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Abhijit Laskar

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Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

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Abhijit Laskar [1]

Abstract

We study ℓ-independence of the conjugacy class of Frobenius, in the the canonical ℓ-adic Galois representation associated to the motives of hyperkähler varieties of $K3$ type, defined over number fields. We also establish a relationship between those results and the conjugacy class of crystalline Frobenius.

Keywords: hyperkähler varieties, motives, Compatible system of Galois representations, Weil-Deligne group.

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[1] Abhijit Laskar: Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany. E-mail: abhijit.laskar@gmail.com, Telephone: +49(0)228-402-369

1 Introduction

Fix a number field F with an embedding $\tau : F \hookrightarrow \mathbb{C}$. Let v be a non-archimedean valuation on F, $\bar{F}$ a fixed algebraic closure of F, $\bar{v}$ an extension of v to $\bar{F}$. We denote F_v the completion of F at v and $\bar{F}_v$ the localization of $\bar{F}$ at $\bar{v}$. The residue fields of F_v and $\bar{F}_v$ are denoted as k_v and $\bar{k}_v$, respectively. Let $p > 0$ be the characteristic of k_v. We denote $\Gamma_v := \mathrm{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \mathrm{Gal}(\bar{F}/F)$, $I_v \subset \Gamma_v$ is the inertia subgroup and $\phi_v \in \mathrm{Gal}(\bar{k}_v/k_v)$ the Frobenius automorphism. Fix an arithmetic Frobenius $\Phi_v \in \Gamma_v$, i.e. an element which induces ϕ_v. The ring of Witt vectors of k_v is denoted by $W(k_v)$.

Consider a proper and smooth algebraic variety X defined over F_v. The group Γ_v acts naturally on the etale cohomology groups $V_\ell^i := H^i_{\acute{e}t}(X_{/\bar{F}_v}, \mathbb{Q}_\ell)$, for each prime number ℓ and all positive integers i. This action gives rise to the representations $\rho_\ell^i : \Gamma_v \to \mathrm{GL}(V_\ell^i)$. It is a major theme in arithmetic geometry to determine to what extent the properties of these representations are independent of ℓ. In order to answer these ℓ-independence questions, one often has to restrict the above representations to the Weil group W_v of F_v. This is the subgroup formed by those elements of Γ_v which induce an integral power of ϕ_v in $\mathrm{Gal}(\bar{k}_v/k_v)$. We endow W_v with the topology determined by the condition that $I_v \subset W_v$ is an open subgroup having the topology inherited from its topology as a Galois group.

In what follows we assume that $\ell \neq p$. Now let X have good reduction at v, i.e that X extends to a proper and smooth scheme over the ring of integers of F_v. This implies that the inertia subgroup I_v acts trivially on the etale cohomology groups V_ℓ^i. It is well known from the works of P. Deligne on the Weil conjectures [9], that in this case the character of the representation of W_v on each V_ℓ^i has values in $\mathbb{Q}$ and is independent of ℓ. Since the action of inertia is trivial, this amounts to a statement on the action of the subgroup of Γ_v, generated by Φ_v. We will summarise this by saying that the ρ_ℓ^i are *defined over* $\mathbb{Q}$ and form a *compatible system* of representations of W_v, for a fixed i and variable ℓ.

If the algebraic variety X is defined over F, then the above situation has a natural generalization. By using the embedding $\tau : F \hookrightarrow \mathbb{C}$ we can consider the i-th singular cohomology $V^i = H^i_B(X(\mathbb{C}), \mathbb{Q})$ of $X(\mathbb{C})$. Then $V_\ell^i = V^i \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Now let $H \subseteq \mathrm{GL}(V^i)$ be a linear algebraic group over $\mathbb{Q}$. Suppose that $\mathrm{Im}(\rho_\ell^i) \subseteq H(\mathbb{Q}_\ell)$. In most of the cases that we consider this H would be the motivic Galois group (see precise definition later) of the absolute Hodge motive $h^i(X)$. Then we can ask if the $H(\overline{\mathbb{Q}_\ell})$-conjugacy class of $\rho_\ell^i(\Phi_v)$ is defined over $\mathbb{Q}$ and if it is independent of ℓ. Consider the case where $H = \mathrm{GL}(V^i)$. Then Deligne's theorem (cited above) becomes a special case of this problem.

We can ask similar questions in the case where the algebraic variety X does not have good reduction. To describe this situation we first need some general notions. Consider any arbitrary quasi-unipotent ℓ-adic representation of the form

$$\xi_\ell : \Gamma_v \to H(\mathbb{Q}_\ell). \tag{1}$$

Grothendieck's ℓ-adic monodromy theorem ([8, 8.2], [12] or [22]) tells us that, for a sufficiently small open subgroup of I_v, this action can be described as exponential of a single endomorphism N'_ℓ, the monodromy operator. The restriction of ξ_ℓ to the Weil group W_v can then be encoded by giving N'_ℓ together with a representation ξ'_ℓ of W_v which is trivial on an open subgroup of the inertia group. We will refer to such a triple $(H_{/\mathbb{Q}_\ell}, \xi'_\ell, N'_\ell)$ as *a representation of the Weil-Deligne group* $'W_v$ of F_v. In fact, $N'_\ell \in \mathrm{Lie}(H_{/\mathbb{Q}_\ell})$. To explain more about ξ'_ℓ we need some preparation. Let $t_\ell : I_v \to \mathbb{Z}_\ell(1)$ be the surjection defined by $\sigma \mapsto \left(\frac{\sigma(\pi^{1/\ell^m})}{\pi^{1/\ell^m}}\right)_m$ for a prime element π of F_v. It is known that t_ℓ is independent of the choice of π and its system of ℓ^m-th roots π^{1/ℓ^m}. Let $w \in W_v$ induce an integral power $\phi_v^{\alpha(w)}$ of ϕ_v in $\mathrm{Gal}(\bar{k}_v/k_v)$. Then, ξ'_ℓ is given as :

