarks on jumping lines of E we may assume that $l \in J^*_k(E)$ for some $0 \leq k \leq 2m$, i.e.

$$h^0(E|l) = 2, \quad h^1(E|l) = 0, \quad \text{if} \quad l \in J^*_0(E),$$

respectively,

$$h^0(E|l) = k+1, \quad h^1(E|l) = k-1, \quad \text{if} \quad l \in J^*_k(E), \quad 1 \leq k \leq 2m.$$

Now for $1 \leq k \leq 2m$ and a given subspace $V_2 \in J^*_k$ set

$$\Sigma_{2,k}(V_2) = \{V_m \in G \mid V_m \supset U_2(x) \text{ for some point } x \in f_2^{-1}(V_2)\}. \tag{37}$$

Then similarly to (26) we have

$$\Sigma_2 = \bigcup_{k=0}^{2m} \bigcup_{V_2 \in J^*_k} \Sigma_{2,k}(V_2).$$

Hence, in view of Lemma 4.3

$$\dim \Sigma_2 \leq \max_{\substack{V_2 \in J^*_k \\ 0 \leq k \leq 2m}} (\dim \Sigma_{2,k}(V_2) + \dim J^*_k). \tag{38}$$

We are going to obtain an estimate for the dimension of $\Sigma_{2,k}(V_2)$ for an arbitrary 2-dimensional subspace V_2 in $J_k^*,\ 0 \le k \le 2m$. This subspace defines a commutative diagram

(39)
$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-2) & \xrightarrow{s} & \Omega_{\mathbb{P}^3} & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_l & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_l & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}
$$

where $l = P(\ker V \twoheadrightarrow V_2^\vee)$ is a line in $\mathbb{P}^3$, $V_2' := V^\vee / V_2$, and $F := \operatorname{coker} s$. Passing to cohomology in diagram (39) twisted by E, we obtain the diagram

(40)
$$
\begin{array}{ccccccc}
 & & & & 0 & & H^0(E|l) \\
 & & & & \downarrow & & \downarrow \\
 & & & & W_{4m+4}^\vee & = & H^1(E \otimes F) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_{2m+1}^\vee \otimes V_2 & \longrightarrow & H_{2m+1}^\vee \otimes V^\vee & \longrightarrow & H_{2m+1}^\vee \otimes V_2' \\
 & & \| & & \downarrow{\scriptstyle mult} & & \downarrow \\
H^0(E|l) & \rightarrowtail & H^1(E \otimes \mathcal{I}_l) & \longrightarrow & H_{4m}^\vee & \longrightarrow & H^1(E|l) \\
 & & & & \downarrow & & \\
 & & & & 0. & &
\end{array}
$$

Assume for definiteness that $1 \le k \le 2m$. (The case $k = 0$ is treated in a similar way.) In this case diagram (40) leads to a diagram

(41)
$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W_{k+1}(V_2) & \xrightarrow{j} & W_{4m+4}^\vee & \longrightarrow & H_{4m-k+3}^\vee \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_{2m+1}^\vee \otimes V_2 & \longrightarrow & H_{2m+1}^\vee \otimes V^\vee & \longrightarrow & H_{2m+1}^\vee \otimes V_2' \\
 & & \downarrow & & \downarrow{\scriptstyle mult} & & \downarrow \\
0 & \longrightarrow & V_{4m-k+1} & \longrightarrow & H_{4m}^\vee & \longrightarrow & W_{k-1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
\end{array}
$$

where we set $W_{k+1}(V_2) := H^0(E|l), \quad W_{k-1} := H^1(E|l), \quad V_{4m-k+1} := H_{2m+1}^\vee \otimes V_2 / W_{k+1}(V_2)$.

Set

$$Z_{2,k}(V_2) := \{x \in P(W_{k+1}(V_2)) \mid \mathrm{rk}(x) = 2\}.$$

The inclusion j in diagram (41) yields the bijection

$$Z_{2,k}(V_2) \xrightarrow{\simeq} f_2^{-1}(V_2). \tag{42}$$

Consider the graph of incidence $\Gamma_{2,k}(V_2) := \{(x, U) \in Z_{2,k}(V_2) \times G(2, H_{2m+1}^\vee) \mid U = U_2(x)\}$ with projections $Z_{2,k}(V_2) \xleftarrow{p_2} \Gamma_{2,k}(V_2) \xrightarrow{q_2} G(2, H_{2m+1}^\vee)$. By construction, $p_2(\Gamma_{2,k}(V_2)) = Z_{2,k}(V_2)$ and the morphism $p_4 : \Gamma_{2,k}(V_2) \to Z_{2,k}(V_2)$ is an isomorphism. Hence

$$\dim q_2(\Gamma_{2,k}(V_2)) \le \dim \Gamma_{2,k}(V_2) = \dim Z_{2,k}(V_2) \le \dim P(W_{k+1}(V_2)) = k. \tag{43}$$

Consider the graph of incidence

$$\Pi_{2,k}(V_2) = \{(U, V_m) \in q_2(\Gamma_{2,k}(V_2)) \times \Sigma_{2,k}(V_2) \mid U \subset V_m\}$$

with projections $q_2(\Gamma_{2,k}(V_2)) \xleftarrow{pr_1} \Pi_{2,k}(V_2) \xrightarrow{pr_2} \Sigma_{2,k}(V_2)$ and a fibre

$$pr_1^{-1}(U) = G(m-2, H_{2m+1}^\vee/U)$$

over an arbitrary point $U \in q_2(\Gamma_{2,k}(V_2))$. The projection $\Pi_{2,k}(V_2) \xrightarrow{pr_2} \Sigma_{2,k}(V_2)$ is surjective in view of (42). Hence using (43) we obtain

$$\dim \Sigma_{2,k}(V_2) \le \dim \Pi_{2,k}(V_2) = \dim q_2(\Gamma_{2,k}(V_2)) + \dim G(m-2, H_{2m+1}^\vee/U) \le k + (m-2)(m+1) =$$

$$= m^2 - m - 2 + k = \dim G - (2m - k + 2), \qquad 1 \le k \le 2m.$$

In a similar way we obtain for $k = 0$

$$\dim \Sigma_{2,0}(V_2) \le 1 + (m-2)(m+1) = m^2 - m - 1 = \dim G - (2m+1).$$

The last two inequalities together with (38), Lemma 4.3 and the assumption $m \ge 3$ yield $\dim \Sigma_2 < \dim G$, i.e. (24) is true for $r = 2$.

(ii) $r = 1$. Consider a morphism $f_1 : Z_1 \to P(V^\vee) = (\mathbb{P}^3)^\vee : x \mapsto V_1(x)$, where the pair of spaces $(U_1(x), V_1(x))$, $\quad U_1(x) \subset H_{2m+1}^\vee$ and $V_1(x) \subset V^\vee$, is determined uniquely by the point x via the condition $x \in P(U_1(x) \otimes V_1(x))$, since $\mathrm{rk}(x) = 1$ (see Definition 4.1). Now for a given subspace $V_1 \in (\mathbb{P}^3)^\vee$ set

$$\Sigma_1(V_1) := \{V_m \in G \mid V_m \supset U_1(x) \text{ for some point } x \in f_1^{-1}(V_1)\}.$$

Then similar to (26) we have

$$\Sigma_1 = \bigcup_{V_1 \in (\mathbb{P}^3)^\vee} \Sigma_1(V_1). \tag{44}$$

Hence,

$$\dim \Sigma_1 \le \dim \Sigma_1(V_1) + 3. \tag{45}$$

We are going to obtain an estimate for the dimension of $\Sigma_1(V_1)$ for an arbitrary 1-dimensional subspace V_1 in $V^\vee$. This subspace defines a commutative diagram

(46)
$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_{\mathbb{P}^3} & = & \Omega_{\mathbb{P}^3} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Note that to the point $V_1 \in (\mathbb{P}^3)^\vee$ there clearly corresponds a projective plane $P(V_1)$ in $\mathbb{P}^3$. Set $B(E) := \{V_1 \in (\mathbb{P}^3)^\vee \mid h^0(E|_{P(V_1)}) \neq 0\}$. It is known that, for $m \geq 1$,

$$\dim B(E) \leq 2.$$

(see [B1]). Moreover, in view of (35)

$$h^0(E|_{P(V_1)}) = 1, \quad V_1 \in B(E).$$

Passing to cohomology in diagram (46) twisted by E and using the equality $h^0(E) = 0$ for $[E] \in I_{2m+1}$ we obtain the diagram

(47)
$$
\begin{array}{ccccccc}
 & & & & 0 & & H^0(E|_{P(V_1)}) \\
 & & & & \downarrow & & \downarrow \\
 & & & & W_{4m+4}^\vee & = & W_{4m+4}^\vee \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_{2m+1}^\vee \otimes V_1 & \xrightarrow{\lambda} & H_{2m+1}^\vee \otimes V^\vee & \longrightarrow & H_{2m+1}^\vee \otimes V_3 \\
 & & \| & & \downarrow{\scriptstyle mult} & & \downarrow \\
H^0(E|_{P(V_1)}) & \rightarrowtail & H_{2m+1}^\vee & \longrightarrow & H_{4m}^\vee & \longrightarrow & H^1(E|_{P(V_1)}) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
\end{array}
$$

Let $V_1 \in B(E)$. Setting $\epsilon := mult \circ \lambda$ and $W_1(V_1) := \ker \epsilon = H^0(E|_{P(V_1)})$, where $\dim W_1(V_1) = 1$, we obtain from (47) a commutative diagram

(48)
$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & W_1(V_1) & \xrightarrow{\ j\ } & W^\vee_{4m+4} & \longrightarrow & W^\vee_{4m+4}/W_1(V_1) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^\vee_{2m+1} \otimes V_1 & \xrightarrow{\ \lambda\ } & H^\vee_{2m+1} \otimes V^\vee & \longrightarrow & H^\vee_{2m+1} \otimes V_3 \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle mult} & & \downarrow \\
0 & \longrightarrow & H^\vee_{2m+1}/W_1(V_1) & \longrightarrow & H^\vee_{4m} & \longrightarrow & H^1(E|_{\mathbb{P}^2(V_1)}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
\end{array}$$

Set

$$Z_1(V_1) := \emptyset \ \text{ if } \ V_1 \neq B(E), \quad \text{resp.,} \ \ Z_1(V_1) := j(W_1(V_1)) \ \text{ if } \ V_1 \in B(E).$$

The diagrams (47) and (48) yield the bijection

(49) $$Z_1(V_1) \xrightarrow{\simeq} f_1^{-1}(V_1), \quad V_1 \in (\mathbb{P}^3)^\vee.$$

The rest argument is completely the same as in cases $r = 3$ and $r = 2$ above. Consider the graph of incidence $\Gamma_1(V_1) := \{(x, U) \in Z_1(V_1) \times P(H^\vee_{2m+1}) \mid U = U_1(x)\}$ with projections $Z_1(V_1) \xleftarrow{p_1} \Gamma_1(V_1) \xrightarrow{q_1} P(H^\vee_{2m+1})$. By construction, $p_1(\Gamma_1(V_1)) = Z_1(V_1)$ and the morphism $p_4 : \Gamma_1(V_1) \to Z_1(V_1)$ is an isomorphism. Hence

(50) $$\dim q_1(\Gamma_1(V_1)) \leq \dim \Gamma_1(V_1) = \dim Z_1(V_1) \leq 0.$$

Consider the graph of incidence

$$\Pi_1(V_1) = \{(U, V_m) \in q_1(\Gamma_1(V_1)) \times \Sigma_1(V_1) \mid U \subset V_m\}$$

with projections $q_1(\Gamma_1(V_1)) \xleftarrow{pr_1} \Pi_1(V_1) \xrightarrow{pr_2} \Sigma_1(V_1)$ and a fibre

$$pr_1^{-1}(U) = G(m-1, H^\vee_{2m+1}/U)$$

over an arbitrary point $U \in q_1(\Gamma_1(V_1))$. The projection $\Pi_1(V_1) \xrightarrow{pr_2} \Sigma_1(V_1)$ is surjective in view of (49). Hence in view of (50) we have

$\dim \Sigma_1(V_1) \leq \dim \Pi_1(V_1) = \dim q_1(\Gamma_1(V_1)) + \dim G(m-1, H^\vee_{2m+1}/U) \leq 0 + (m-1)(m+1) =$ $= m^2 - 1$. This together with (45) and the assumption $m \geq 3$ yields $\dim \Sigma \leq m^2 + 2 = \dim G + 2 - m < \dim G$, i.e. (24) holds for $r = 1$. Theorem is proved. □

