Generalized Conner-Floyd conjecture and non-immersions of low 2-torsion lens spaces

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Abstract

We use a partial generalization of the Brown-Peterson version for the Conner-Floyd conjecture to detect Euler-type obstructions for the existence of Euclidean immersions of low 2-torsion lens spaces. With an extra technical hypothesis, our main result (Theorem 1.4) generalizes Davis' non-immersion theorem for real projective spaces. Indeed, our non-immersion condition relates the cellular dimension of the lens space to its 2-torsion.

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1 Introduction

For a prime p, the study of free $(\mathbb{Z}/p)^n$-actions on oriented manifolds led Conner and Floyd to consider the MSO homology of $(B\mathbb{Z}/p)^{\wedge n}$ —the iterated n-fold smash product of $B\mathbb{Z}/p$. As they noticed, the bottom "toral" class in these groups plays a fundamental role in the problem, and they made a conjecture on what the MSO_*-module generated by this element should look like [4]. At odd primes the situation can be restated in terms of the Brown-Peterson spectrum BP. The conjecture (established in [17, 20]) boils down to the fact that the annihilator ideal of the toral class in the BP-homology of $(B\mathbb{Z}/p)^{\wedge n}$ is $I_n = (p, v_1, v_2, \ldots, v_{n-1})$. As showed by tomDieck [9], the original geometric problem has the same solution if $(\mathbb{Z}/p)^n$ is replaced by any finite Abelian p-group of rank n. However, the annihilator ideal of the toral class in $BP_*(B\mathbb{Z}/p^{k_1} \wedge \cdots \wedge B\mathbb{Z}/p^{k_n})$ is in general smaller than I_n. For

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instance, in the case $n = 1$, the annihilator is (p^{k_1}). This raises two problems. On the one hand, describing the annihilator in the general case, and on the other and perhaps more importantly, determining geometrical interpretations for this ideal. We refer to the first problem as the generalized Conner-Floyd conjecture. This paper deals with the second problem. Indeed, inside the Brown Peterson homology of $B\mathbb{Z}/2^k \wedge B\mathbb{Z}/2$ we identify obstructions for the existence of Euclidean immersions of 2^k-lens spaces. The relation to the generalized Conner-Floyd conjecture comes from the fact that the obstructions are BP_*-multiples of the toral class. This is the reason for our interest in its annihilator ideal. The first systematic study of the generalized Conner-Floyd conjecture known to the author was made by G. Nakos in his Ph.D. thesis [18], where the next result is proved.

Proposition 1.1 ([18],[19]) *For a prime p, the annihilator ideal of the bottom class in the reduced Brown-Peterson homology of $B\mathbb{Z}/p^k \wedge B\mathbb{Z}/p$ is (p, v_1^k).*

Our results on the immersion of lens spaces are best understood within the context of the corresponding immersion situation for projective spaces, which we now briefly outline. We begin with the immersion problem for complex projective spaces (CPS's), which has a more regular behavior. It is known that $\mathbb{C}P^n$ immerses in $\mathbb{R}^{4n-\alpha(n)}$ [3, 16], but it cannot immerse in $\mathbb{R}^{4n-2\alpha(n)-1}$ [8, 11, 15, 21]. This limits the optimal immersion within a range of $\alpha(n)+1$ dimensions. A long standing conjecture of Davis and Mahowald [8] claims that the optimal immersion of $\mathbb{C}P^n$ is much closer to $4n-2\alpha(n)$ than to $4n-\alpha(n)$. An updated version of the conjecture is given in [7], taking into consideration the results in [5].