$$\xi'_\ell(w) = \xi_\ell(w)\exp(-N'_l t_\ell(\Phi_v^{-\alpha(w)} w)). \tag{2}$$

For a fixed ℓ, we say that the representation $(H_{/\mathbb{Q}_\ell}, \xi'_\ell, N'_\ell)$ is *defined over* $\mathbb{Q}$, if for every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$, and every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$, there exists a $g \in H(\Omega)$ such that

$$\sigma\xi'_{\ell/\Omega} = g \cdot \xi'_{\ell/\Omega} \cdot g^{-1} \quad \text{and} \quad \sigma(N'_\ell \otimes_{\mathbb{Q}_\ell} 1) = \mathrm{Ad}(g)(N'_\ell \otimes_{\mathbb{Q}_\ell} 1) \tag{3}$$

where $\xi'_{\ell/\Omega} : W_K \to H_{/\mathbb{Q}_\ell}(\Omega)$ is the extension of scalars and

$$N'_\ell \otimes_{\mathbb{Q}_\ell} 1 \in (\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \Omega = \mathfrak{h} \otimes \Omega,$$

$\mathfrak{h}$ being the Lie-algebra of H.

Varying ℓ over all primes different from p, we say that the representations (N'_ℓ, ξ'_ℓ) form a *compatible system of representations of* $'W_v$ if for every pair (ℓ, ℓ') of prime numbers different fro p and every algebraically closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$, there exists some $g \in H(\Omega)$ such that

$$\xi'_{\ell/\Omega} = g \cdot \xi'_{\ell'/\Omega} \cdot g^{-1} \quad \text{and} \quad N'_\ell \otimes_{\mathbb{Q}_\ell} 1 = \mathrm{Ad}(g)(N'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} 1) \in \mathfrak{h} \otimes \Omega. \tag{4}$$

These notions originated in the work of P. Deligne (see [8]). See also [11].

Now return to the case of the algebraic variety X and the representations ρ_ℓ^i. We want to use the above notions to study the case where X does not have good reduction at v. It is known that the representations ρ_ℓ^i are quasi-unipotent [1]. Suppose again that X is defined over F and in (1) take H as $\mathrm{GL}(V^i)$ and ξ_ℓ as ρ_ℓ^i. Then it is conjectured by J-M Fontaine that for a fixed i, the representations $(\mathrm{GL}(V_\ell^i), \rho_\ell'^i, N_\ell')$ form a compatible system of representations of $'W_v$ defined over $\mathbb{Q}$. We refer the reader to [11, 2.4.3] conjecture C_{WD}, for a statement also covering p-adic representations.

In a similar spirit, one can raise these questions for motives instead of algebraic varieties. In Grothendieck's category of motives the morphisms are defined by algebraic cycles. However, many of the desired properties of this category depend on unknown properties (standard conjectures) of algebraic cycles. P. Deligne gave a construction of a category of motives, where morphisms are defined using absolute Hodge cycles. Here we shall consider this unconditional theory of motives.

Let $\mathcal{M}_{AH}(F)$ denote the $\mathbb{Q}$-linear semismple Tannakian category of absolute Hodge motives over F [10]. The questions that we asked before are now stated in terms of the motivic Galois groups of motives in $\mathcal{M}_{AH}(F)$. The *motivic Galois group* G_M of any object M in $\mathcal{M}_{AH}(F)$ is defined as the automorphism group of the Betti realization functor H_τ, restricted to the Tannakian category generated by M and the Tate motive. It is a linear algebraic group over $\mathbb{Q}$. Now for each prime number ℓ, let $H_\ell(M)$ denote the ℓ-adic realization of the motive M. Then the action of the Galois group Γ_F on $H_\ell(M)$ gives us a ℓ-adic Galois representation

$$\rho_{M,\ell} : \Gamma_F \to \mathrm{GL}(H_\ell(M)).$$

It is known that $\rho_{M,\ell}$ factors through $G_M(\mathbb{Q}_\ell)$.

We say that a motive $M \in \mathrm{Ob}(\mathcal{M}_{AH}(F))$ has good reduction at v if $\rho_{M,\ell}$ is trivial on the inertia subgroup for every $\ell \neq p$. Let $\mathrm{Conj}(G_M)$ be the universal categorical quotient of G_M for its action on itself by conjugation and

$$\mathrm{Cl} : G_M \to \mathrm{Conj}(G_M)$$

be the corresponding quotient map. The image of the arithmetic Frobenius $\rho_{M,\ell}(\Phi_v)$ defines an element $\mathrm{Cl}(\rho_{M,\ell}(\Phi_v)) \in \mathrm{Conj}(G_M)(\mathbb{Q}_\ell)$, which depends only on the valuation v.

In the good reduction case, a motivic version of the previously stated questions on varieties, is formulated by J-P Serre [21, 12.6].

Conjecture 1.1 (Serre)**.** *If M is a motive with good reduction at v. Then the conjugacy class*

$$\mathrm{Cl}(\rho_{M,\ell}(\Phi_v)) \in \mathrm{Conj}(G_M)(\mathbb{Q}), \quad \forall \quad \textit{prime numbers} \quad \ell \neq p$$

and is independent of ℓ.

Serre conjectures this for any unconditional category of motives. As the category of motives defined in terms of absolute Hodge classes have more morphisms then the category of Grothendieck motives, so the absolute Hodge version of the conjecture is a priori stronger then the original conjecture. The Hodge conjecture implies that both versions should be equivalent.

Let us now study the bad reduction case. Let M be a motive and $\rho_{M,\ell} : \Gamma_F \to G_M(\mathbb{Q}_\ell)$ be the corresponding ℓ-adic representation. We say that a motive $M \in \mathrm{Ob}(\mathcal{M}_{AH}(F))$ has *semi-stable reduction* at v if $\rho_{M,\ell}$ is unipotent on the inertia subgroup for every $\ell \neq p$. By an earlier discussion we know that $\rho_{M,\ell}$ induces a representation $(G_{M/\mathbb{Q}_\ell}, \rho'_{M,\ell}, N'_{M,\ell})$ of the Weil-Deligne group $'W_v$, where $N'_{M,\ell}$ is the monodromy operator associated to $\rho_{M,\ell}$ and

$$\rho'_{M,\ell}(w) = \rho_{M,\ell}(w)\exp(-N'_{M,\ell}t_\ell(\Phi_v^{-\alpha(w)}w)) \quad \forall\, w \in W_v,$$

as in (2).