5. Decomposition $\mathbf{k}^{2m+1} \simeq \mathbf{k}^{m+1} \oplus \mathbf{k}^m$ and related constructions

5.1. Decomposition $\mathbf{k}^{2m+1} \simeq \mathbf{k}^{m+1} \oplus \mathbf{k}^m$.

Fix an isomorphism

(51) $$\xi : \mathbf{k}^{m+1} \oplus \mathbf{k}^m \xrightarrow{\simeq} \mathbf{k}^{2m+1}$$

and let

(52) $$\mathbf{k}^{m+1} \overset{i_{m+1}}{\hookrightarrow} \mathbf{k}^{m+1} \oplus \mathbf{k}^m \overset{i_m}{\hookleftarrow} \mathbf{k}^m$$

be the injections of direct summands. For a given $(2m+1)$-instanton vector bundle E, $[E] \in I'_{2m+1}$, fix an isomorphism $f : \mathbf{k}^{2m+1} \xrightarrow{\simeq} H^2(E(-3)) = H_{2m+1}$ and a symplectic structure $j : E \xrightarrow{\simeq} E^\vee$. The data $[E, f, j]$ define a net of quadrics $A \in MI'_{2m+1}$ (see section 3), and the exact triple (20) is naturally identified with the dual to the triple $0 \to \ker A \to \mathbf{k}^{2m+1} \otimes V \to W_A \to 0$ and fits in diagram (9) for $n = 2m+1$

(53)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker A & \longrightarrow & \mathbf{k}^{2m+1} \otimes V & \xrightarrow{c_A} & W_A & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle A} & & \cong \downarrow{\scriptstyle q_A} & & \\
0 & \longleftarrow & \ker A^\vee & \longleftarrow & (\mathbf{k}^{2m+1})^\vee \otimes V^\vee & \xleftarrow{c_A^\vee} & W_A^\vee & \longleftarrow & 0.
\end{array}$$

Consider the composition

(54)
$$j_{\xi,A} : \mathbf{k}^{m+1} \otimes V \overset{i_{m+1}}{\hookrightarrow} \mathbf{k}^{m+1} \otimes V \oplus \mathbf{k}^m \otimes V \underset{\simeq}{\overset{\xi}{\to}} \mathbf{k}^{2m+1} \otimes V \xrightarrow{c_A} W_A.$$

Under these notations Theorem 4.2 can be reformulated in the following way:

(*) *Assume $m \geq 3$ and let A be an arbitrary $(2m+1)$-net from MI'_{2m+1}. Then for a generic isomorphism $\xi : \mathbf{k}^{2m+1} \xrightarrow{\simeq} \mathbf{k}^{m+1} \oplus \mathbf{k}^m$ one has*

(55)
$$\ker A \cap \xi \circ i_{m+1}(\mathbf{k}^{m+1} \otimes V) = \{0\}.$$

Equivalently, $j_{\xi,A} :\ \mathbf{k}^{m+1} \otimes V \to W_A$ is an isomorphism.

Consider the direct sum decomposition corresponding to the isomorphism (51)

(56)
$$\widetilde{\xi} : \mathbf{S}_{m+1} \quad \oplus \quad (\mathbf{k}^m)^\vee \otimes (\mathbf{k}^{m+1})^\vee \otimes \wedge^2 V^\vee \quad \oplus \quad \mathbf{S}_m \xrightarrow{\sim} \mathbf{S}_{2m+1}$$

and let

(57)
$$\begin{aligned}
&\xi_1 : \mathbf{S}_{2m+1} \twoheadrightarrow \mathbf{S}_{m+1}, \\
&\xi_2 : \mathbf{S}_{2m+1} \twoheadrightarrow (\mathbf{k}^m)^\vee \otimes (\mathbf{k}^{m+1})^\vee \otimes \wedge^2 V^\vee, \\
&\xi_3 : \mathbf{S}_{2m+1} \twoheadrightarrow \mathbf{S}_m
\end{aligned}$$

be projections onto summands. By definition, $\xi_1(A)$ considered as a skew-symmetric homomorphism $\mathbf{k}^{m+1} \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee$ coincides with the composition

(58)
$$\xi_1(A) :\ \mathbf{k}^{m+1} \otimes V \xrightarrow{j_{\xi,A}} W_A \underset{\simeq}{\xrightarrow{q_A}} W_A^\vee \xrightarrow{j_{\xi,A}^\vee} (\mathbf{k}^{m+1})^\vee \otimes V^\vee.$$

This means that assertion (*) can be reformulated as:

(**) *Assume $m \geq 3$ and let A be an arbitrary $(2m+1)$-net from MI'_{2m+1}. Then for a generic isomorphism ξ in (51) the skew-symmetric homomorphism $\xi_1(A) :\ \mathbf{k}^{m+1} \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee$ is invertible.*

For A and ξ from (**) we have the commutative diagram

(59)
$$\begin{array}{ccccccc}
 & \mathbf{k}^{m+1} \otimes V & \xrightarrow[\simeq]{\xi_1(A)} & (\mathbf{k}^{m+1})^\vee \otimes V^\vee & \\
\swarrow{\scriptstyle i_{m+1}} & & & & \nwarrow{\scriptstyle i_{m+1}^\vee} \\
\mathbf{k}^{m+1} \otimes V \oplus \mathbf{k}^m \otimes V & & \xrightarrow{\xi(A)} & & (\mathbf{k}^{m+1})^\vee \otimes V^\vee \oplus (\mathbf{k}^m)^\vee \otimes V^\vee \\
{\scriptstyle \xi} \downarrow \simeq & {\scriptstyle j_{\xi,A}} \downarrow \simeq & & \simeq \uparrow {\scriptstyle j_{\xi,A}^\vee} & \simeq \uparrow {\scriptstyle \xi^\vee} \\
\mathbf{k}^{2m+1} \otimes V & & \xrightarrow{A} & & (\mathbf{k}^{2m+1})^\vee \otimes V^\vee \\
\searrow{\scriptstyle c_A} & & & & \nearrow{\scriptstyle c_A^\vee} \\
 & W_A & \xrightarrow[\simeq]{q_A} & W_A^\vee, &
\end{array}$$

where $\xi(A)$ is the matrix $\begin{pmatrix} \xi_1(A) & \xi_2(A)^\vee \\ \xi_2(A) & \xi_3(A) \end{pmatrix}$. As $j_{\xi,A}$ in this diagram is invertible, the composition

$$g_{\xi,A} = j_{\xi,A}^{-1} \circ c_A \circ \xi \circ i_m$$

is well-defined, and we obtain a commutative diagram

(60)
$$\begin{array}{ccc} \mathbf{k}^m \otimes V & \xrightarrow{\xi_3(A)} & (\mathbf{k}^m)^\vee \otimes V^\vee \\ \downarrow g_\xi & \searrow^{\xi_2(A)} \nearrow_{\xi_2(A)^\vee} & \uparrow g_\xi^\vee \\ \mathbf{k}^{m+1} \otimes V & \underset{\simeq}{\xrightarrow{\xi_1(A)}} & (\mathbf{k}^{m+1})^\vee \otimes V^\vee. \end{array}$$

In particular,

(61)
$$\xi_3(A) = \xi_2(A)^\vee \circ \xi_1(A)^{-1} \circ \xi_2(A).$$

For $m \geq 1$ let

$$\mathrm{Isom}_{2m+1}$$

be the set of all isomorphisms ξ in (51). Consider the open subset MI'_{2m+1} of MI_{2m+1} defined in (17) and set

(62)
$$MI_{2m+1}(\xi) := \{A \in MI'_{2m+1} \mid \text{the skew}-\text{symmetric homomorphism } \xi_1(A) \text{ in (58) is invertible}\}, \quad \xi \in \mathrm{Isom}_{2m+1}.$$

The relation (61) together with (**) implies the following corollary of Theorem 4.2.

Theorem 5.1. *Fom $m \geq 3$ the following statements hold.*

(i) The sets $MI_{2m+1}(\xi)$, $\xi \in \mathrm{Isom}_{2m+1}$, are dense open subsets of the set MI'_{2m+1} constituting its open cover.

(ii) For any $\xi \in \mathrm{Isom_{2m+1}}$ and any $A \in MI_{2m+1}(\xi)$ the relation (61) is true.

We will need below the following lemma.

Lemma 5.2. *Let ξ and $A \in MI_{2m+1}(\xi)$ be as in Theorem 5.1 and set*

(63)
$$B := \xi_1(A), \quad C := \xi_2(A).$$

Then the following statements hold.

(i) Consider a subbundle morphism

(64)
$$\alpha_{\xi,A} := j_\xi^{-1} \circ a_A \circ \xi : (\mathbf{k}^{m+1} \oplus \mathbf{k}^m) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathbf{k}^{m+1} \otimes V \otimes \mathcal{O}_{\mathbb{P}^3}.$$

Then there exists an epimorphism

(65)
$$\lambda_{\xi,A} : \mathrm{coker}(B \circ \alpha_{\xi,A}) \twoheadrightarrow (\mathbf{k}^{m+1})^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1).$$

making commutative the diagram

(66)
$$\begin{array}{ccc} (\mathbf{k}^{m+1})^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{can} & \mathrm{coker}(B \circ \alpha_{\xi,A}) \\ & \searrow^{u^\vee} & \downarrow \lambda_{\xi,A} \\ & & (\mathbf{k}^{m+1})^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1), \end{array}$$

where can is a canonical surjection.

(ii) Consider the commutative diagram

(67)

$$\begin{array}{ccccccccc}
 & & \mathbf{k}^m\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & (\mathbf{k}^{m+1}\oplus\mathbf{k}^m)\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{B\circ\alpha_{\xi,A}} & (\mathbf{k}^{m+1})^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{can} & \operatorname{coker}(B\circ\alpha_{\xi,A}) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle i_{m+1}} & & \| & & \uparrow{\scriptstyle \epsilon_{\xi,A}} & & \\
0 & \longrightarrow & \mathbf{k}^{m+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{B\circ u} & (\mathbf{k}^{m+1})^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{v\circ B^{-1}} & \mathbf{k}^{m+1}\otimes T_{\mathbb{P}^3}(-1) & \longrightarrow & 0\\
 & & & & & & \uparrow{\scriptstyle \tau_{\xi,A}} & & \\
 & & & & & & \mathbf{k}^m\otimes\mathcal{O}_{\mathbb{P}^3}(-1), & &
\end{array}$$

where $\tau_{\xi,A}$ and $\epsilon_{\xi,A}$ are the induced morphisms. Then the morphism $\tau_{\xi,A}$ is a subbundle morphism fitting in a commutative diagram

(68)
$$\begin{array}{ccc}
(\mathbf{k}^{m+1})^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{v\circ B^{-1}} & \mathbf{k}^{m+1}\otimes T_{\mathbb{P}^3}(-1)\\
\uparrow{\scriptstyle C\circ u} & & \uparrow{\scriptstyle \tau_{\xi,A}}\\
\mathbf{k}^m\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & = & \mathbf{k}^m\otimes\mathcal{O}_{\mathbb{P}^3}(-1).
\end{array}$$

Доказательство. (i) Consider the commutative diagram

(69)
$$\begin{array}{ccccccc}
\mathbf{k}^{2m+1}\otimes\mathcal{O}(-1) & \xrightarrow{a_A} & W_A\otimes\mathcal{O} & \xrightarrow[\simeq]{q_A} & W_A^\vee\otimes\mathcal{O} & \xrightarrow{a_A^\vee} & (\mathbf{k}^{2m+1})^\vee\otimes\mathcal{O}(1)\\
\uparrow{\scriptstyle \xi}\ \simeq & & \uparrow{\scriptstyle j_{\xi,A}}\ \simeq & & \simeq\ \downarrow{\scriptstyle j^\vee_{\xi,A}} & & \simeq\ \downarrow{\scriptstyle \xi^\vee}\\
(\mathbf{k}^{m+1}\oplus\mathbf{k}^m)\otimes\mathcal{O}(-1) & \xrightarrow{\alpha_{\xi,A}} & (\mathbf{k}^{m+1})\otimes V\otimes\mathcal{O} & \xrightarrow[\simeq]{B} & (\mathbf{k}^{m+1})^\vee\otimes V^\vee\otimes\mathcal{O} & \xrightarrow{\alpha^\vee_{\xi,A}} & (\mathbf{k}^{m+1}\oplus\mathbf{k}^m)^\vee\otimes\mathcal{O}(1)\\
\uparrow{\scriptstyle i_{m+1}} & \nearrow{\scriptstyle u} & & & & \searrow{\scriptstyle u^\vee} & \downarrow{\scriptstyle i^\vee_{m+1}}\\
\mathbf{k}^{m+1}\otimes\mathcal{O}(-1) & & & & & & (\mathbf{k}^{m+1})^\vee\otimes\mathcal{O}(1)
\end{array}$$

Here the upper triple is the monad (11) for $n = 2m+1$. Whence the statement (i) follows.