The immersion problem for real projective spaces (RPS's) is far more complex. Finding the smallest Euclidean space in which P^n immerses is a very challenging and tantalizing problem which has been considered by many people using an even larger amount of mathematical techniques. Despite of this, the known information indicates only that a precise solution to this problem is still far away. It is even hard to make a guess on what kind of solution we should expect. A common belief seems to be that an m-dimensional Euclidean (non)immersion of P^n is considered to be close to optimal if m lies within a certain small neighborhood of $2n-2\alpha(n)$, where $\alpha(n)$ is the number of ones in the dyadic expansion of n. This neighborhood is not expected to be centered around $2n-2\alpha(n)$ in any particular way and, although it should be generally smaller than $2\alpha(n)$, the only concrete indication of its size is given by its behavior in some particular known cases. Thus, unlike the complex case, finding bounds for the optimal immersion of a RPS is a less understood problem. Although optimal immersions for CPS's give a satisfactory upper bound for optimal immersions of RPS's (see Remark 2.1 in [11]), setting lower bounds in the real case (that is, finding non-immersions for RPS's) seems to be a very subtle problem. The best general result in this direction was proved by D. M. Davis.

Theorem 1.2 ([6]) *$P^{2(n+\alpha(n)-1)}$ does not immerse in $\mathbb{R}^{4n-2\alpha(n)}$.*

In an precise sense, Davis' theorem is very close to the best known non-immersion result for a given RPS. But perhaps the most important feature in Theorem 1.2 is the suggestion that the immersion problem for RPS's may benefit from adopting slightly different points of view. Indeed, key (non)immersions could be obtained if we are willing to vary both the Euclidean and the projective space dimensions. Interesting general patterns could be simpler to describe (and prove) with such an approach. In this paper we go on in that direction and introduce a third related variable — the 2-torsion — through the use of lens spaces, which in a natural way fill in the gap between real and complex projective spaces. In particular, Theorem 1.2 is embedded in a more general setting where the type of torsion is also an important factor. We conjecture the following non-immersion result.

Conjecture 1.3 *Let n, k and s be nonnegative integers with n, $k > 0$ and $k + s \geq \alpha(n)$. Then $L(2^k)^{2(n+s)+1}$ does not immerse in $\mathbb{R}^{4n-2\alpha(n)}$.*

Here $L(2^k)^m$ is the m-dimensional skeleton of the classifying space $B\mathbb{Z}/2^k$. We remark that the case $s = 0$ in the above conjecture is settled by [11, Theorem 1.1] and, in view of Proposition 2.1 in the next section, the case $s \geq \alpha(n) - 1$ is equivalent to Davis' Theorem 1.2. Thus, while [11] suggests a close relationship between immersions for CPS's and those for "high" 2-torsion lens spaces, this paper takes advantage of the similarities in the immersion problems for RPS's and "low" 2-torsion lens spaces. Heuristically, the lower the 2-torsion is, the more irregular the immersions behave. Therefore a knowledge of critical (non)immersions of low 2-torsion lens spaces could share some light on the type of solution we should expect for the immersion problem of RPS's.

The calculations in this paper show that Conjecture 1.3 is true provided an extra condition is assumed. Our main result is then:

Theorem 1.4 *The non-immersion in Conjecture 1.3 holds if, in addition, $2^{\nu(n)} > \alpha(n) + 2s + 1$, where $\nu(n)$ is the highest power of 2 dividing n.*

The approach to Theorem 1.4 uses a standard geometrical argument describing a certain bundle η which should have an everywhere nontrivial section if $L(2^k)^{2(n+s)+1}$

were to immerse in $\mathbb{R}^{4n-2\alpha(n)}$. Obstructions to such an immersion are given by the Euler classes of η with respect to any oriented cohomology theory. We use the Brown-Peterson spectrum for our calculations. In Section 3 some algebraic manipulations are performed to reduce this BP-Euler class to a simple form. The resulting element is related in Section 4 to the generalized Conner-Floyd conjecture as stated in Proposition 1.1.

This paper owes its existence to [24] where this type of calculations first appeared in print. In turn the idea goes back to [23]. On the other hand we were motivated by the questions in the introduction to [13]. The present work gives an indication of the type of arguments and modifications needed to prove a general Conner-Floyd conjecture.

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2 Geometric and algebraic preliminaries

Let $L(2^k)^m$ be the m-dimensional lens space modulo 2^k. For $m = 2n+1$ this is a manifold given as the orbit space of the usual $\mathbb{Z}/2^k$ action on the unit vectors in $\mathbb{C}^{n+1}$. The effect of the 2-torsion on the optimal immersion of a lens space is indicated in the next result. It appears as Lemma 1.3 in [11]. We omit the proof.