Then, in analogy with Serre's conjecture 1.1, we may formulate a motivic version of Fontaine's conjecture (cited before), as following:

Question 1.2. *Let $M \in \mathrm{Ob}(\mathcal{M}_{AH}(F))$ be a motive with semi-stable reduction at v and G_M be the motivic Galois group associated to M. Then, for $\ell \neq p$ the representations*

$$(G_{M/\mathbb{Q}_\ell}, \rho'_{M,\ell}, N'_{M,\ell})$$

of $'W_v$ induced by $\rho_{M,\ell}$'s, forms a compatible system of representations of $'W_v$ defined over $\mathbb{Q}$ with values in G_M.

Now recall that a hyperkähler variety over any field K of characteristic 0 is a simply connected smooth projective K-variety Y of even dimension $2r$, with the property that there exists a section ω of Ω^2_Y, unique up to multiplication by a constant, such that ω^r vanishes nowhere. They are the generalizations of $K3$ surfaces in higher dimensions.

In the first part of the paper we answer the aforementioned question 1.2 and the conjecture of Serre 1.1, for motives of a class of hyperkähler varieties, which we call of $K3$ type (see §2 for precise definition). These forms theorems 3.10 and 3.11 of the article. On our way we prove few other important results;(i) for any hyperkähler variety of $K3$ type, the coefficients of characteristic polynomial for the action of Frobenius Φ_v on the the ℓ-adic cohomology groups, are in $\mathbb{Q}$ and is independent of ℓ, (ii) After a finite base extension, the action of Φ_v on the the ℓ-adic cohomology groups, are semisimple. These forms theorems 2.1 and 2.3 of the article.

In the second part of the paper we pass on to the situation when $\ell = p$. In particular, we prove a crystalline version of the conjecture of Serre, for those algebraic vareities whose motives belong to $\mathcal{M}^{ab}_{AH}(F)$, the Tannakian subcatgeory of $\mathcal{M}_{AH}(F)$ generated by the motives of abelian varieties and Artin motives. This is theorem 4.3 of the paper. This result also holds for motives of hyperkähler varieties of $K3$ *type*. Recall that $\mathcal{M}^{ab}_{AH}(F)$ contains the motives of $K3$-surfaces, unirational varieties of dimension ≤ 3, curves and Fermat hypersufaces (see [10, II.6.26]). *It is unknown what are all the algebraic varieties, whose motives belongs to* $\mathcal{M}^{ab}_{AH}(F)$.

There are two types of extra hypotheses that we need in proving theorems 3.10, 3.11 and 4.3. First of all we assume that the base field is sufficiently large, to guarantee that the motivic galois groups are connected and that the Frobenius elements at the given place of F is *weakly neat* (see defn. 3.8). *It can be shown that over a finite extension these theorems holds without any of these extra assumptions.*

The second hypothesis is that in certain cases we prove the conjugacy only in a group $G^{\natural}_M$ which is larger than the motivic Galois group G_M. Only certain factors of $G^{der}_{M/\bar{\mathbb{Q}}}$ of type D are affected by this modification. Enlarging the groups obviously weakens the notion of conjugacy (see §§3.2).

2 Arithmetic-alegbraic properties of hyperkähler varieties

In this section we shall follow the notations as set out at beginning of the introduction, unless the contrary is stated.

In [4] Beauville constructs the following classes of hyperkähler varieties Y in any dimension $2r \geq 2$:

(i) For any $K3$ surface S, take $Y = S^{[r]}$ the punctual Hilbert scheme which parametrises finite closed subschemes of S of length r; thus for $r = 1$, $Y = S$;

(ii) For any abelian surface, form in the same way as in (i) $A^{[r+1]}$ and take $Y = K_r :=$ the fibre above 0 of the "summation" morphism $A^{[r+1]} \to A$; thus for $r = 1$, Y is the Kummer surface of A.

(iii) Any projective deformation Y of a hyperkähler variety of type $S^{[r]}$.

We shall call these varieties to be of '*K3 type*'.

For any algebraic variety X/F and $\ell \neq p$ a prime number, we denote by $P_{\ell,T,i} = \det(1 - {\Phi_v}^*T; H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_\ell))$ the characteristic polynomial of the $\mathbb{Q}_\ell$-linear map induced by the action of the arithmetic Frobenius element Φ_v, on the ℓ-adic cohomology groups $V^i_\ell = H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_\ell) \cong H^i_{et}(X_{/\bar{F}_v}, \mathbb{Q}_\ell)$.

Theorem 2.1. *Let X/F be a hyperkähler variety of $K3$ type . Then $P_{\ell,T,i}$ has coefficients in the rational numbers and is independent of ℓ.*

Proof. Let us denote $\tau X = X \times_{F,\tau} \mathbb{C}$, $Z = \tau X \times \tau X$ and $\Delta \subset Z$ the diagonal subvariety. Let $\gamma_Z : \mathrm{CH}^d(Z) \to \mathrm{H}^{2d}_\tau(Z)$ be the cycle class map to the Betti cohomology of the complex algebraic variety Z.