(ii) Standard diagram chasing using (63) and diagrams (59) and (67). □

5.2. Remarks on t'Hooft instantons.

Consider the set

$$I^{tH}_{2m+1} := \{[E] \in I_{2m+1} \mid h^0(E(1)) \neq 0\},$$

of *t'Hooft instanton bundles* and the corresponding set of *t'Hooft instanton nets*

$$MI^{tH}_{2m+1} := \pi_n^{-1}(I^{tH}_{2m+1}).$$

We collect some well-known facts about I^{tH}_{2m+1} in the following lemma - see [BT], [NT], [T2, Prop. 2.2].

Lemma 5.3. *Let $m \geq 1$. Then the following statements hold.*

(i) I^{tH}_{2m+1} is an irreducible $(10m+9)$-dimensional subvariety of I_{2m+1}. Respectively, MI^{tH}_{2m+1} is an irreducible $(4m^2+14m+10)$-dimensional subvariety of I_{2m+1}.

(ii) $I^{tH*}_{2m+1} := I^{tH}_{2m+1} \cap I'_{2m+1}$ is a smooth dense open subset of I^{tH}_{2m+1} and

(70)
$$h^0(E(1)) = 1, \qquad [E] \in I^{tH*}_{2m+1}.$$

(iii) MI^{tH}_{2m+1} *is a smooth dense open subset of the set*

$$TH_{2m+1} := \{A \in \mathbf{S}_{2m+1} | A = \sum_{i=1}^{2m+2} h^2 \otimes w, \text{ where } h \in (\mathbf{k}^{2m+1})^\vee, w \in \wedge^2 V^\vee, \ w \wedge w = 0\}. \tag{71}$$

We are going to extend the statement of Theorem 5.1 to the cases $m = 1$ and 2. To this end, for $m = 1, 2$ and $\xi \in \mathrm{Isom}_{2m+1}$ consider the sets $MI_{2m+1}(\xi)$ defined in (62) and set

$$MI''_{2m+1} := \underset{\xi \in \mathrm{Isom}_{2m+1}}{\cup} MI_{2m+1}(\xi), \quad m = 1, 2. \tag{72}$$

For $m \geq 1$ let $\xi^0 \in \mathrm{Isom}_{2m+1}$ be the standard isomorphism $\mathbf{k}^{m+1} \oplus \mathbf{k}^m \xrightarrow{\sim} \mathbf{k}^{m+1}$: $((a_1, ..., a_{m+1}), (a_{m+2}, ..., a_{2m+1})) \mapsto (a_1, ..., a_{2m+1})$. Let $\{h_1 = (1, 0, ..., 0), ..., h_{2m+1}(0, ..., 0, 1)$ be the standard basis in $(\mathbf{k}^{2m+1})^\vee$ and let $e_1, ..., e_4$ be some fixed basis in $V^\vee$. Consider the nets $A_{(m)} \in TH_{2m+1}, \quad m = 1, 2$, defined as follows

$$A_{(1)} = h_1^2 \otimes (e_1 \wedge e_2 + e_3 \wedge e_4) + h_2^2 \otimes (e_1 \wedge e_3 + e_4 \wedge e_2), \tag{73}$$

$$A_{(2)} = h_1^2 \otimes (e_1 \wedge e_2 + e_3 \wedge e_4) + h_2^2 \otimes (e_1 \wedge e_3 + e_4 \wedge e_2) + h_3^2 \otimes (e_1 \wedge e_4 + e_2 \wedge e_3).$$

It is an exercise to show that the homomorphisms

$$\xi_1^0(A_{(m)}) : \mathbf{k}^{m+1} \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee, \quad m = 1, 2,$$

are invertible. On the other hand, for a given $\xi \in \mathrm{Isom}_{2m+1}$ the condition that a homomorphism $\xi_1(A) : \mathbf{k}^{m+1} \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee$ is invertible is an open condition on the net $A \in TH_{2m+1}$. Hence, since the sets $MI'_{2m+1}, \ m = 1, 2$, are irreducible, Lemma 5.3 yields the following corollary.

Corollary 5.4. *Let* $1 \leq m \leq 2$.

(i) For $m = 1, 2$ *the set* MI''_{2m+1} *is a dense open subset of* MI'_{2m+1} *and of* MI_{2m+1}*, and the statement of Theorem 5.1 extends to the cases* $m = 1$ *and 2, with* MI'_{2m+1} *being substituted by* MI''_{2m+1}.

(ii) Let $m \geq 1$. *The set*

$$MI^{tH**}_{2m+1} := \begin{cases} MI^{tH*}_{2m+1}, & m \geq 3, \\ MI''_{2m+1} \cap MI^{tH*}_{2m+1}, & m = 1, 2, \end{cases}$$

is a dense open subset of MI^{tH*}_{2m+1} *and of* MI^{tH}_{2m+1} *covered by dense open subsets*

$$MI^{tH}_{2m+1}(\xi) := MI^{tH**}_{2m+1} \cap MI_{2m+1}(\xi), \quad \xi \in \mathrm{Isom}_{2m+1}. \tag{74}$$

Note that (18), Theorem 5.1 and Corollary 5.4 yield

Corollary 5.5. *Let* $m \geq 1$. *Then for any* $\xi \in \mathrm{Isom}_{2m+1}$ *the scheme* $(MI_{2m+1}(\xi))_{red}$ *is dense open in* $(MI_{2m+1})_{red}$. *In particular,*

$$\dim MI_{2m+1}(\xi) = \dim MI_{2m+1}. \tag{75}$$

5.3. Invertible nets of quadrics from $\mathbf{S}_{m+1}$ and symplectic rank-$(2m + 2)$ bundles.

Introduce more notations. Set

$$N_{m+1} := \{B \in \mathbf{S}_{m+1} \mid B : \mathbf{k}^{m+1} \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee \text{ is an invertible homomorphism}\}. \tag{76}$$

The set N_{m+1} is a dense open subset of the vector space $\mathbf{S}_{m+1}$, and it is easy to see that for any $B \in N_{m+1}$ the following conditions are satisfied.

(1) The morphism $\widetilde{B} : \mathbf{k}^{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to (\mathbf{k}^{m+1})^\vee \otimes \Omega_{\mathbb{P}^3}(1)$ induced by the homomorphism $B : \mathbf{k}^{m+1} \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee$ is a subbundle morphism, i.e.

$$E_{2m+2}(B) := \mathrm{coker}(\widetilde{B}) \tag{77}$$

is a vector bundle of rank $2m+2$ на $\mathbb{P}^3$. This follows from the diagram

(78)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbf{k}^{m+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\widetilde{B}} & (\mathbf{k}^{m+1})^\vee\otimes\Omega_{\mathbb{P}^3}(1) & \xrightarrow{e} & E_{2m+2}(B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v^\vee} & & & & \\
 & & \mathbf{k}^{m+1}\otimes V\otimes\mathcal{O}_{\mathbb{P}^3} & \underset{\simeq}{\xrightarrow{B}} & (\mathbf{k}^{m+1})^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & & & & \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u^\vee} & & & & \\
0\to E_{2m+2}(B)^\vee & \longrightarrow & \mathbf{k}^{m+1}\otimes T_{\mathbb{P}^3}(-1) & \xrightarrow{\widetilde{B}^\vee} & (\mathbf{k}^{m+1})^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

(2) The homomorphism $^\sharp B:\mathbf{k}^{m+1}\to(\mathbf{k}^{m+1})^\vee\otimes\wedge^2V^\vee$ induced by $B:\mathbf{k}^{m+1}\otimes V\to(\mathbf{k}^{m+1})^\vee\otimes V^\vee$ is injective. This follows from the commutative diagram extending the upper horizontal triple in (78)

(79)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\mathbf{k}^{m+1})^\vee\otimes T_{\mathbb{P}^3}(-2) & = & (\mathbf{k}^{m+1})^\vee\otimes T_{\mathbb{P}^3}(-2) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{k}^{m+1}\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{^\sharp B} & (\mathbf{k}^{m+1})^\vee\otimes\wedge^2V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{can} & H^0(E_{2m+2}(B)(1))\otimes\mathcal{O}_{\mathbb{P}^3} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle ev} & & \\
0 & \longrightarrow & \mathbf{k}^{m+1}\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\widetilde{B}} & (\mathbf{k}^{m+1})^\vee\otimes\Omega_{\mathbb{P}^3}(2) & \xrightarrow{e} & E_{2m+2}(B)(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}
$$

where w is the morphism induced by the morphism v from the Euler exact sequence in (78). From this diagram we obtain the isomorphism

$$\operatorname{coker}(^\sharp B)\simeq H^0(E_{2m+2}(B)(1)). \tag{80}$$

(3) Diagram (78) and the Five-Lemma yield an isomorphism

$$\theta: E_{2m+2}(B)\xrightarrow{\sim}E_{2m+2}(B)^\vee \tag{81}$$

which is in fact symplectic,

$$\theta^\vee=-\theta,$$

since the homomorphism $B : \mathbf{k}^m \otimes V \to (\mathbf{k}^m)^\vee \otimes V^\vee$ is skew-symmetric. The isomorphism θ together with the upper triple from (78) and its dual fits in the commutative diagram

(82)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{k}^{m+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\widetilde{B}} & (\mathbf{k}^{m+1})^\vee\otimes\Omega_{\mathbb{P}^3}(1) & \xrightarrow{e} & E_{2m+2}(B) & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle v^\vee} & & \downarrow{\scriptstyle e^\vee\circ\theta} & & \\
0 & \to & \mathbf{k}^{m+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{B\circ u} & (\mathbf{k}^{m+1})^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{v\circ B^{-1}} & \mathbf{k}^{m+1}\otimes T_{\mathbb{P}^3}(-1) & \to & 0 \\
 & & & & \downarrow{\scriptstyle u^\vee} & & \downarrow{\scriptstyle \widetilde{B}^\vee} & & \\
 & & & & (\mathbf{k}^{m+1})^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & = & (\mathbf{k}^{m+1})^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

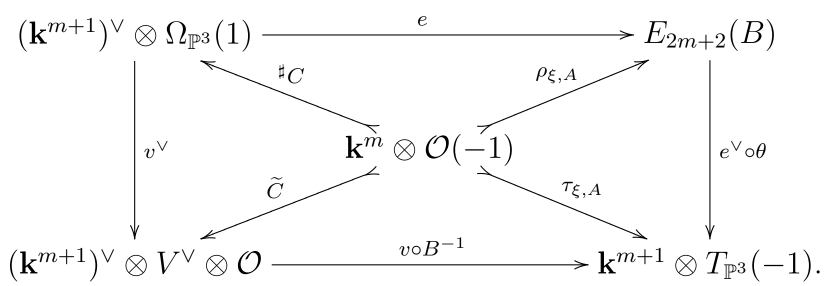
Note that this diagram immediately implies that

(83) $$h^0(E_{2m+2}(B)) = h^i(E_{2m+2}(B)(-2)) = 0, \quad i \geq 0.$$

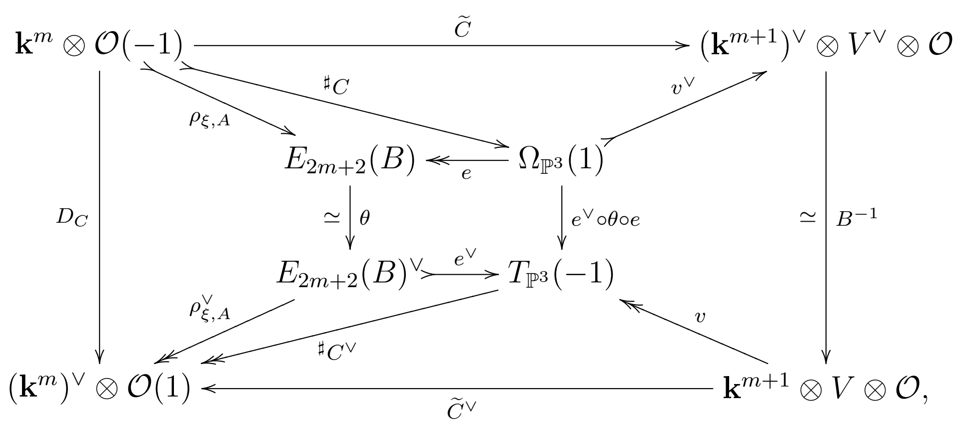
Let ξ and $A \in MI_{2m+1}(\xi)$ be as in Theorem 5.1 for $m \geq 3$, respectively, in Corollary 5.4 for $m = 1, 2$. Then the homomorphism $B : \mathbf{k}^{m+1} \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee$ defined in (63) by definition lies in N_{m+1}. Hence by Lemma 5.2 diagrams (66) and (66) hold. These diagrams together with (82) imply $\widetilde{B}^\vee \circ \tau_{\xi,A} = 0$, so that there exists a morphism