Proposition 2.1 *If $L(2^{k+1})^{2\ell+1}$ immerses in $\mathbb{R}^N$, then so does $L(2^k)^{2\ell+1}$.*

Next, let μ_{2n+1} stand for the canonical complex line bundle over $L(2^k)^{2n+1}$ and let λ_{2n+1} denote its realification. For $k=1$ the restriction of λ_{2n+1} to $L(2^k)^{2n}$ agrees with $2\xi_{2n}$, where ξ_{2n} is the canonical real line bundle over P^{2n}. The starting point in our discussion is the following standard result [2, 6, 11, 24].

Proposition 2.2 *If $L(2^k)^{2\ell+1}$ immerses in Euclidean dimension $2M$, then the bundle $(2^L-\ell-1)\lambda_{2\ell+1}\otimes\xi_{2t}$ has an everywhere nontrivial section, where $t = 2^L - M - 1$ and $L \gg 0$.*

Proof. The stable tangent bundle of $L(2^k)^{2\ell+1}$ is $(\ell+1)\lambda_{2\ell+1}$ so that, in view of Hirsch's theorem [12], the geometrical dimension of $-(\ell+1)\lambda_{2\ell+1}$ is at most $2(M-\ell)-1$. Since the virtual bundle $\lambda_{2\ell+1}$ is of 2-torsion, the above means that, for large L, $(2^L-\ell-1)\lambda_{2\ell+1}$ has $2t+1$ everywhere linearly independent sections, where $t=2^L-M-1$. The desired conclusion follows from [2, Proposition 2.1]. ∎

Let BP stand for the 2-primary Brown-Peterson spectrum. This is a complex oriented theory with $BP^*(\mathbb{C}P^\infty) = BP^*[[x]]$, where x is the Euler class of the canonical complex line bundle of $\mathbb{C}P^\infty$. The 2^k-power map in the multiplication of $\mathbb{C}P^\infty$ defines the 2^k-series for BP:

$$[2^k](x) = \left(2^k\right)^*(x) = \sum_{s\geq 0} a_{k,s}x^{s+1} \tag{1}$$

for coefficients $a_{k,s}\in BP^{-2s}$. The calculations in Section 3 of [11] show that the even dimensional Brown-Peterson cohomology of the Cartesian product $L(2^k)^{2\ell+1}\times P^{2t}$ is given by the ring

$$BP^*[x,y]\Big/(x^{\ell+1},y^{t+1},[2^k](x),[2](y)) \tag{2}$$

where x and y are the Euler classes of $\lambda_{2\ell+1}\otimes 1$ and $1\otimes 2\xi_{2t}$, respectively. Moreover, Lemma 2.3 and Proposition 2.4 in [11] show that the BP-Euler class of the bundle in Proposition 2.2 is given, up to a unit, by

$$(x-y)^{2^L-\ell-1}\in BP^{2(2^L-\ell-1)}\left(L(2^k)^{2\ell+1}\times P^{2t}\right). \tag{3}$$

3 The BP-Euler class

As observed in the Introduction, we may restrict to $1\leq k<\alpha(n)$ in Conjecture 1.3. Then, in view of Proposition 2.1, the required non-immersion would follow from the simplified case:

$$L(2^k)^{2(n+\alpha(n)-k)+1} \text{ does not immerse in } \mathbb{R}^{4n-2\alpha(n)}.$$

The validness of this statement is established under the hypothesis of Theorem 1.4 by showing the non-triviality of the corresponding Euler class (3). This section is devoted to the proof of the next result which, as a first step in the above direction, gives a simple description for (3).