For $j = 1, 2 \cdots 2r$, where $r = \dim X$, form the Künneth components of $\gamma_Z \Delta$

$$\pi^j_\tau \in H^{2r-j}_\tau(\tau X) \otimes H^j_\tau(\tau X).$$

By [7, cor. 1.2] and [3, 7.5] the standard conjectures are true for τX. This implies [14, thm. 4-1] that all the π^j_τ's are algebraic. Now by the compatibility of the cycle class maps with comparison isomorphisms, between ℓ-adic cohomology and betti cohomology we get that, the Künneth components $\pi^j_\ell \in V^{2r-j}_\ell(X) \otimes V^j_\ell(X)$ of the diagonal subvariety of $X_{/\bar{F}} \times X_{/\bar{F}}$ are also algebraic. Then by [19, cor. 0.6] it follows that $P_{\ell,T,i}$ has rational coefficients and are independent of ℓ. □

Proposition 2.2. *Let X/F be a hyperkähler variety of $K3$ type. Then there exsits a finite extension F' of F such that $h(X_{F'}) \in \mathrm{Ob}(\mathcal{M}^{ab}_{AH}(F'))$.*

Proof. Let $\mathcal{M}_{mot}(\bar{F})$ be the category of André motives [2] defined over $\bar{F}$. As motivated cycles are also absolute Hdoge cylces, thus $\mathcal{M}_{mot}(\bar{F})$ is a subcategory of $\mathcal{M}_{AH}(\bar{F})$. Let $i : \mathcal{M}_{mot}(\bar{F}) \to \mathcal{M}_{AH}(\bar{F})$ denote the inclusion functor. Denote $\mathcal{M}^{ab}_{mot}(\bar{F})$ to be the Tannakian subcategory of $\mathcal{M}_{mot}(\bar{F})$, generated by abelian varieties, artin motives and Tate motives.

As $\mathcal{M}_{mot}(\bar{F})$ is a semisimple abelian category, thus $h(X_{\bar{F}})$ can be decomposed into sum of indecomposable objects in $\mathcal{M}_{mot}(\bar{F})$. Let L be any such indecomposable component of $h(X_{\bar{F}})$. From [20] and [3, 7.4] it follows that the motive $h(X_{\bar{F}}) \in \mathrm{Ob}\mathcal{M}^{ab}_{mot}(\bar{F})$, thus L is a direct summand of a motive $h(Y)(s)$, where Y is a finite product of abelian varieties and zero-dimensional varieties and s is an integer. Thus, we have a monomorphism $f : L \to h(Y)(s)$ in the category $\mathcal{M}_{mot}(\bar{F})$.

Denote PHS the category of polarized rational hodge structures. Betti cohomology induces faithful additive exact functors $H_\tau : \mathcal{M}_{mot}(\bar{F}) \to PHS$ and $H_\tau : \mathcal{M}_{AH}(\bar{F}) \to PHS$. The faithfulness can be checked by Manin's identity principle and the exactness follows from the fact that $\mathcal{M}_{mot}(\bar{F})$ and $\mathcal{M}_{AH}(\bar{F})$ are semisimple abelian categories. Exactness of H_τ implies that $H_\tau(f) = H_\tau(i(f))$ is a monomorphism in PHS. The faithfulness of H_τ implies that $i(f)$ is a monomorphism in $\mathcal{M}_{AH}(\bar{F})$. As, $h(Y)(s) \in \mathrm{Ob}(\mathcal{M}^{ab}_{AH}(\bar{F}))$ and $\mathcal{M}^{ab}_{AH}(\bar{F})$ is closed under subobjects, so $L \in \mathrm{Ob}(\mathcal{M}^{ab}_{AH}(\bar{F}))$. As $\mathcal{M}_{mot}(\bar{F})$ is a semisimple abelian category, thus we conclude that $h(X_{\bar{F}}) \in \mathrm{Ob}\mathcal{M}^{ab}_{A}(\bar{F})$. Finally using [10, I,2.9] we conclude the statement of the theorem. □

Theorem 2.3. *Let X/F be a hyperkähler variety of $K3$ type. Then after a finite base extension, the arithmetic Frobenius Φ_v acts semisimply on V^i_ℓ.*

Proof. For any fixed $i \in \mathbb{N}$ we denote the absolute Hodge motive $h^i(X)$ by M. By the previous proposition there exists a finite extension F' of F such that $h(X_{F'}) \in \mathrm{Ob}(\mathcal{M}_{AH}^{ab}(F'))$. So, we may apply [15, eqn.4.3] and [17, 3.6], to obtain an abelian variety A defined over F' such that $M_{F'} \in \langle h^1(A_{/F'}), \mathbb{Q}(1)\rangle$. From this fact and properties of Tannakian categories [10, II,2.9] we get a homomorphism of algebraic groups $\theta : G_A \to G_{M_{F'}}$, where G_A and $G_{M_{F'}}$ denotes the motivic Galois groups associated to the motives $h(A)$ and $M_{F'}$ respectively.

Let us prove that the map θ is compatible with the action of the Galois group $\Gamma_{F'}$. Denote by $\mathcal{T}$ the Tannakian subcategory generated by M in $\mathcal{M}_{AH}^{ab}(F)$ and $\mathcal{T}'$ the Tannakian category generated there by $N := h^1(A)$. Then by definition we have

$G_{M_{F'}}(\mathbb{Q}_\ell) \ = \ \mathrm{Aut}^{\otimes}(H_\tau \mid_{\mathcal{T}})(\mathbb{Q}_\ell)$

$= \ \{\phi \mid \phi : H_\tau \mid_{\mathcal{T}} \otimes \mathbb{Q}_\ell \longrightarrow H_\tau \mid_{\mathcal{T}} \otimes \mathbb{Q}_\ell$ is ismorphism of functors$\}$

and $G_A(\mathbb{Q}_\ell) := G_N(\mathbb{Q}_\ell) \ = \ \mathrm{Aut}^{\otimes}(H_\tau \mid_{\mathcal{T}'})(\mathbb{Q}_\ell)$

$= \ \{\phi \mid \phi : H_\tau \mid_{\mathcal{T}'} \otimes \mathbb{Q}_\ell \longrightarrow H_\tau \mid_{\mathcal{T}'} \otimes \mathbb{Q}_\ell$ is ismorphism of functors$\}$