(84) $$\rho_{\xi,A} : \mathbf{k}^m \otimes \mathcal{O}(-1) \to E_{2m+2}(B)$$

such that $\tau_{\xi,A} = e^\vee \circ \theta \circ \rho_{\xi,A}$. Since $\tau_{\xi,A}$ is a subbundle morphism, $\rho_{\xi,A}$ is also a subbundle morphism. Moreover, diagrams (68) and (82) yield the commutative diagram

(85)


Diagrams (82) and (85) yield the commutative diagram

(86)


where $D_C := \widetilde{C}^\vee \circ B^{-1} \circ \widetilde{C} = u^\vee \circ (C^\vee \circ B^{-1} \circ C) \circ u$ is the zero map. In fact, by (61) and (63) we have $D_C = p_2(\xi_3(A))$, where $p_2 : \wedge^2((\mathbf{k}^n)^\vee \otimes V^\vee) \to \wedge^2(\mathbf{k}^n)^\vee \otimes S^2V^\vee$ is the projection onto the second direct summand of the decomposition (8). Since by (57) $\xi_3(A)$ lies in the first direct summand of (8) it follows that $D_C = 0$. We thus obtain the monad

$$0 \to \mathbf{k}^m \otimes \mathcal{O}(-1) \xrightarrow{\rho_{\xi,A}} E_{2m+2}(B) \xrightarrow{\theta\circ\rho^\vee_{\xi,A}} (\mathbf{k}^m)^\vee \otimes \mathcal{O}(1) \to 0 \tag{87}$$

with the cohomology sheaf

$$E_2(\xi, A) := \ker(\theta \circ \rho^\vee_{\xi,A})/\operatorname{Im}\rho_{\xi,A} \tag{88}$$

which is a vector bundle since $\rho_{\xi,A}$ is a subbundle morphism. Furthermore, by (83) it follows from the monad (87) that $E_2(\xi, A)$ is a $(2m + 1)$-instanton,

$$[E_2(\xi, A)] \in I_{2m+1}. \tag{89}$$

Lemma 5.6. *$E_2(\xi, A) \simeq E(A)$, where the sheaf $E(A)$ is defined in (12).*

Доказательство. Diagram chasing using (59), (60), (67)-(69), (78)-(79) and (82). □

6. Scheme X_m. An isomorphism between X_m and an open subset of the space MI_{2m+1}

6.1. Space X_m. Consider the vector space $\mathbf{S}_{m+1}$, respectively, its dual space $\mathbf{S}^\vee_{m+1}$ and set

$$(\mathbf{S}^\vee_{m+1})^0 := \{B \in \mathbf{S}^\vee_{m+1} \mid D : (\mathbf{k}^{m+1})^\vee \otimes V^\vee \to \mathbf{k}^{m+1} \otimes V \text{ is an invertible homomorphism}\}, \tag{90}$$

$$\mathbf{\Sigma}_{m+1} := \operatorname{Hom}(\mathbf{k}^m, (\mathbf{k}^{m+1})^\vee \otimes \wedge^2V^\vee) \tag{91}$$

According to our convention on notations we will understand an arbitrary point $C \in \mathbf{\Sigma}_{m+1}$ either as a homomorphism

$$C : \mathbf{k}^m \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee,$$

or as a homomorphism

$$^\sharp C : \mathbf{k}^m \to (\mathbf{k}^{m+1})^\vee \otimes \wedge^2V^\vee,$$

or as an induced morphism

$$\widetilde{C} : \mathbf{k}^m \otimes \mathcal{O}(-1) \to (\mathbf{k}^{m+1})^\vee \otimes \Omega(1).$$

Note also that the set $(\mathbf{S}^\vee_{m+1})^0$ is a dense open subset of the vector space $\mathbf{S}^\vee_{m+1}$.

Consider the set

(92)

$$X_m := \left\{ (D, C) \in (\mathbf{S}^\vee_{m+1})^0 \times \mathbf{\Sigma}_{m+1} \;\middle|\; \begin{array}{c} (i)\ (C^\vee \circ D \circ C : \mathbf{k}^m \otimes V \to (\mathbf{k}^m)^\vee \otimes V^\vee) \in \mathbf{S}_m, \\ (ii) \text{ the map } (\mathbf{k}^{m+1} \oplus \mathbf{k}^m) \otimes \mathcal{O} \xrightarrow{(D^{-1},C)\circ u} (\mathbf{k}^{m+1})^\vee \otimes V^\vee \otimes \mathcal{O}(1) \\ \text{is a subbundle morphism,} \\ (iii) \text{ the composition } \hat{C} : \mathbf{k}^m \xrightarrow{^\sharp C} (\mathbf{k}^{m+1})^\vee \otimes \wedge^2V^\vee \overset{can}{\twoheadrightarrow} \\ (\mathbf{k}^{m+1})^\vee \otimes \wedge^2V^\vee/\operatorname{Im}(^\sharp D^{-1}) \simeq H^0(E_{2m+2}(D^{-1})(1)) \text{ yields} \\ \text{a subbundle morphism} \\ \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\rho_{D,C}} E_{2m+2}(D^{-1}), \\ \text{i.e. } \rho^\vee_{D,C} \text{ is surjective and } E_2(D, C) := \operatorname{Ker}(^t\rho_{D,C})/\operatorname{Im}(\rho_{D,C}) \\ \text{is locally free} \end{array} \right\}.$$

By definition X_m is a locally closed subset of $(\mathbf{S}^\vee_{m+1})^0 \times \mathbf{\Sigma}_{m+1}$. Hence it is naturally supplied with the structure of a reduced scheme.

Note that in the condition (iii) of the definition of X_m we set ${}^t\rho_{D,C} := \theta \circ \rho^\vee_{D,C}$, where $\theta : E_{2m+2}(D^{-1}) \xrightarrow{\simeq} E^\vee_{2m+2}(D^{-1})$ is a natural symplectic structure on $E_{2m+2}(D^{-1})$ defined in (81).

Theorem 6.1. *Let $m \geq 1$ and let ξ be as in Theorem 5.1 and Corollary 5.4.*

(i) There is an isomorphism of reduced schemes

$$f_m : \ (MI_{2m+1}(\xi))_{red} \xrightarrow{\simeq} X_m : \ A \mapsto (\xi_1(A)^{-1}, \xi_2(A)). \tag{93}$$

(ii) The inverse isomorphism is given by the formula

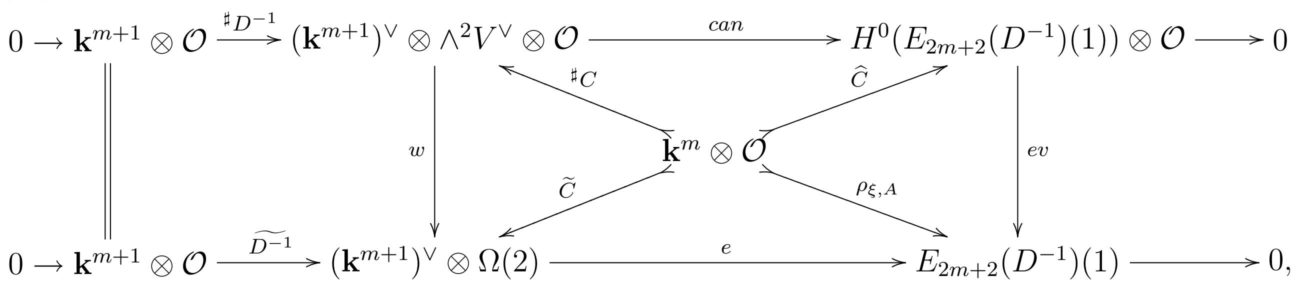
$$g_m : \ X_m \xrightarrow{\simeq} (MI_{2m+1}(\xi))_{red} : \ (D, C) \mapsto \ \widetilde{\xi}(D^{-1}, \ C, \ C^\vee \circ D \circ C).^1 \tag{94}$$

Доказательство. (i) We first show that the image of the map $f_m : (MI_{2m+1}(\xi))_{red} \to (\mathbf{S}^\vee_{m+1})^0 \times \Sigma^{in}_{m,m+1}$ lies in X_m, i.e. satisfies the conditions (i)-(iii) in the definition of X_m. Indeed, the condition (i) is automatically satisfied, since (57) and (61) give $C^\vee \circ D \circ C = \xi_2(A)^\vee \circ \xi_1(A)^{-1} \circ \xi_2(A) = \xi_3(A) \in S^2(\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee$. Next, the morphism $\rho_{D,C}$ defined in (iii) above coincides by its definition with the morphism $\rho_{\xi,A}$ defined in (84). In fact, the upper triangle of the diagram (85) twisted by $\mathcal{O}(1)$ and the lower part of the diagram (79) in which we put

$$B = D^{-1} \tag{95}$$

(note that D is invertible) fit in the diagram

(96)



where the composition $\widehat{C} = can \circ C$ is defined in the condition (iii) of the definition of X_m. Whence

$$\rho_{D,C} = \rho_{\xi,A}. \tag{97}$$

Since $\rho_{\xi,A}$ is a subbundle morphism, the condition (iii) is satisfied and, moreover, $\widehat{C}$ is a subbundle morphism as well. Thus, the lower part of the diagram (96) shows that the morphism $(\widetilde{D^{-1}}, \widetilde{C}) : (\mathbf{k}^{m+1} \oplus \mathbf{k}^m) \otimes \mathcal{O} \to (\mathbf{k}^{m+1})^\vee \otimes \Omega(2)$ is a subbundle morphism. Hence its composition with the subbundle morphism $v^\vee : (\mathbf{k}^{m+1})^\vee \otimes \Omega(2) \hookrightarrow (\mathbf{k}^{m+1})^\vee \otimes V \otimes \mathcal{O}(1)$ is a subbundle morphism as well. By definition, this composition coincides with $(D^{-1}, C) \circ u$. Hence the condition (ii) in the definition of X_m is satisfied.

This shows that $f_m((MI_{2m+1}(\xi))_{red})$ lies in X_m. Last, the equality $g_m \circ f_m = id$ follows directly from (57) and (61).

(ii) We first prove that the image of the map

$$g_m : X_m \to \mathbf{S}_{2m+1} : \ (D, C) \mapsto (D^{-1}, \ C, \ C^\vee \circ D \circ C) \ ^2 \tag{98}$$

---

[1] Here we use the decomposition (56) fixed by the choice of ξ.