Proposition 3.1 *Let $1\leq k<\alpha(n)$ and assume $2^{\nu(n)}>3\alpha(n)-2k+1$. For $\ell=n+\alpha(n)-k$ and $t=2^L-2n+\alpha(n)-1$ ($L\gg 0$), the element in (3) agrees up to a unit with*

$$v_1^{k-1}v_2^{\alpha(n)-k}x^{\ell-1}y^t\in BP^{even}\left(L(2^k)^{2\ell+1}\times P^{2t}\right).$$

The formal sum for BP is, by definition, the BP-Euler class of the tensor product of the two canonical complex lines bundles over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. This is a power series of the form

$$F(x, y) = x + y + xy \sum e_{ij} x^i y^j \in BP^*[[x, y]]. \tag{4}$$

(Note $F(x, y) = x + \omega y$, where $\omega = 1 + x \sum e_{ij} x^i y^j$ is a unit). The formal sum of two series $\gamma, \lambda \in BP^*[[x]]$ with trivial independent terms is defined by $\gamma +_F \lambda = F(\gamma, \lambda)$. Likewise, the formal series $\sum^F \gamma_i$ is defined for sequences γ_i converging to zero in the filtration topology for the ideal (x) (see [1]). Since F is p-typical, the formula

$$[2^k](x) = \sum_{s \geq 0}^F v_{k,s} x^{2^s} \tag{5}$$

defines Araki type elements $v_{k,s} \in BP^{-2^{s+1}+2}$ (for $k = 1$ these are the usual Araki generators v_s). The 2-divisibility properties we need for the coefficients in (1) and (5) are summarized in the next result. Here $\nu(a)$ stands for the highest power of 2 dividing the element $a \in BP^* = \mathbb{Z}_{(2)}[v_1, v_2, \ldots]$, and the expression $a =_u b$ means $a = \mu b$ for some unit $\mu \in BP^*$.

Lemma 3.2 *a)* $a_{k,0} = v_{k,0} = 2^k$.

b) $a_{k,1} =_u v_{k,1} =_u 2^{k-1} v_1$.

c) $\nu(a_{k,2}) \geq 2k - 1$.

d) For $s \geq 3$, $\nu(a_{k,s}) \geq k - s + 1$.

e) For $k \geq s$, $v_{k,s} \equiv 2^{k-s} v_1^{2^s - 1} \mod 2^{k-s+1} \cdot BP^*$.

f) For $k \geq 2$, *there is a unit* $\omega \in BP^*$ *such that* $v_{k,2} = 2^{k-2} v_1^3 + \omega 2^{k-1} v_2$.

g) For $k \geq 3$, $v_{k,3} \in (2^{k-3} v_1^7, 2^{k-2} v_1, 2^{k-1})$.

Proof. Parts $a)$ through $e)$ are contained or follow easily from [10]. The last two relations can be deduced from the formula [10, Lemma 2.1]

$$v_{k,s} = 2^k u m_s - \sum_{i=1}^{s-1} m_i v_{k,s-i}^{2^i} \tag{6}$$

where $m_s \in H_{2^{s+1}-2}(BP)$ are the usual polynomial generators [25] and $u \in BP^*$ is a unit (depending on k and s). We only indicate the details for $g)$ as $f)$ is similar and simpler. By e) and dimensional arguments, it suffices to show that, in the expression for $v_{k,3}$, the coefficient of the monomial v_3 is, up to a unit, 2^{k-1}. But since $2^s m_s \in BP^*$, equation (6) gives, modulo decomposable elements and up to units:

$$v_{k,3} \equiv 2^k m_3 = 2^{k-3}(2^3 m_3) \equiv 2^{k-3}(2^2 v_3) = 2^{k-1} v_3. \qquad \blacksquare$$

Corollary 3.3 *For $0 \leq i < \ell$, the order of $x^{\ell-i}$ in $BP^*[x] \big/ (x^{\ell+1}, [2^k](x))$ is 2^{k+i}.*

Proof. We induct on i. The case $i = 0$ follows from Lemma 3.2 a), since $0 = 2^k x^\ell$ is the only relation in the top dimension. For $1 \leq i < \ell$, induction and Lemma 3.2 a) through d) show that the relation $0 = 2^{i-1} x^{\ell-i-1} \cdot [2^k](x)$ reduces to $0 = 2^{k+i-1} x^{\ell-i} + 2^{k+i-2} v_1 x^{\ell-i+1}$ (the unit disappears because the second term has order ≤ 2). This gives $0 = 2^{k+i} x^{\ell-i}$. If $2^{k+i-1} x^{\ell-i}$ were trivial, then the following relation would hold in $BP^*[[x]]$:

$$2^{k+i-1} x^{\ell-i} = A \cdot x^{\ell+1} + (a_{k,0} x + a_{k,1} x^2 + \cdots)(c_0 x^{\ell-i-1} + c_1 x^{\ell-i} + \cdots).$$