Let $\gamma \in \Gamma_{F'}$. Then we claim that γ induces an isomorphism of functors $\gamma_N^* : H_\tau|_{\mathcal{T}'} \otimes \mathbb{Q}_\ell \to H_\tau|_{\mathcal{T}'} \otimes \mathbb{Q}_\ell$ and similarly $\gamma_M^* : H_\tau|_{\mathcal{T}} \otimes \mathbb{Q}_\ell \to H_\tau|_{\mathcal{T}} \otimes \mathbb{Q}_\ell$. This is true since for any motive $N' \in \mathrm{Ob}(\mathcal{T}')$, the action of $\Gamma_{F'}$ on the ℓ-adic realization $H_\ell(N')$ induces an isomorphism of vector spaces $\gamma^*(N') : H_\ell(N') \to H_\ell(N')$. Similarly for $M' \in \mathrm{Ob}(\mathcal{T})$, we get an isomorphism $\gamma^*(M') : H_\ell(M') \to H_\ell(M')$. Now, using the comparison isomorphism §1(v) we see that

$$H_\ell(N') = H_\tau(N') \otimes \mathbb{Q}_\ell$$

and thus we have an isomorphism of vector space

$$\gamma^*(N') : H_\tau(N') \otimes \mathbb{Q}_\ell \to H_\tau(N') \otimes \mathbb{Q}_\ell.$$

If we have a morphism of motives $N_1 \xrightarrow{f} N_2$ for $N_1, N_2 \in \mathrm{Ob}(\mathcal{T}')$, then by §1(vi), we know that the morphism f commutes with the action of the Galois group $\Gamma_{F'}$ on the ℓ-adic realizations of motives. Thus we get a commutative diagram:

$$\begin{array}{ccc} H_\tau(N_1) \otimes \mathbb{Q}_\ell & \xrightarrow{\gamma^*(N_1)} & H_\tau(N_1) \otimes \mathbb{Q}_\ell \\ \downarrow f_\ell & & \downarrow f_\ell \\ H_\tau(N_2) \otimes \mathbb{Q}_\ell & \xrightarrow{\gamma^*(N_2)} & H_\tau(N_2) \otimes \mathbb{Q}_\ell \end{array} \qquad (5)$$

This shows that the natural transformation of functors defined as $\gamma_N^*(N') = \gamma^*(N')$ is an isomorphism of functors $H_\tau|_{\mathcal{T}'} \otimes \mathbb{Q}_\ell \longrightarrow H_\tau|_{\mathcal{T}'} \otimes \mathbb{Q}_\ell$. Hence, $\gamma_N^* \in G_A(\mathbb{Q}_\ell)$ and similarly $\gamma_M^* \in G_{M_{F'}}(\mathbb{Q}_\ell)$.

The map θ sends an isomorphism of functors $\phi \in G_A(\mathbb{Q}_\ell)$ to the isomorphism of the same functors restricted to the full subcategory $\mathcal{T}$ of $\mathcal{T}'$ [10, II,2.9]. Thus, $\theta(\gamma_N^*) = \gamma_M^*$.

Now it is well known that $\rho_{A,\ell}(\Phi_{v'})$ is semisimple. As $\theta : G_A \to G_{M_{F'}}$ is a homomorphism of algebraic groups, thus by the preceding discussion we see that

$$\theta(\rho_{A,\ell}(\Phi_{v'})) = \rho_{M,\ell}(\Phi_{v'})$$

is semisimple. Thus $\Phi_{v'}$ acts semisimply on V_ℓ^i.

□

Remark 2.4. This shows that over finite extension of the base field, the conejcture of Serre [21, 12.4] holds true for any hyperkähler variety of $K3$ type defined over number field.

3 Compatible systems of ℓ-adic Galois representations

We introduce some new notations for this section. Let E be a finite extension of $\mathbb{Q}_p$, k be its residue field and $|k| = q$. Let $\bar{E}$ be a fixed algebraic closure of E and $\bar{k}$ its residue field. The *Weil-Deligne group* $'W_E$ of E is the group scheme over $\mathbb{Q}$ defined as the semi-direct product of the Weil-group W_E of E, with the additive group $\mathbb{G}_a$ over $\mathbb{Q}$, on which W_E acts as :

$$w \cdot x \cdot w^{-1} = q^{\alpha(w)} \cdot x.$$

Definition 3.1. Let L be any field of characteristic 0. An *L-algebraic representation* of $'W_E$ is a triple (H, ξ', N), where H is a linear algebraic group over L, $\xi' : W_E \to H(L)$ a linear representation with an open subgroup of the inertia group $I_E \subset \Gamma_E$ in its kernel and $N \in \mathrm{Lie}(H)$ satisfying:

$$\mathrm{Ad}(\rho'(w))(N) = q^{\alpha(w)} \cdot N, \text{ for all } w \in W_E$$

We are interested here in the representations of the Weil-Deligne group with values in linear algebraic groups defined over $\mathbb{Q}_\ell$. If H is a linear algebraic group over $\mathbb{Q}_\ell$ and $\xi_\ell : \mathrm{Gal}(\bar{E}/E) \to H(\mathbb{Q}_\ell)$ is a ℓ-adic repesentaton, then we can construct a $\mathbb{Q}_\ell$-algebraic representation of $'W_E$ from ξ_ℓ. Fix an arithmetic Frobenius $\Phi \in \mathrm{Gal}(\bar{E}/E)$, i.e a lift of the Frobenius automorphism of $\bar{k}$. Then we define

$$\xi'_\ell(w) = \xi_\ell(w)\mathrm{exp}(-N'_l t_\ell(\Phi^{-\alpha(w)} w)) \quad \forall\, w \in W_E. \tag{6}$$

Here N'_ℓ is the ℓ-adic monodromy operator associated to ξ_ℓ by Grothendieck's ℓ-adic monodromy theorem (see [22, Appendix] or [8, 8.2]). It is identified as an element of the $\mathrm{Lie}(H)$ by fixing an isomorphism $\mathbb{Q}_\ell \simeq \mathbb{Q}_\ell(1)$. The ℓ-adic monodromy theorem also implies that ξ'_ℓ is trivial on some open subgroup $J \subset I_E$. Thus, (H, ξ'_ℓ, N'_ℓ) indeed gives us a $\mathbb{Q}_\ell$-algebraic representation of $'W_E$. Moreover, according to [8, 8.11] the geometric isomorphism class of this representation is independent of the choice of Φ and the chosen isomorphism $\mathbb{Q}_\ell \simeq \mathbb{Q}_\ell(1)$.