[2] We identify here the triple $(D^{-1}, \ C, \ C^\vee \circ D \circ C)$ with a point in $S^2(\mathbf{k}^{2m+1})^\vee \otimes \wedge^2 V^\vee$ via the decomposition (56).

lies in $(MI_{2m+1}(\xi))_{red}$. In fact, the subbundle morphism $\mathcal{A} := (D^{-1}, C)\circ u : (\mathbf{k}^{m+1}\oplus\mathbf{k}^m)\otimes\mathcal{O} \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee \otimes \mathcal{O}(1)$ and its dual extend to the right and left exact sequence

(99) $$0 \to (\mathbf{k}^{m+1} \oplus \mathbf{k}^m) \otimes \mathcal{O}(-1) \xrightarrow{\mathcal{A}} (\mathbf{k}^{m+1})^\vee \otimes V^\vee \otimes \mathcal{O} \xrightarrow{\mathcal{A}^\vee\circ D} (\mathbf{k}^{m+1} \oplus \mathbf{k}^m)^\vee \otimes \mathcal{O}(1) \to 0.$$

Furthermore, by definition $\mathcal{A}^\vee\circ D\circ\mathcal{A} = u^\vee\circ A\circ u$, where A is the matrix $\begin{pmatrix} D^{-1} & C \\ C^\vee & C^\vee \circ D \circ C \end{pmatrix}$. Since the condition (i) is satisfied, under the direct sum decomposition (56) this matrix A can be treated an element of $\mathbf{S}_{2m+1}$. Hence $u^\vee \circ A \circ u = 0$, i.e. (99) is a monad. Show that its cohomology bundle

$$E(D, C) := \ker(\mathcal{A}^\vee \circ D)/\operatorname{Im}\mathcal{A}$$

is an $(2m + 1)$-instanton, this giving the desired inclusion $g(X_m) \subset (MI_{2m+1}(\xi))_{red}$. For this, consider the diagram (67) in which we substitute $B\circ\alpha_{\xi,A}$ by $\mathcal{A}$, respectively, B by D^{-1}, denote $\mathcal{G} := \operatorname{coker}\mathcal{A}$, and change the notation for $\tau_{\xi,A}$ and $\epsilon_{\xi,A}$, respectively, to $\tau_{D,C}$ and $\epsilon_{D,C}$

(100)
$$\begin{array}{ccccccccc}
 & & \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \to & (\mathbf{k}^{m+1} \oplus \mathbf{k}^m) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\mathcal{A}} & (\mathbf{k}^{m+1})^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{can} & \mathcal{G} & \to & 0 \\
 & & \uparrow{\scriptstyle i_{m+1}} & & \| & & \uparrow{\scriptstyle \epsilon_{D,C}} & & \\
0 & \to & \mathbf{k}^{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{D^{-1}\circ u} & (\mathbf{k}^{m+1})^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{v\circ D} & \mathbf{k}^{m+1} \otimes T_{\mathbb{P}^3}(-1) & \to & 0 \\
 & & & & & & \uparrow{\scriptstyle \tau_{D,C}} & & \\
 & & & & & & \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3}(-1). & &
\end{array}$$

In these notations the diagram (82) becomes the display of the antiselfdual monad

(101) $$0 \to \mathbf{k}^{m+1} \otimes \mathcal{O}(-1) \xrightarrow{D^{-1}\circ u} (\mathbf{k}^{m+1})^\vee \otimes V^\vee \otimes \mathcal{O} \xrightarrow{u^\vee} (\mathbf{k}^{m+1})^\vee \otimes \mathcal{O}(1) \to 0$$

with the symplectic cohomology sheaf $E_{2m+2}(D^{-1})$:

(102) $$E_{2m+2}(D^{-1}) = \ker(u^\vee)/\operatorname{Im}(D^{-1} \circ u).$$

Moreover, as in (84) and (85) we obtain a subbundle morphism

(103) $$\rho_{D,C} : \mathbf{k}^m \otimes \mathcal{O}(-1) \to E_{2m+2}(D^{-1})$$

such that

(104) $$\tau_{D,C} = e^\vee \circ \theta \circ \rho_{D,C},$$

where $\theta : E_{2m+2}(D^{-1}) \xrightarrow{\simeq} E_{2m+2}(D^{-1})$ is a symplectic structure on $E_{2m+2}(D^{-1})$. Besides, as in (83) we have

(105) $$h^0(E_{2m+2}(D^{-1})) = h^i(E_{2m+2}(D^{-1})(-2)) = 0, \quad i \geq 0.$$

Furthermore, as before, the antiselfdual monads (99) and (101) imply the (antiselfdual) monad (87)

(106) $$0 \to \mathbf{k}^m \otimes \mathcal{O}(-1) \xrightarrow{\rho_{D,C}} E_{2m+2}(D^{-1}) \xrightarrow{\theta\circ\rho_{D,C}^\vee} (\mathbf{k}^m)^\vee \otimes \mathcal{O}(1) \to 0$$

with the cohomology sheaf $E(D, C)$,

(107) $$E(D, C) = \ker(\theta \circ \rho_{D,C}^\vee)/\operatorname{Im}(\rho_{D,C}).$$

Now (105) and (106) yield $h^0(E(D, C)) = h^i(E(D, C)(-2)) = 0, \quad i \geq 0$, i.e. $E(D, C)$ is an $(2m + 1)$-instanton.

Thus $\operatorname{Im} g_m \subset I_{2m+1}(\xi)$. The fact that $f_m \circ g_m = id$ follows directly from (93) and (94). □

7. Variety Z_m

7.1. Scheme Z_m. Set

$$\mathbf{\Lambda}_m := \wedge^2(\mathbf{k}^m)^\vee \otimes S^2V^\vee, \quad \mathbf{\Phi}_m := \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k}^m)^\vee) \otimes \wedge^2 V^\vee, \tag{108}$$

and consider the set

$$Z_m := \left\{ (D,\phi) \in \mathbf{S}_m^\vee \times \mathbf{\Phi}_m \;\middle|\; \begin{array}{c} \Theta_m(D,\phi) := \phi^\vee \circ D \circ \phi : \mathbf{k}^m \otimes V \to \\ \to (\mathbf{k}^m)^\vee \otimes V^\vee \text{ satisfies the condition} \\ \Theta_m(D,\phi) \in \mathbf{S}_m \end{array} \right\}. \tag{109}$$

(Here, as in (90), we understand a point $D \in \mathbf{S}_m^\vee$ as a homomorphism $(\mathbf{k}^m)^\vee \otimes V^\vee \to \mathbf{k}^m \otimes V$.) Consider the standard decomposition

$$\wedge^2((\mathbf{k}^m)^\vee \otimes V^\vee) = \mathbf{S}_m \oplus \mathbf{\Lambda}_m$$

with the induced projections

$$\mathbf{S}_m \overset{pr_1}{\leftarrow} \wedge^2((\mathbf{k}^m)^\vee \otimes V^\vee) \overset{pr_2}{\to} \mathbf{\Lambda}_m.$$

We have a morphism $h_m : \mathbf{S}_m \times \mathbf{\Phi}_m \to \mathbf{\Lambda}_m : (A_m, \phi_m) \mapsto pr_2(\Theta(A_m, \phi_m))$. By the definition Z_m we have

$$Z_m = h_m^{-1}(0). \tag{110}$$

Convention: If Z_m is nonempty, we supply Z_m with a scheme structure of a scheme-theoretic fibre $h_m^{-1}(0)$ of the morphism h_m.

Assume that

$$Z_m \neq \emptyset. \tag{111}$$

Then from the definition of Z_m we obtain the estimate for the dimension of Z_m at each point $z \in Z_m$

$$\begin{aligned} \dim_z Z_m = \dim h_m^{-1}(0) \geq \dim(\mathbf{S}_m \times \Phi_m) - \dim \wedge^2(\mathbf{k}^m)^\vee \otimes S^2V^\vee = \\ = 3m(m+1) + 6m^2 - 5m(m-1) = 4m(m+2). \end{aligned} \tag{112}$$

Consider the open dense subset $\mathbf{\Phi}_m^0 := \{\phi \in \mathbf{\Phi}_m | {}^\sharp\phi : \mathbf{k}^m \to (\mathbf{k}^m)^\vee \otimes \wedge^2V^\vee) \text{ is injective}\}$ of $\mathbf{\Phi}_m$ and set

(113)

$$Z'_m := \left\{ (D,\phi) \in Z_m \cap (\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m^0 \;\middle|\; \mathrm{Im}({}^\sharp\phi) \cap \mathrm{Im}({}^\sharp(D^{-1}) : \mathbf{k}^m \to (\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee) = \{0\} \right\}$$

The set Z'_m is by definition an open subset in Z_m.

Assume $Z'_m \neq \emptyset$. Pick a point $z = (D,\phi) \in Z'_m$ and set

$$W_{5m} := (\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee / \mathrm{Im}({}^\sharp(D^{-1})), \quad \dim W_{5m} = 5m.$$

Let $i(z)$ be the composition in the diagram

$$\begin{array}{ccccccccc} & & & & \mathbf{k}^m & & & & \\ & & & & \downarrow \phi & \searrow i(z) & & & \\ 0 & \longrightarrow & \mathbf{k}^m & \xrightarrow{{}^\sharp(D^{-1})} & (\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee & \xrightarrow{can} & W_{5m} & \longrightarrow & 0 \end{array} \tag{114}$$

The lower horizontal triple in (114) yields the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{{}^\sharp(D^{-1})} & (\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{can} & W_{5m} \otimes \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & 0 \\ & & \| & & \downarrow ev & & \downarrow ev & & \\ 0 & \longrightarrow & \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\widetilde{D^{-1}}} & (\mathbf{k}^m)^\vee \otimes \Omega_{\mathbb{P}^3}(2) & \xrightarrow{can} & E_{2m}(D^{-1})(1) & \longrightarrow & 0, \end{array} \tag{115}$$

where $E_{2m}(D^{-1})$ is a symplectic bundle (see (81)). From this diagram we deduce the equalities

$$h^i(E_{2m}(D^{-1})(-2)) = 0, \quad i \geq 0, \tag{116}$$

and the isomorphism

$$h^0(ev) : W_{5m} \xrightarrow{\sim} H^0(E_{2m}(D^{-1})), \quad i \geq 0, \tag{117}$$

Moreover, the diagrams (114) and (115) define the composition

$$i_z : \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{i(z)} W_{5m} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{ev}{\twoheadrightarrow} E_{2m}(D^{-1}). \tag{118}$$

Note that from the definition of the set Z_m it follows that

$${}^t i_z \circ i_z = 0, \tag{119}$$

where ${}^t i_z := i_z^\vee \circ \theta$ and $\theta : E_{2m}((D^{-1})) \xrightarrow{\sim} E_{2m}((D^{-1}))^\vee$ is the symplectic structure on $E_{2m}((D^{-1}))$ mentioned above, i.e. we have an antiselfdual complex

$$0 \to \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{i_z} E_{2m}(D^{-1}) \xrightarrow{{}^t i_z} (\mathbf{k}^m)^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{120}$$

(*Warning*: this complex is not right exact.)

Twisting the sequence (118) by $\mathcal{O}_{\mathbb{P}^3}(1)$ and passing to sections, we obtain in view of

Furthermore, the standard embedding

$$j : \mathbf{k}^{m-1} \hookrightarrow \mathbf{k}^m : (a_1, ..., a_{m-1}) \mapsto (a_1, ..., a_{m-1}, 0) \tag{121}$$

and the morphism i_z from (118) define the composition

$$j_z : \mathbf{k}^{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{j} \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{i_z} E_{2m}(D^{-1}) \tag{122}$$

7.2. Varieties Z_m^* and N_{2m-1}^{tH}.

Assume, as above, that $Z_m' \neq \emptyset$ and set

$$Z_m^* = \{z = (D, \phi) \in Z_m' \mid j_z : \mathbf{k}^{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to E_{2m}(D^{-1}) \text{ is a subbundle morphism}\}. \tag{123}$$

By definition, Z_m^* is an open subset of Z_m', hence also of Z_m. If $Z_m^* \neq \emptyset$, then for any point $z = (D, \phi) \in Z_m^*$ we obtain from (119) that ${}^t j_z \circ j_z = 0$, where ${}^t j_z := j_z^\vee \circ \theta$. Thus j_z defines a monad

$$0 \to \mathbf{k}^{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{j_z} E_{2m}(D^{-1}) \xrightarrow{{}^t j_z} (\mathbf{k}^{m-1})^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{124}$$

and in view of (116) the cohomology sheaf of this monad is an instanton bundle

$$E_2(z) := \mathrm{Ker}({}^t j_z)/\,\mathrm{Im}(j_z), \quad [E_2(z)] \in I(2m-1). \tag{125}$$

Consider the subvariety $I_{2m-1}^{tH} \subset I_{2m-1}$ of *t'Hooft instanton bundles*