Comparing coefficients of $x^{\ell-i+j}$ $(0 \leq j \leq i)$ this would give the equations

$$a_{k,0} c_j + a_{k,1} c_{j-1} + \cdots + a_{k,j} c_0 = \begin{cases} 2^{k+i-1} & \text{if } j = 0; \\ 0 & \text{if } 1 \leq j \leq i. \end{cases}$$

But then a simple inductive argument on j using Lemma 3.2 a) through d) would imply $\nu(c_j) = i - 1 - j$ for $0 \leq j \leq i$. This is impossible for $j = i$. $\blacksquare$

For reference we record the following consequence of the proof of Corollary 3.3

Corollary 3.4 *For $0 \leq i, j$ and $i + j < \ell$ the relation $2^{k+i-1} v_1^j x^{\ell-i} = 2^{k+i+j-1} x^{\ell-i-j}$ holds in the ring $BP^*[x] \big/ (x^{\ell+1}, [2^k](x))$.*

The simplification we need for the Euler class in (3) starts with the following result whose validness depends on the extra condition in Theorem 1.4. Through the rest of the section we assume the notation and hypothesis of Proposition 3.1.

Lemma 3.5 *For $0 \leq j \leq 3\alpha(n) - 2k$, $\nu \binom{2^L - \ell - 1}{\ell - j} \geq \alpha(n) - 1$. Moreover, the inequality is in fact*

a) an equality for $j = 2\alpha(n) - 2k + 1$.

b) strict for $0 \leq j \leq 2\alpha(n) - 2k$.

Proof. Take $s = \alpha(n) - k + 1$ in [24, Proposition 5.1]. ∎

Our first goal in the proof of Proposition 3.1 is to show the triviality of all the summands but one in the expression

$$\sum_{j=0}^{3\alpha(n)-2k} \pm \binom{2^L - \ell - 1}{\ell - j} x^{\ell-j} y^{t-3\alpha(n)+2k+j} \tag{7}$$

for the element in (3). Section 4 deals with the exceptional summand having $j = 2\alpha(n) - 2k + 1$. The summands with $j \leq \alpha(n) - k$ or $j > 2\alpha(n) - 2k + 1$ are trivial in view of Corollary 3.3 and Lemma 3.5. The summands with $\alpha(n) - k < j \leq 2\alpha(n) - 2k$ could be treated in a similar way by noticing that the corresponding binomial coefficients $\binom{2^L-\ell-1}{\ell-j}$ can be made more divisible by 2 than that asserted by Lemma 3.5. This can be proved at the cost of strengthening the condition $2^{\nu(n)} > 3\alpha(n) - 2k + 1$. We prefer the following deeper analysis of the relations in the ring (2) because it will allow us to relate (3) with the generalized Conner-Floyd conjecture in Proposition 1.1.

The next results are our version of the "algorithm" used in the proof of [24, Lemma 2.2] (see also [13, Theorem 3.2] and [14, Lemma 2.6]). The calculations below are however a bit more elaborated due to the 2-divisibility properties of the coefficients $v_{k,s}$ in (5).

Lemma 3.6 *In the ring (2) we have* $2^a x^b y^c = 0$ *provided* $3a + b + c \geq \ell + t + 2k - 1$, $b > 0$ *and* $b + c > \ell + 1$.