3.2 The Algebraic Group $G^{\natural ad}$

Let G be a reductive algebraic group over field K of characteristic zero; the adjoint group G^{ad} is also reductive. The algebraic group G^{ad} acts on G by conjugation action. Thus by theorem [16, 1.1] we have an universal categorical quotient $(\mathrm{Conj}(G), \mathrm{Cl})$. The pair $(\mathrm{Conj}(G), \mathrm{Cl})$ is also the categorical quotient for the action of G on itself by conjugation.

In [18, 1.5] R.Noot constructs an algebraic group $G^{\natural ad}$. We recall here this construction. We assume that G is connected. Let $\bar{K}$ be a separable algebraic closure of K. Then the derived algebraic group $G^{der}_{\bar{K}}$ over $\bar{K}$ is an almost direct product of almost simple subgroups G_i, for $i \in I$ a finite indexing set. Let $J \subset I$ such that for $i \in J$, $G_i \cong \mathrm{SO}(2m_i)_{\bar{K}}$ with $m_i \geq 4$.

Denote $G_i^\natural := \mathrm{O}(2m_i)_{\bar{K}}$, so that G_i is the identity component of $G_i^\natural$. The constant algebraic group $\Lambda(G)_{\bar{K}} = \prod_{i\in J} G_i^\natural / G_i$ descends to an algebraic group $\Lambda(G)$ over K. Now define

$$G^{\natural ad} = \prod_{i\in J} G_i^{\natural ad} \times \prod_{i\in I\setminus J} G_i^{ad} \supset G^{ad}_{\bar{K}}$$

As this algebraic group operates trivially on the centre of $G^{der}_{\bar{K}}$, we can define an action of $G^{\natural ad}$ on $G_{\bar{K}}$ extending the adjoint action on $G^{der}_{\bar{K}}$ and with $G^{\natural ad}$ acting trivially on the centre of $G_{\bar{K}}$. It is clear that, through the adjoint representation, the group $G^{\natural ad}$ also acts on the Lie algebra $\mathfrak{g} \otimes \bar{K}$. We denote by $\mathrm{Conj}'(G)_{\bar{K}}$ the categorical quotient of $G_{\bar{K}}$ under this action of $G^{\natural ad}$. By the properties of categorical quotients ([6, VI,6.10 & 6.16], this action induces an action of the constant algebraic group $\Lambda(G)_{\bar{K}} = G^{\natural ad}/G^{ad}_{\bar{K}}$ on $\mathrm{Conj}(G)_{\bar{K}}$, such that quotient for this action is isomorphic to $\mathrm{Conj}'(G)_{\bar{K}}$. This in turn gives us an action of $\Lambda(G)$ on $\mathrm{Conj}(G)$. The quotient of $\mathrm{Conj}(G)$ under this action, is denoted as $\mathrm{Conj}'(G)$ and the 'quotient' map thus obtained is denoted as $\mathrm{Cl} : G \to \mathrm{Conj}'(G)$.

3.3 Reprsentations with values in $G^{\natural ad}$

We follow the notations as in the beginning of this section. Let E be a finite extension of $\mathbb{Q}_p$ and $G/\mathbb{Q}$ be a reductive algebraic group.

Definition 3.4. Let $(G_{/\mathbb{Q}_\ell}, \xi', N)$ be a $\mathbb{Q}_\ell$-algebraic representation of $'W_E$, for some prime number ℓ. For a fixed ℓ we say that this representation is *defined over* $\mathbb{Q}$ *modulo the action of* $G^{\natural ad}$, if for every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$, the base extension of $(G_{/\mathbb{Q}_\ell}, \xi', N)$ to Ω is conjugate under $G^{\natural ad}(\Omega)$ to all its images under $\mathrm{Aut}_{\mathbb{Q}}(\Omega)$.

More precisely this means that for every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$, there exists a

$$g \in G^{\natural ad}(\Omega)$$

such that

$$\sigma\xi'_\Omega = g \cdot \xi'_\Omega \cdot g^{-1} \quad \text{and} \quad \sigma(N \otimes_{\mathbb{Q}_\ell} 1) = \mathrm{Ad}(g)(N \otimes_{\mathbb{Q}_\ell} 1) \tag{7}$$

Definition 3.5. For any two prime numbers ℓ_1 and ℓ_2, we say that two representations $(G_{/\mathbb{Q}_{\ell_i}}, \xi'_i, N)$ (for $i = 1, 2$) are *compatible modulo the action of* $G^{\natural ad}$ if for every algebraically closed field $\Omega \supset \mathbb{Q}_{\ell_1}, \mathbb{Q}_{\ell_2}$, there exists some $g \in G^{\natural ad}(\Omega)$ such that

$$\xi'_{1/\Omega} = g \cdot \xi'_{2/\Omega} \cdot g^{-1} \quad \text{and} \quad N_1 \otimes_{\mathbb{Q}_{\ell_1}} 1 = \mathrm{Ad}(g)(N_2 \otimes_{\mathbb{Q}_{\ell_2}} 1) \in \mathfrak{g} \otimes \Omega. \tag{8}$$

If the above relations holds for all prime numbers ℓ different from the residual characteristic of E, then we say that they form a *compatible system of representation modulo the action of* $G^{\natural ad}$.