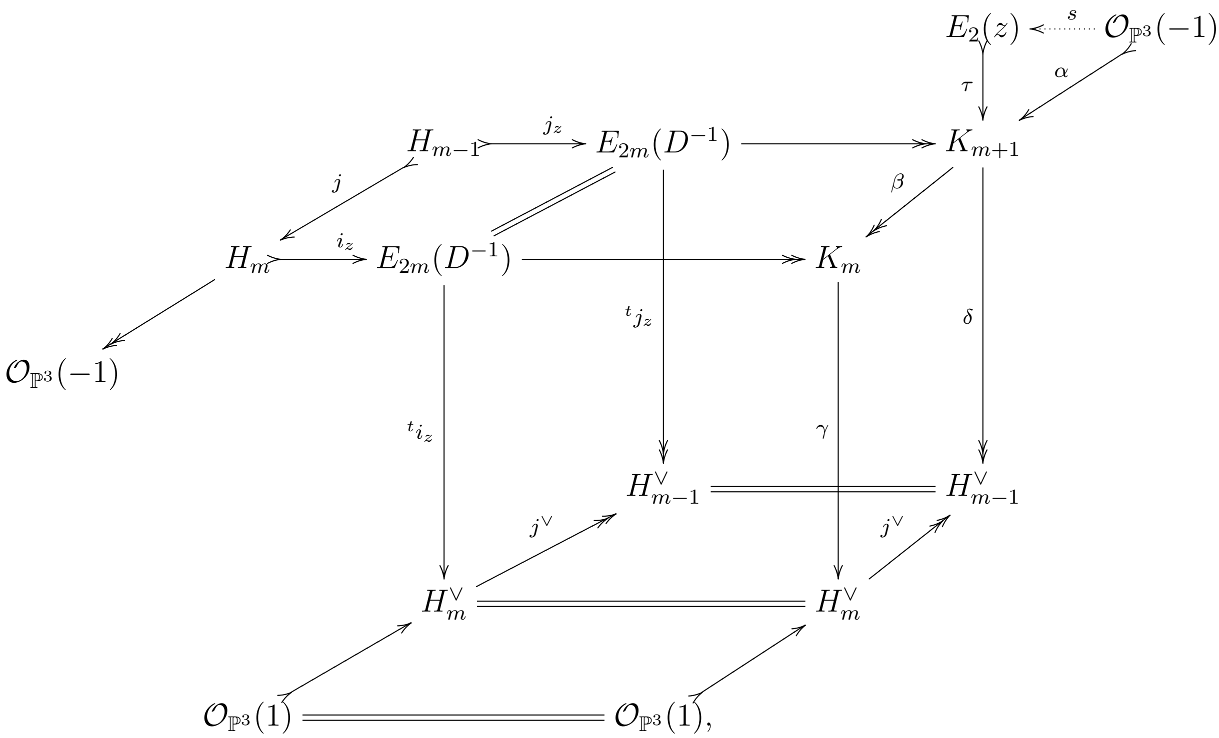
$$I_{2m-1}^{tH} := \{[E] \in I_{2m-1} \mid h^0(E(1)) \neq 0\}.$$

Lemma 7.1. *Assume $Z_m^* \neq \emptyset$. Then for any $z = (D, \phi) \in Z_m^*$ the bundle $E_2(z)$ is a t'Hooft instanton bundle, i.e. $[E_2(z)] \in I_{2m-1}^{tH}$.*

Proof. Consider the complexes (120) and (124) and set

$$H_{m-1} := \mathbf{k}^{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad H_m := \mathbf{k}^m \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad K_{m+1} := \mathrm{coker}\, j_z, \quad K_m := \mathrm{coker}\, i_z.$$

The complexes (120) and (124) are antiselfdual, hence they extend to a commutative diagram (126)



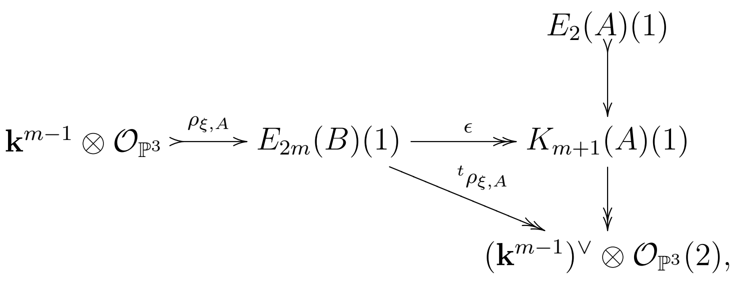
in which $\alpha, \beta, \gamma, \delta$ and τ are the induced morphisms. In this diagram we have $\beta \circ \alpha = 0$ and $j^\vee \circ \gamma \circ \beta = \delta$. Hence $\delta \circ \alpha = 0$. This implies that α factors through the morphism τ, i.e. there exists an injection $s : \mathcal{O}_{\mathbb{P}^3}(-1) \to E_2(z)$ such that $\alpha = \tau \circ s$. This injection s is a nonzero section $s \in H^0(E_2(z)(1))$. Hence $E_2(z)$ is a t'Hooft bundle. $\square$

We will show that Z_m^* is an irreducible variety of dimension $4m(m+2)$, hence it is nonempty. For this, fix an isomorphism

$$\xi : \mathbf{k}^m \oplus \mathbf{k}^{m-1} \xrightarrow{\simeq} \mathbf{k}^{2m-1} \tag{127}$$

and consider the variety $MI^{tH}_{2m-1}(\xi)$ defined in (74). Take an arbitrary point $A \in MI^{tH}_{2m-1}(\xi)$. The point A defines a point $B = \xi_1(A)$ and a monad $0 \to \mathbf{k}^{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\rho_{\xi,A}} E_{2m}(B) \xrightarrow{{}^t\rho_{\xi,B}} (\mathbf{k}^{m-1})^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$ with the cohomology bundle $[E_2(A)] = \pi_{2m-1}(A)$ (see subsection 5.3). The display of this monad twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$ is

(128)



where $K_{m+1}(A) := \operatorname{coker} \rho_{\xi,A}$.

Note that from (70) and the definition of $MI^{tH}_{2m-1}(\xi)$ it follows that $h^0(E_2(A)(1)) = 1$. Hence, passing to sections in the diagram (128) we obtain a well defined epimorphism

$$b(\xi, A) : H^0(E_{2m}(B)(1)) \xrightarrow{h^0(\epsilon)} H^0(K_{m+1}(A)(1)) \xrightarrow{can} H^0(K_{m+1}(A)(1))/H^0(E_2(A)(1)) \simeq \mathbf{k}^{4m}. \tag{129}$$

On the other hand, similar to (115) and (117) we obtain the exact triple

$$0 \to \mathbf{k}^m \xrightarrow{\sharp B^{-1}} (\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee \xrightarrow{c(A)} H^0(E_{2m}(B)(1)) \to 0. \tag{130}$$

Denote by $c(A)$ the epimorphism $(\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee \twoheadrightarrow H^0(E_{2m}(B)(1))$ in this triple and set

$$V_{2m}(\xi, A) := c(A)^{-1}(\ker b(\xi, A)) \simeq \mathbf{k}^{2m}, \tag{131}$$

$$V^*_{2m}(\xi, A) := \{v \in V_{2m}(\xi, A) \mid \mathrm{Span}(\mathrm{Im}\,^\sharp(\xi_1(A)^{-1}),\ \mathrm{Im}\,^\sharp(\xi_2(A)), \mathbf{k}v) = V_{2m}(\xi, A)\},$$

$$V_{2m}(\xi) := \{(A, v) \mid A \in MI^{tH}_{2m-1}(\xi),\ v \in V_{2m}(\xi, A)\}. \tag{132}$$

Here the projection $V_{2m}(\xi) \to MI^{tH}_{2m-1}(\xi) : (A, v) \mapsto A$ is a $\mathbf{k}^{2m}$-bundle over $MI^{tH}_{2m-1}(\xi)$, hence by Lemma 5.3 and Corollary 5.4 $V_{2m}(\xi)$ is irreducible of dimension

$$\dim V_{2m}(\xi) = \dim MI^{tH}_{2m-1}(\xi) + 2m = 4m(m + 2). \tag{133}$$

Besides, $V^*_{2m}(\xi, A)$ is a dense open subset of $V_{2m}(\xi, A)$ for each $A \in MI^{tH}_{2m-1}(\xi)$,

$$V^*_{2m}(\xi, A) \xhookrightarrow{\text{dense open}} V_{2m}(\xi, A) \simeq \mathbf{k}^{2m}\ . \tag{134}$$

Next, set $\Pi_m := \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k}^m)^\vee \otimes \wedge^2 V)$ and

(135)

$$N(\xi, A) := \left\{ (\phi : \mathbf{k}^m \otimes V \xrightarrow{\sim} (\mathbf{k}^m)^\vee \otimes V^\vee) \in \Pi_m \ \middle|\ \begin{array}{l} (i)\ \mathrm{Span}(\mathrm{Im}\,^\sharp(\xi_1(A)^{-1}),\ \mathrm{Im}\,^\sharp\phi) = V_{2m}(\xi, A), \\ (ii)\ \phi \circ j = \xi_2(A), \\ (iii)\ \phi^\vee \circ (\xi_1(A)^{-1}) \circ \phi \in \mathbf{S}_m \end{array} \right\},$$

$$N^{tH}_{2m-1}(\xi) := \{(A, \phi) \mid A \in MI^{tH}_{2m-1}(\xi),\ \phi \in N(\xi, A)\}. \tag{136}$$

Consider the standard decomposition $\mathbf{k}^m = \mathbf{k}^{m-1} \oplus \mathbf{k}$, so that the injection j in (121) is an embedding of the left direct summand of this decomposition. Then each monomorphism $(^\sharp\phi : \mathbf{k}^m \to (\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee) \in N(\xi, A)$ in view of the conditions (i)-(iii) of (135) is uniquely determined by its restriction onto the right direct summand $\mathbf{k}$ of the standard decomposition,

$$^\sharp\phi|_{\mathbf{k}} :\ \mathbf{k} \to V_{2m}(\xi, A) \subset (\mathbf{k}^m)^\vee \otimes \wedge^2 V^\vee :\ 1 \mapsto v$$

satisfying the conditions

$$\mathrm{Span}(\mathrm{Im}\,^\sharp(\xi_1(A)^{-1}),\ \mathrm{Im}\,^\sharp\phi) = \mathrm{Span}(\mathrm{Im}\,^\sharp(\xi_1(A)^{-1}),\ \mathrm{Im}\,^\sharp(\xi_2(A)), \mathbf{k}v) = V_{2m}(\xi, A).$$

and

$$(\xi_2(A) + \phi|_{\mathbf{k}\otimes V})^\vee \circ (\xi_1(A)^{-1}) \circ (\xi_2(A) + \phi|_{\mathbf{k}\otimes V}) \in \mathbf{S}_m.$$

These conditions and the definition of $V^*_{2m}(\xi, A)$ mean that $N(\xi,)$ is a closed subset of $V^*_{2m}(\xi, A)$, hence by (134) it is a locally closed subset of $V_{2m}(\xi, A)$. As a result, we have

$$N^{tH}_{2m-1}(\xi) \xhookrightarrow{\text{locally closed}} V_{2m}(\xi)\ . \tag{137}$$

In particular,

$$\dim N^{tH}_{2m-1}(\xi) \le \dim V_{2m}(\xi) = 4m(m + 2). \tag{138}$$

Now consider the map

$$h_m :\ N^{tH}_{2m-1}(\xi) \to Z^*_m : (A, \phi) \mapsto (D := \xi_1(A)^{-1}, \phi). \tag{139}$$

This map is well defined. In fact, take any point $(A, \phi) \in N^{tH}_{2m-1}(\xi)$. Since $A \in MI^{tH}_{2m-1}(\xi)$, we have $D \in (\mathbf{S}^\vee_m)^0$, so that the vector bundle $E_{2m}(D^{-1})$ is well-defined. Next, since $\phi \circ j = \xi_2(A)$ (see condition (ii) in (135)), it follows from Theorem 6.1 that the morphism

$$j_z : \mathbf{k}^{m-1} \otimes \mathcal{O}(-1) \to E_{2m}(D^{-1})$$

for $z = (D, \phi)$ coincides with the subbundle morphism $\rho_{\xi,A}$ satisfying diagram (96). Note that in view of (97) we can rewrite this also as

$$j_z = \rho_{D,C}, \quad C = \phi \circ j. \tag{140}$$

The diagram (96), in turn, implies that the condition $\mathrm{Im}(^\sharp D) \cap \mathrm{Im}(^\sharp \phi) = \{0\}$ is satisfied. This together with the injectivity of j_z and the condition (iii) in (135) precisely means that $z \in Z_m^*$.

