A remark on matrix Airy function

A.M. Perelomov [1]

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

In Memoriam M.V. Saveliev

Abstract

An integral representation for matrix Airy function is presented .

1. The well-known Witten's conjecture [Wi 1991a], that the logarithm of the partition function of general one-matrix model may be considered as the generating function for intersection numbers on the moduli space of algebraic curves, was proved by Kontsevich [Ko 1991], [Ko 1992], who has reduced this problem to the computation of the matrix Airy function. See papers [Wi 1991b], [Wi 1992], [IZ 1992], where many aspects of this problem have been discussed.

The matrix Airy function is defined by integral representation. Here we give once more integral representation for this function.

2. Following to [Ko 1992], we define the matrix Airy function on the space $\mathcal{H}_N$ of hermitian $N \times N$ matrices by the integral

$$\mathrm{A}(X) = \int \exp\{i(\frac{1}{3}\mathrm{tr}\,(Y^3) - \mathrm{tr}(XY))\}dY \tag{1}$$

over the space $\mathcal{H}_N$, where dY is $U(N)$-invariant measure on this space, $X, Y \in \mathcal{H}_N$.

Note also that the function $\mathrm{A}\,(X)$ satisfies to the equation

$$\Delta\,\mathrm{A}(X) + (\mathrm{tr}\,X)\mathrm{A}\,(X) = 0, \tag{2}$$

where Δ is the Laplacian.

Let us remind that the classical Airy function

$$\mathrm{Ai}\,(x) = \int_{-\infty}^{+\infty} \exp(i(y^3/3 - xy))dy \tag{3}$$

is the unique (up to scalar factor) bounded on real axis solution of the differential equation

$$\mathrm{Ai}^{''}(x) + x\,\mathrm{Ai}(x) = 0. \tag{4}$$

[1] On leave of absence from Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia. Current e-mail address: perelomo@posta.unizar.es

It follows from Eq.(1) that the dependence of $\mathrm{A}(X)$ from variables $\tilde{X}$ and $\xi = N^{-1}\mathrm{tr}X$ ($X = \tilde{X} + \xi I$, $\mathrm{tr}\tilde{X} = 0$) may be separated.

Indeed we have

$$X = \tilde{X} + \xi I, Y = \tilde{Y} + \eta I, \mathrm{tr}\,\tilde{X} = \mathrm{tr}\,\tilde{Y} = 0,$$

$$\mathrm{tr}\,(Y^3) = \mathrm{tr}\,(\tilde{Y}^3) + 3\eta\,\mathrm{tr}\,(\tilde{Y}^2) + N\eta^3,$$

$$\mathrm{tr}(XY) = \mathrm{tr}(\tilde{X}\tilde{Y}) + N\xi\eta.$$

Theorem 1. *For the matrix Airy function (1) the following integral representation is valid*

$$\mathrm{A}\,(X) = \mathrm{A}\,(\tilde{X}, \xi) = \mathrm{A}\,(Q, \xi)$$

$$= CN^{-1/3} \int \exp\{\frac{i}{3}\mathrm{tr}(P^3)\}\,\mathrm{Ai}\,(N^{-1/3}(N\xi - \mathrm{tr}\,(P^2)))\,\Phi(Q|P)\,d\mu(P). \quad (5)$$

Here Q and P are diagonal matrices $\tilde{X} = UQU^{-1}, \tilde{Y} = U_1PU_1^{-1}$, U and U_1 are unitary matrices, $\Phi(Q|P)$ is zonal spherical function

$$\Phi(Q|P) = \frac{\det\{\exp(i(q_j p_k))\}}{\det\{(q_j p_k)\}}, \quad (6)$$

$$Q = \mathrm{diag}\,(q_1, ..., q_N), P = \mathrm{diag}\,(p_1, ..., p_N), d\mu(P) = (V(P))^2 dP, \quad (7)$$

C=const , *$V(P)$ is the Vandermonde determinant,*

$$V(P) = \prod_{j<k}(p_j - p_k) = \det\,(p_j^{k-1}). \quad (8)$$

Proof . *This follows immediately from the Harish - Chandra formula.*

Theorem 2 [HC 1957]. *If Φ is a conjugacy invariant function on the space of hermitian $N \times N$ matrices, then for any diagonal hermitian matrix Q*

$$\int \Phi(Y)\exp(-i\,\mathrm{tr}(QY))dY$$

$$= (-2\pi i)^{N(N-1)/2}(V(Q))^{-1}\int \Phi(P)\exp(-i\,\mathrm{tr}(QP))(V(P))^2 dP, \quad (9)$$

where $Y = UPU^{-1}$ and last integral is taken over the space of diagonal hermitian matrices P .

So we obtain the integral representation of matrix Airy such that the dependence from variable ξ contains only in standard one-dimensional Airy function and this representation may be useful for investigation of properties of matrix Airy function.

3. For the case $N = 2$, $\mathrm{tr}(P^3) = 0$ and the integral representation (5) takes more simple form

$$\mathrm{A}\,(X) \equiv \mathrm{A}\,(\xi, r) = \int \exp\{\frac{2i}{3}(\eta^3 + 3\eta\, p^2 - 3\,\xi\,\eta)\}\Phi(r|p)p^2 dp\, d\eta, \tag{10}$$

where

$$\Phi(r|p) = \frac{\sin pr}{pr} \tag{11}$$

is the zonal spherical function for the matrix space $N = 2$. The integration on η gives

$$\mathrm{A}\,(\xi, r) = C \int \mathrm{Ai}\,((3/2)^{1/3}(\xi - p^2))\Phi(r|p)\, p^2 dp, C = 2(3/2)^{1/3}. \tag{12}$$

If we take in (10) the integral on p, we get another integral representation

$$\mathrm{A}\,(\xi, r) = C_1 \int \exp\{\frac{2i}{3}(\tau^3 - 3\,\xi\tau)\}G(r, \tau)d\tau,\ C_1 = \mathrm{const}, \tag{13}$$

where

$$G(r, \tau) = \tau^{-3/2}\exp\{i\tau^{-1}r^2\} \tag{14}$$

is the Green function of the Schrödinger equation for free motion in three-dimensional space.

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