Remark 3.6. We note that if $G^{der}_{\bar{K}}$ does not have any almost direct factor of type $\mathrm{SO}(2m)$ with $m \geq 4$, then the above definition is the same as one given by Deligne in [8, 8.11]

3.7 ℓ-independence of the conjugacy class of Frobenius

Definition 3.8. Let G be a reductive algebraic group over a field K of characteristic 0. Fix a faithful K-linear representation V of G. A semisimple element $g \in G(\Omega)$ is said to be *weakly neat* if the the only root of unity amongst the quotients $\lambda\mu^{-1}$, with λ, μ being the eigenvalues of g, is 1. Note that if g is neat then its weakly neat.

Lemma 3.9. *let Ω be an algebraically closed field, d a positive integer. Let $x, y \in \mathrm{GL}_d(\Omega)$ be two weakly neat elements such that $x^n = y^n$ for some positive integer n, and x and y have same characteristic polynomial. Then $x = y$*

Henceforth we follow the notations of the introduction to the article. The action of Γ_F on the ℓ-adic realization of an absolute Hodge motive $M \in \mathrm{Ob}(\mathcal{M}_{AH}(F))$ induces a map

$$\rho_{M,\ell} : \Gamma_F \to \mathrm{GL}(H_\ell(M)).$$

It is known that that $Im(\rho_{M,\ell}) \subset G_M(\mathbb{Q}_\ell)$, where G_M is the motivic galois group associate to M. As G_M is reductive group, the construction in the §§3.2 gives us an algebraic variety $\mathrm{Conj}'G_M$ over $\mathbb{Q}$. Recall that we also have the quotient map

$$\mathrm{Cl} : G_M \to \mathrm{Conj}'G_M.$$

In what follows we will suppose that G_M is connected. Note that for any motive $M \in \mathrm{Ob}(\mathcal{M}_{AH}^{ab}(F))$, there exists a finite extension of F' such that $G_{M_{F'}}$ is connected. The following theorem is a particular instance of the conjecture of Serre 1.1, which was stated in the introduction.

Theorem 3.10. *Let X/F be a hyperkähler variety of K3 type. For any fixed positive integer i, let M denote the absolute Hodge motive $h^i(X)$. Suppose X has good reduction at v and $\rho_{M,\ell}(\Psi_v)$ is weakly neat for some $\ell \neq p$. Then the conjugacy class*

$$\mathrm{Cl}(\rho_\ell(\Phi_v)) \in \mathrm{Conj}'(G)(\mathbb{Q}), \quad \forall \quad \textit{prime numbers} \quad \ell \neq p$$

and is independent of ℓ.

Proof 3.10. By [15, Thm. 4.2.6] the statement of the theorem holds for any smooth projective variety Y/F such that $h(Y) \in \mathrm{Ob}(\mathcal{M}_{AH}^{ab}(F)$. Essentially the same arguments show that, the conclusion of the theorem continue to hold even if we just had $h(Y_{F'}) \in \mathrm{Ob}(\mathcal{M}_{AH}^{ab}(F'))$ instead of $h(Y) \in \mathrm{Ob}(\mathcal{M}_{AH}^{ab}(F))$.

By 2.2 we know the the motive $M_{F'}$ is an object of $\mathcal{M}_{AH}^{ab}(F')$, whence the theorem.

□

Now we move on to the bad reduction case. The following theorem answers the question 1.2, which was raised in the introduction, for motives of hyperkähler varieties of K3 type.

Theorem 3.11. *Let X/F be a hyperkähler variety of K3 type and $i \in \mathbb{N}$. Let M be the absolute Hodge motive $h^i(X)$. Assume that M has semis-table reduction at v and $\rho_{M,\ell}(\Phi_v)$ is weakly neat for some prime number $\ell \neq p$. Then*

(i) for every $\ell \neq p$, the representation $(G_{M/\mathbb{Q}_\ell}, \rho'_{M,\ell}, N'_{M,\ell})$ of $'W_v$ is defined over $\mathbb{Q}$ modulo the action of $G_M^{\natural ad}$ and

(ii) for ℓ running through primes different from p, these representations form a compatible system of representations of $'W_v$ modulo the action of $G_M^{\natural ad}$.

Proof. The hypothesis of semi-stability means that the inertia group I_v acts unipotently on $H_\ell(M)$ for all $\ell \neq p$. This implies that $\rho'_{M,\ell}(I_v) = 1$. We know that $W_v/I_v \simeq \mathbb{Z}$, and Φ_v is a generator of W_v/I_v. This means that to verify the first equality of the equations (7) and (8) on $\rho'_{M,\ell/\Omega}$, it suffices to do so at the arithmetic Frobenius element $\Phi_v \in W_v$.

By [15, 5.5.8] we know that there exists a finite extension F' of F such that the statement of the theorem holds true after passing over to F'. Let v' be the extension of v to F'. Denote by $\Phi_{v'} = \Phi_v^n$ the corresponding arithmetic Frobenius at v', where n is the residual degree of the extension $F'_{v'}/F_v$. This implies that

$$\rho'_{M,\ell}(\Phi_{v'}) = \rho'_{M,\ell}(\Phi_v^n).$$

The monodromy operator $N'_{M,\ell}$ is unchanged by base extension.

Fix a prime number $\ell \neq p$. We want to show that the representation $(G_{M/\mathbb{Q}_\ell}, \rho'_{M,\ell}, N'_{M,\ell})$ of $'W_v$ is defined over $\mathbb{Q}$ modulo the action of $G_M^{\natural ad}$. Let $\Omega \supset \mathbb{Q}_\ell$ be an algebraically closed field and $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$. The theorem [15, 5.5.8] gives us a $g_\sigma \in G_M^{\natural ad}(\Omega)$ such that equation (7) holds over F' i.e.

$$\sigma(\rho'_{M,\ell/\Omega}(\Phi_v^n)) = g_\sigma \cdot (\rho'_{M,\ell/\Omega}(\Phi_v^n)) \cdot g_\sigma^{-1} \text{ and}$$

$$\sigma(N'_{M,\ell} \otimes_{\mathbb{Q}_\ell} 1) = \mathrm{Ad}(g_\sigma)(N'_{M,\ell} \otimes_{\mathbb{Q}_\ell} 1).$$