As a result, it follows that Z_m^* and, respectively, Z_m is nonempty. Moreover, since Z_m^* is supplied with the structure of a reduced scheme and $N_{2m-1}^{tH}(\xi)$ is smooth (hence reduced) it follows that the map h_m given by formula (139) is a morphism of reduced schemes. Next, consider the set

$$Z_m^*(\xi) := \{z \in Z_m^* \mid z = (D, \phi) \text{ satisfies the condition } (*)\}$$

where

$$(D^{-1}, \phi \circ j) \circ u : (\mathbf{k}^m \oplus \mathbf{k}^{m-1}) \otimes \mathcal{O}(-1) \to (\mathbf{k}^m)^\vee \otimes V^\vee \otimes \mathcal{O} \text{ is a subbundle morphism.} \quad (*)$$

Since the condition (*) is open and $Z_m^*(\xi)$ contains a subset $h_m(N_{2m-1}^{tH}(\xi))$, it follows that $Z_m^*(\xi)$ is a nonempty open subset of Z_m^*.

Consider the map

$$\lambda_m : \ Z_m^*(\xi) \to \mathbf{S}_{2m-1} : \ z = (D, \phi) \mapsto A := \tilde{\xi}(D^{-1}, \phi \circ j, (\phi \circ j)^\vee \circ D \circ (\phi \circ j)). \tag{141}$$

Since $(\phi^\vee \circ D \circ \phi) \in \mathbf{S}_m$ by the definition of Z_m, it follows that

$$(\phi \circ j)^\vee \circ D \circ (\phi \circ j) \in \mathbf{S}_{m-1}, \tag{142}$$

i.e. the map λ_m in (141) is well-defined. Moreover, since $Z_m^*(\xi)$ is a reduced scheme, the map λ_m is a morphism of reduced schemes.

Theorem 7.2. *Let $m \geq 1$ and ξ be a fixed isomorphism (127). Then $Z_m^*(\xi)$ is a smooth irreducible variety of dimension $4m(m+2)$ and there is an isomorphism of smooth varieties*

$$\nu_m : Z_m^*(\xi) \xrightarrow{\sim} N_{2m-1}^{tH}(\xi) : \ (D, \phi) \mapsto (A, \phi), \tag{143}$$

where A is given by (141).

Proof. Consider the set X_{m-1} defined in (92) and the morphism of reduced schemes

$$\eta_m : \ Z_m^*(\xi) \to X_{m-1} : z = (D, \phi) \mapsto (D, \phi \circ j). \tag{144}$$

This morphism is well-defined since (142), (*) and (140) are precisely the conditions (i), (ii) and (iii) of the definition of X_{m-1}. Next, comparing (94), (141) and (144) we obtain that $\lambda_m = g_{m-1} \circ \eta_m$ for $m \geq 1$. Whence $\mathrm{Im}\, \lambda_m \subset MI_{2m-1}(\xi)$. Moreover, for any point $z = (D, \phi)$ the diagram (126) defines a section $s \in E_2(A)(1)$ for $A = \lambda_m(z)$, so that $[E_2(A)] \in I_{2m-1}^{tH}$, i.e. $A \in MI_{2m-1}^{tH}(\xi)$. Hence $(A, \phi) \in N_{2m-1}^{tH}(\xi)$, and the morphism ν_m in (143) is well-defined. Comparing now (139) and (143), we obtain that $h_m = \nu_m^{-1}$, i.e. ν_m is an isomorphism of reduced schemes.

Next, since by definition $Z_m^*(\xi)$ is an open subset of Z_m, it follows from (112) that $\dim Z_m^*(\xi) \geq 4m(m+2)$. This together with (138) and the isomophism ν_m shows that

$$\dim Z_m^*(\xi) = \dim N_{2m-1}^{tH}(\xi) = \dim V_{2m}(\xi) = 4m(m+2).$$

Whence by (137) and the irreducibility and smoothness of $V_{2m}(\xi)$ we obtain that $Z_m^*(\xi) \simeq N_{2m-1}^{tH}(\xi)$ is a dense open subset of $V_{2m}(\xi)$, so that $Z_m^*(\xi)$ is smooth and irreducible of dimension $4m(m+2)$. ∎

7.3. Irreducibility of Z_m.

Consider the standard isomorphism

$$\mathbf{k}^{m-1} \oplus \mathbf{k} \xrightarrow{\sim} \mathbf{k}^m : ((a_1, ..., a_{m-1}), a_m) \mapsto (a_1, ..., a_m). \tag{145}$$

Under this isomorphism any homomorphism

$$\phi : \mathbf{k}^m \otimes V \to (\mathbf{k}^m)^\vee \otimes V^\vee, \qquad \phi \in \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k}^m)^\vee) \otimes \wedge^2 V^\vee. \tag{146}$$

can be represented as a homomorphism

$$\phi : \mathbf{k}^{m-1} \otimes V \oplus \mathbf{k} \otimes V \to (\mathbf{k}^{m-1})^\vee \otimes V^\vee \oplus \mathbf{k}^\vee \otimes V^\vee, \tag{147}$$

i.e. as a matrix

$$\phi = \left(\begin{array}{c|c} \phi_1 & \chi_1 \\ \hline \psi_1 & \theta_1 \end{array}\right), \tag{148}$$

where

$$\phi_1 \in \mathrm{Hom}(\mathbf{k}^{m-1}, (\mathbf{k}^{m-1})^\vee) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_{m-1}, \quad \psi_1 \in \mathbf{\Psi}_{m-1} := \mathrm{Hom}(\mathbf{k}^{m-1}, (\mathbf{k})^\vee) \otimes \wedge^2 V^\vee, \tag{149}$$
$$\chi_1 \in \mathbf{B}_\chi := \mathrm{Hom}(\mathbf{k}, (\mathbf{k}^{m-1})^\vee) \otimes \wedge^2 V^\vee, \qquad \theta_1 \in \mathbf{B}_\theta := \mathrm{Hom}(\mathbf{k}, \mathbf{k}^\vee) \otimes \wedge^2 V^\vee = \mathbf{S}_1.$$

Respectively, a homomorphism

$$D \in \mathbf{S}_m^\vee \subset \mathrm{Hom}((\mathbf{k}^m)^\vee \otimes V^\vee, \mathbf{k}^m \otimes V) \tag{150}$$

can be represented as a matrix

$$D = \left(\begin{array}{c|c} D_1 & a_1 \\ \hline -a_1^\vee & \alpha_1 \end{array}\right), \tag{151}$$

where

$$D_1 \in \mathbf{S}_{m-1}^\vee \subset \mathrm{Hom}((\mathbf{k}^{m-1})^\vee \otimes V^\vee, \mathbf{k}^{m-1} \otimes V), \tag{152}$$
$$a_1 \in \mathrm{Hom}((\mathbf{k})^\vee, \mathbf{k}^{m-1}) \otimes \wedge^2 V = \mathbf{\Psi}_{m-1}^\vee, \qquad \alpha_1 \in \mathrm{Hom}((\mathbf{k})^\vee, \mathbf{k}) \otimes \wedge^2 V = \mathbf{B}_\theta^\vee.$$

From (148) and (151) it follows that the homomorphism

$$\Theta(D, \phi) := \phi^\vee \circ D \circ \phi : \mathbf{k}^m \otimes V \to (\mathbf{k}^m)^\vee \otimes V^\vee, \qquad \Theta(D, \phi) \in \wedge^2((\mathbf{k}^m)^\vee \otimes V^\vee),$$

can be represented as a matrix

$$\Theta(D, \phi) = \left(\begin{array}{c|c} \Theta_1(D, \phi) & b_1(D, \phi) \\ \hline -b_1(D, \phi)^\vee & \beta_1(D, \phi) \end{array}\right), \tag{153}$$

where

$$\Theta_1(D, \phi) := \phi_1^\vee \circ D_1 \circ \phi_1 + \phi_1^\vee \circ a_1 \circ \psi_1 - \psi_1^\vee \circ a_1^\vee \circ \phi_1 + \psi_1^\vee \circ \alpha_1 \circ \psi_1 \in \tag{154}$$
$$\in \wedge^2((\mathbf{k}^{m-1})^\vee \otimes V^\vee) \subset \mathrm{Hom}((\mathbf{k}^{m-1})^\vee \otimes V^\vee, \mathbf{k}^{m-1} \otimes V),$$
$$b_1(D, \phi) := \phi_1^\vee \circ D_1 \circ \chi_1 + \phi_1^\vee \circ a_1 \circ \theta_1 - \psi_1^\vee \circ a_1^\vee \circ \chi_1 + \psi_1^\vee \circ \alpha_1 \circ \theta_1 \in$$
$$\in \mathrm{Hom}(\mathbf{k}^{m-1} \otimes V, \mathbf{k}^\vee \otimes V^\vee),$$
$$\beta_1(D, \phi) := \chi_1^\vee \circ D_1 \circ \chi_1 + \chi_1^\vee \circ a_1 \circ \theta_1 - \theta_1^\vee \circ a_1^\vee \circ \chi_1 + \theta_1^\vee \circ \alpha_1 \circ \theta_1 \in \mathbf{B}_\theta.$$

In these notations Z_m can be described as

$$Z_m = \left\{ (D, \phi) \in \mathbf{S}_m^\vee \times \mathbf{\Phi}_m \;\middle|\; \begin{array}{l} (i)\ \Theta_1(D, \phi) \in \mathbf{S}_{m-1}, \\ (ii)\ b_1(D, \phi) \in \mathbf{\Psi}_{m-1} \end{array} \right\}. \tag{155}$$

Let Z_m^0 be any irreducible component of $(Z_m)_{red}$. Take an arbitrary point

$$z = (D, \phi) = (D_1, a_1, \alpha_1, \phi_1, \chi_1, \psi_1, \theta_1) \in Z_m^0 \tag{156}$$

and consider the morphism

$$f_m : \mathbb{A}^1 \to Z_m^0 : \ t \mapsto (tD_1, ta_1, t\alpha_1, \phi_1, t\chi_1, \psi_1, t\theta_1). \tag{157}$$

This morphism is well-defined in view of (152) and (154)-(155). We have

$$f_m(0) = (0, 0, 0, \phi_1, 0, \psi_1, 0). \tag{158}$$

Consider the projection

$$\pi_m : Z_m \to \mathbf{B}_\psi^\vee \times \mathbf{B}_\theta^\vee \times \mathbf{B}_\chi \times \mathbf{B}_\theta : \tag{159}$$
$$(D_1, a_1, \alpha_1, \phi_1, \chi_1, \psi_1, \theta_1) \mapsto (a_1, \alpha_1, \chi_1, \theta_1).$$

The equality (158) means that there is a scheme-theoretic inclusion

$$\emptyset \neq Y_m^0 := (\pi_m|Z_m^0)^{-1}(0, 0, 0, 0) \subset Y_m := \pi_m^{-1}(0, 0, 0, 0), \tag{160}$$

where by (154)-(155) and (109)

$$Y_m = \{(D_1, \phi_1, \psi_1) \in \mathbf{S}_{m-1}^\vee \times \mathbf{\Phi}_{m-1} \times \mathbf{\Psi}_{m-1} \mid \phi_1^\vee D_1 \phi_1 \in \mathbf{S}_{m-1}\} = \tag{161}$$
$$= Z_{m-1} \times \mathbf{\Psi}_{m-1}.$$

Now let $(Z_m)_{red} = \underset{j}{\cup} Z_m^j$ be the decomposition of Z_m into irreducible components. The inclusion (160) means that

(i) $Z_m^j \cap Y_m \neq \emptyset$ for any irreducible component Z_m^j of Z_m, and

(ii) set-theoreticlly $Y_m = \underset{j}{\cup} (Y_m \cap Z_m^j)$, where the union is taken over all irreducible components Z_m^j of Z_m.

We now proceed to the proof of the irreducibility of Z_m by increasing induction on m. For $m = 1$ clearly $\mathbf{\Lambda}_m = 0$, so that the equations $\{\Theta_1(D_1, \phi_1) \in \mathbf{S}_1\}$ of Z_1 in $\wedge^2((\mathbf{k}^1)^\vee \otimes V^\vee)$ are empty, i.e. scheme-theoretically we have

$$Z_1 = \wedge^2(\mathbf{k}^\vee \otimes V^\vee) \simeq \mathbf{k}^6.$$

Thus $Z_1 \simeq \mathbb{A}^6$ is reduced and irreducible.