Proof. We make a decreasing induction on b (the lemma certainly holds for $b > \ell$). The orders of the elements (Corollary 3.3) give the result if $a \leq k - 1$. Assume $a \geq k$. Using the expression (4) for the formal sum we see that, in the ring (2), the relation $0 = [2^k](x)$, given as in (5), takes the form $\omega 2^k x = \sum^F v_{k,s} x^{2^s}$, with ω a unit and the (formal) sum running only over $s \geq 1$. Therefore it suffices to verify the triviality of

$$2^{a-k} v_{k,s} x^{b+2^s-1} y^c \tag{8}$$

for $s \geq 1$. The case $k < s$ is easy because $2^{a-k} x^{b+2^s-1} y^c = 0$ by induction, since $2^s - 1 \geq 3k$. Assume $k \geq s$. By Lemma 3.2 *e*), the element in (8) lies in the ideal $(2^{a-s} v_1^{2^s-1} x^{b+2^s-1} y^c,\ 2^{a-s+1} x^{b+2^s-1} y^c)$ whose second generator is in fact trivial by induction. To analyze the first generator we apply a substitution using the relation $0 = [2](y)$ as in (1). Namely, by Lemma 3.2 *b*) and *c*), we have

$$v_1 y^2 \in (2y, y^4) \tag{9}$$

which shows that $2^{a-s} v_1^{2^s-1} x^{b+2^s-1} y^c$ lies in the ideal

$$(2^{a-s+1} v_1^{2^s-2} x^{b+2^s-1} y^{c-1},\ 2^{a-s} v_1^{2^s-2} x^{b+2^s-1} y^{c+2}). \tag{10}$$

By induction both generators are trivial provided $s \neq 2$. For $s = 2$ we need to apply the substitution (9) once more to see that (10) is contained in the ideal

$$(2^{a-s+2} x^{b+2^s-1} y^{c-2},\ 2^{a-s+1} x^{b+2^s-1} y^{c+1},\ 2^{a-s} x^{b+2^s-1} y^{c+4})$$

all whose generators are trivial by induction. ∎

Remark 3.7 The condition $b + c > \ell + 1$ in Lemma 3.6 is used in case we run out of powers of y in the process of applying the substitutions in (9). If that is the case, the element under consideration is trivial because the power on x will be too high. For our application, this condition is guaranteed by the fact that $L \gg 0$.

It follows from Lemmas 3.5 and 3.6 that the element in (7) is up to a unit $2^{\alpha(n)-1} x^{\ell-2\alpha(n)+2k-1} y^{t-\alpha(n)+1}$. Now Proposition 3.1 follows from Corollary 3.4 and the next result.

Lemma 3.8 *In the ring (2) the equation* $2^a x^b y^{t-a} = 2^{a-1} v_2 x^{b+2} y^{t-a+1}$ *holds up to units provided* $a \geq k$, $b \geq 2$, $t - a \geq 3$, $2a + b \geq \ell + 2k - 3$ *and* $t - a + b > \ell + 1$.

Proof. We now write the relation $0 = [2^k](x)$ as

$$0 = \omega_1 2^k x + \omega_2 2^{k-1} v_1 x^2 + \omega_3 v_{k,2} x^4 + \sum_{s\geq 3}^F v_{k,s} x^{2^s}$$

for units ω_1, ω_2 and ω_3. Together with Corollary 3.4 this gives

$$\begin{aligned}
2^a x^b y^{t-a} &= x^b (2^a y^{t-a}) = x^b \left(2^{a-1} v_1 y^{t-a+1}\right) \\
&= \left(2^{k-1} v_1 x^b\right) \left(2^{a-k} y^{t-a+1}\right) \\
&= -\omega_2^{-1} \left(\omega_1 2^k x^{b-1} + \omega_3 v_{k,2} x^{b+2} + x^{b-2} \sum\nolimits^F v_{k,s} x^{2^s}\right) \left(2^{a-k} y^{t-a+1}\right)
\end{aligned}$$

where the formal sum in the third line runs over $s \geq 3$. The first (regular) summand in that line is trivial in view of Corollary 3.3. The following claims take care of the other summands.

A. Up to a unit, $2^{a-k}v_{k,2}x^{b+2}y^{t-a+1}$ is the required element $2^{a-1}v_2x^{b+2}y^{t-a+1}$.

B. For $s \geq 3$, $2^{a-k}v_{k,s}x^{b+2^s-2}y^{t-a+1} = 0$.

The claims follow from straightforward calculations using Lemma 3.2 f) and g) respectively. In some cases the substitution in (9) needs to be applied up to three consecutive times before using Lemma 3.6 to conclude the triviality of a particular element (this happens both in **A** and **B**). We only give the details for **B**. When $k < s$, Lemma 3.6 can be used right away to see $2^{a-k}x^{b+2^s-2}y^{t-a+1} = 0$, except for $s = 3$ and $k = 2$, in which case we first make an application of (9) using the easy to verify relation $v_{2,3} \in (2, v_1)$. When $s \leq k$, Lemma 3.2 e) shows that the element in **B** is divisible by $2^{a-s}x^{b+2^s-2}y^{t-a+1}$ which, by Lemma 3.6, is trivial for $s \geq 4$. If $s = 3$, Lemma 3.2 g) shows that the element in **B** lies in fact in the ideal

$$(2^{a-s}v_1^3x^{b+2^s-2}y^{t-a+1},\ 2^{a-s+1}v_1x^{b+2^s-2}y^{t-a+1},\ 2^{a-s+2}x^{b+2^s-2}y^{t-a+1}).$$

After applying the substitution in (9) (once in the second generator and three times in the first), Lemma 3.6 shows that all the summands involved are trivial. ■

4 Proof of Theorem 1.4

We start with some well known topological considerations. For a positive integer ℓ let $S^1 \to L(2^k)^{2\ell+1} \to \mathbb{C}P^\ell$ be the spherical fibration $S(H^{\otimes 2^k})$ associated to the 2^k-fold tensor product of the canonical complex line bundle H over $\mathbb{C}P^\ell$. When $\ell = \infty$ the fibration extends one step to the right by means of the 2^k-power map in the Hopf multiplication of $\mathbb{C}P^\infty$. In particular, the BP-Euler class of $S(H^{\otimes 2^k})$ is (the restriction of) the $[2^k]$-series. The corresponding BP-Gysin sequences split into short exact sequences and describe the even dimensional BP-cohomology of $L(2^k)^{2\ell+1}$ (or equivalently $BP^*(L(2^k)^{2\ell})$) as

$$BP^*[[x]] \big/ (x^{\ell+1}, [2^k](x)) ,$$

and the *reduced* BP-homology of $L(2^k)^{2\ell}$ as the free BP_*-module on generators d_i ($1 \leq i \leq \ell$), of degree $2i+1$, subject to the relations

$$\sum_{s=0}^{i-1} a_{k,s}d_{i-s} \qquad (1 \leq i \leq \ell)$$

where, as in (1), the $a_{k,s}$ are the coefficients of the 2^k-series. In particular, we have a BP_*-isomorphism

$$\phi_k : \widetilde{BP}^{even}(L(2^k)^{2\ell+1}) \to \widetilde{BP}_*(L(2^k)^{2\ell}) \tag{11}$$

with $\phi(x^\ell) = d_1$.

Proof of Theorem 1.4. As observed in Section 3, it suffices to verify the non-triviality of the element in Proposition 3.1 or, for that case, the non-triviality of

$$v_1^{k-1} v_2^{\alpha(n)-k} x^\ell y^t \in \widetilde{BP}^{even}\left(L(2^k)^{2\ell+1} \wedge P^{2t}\right).$$

The considerations in (2) amount to saying that the product map

$$\widetilde{BP}^{even}\left(L(2^k)^{2\ell+1}\right) \otimes_{BP_*} \widetilde{BP}^*\left(P^{2t}\right) \to \widetilde{BP}^{even}\left(L(2^k)^{2\ell+1} \wedge P^{2t}\right) \tag{12}$$

is a monomorphism. Since $x^\ell y^t$ lies in the image, we can restrict attention to the left hand side of (12) or, by means of $\phi_k \otimes \phi_1$, to the bottom class (multiplied by $v_1^{k-1} v_2^{\alpha(n)-k}$) in $\widetilde{BP}_*(L(2^k)^{2\ell}) \otimes_{BP_*} \widetilde{BP}_*(P^{2t})$. The desired conclusion follows from Proposition 1.1 and the obvious BP_*-morphism from the latter module into the reduced BP-homology of $B\mathbb{Z}/2^k \wedge B\mathbb{Z}/2$. ■

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