To perform the induction step, assume that Z_{m-1} is an irreducible and reduced scheme given by definition via the equations $\{\phi_1^\vee \circ D_1 \circ \phi_1 \in \mathbf{S}_{m-1}\}$ in $\mathbf{S}_{m-1}^\vee \times \mathbf{\Phi}_{m-1}$. Comparing this with (161) we see that $Y_m = Z_{m-1} \times \mathbf{\Psi}_{m-1}$ is reduced and irreducible as a scheme-theoretic fibre $\pi_m^{-1}(0, 0, 0, 0)$. Hence the properties (i) and (ii) above clearly imply that

(a) $(Z_m)_{red}$ is irreducible and

(b) Z_m is *generically reduced* in the sense that

$$Nil(Z_m) := \{x \in (Z_m)_{red} \mid Z_m \text{ is not reduced at the point } x\}$$

is a proper closed subset of $(Z_m)_{red}$, i.e.

$$Nil(Z_m) \underset{\neq}{\subset} (Z_m)_{red}. \tag{162}$$

On the other hand, by Theorem 7.2 $(Z_m)_{red}$ contains an open subset $Z_m^*(\xi)$ of dimension $4m(m+2)$. This together with (110) and (112) implies that Z_m is a locally complete intersection subscheme of dimension $4m(m+2)$ of the smooth variety $\mathbf{S}_m^\vee \times \mathbf{\Phi}_m$. Now we invoke the following easy lemma from commutative algebra.

Lemma 7.3. *Let $\mathcal{X}$ be a locally complete intersection subscheme of a smooth irreducible variety such that*

(a) $\mathcal{X}_{red}$ is irreducible and

(b) $Nil(\mathcal{X}) := \{x \in (\mathcal{X})_{red} \mid \mathcal{X}$ is not reduced at $x\} \underset{\neq}{\subset} (\mathcal{X})_{red}$.

Then $\mathcal{X}$ is irreducible and reduced.

Applying this Lemma to $\mathcal{X} = Z_m$ we obtain that Z_m is irreducible and reduced. Hence we obtain the following result.

Theorem 7.4. *Z_m is irreducible and reduced locally complete intersection scheme of dimension $4m(m+2)$.*

8. Irreducibility of I_{2m+1}

In this section we give the proof of Theorem 1.1. Set

(163) $$\widetilde{X}_m := \{(D,C) \in \mathbf{S}^\vee_{m+1} \times \mathbf{\Sigma}_{m+1} \mid \ (C^\vee \circ D \circ C : \mathbf{k}^m \otimes V \to (\mathbf{k}^m)^\vee \otimes V^\vee) \in \mathbf{S}_m\}.$$

The set $\widetilde{X}_m$ has a natural structure of a closed subscheme of $\mathbf{S}^\vee_{m+1} \times \mathbf{\Sigma}_{m+1}$ defined by the equations

(164) $$C^\vee \circ D \circ C \in \mathbf{S}_m.$$

Since $(\mathbf{S}^\vee_{m+1})^0$ is a dense open subset of $\mathbf{S}^\vee_{m+1}$ and the conditions (ii) and (iii) in the definition (92) of X_m are open and X_m is nonempty (see Theorem 6.1) it follows immediately that X_m is a nonempty open subset of $\widetilde{X}_m$,

(165) $$\emptyset \neq X_m \overset{\text{open}}{\hookrightarrow} (\widetilde{X}_m)_{red}.$$

Thus, to prove the irreducibility of X_m it is enough to prove the irreducibility of $\widetilde{X}_m$.

For this, consider the standard direct sum decomposition

$$\mathbf{k}^{m+1} \xrightarrow{\sim} \mathbf{k}^m \oplus \mathbf{k} : (a_1, ..., a_{m+1}) \mapsto ((a_1, ..., a_m), a_{m+1}).$$

Under this isomorphism any homomorphism

(166) $$C \in \mathbf{\Sigma}_{m+1} = \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k}^{m+1})^\vee) \otimes \wedge^2 V^\vee, \quad C : \mathbf{k}^m \otimes V \to (\mathbf{k}^{m+1})^\vee \otimes V^\vee,$$

can be represented as a homomorphism

(167) $$C : \mathbf{k}^m \otimes V \oplus \mathbf{k} \otimes V \to (\mathbf{k}^m)^\vee \otimes V^\vee \ \oplus \ \mathbf{k}^\vee \otimes V^\vee,$$

i.e. as a matrix

(168) $$C = \left(\frac{\phi}{\psi}\right),$$

where

(169) $$\phi \in \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k}^m)^\vee) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_m, \quad \psi \in \mathbf{\Psi}_m := \mathrm{Hom}(\mathbf{k}^m, (\mathbf{k})^\vee) \otimes \wedge^2 V^\vee.$$

Respectively, any homomorphism $D \in (\mathbf{S}^\vee_{m+1})^0 \subset S^2(\mathbf{k}^{m+1}) \otimes \wedge^2 V = \mathbf{S}^\vee_{m+1} \subset \mathrm{Hom}((\mathbf{k}^{m+1})^\vee \otimes V^\vee, \mathbf{k}^{m+1} \otimes V)$ can be represented as a matrix

(170) $$D = \left(\begin{array}{c|c} D_1 & \lambda \\ \hline -\lambda^\vee & \mu \end{array}\right),$$

where

(171) $$D_1 \in \mathbf{S}^\vee_m \subset \mathrm{Hom}((\mathbf{k}^m)^\vee \otimes V^\vee, \mathbf{k}^m \otimes V),$$
$$\lambda \in \mathbf{L}_m := \mathrm{Hom}(\mathbf{k}^\vee, \mathbf{k}^m) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_m := \mathrm{Hom}(\mathbf{k}^\vee, \mathbf{k}) \otimes \wedge^2 V.$$

From (168) and (170) it follows that the homomorphism

$$C^\vee \circ D \circ C : \mathbf{k}^m \otimes V \to (\mathbf{k}^m)^\vee \otimes V^\vee, \qquad C^\vee \circ D \circ C \in \wedge^2((\mathbf{k}^m)^\vee \otimes V^\vee),$$

can be represented as

(172) $$C^\vee \circ D \circ C = \phi^\vee \circ D_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda \circ \phi + \psi^\vee \circ \mu \circ \psi.$$

Let $\overline{X}_m$ be the closure of $(\widetilde{X}_m)_{red}$ in $\mathbf{S}^\vee_{m+1} \times \mathbf{\Sigma}_{m+1}$. and let X^0 be any irreducible component of $\overline{X}_m$. By (168)-(171) we have

$$\mathbf{S}^\vee_{m+1} \times \mathbf{\Sigma}_{m+1} = \mathbf{S}^\vee_m \times \mathbf{\Phi}_m \times \mathbf{\Psi}_m \times \mathbf{L}_m \times \mathbf{M}_m,$$

and we have well-defined projections

$$p_m : \widetilde{X}_m \to \mathbf{L}_m \times \mathbf{M}_m : (A, \phi, \psi, \lambda, \mu) \mapsto (\lambda, \mu).$$

and

$$\overline{p}_m := p_m|\overline{X}_m : \overline{X}_m \to \mathbf{L}_m \times \mathbf{M}_m.$$

Take an arbitrary point $z = (D_1, \phi, \psi, \lambda, \mu) \in X^0$ and consider the morphism

$$f^0 : \mathbb{A}^1 \to X^0 : t \mapsto (tA, \phi, \psi, t\lambda, t\mu). \tag{173}$$

(This morphism is well-defined by (172.) By definition, the point $f^0(0) = (0, \phi, \psi, 0, 0)$ lies in the fibre $p_m^{-1}(0,0)$. Hence,

$$\overline{p}_m^{-1}(0,0) \cap X^0 \neq \emptyset. \tag{174}$$

Now from (172) and the definition of $\widetilde{X}_m$ it follows that

$$p_m^{-1}(0,0) = \{(D_1, \phi, \psi) \in \mathbf{S}_m^\vee \times \mathbf{\Phi}_m \times \mathbf{\Psi}_m \mid \phi^\vee \circ A \circ \phi \in \mathbf{S}_m\}. \tag{175}$$

Comparing this with the definition (109) of Z_m we see that, set-theoretically,

$$\overline{p}_m^{-1}(0,0) \overset{\text{sets}}{=} p_m^{-1}(0,0) \overset{\text{sets}}{=} Z_m \times \mathbf{\Psi}_m. \tag{176}$$

Respectively, scheme-theoretically we have the inclusion of schemes

$$\overline{p}_m^{-1}(0,0) \overset{\text{schemes}}{\subset} p_m^{-1}(0,0) \overset{\text{schemes}}{=} Z_m \times \mathbf{\Psi}_m. \tag{177}$$

Assume now that $\overline{X}_m$ is not irreducible and let

$$\overline{X}_m = \cup_{i=1}^r X^i, \quad r \geq 2, \tag{178}$$

be its decomposition into irreducible components. In view of (174) each irreducible component X^i of $\overline{X}_m$ has a nonempty intersection with $p_m^{-1}(0,0)$. Hence, since $r \geq 2$, $p_m^{-1}(0,0)$ as a scheme-theoretic fibre is either reducible or non-reduced. Hence by (176) and (177) $Z_m \times \mathbf{\Psi}_m$ is either reducible or nonreduced. This, however, contradicts to Theorem 7.4. Thus $\overline{X}_m$ is irreducible.

Moreover, Theorem 7.4 implies that the scheme-theoretic inclusion of fibres in (177) becomes an isomorphism of reduced irreducible schemes

$$\overline{p}_m^{-1}(0,0) \overset{\text{schemes}}{=} p_m^{-1}(0,0) \overset{\text{schemes}}{=} Z_m \times \mathbf{\Psi}_m. \tag{179}$$

In particular, $p_m^{-1}(0,0)$ is a reduced and irreducible scheme and, since $\overline{X}_m$ is reduced, $\widetilde{X}_m$ is generically reduced. Furthermore, applying theorem on fibres of a morphism to the projection $\overline{p}_m : \overline{X}_m \to \mathbf{L}_m \times \mathbf{M}_m$ and using (179) and Theorem 7.4, we obtain

$$\begin{aligned} \dim \widetilde{X}_m = \dim \overline{X}_m \leq \dim \overline{p}^{-1}(0,0) + \dim(\mathbf{L}_m \times \mathbf{M}_m) = \dim Z_m + \dim \mathbf{\Psi}_m + \\ + \dim \mathbf{L}_m + \dim \mathbf{M}_m = 4m(m+2) + 6m + 6m + 6 = 4m^2 + 20m + 6. \end{aligned} \tag{180}$$

On the other hand, formula (15) for $n = 2m+1$, equality (75), Theorem 6.1 and the open inclusion (165) show that

$$\begin{aligned} 4m^2 + 20m + 6 = (2m+1)^2 + 8(2m+1) - 3 \leq \dim MI_{2m+1} = \dim MI_{2m+1}(\xi) = \\ = \dim X_m = \dim \widetilde{X}_m. \end{aligned} \tag{181}$$

Comparing (180) with (181) we see that all inequalities here are equalities. In particular, X_m is a $(4m^2+20m+6)$-dimensional locally closed locally complete intersection subscheme of $\mathbf{S}_{m+1}^\vee \times \mathbf{\Sigma}_{m+1}$ and $(X_m)_{red}$ is irreducible as an open part of the irreducible scheme $\overline{X}_m$. Hence by Lemma 7.3 X_m is reduced and irreducible. It follows now from Corollary 5.5 and Theorem 6.1 that $(MI_{2m+1})_{red}$ is irreducible of dimension $4m^2+20m+6 = n^2+8n-3$ for $n = 2m+1$, i.e. the inequality (15) becomes the strict equality. This together with Theorem 3.1 implies that MI_{2m+1} is a locally complete intersection subscheme of the vector space $\mathbf{S}_{2m+1}$. As a result, by Lemma 7.3 MI_{2m+1} is reduced. Since $\pi_{2m+1} : MI_{2m+1} \to I_{2m+1} : A \mapsto [E(A)]$ is a principal $GL(\mathbf{k}^{2m+1})/\{\pm id\}$-bundle in the étale topology (see section 3), it follows that I_{2m+1} is reduced and irreducible of dimension $16m+5 = 8n-3$ for $n = 2m+1$. This finishes the proof of Theorem 1.1.

Remark 8.1. Note that Theorem on fibres of a morphism together with the fact that all inequalities in (180) with (181) are equalities also implies that the projection $X_m \to \mathbf{S}^\vee_{m+1}$: $(D, C) \mapsto D$ is dominating. In view of Theorem 6.1 this is equivalent to the fact that that the restriction onto MI_{2m+1} of the linear projection $\mathbf{S}_{2m+1} \to \mathbf{S}_{m+1}$ induced by a generic embedding $\mathbf{k}^{m+1} \hookrightarrow \mathbf{k}^{2m+1}$ is dominating.

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