So we have $(\sigma(\rho'_{M,\ell/\Omega}(\Phi_v)))^n = (g_\sigma \cdot (\rho'_{M,\ell/\Omega}(\Phi_v)) \cdot g_\sigma^{-1})^n$. By our hypothesis $\rho'_{M,\ell/\Omega}(\Phi_v)$ is weakly neat. So if we prove that $\sigma(\rho'_{M,\ell/\Omega}(\Phi_v))$ and $g_\sigma \cdot (\rho'_{M,\ell/\Omega}(\Phi_v)) \cdot g_\sigma^{-1}$ have same the characteristic polynomial, then we can use proposition 3.9 to conclude that $\sigma(\rho'_{M,\ell/\Omega}(\Phi_v)) = g_\sigma \cdot (\rho'_{M,\ell/\Omega}(\Phi_v)) \cdot g_\sigma^{-1}$. As the monodromy operator $N'_{M,\ell}$ is unchanged by base extension therefore $\sigma(N'_{M,\ell} \otimes_{\mathbb{Q}_\ell} 1) = \mathrm{Ad}(g_\sigma)(N'_{M,\ell} \otimes_{\mathbb{Q}_\ell} 1)$. Thus, we would be able to conclude that the representation $(G_{M/\mathbb{Q}_\ell}, \rho'_{M,\ell}, N'_{M,\ell})$ of $'W_v$ is defined over $\mathbb{Q}$ modulo the action of $G_M^{\natural ad}$.

We begin by noting that $g_\sigma \cdot \rho'_{M,\ell/\Omega}(\Phi_v) \cdot g_\sigma^{-1}$ and $\rho'_{M,\ell/\Omega}(\Phi_v)$ are conjugate under the action of $G_M^{\natural ad}$ and so they have same characteristic polynomial. Let $\mathcal{B}$ be an ordered basis of the $\mathbb{Q}$-vector space $H_\tau(M)$ and let $\mathcal{B}_{\mathbb{Q}_\ell}$ denote the corresponding basis of

$$V_\ell := H_\tau(M) \otimes \mathbb{Q}_\ell.$$

If (a_{ij}) is the matrix of the transformation $\rho'_{M,\ell}(\Phi_v) : V_\ell \to V_\ell$ in the ordered basis $\mathcal{B}_{\mathbb{Q}_\ell}$, then

$$\sigma(\rho'_{M,\ell/\Omega}(\Phi_v)) : V_\ell \otimes_{\mathbb{Q}_\ell} \Omega \to V_\ell \otimes_{\mathbb{Q}_\ell} \Omega$$

is the linear transformation whose matrix in the basis $\mathcal{B}_\Omega$ is $(\sigma(a_{ij}))$. Using proposition 2.1 we conclude that $\sigma(\rho'_{M,\ell/\Omega}(\Phi_v))$ and $\rho'_{M,\ell/\Omega}(\Phi_v)$ have the same characteristic polynomial with coefficients in $\mathbb{Q}$. Thus $\sigma(\rho'_{M,\ell/\Omega}(\Phi_v))$ and $g_\sigma \cdot \rho'_{M,\ell/\Omega}(\Phi_v) \cdot g_\sigma^{-1}$ also have the same characteristic polynomial. This implies that

$$\sigma(\rho'_{M,\ell/\Omega}(\Phi_v)) = g_\sigma \cdot \rho'_{M,\ell/\Omega}(\Phi_v) \cdot g_\sigma^{-1}.$$

Thus, we have shown assertion (i) of the theorem.

Finally we want to show that for varying ℓ the representations

$$(G_{M/\mathbb{Q}_\ell}, \rho'_{M,\ell}, N'_{M,\ell})$$

of W_v form a compatible system modulo the action of $G_M^{\natural ad}$. For this, let ℓ and ℓ' be two different primes and Ω an algebraically closed field containing both $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$. Then by theorem [15, 5.5.8] we know that there exists a $g \in G_M^{\natural ad}(\Omega)$, such that

$$\rho'_{M,\ell/\Omega}(\Phi_v^n) = g \cdot \rho'_{M,\ell'/\Omega}(\Phi_v^n) \cdot g^{-1} \text{ and} \tag{9}$$

$$N'_{M,\ell} \otimes_{\mathbb{Q}_\ell} 1 = \mathrm{Ad}(g)(N'_{M,\ell'} \otimes_{\mathbb{Q}_{\ell'}} 1).$$

As the mondoromy operators are unchanged by base extensions thus we just need to verify the first equality of equation (8). By using proposition 2.1 we know that $\rho'_{M,\ell/\Omega}(\Phi_v)$ and $\rho'_{M,\ell'/\Omega}(\Phi_v)$ have same characteristic polynomial with coefficients in $\mathbb{Q}$ and it is independent of the choice of ℓ or ℓ'. As $\rho'_{M,\ell/\Omega}(\Phi_v)$ and $\rho'_{M,\ell'/\Omega}(\Phi_v)$ are weakly neat, by proposition 3.9 and equation (9) we conclude that

$$\rho'_{M,\ell/\Omega}(\Phi_v) = g \cdot \rho'_{M,\ell'/\Omega}(\Phi_v) \cdot g^{-1}.$$

This establishes assertion (ii) of the theorem. □

4 Conjugacy class of the Crystalline Frobenius

Let X/F be an algebraic variety with good reduction at v. Let X_v denote the special fibre of the smooth model of X. Then, X_v is an algebraic variety over the finite field k_v. Let $\mathrm{Frob}_{X,abs} : X_v \to X_v$ be the absolute Frobenius. Let $\sigma : W(k_v) \to W(k_v)$ be the morphism induced by the Frobenius automorphism of k_v. We have a σ-linear morphism

$$\mathrm{Frob}_X : H^i_{\mathrm{Cris}}(X_v/W(k_v)) \to H^i_{\mathrm{Cris}}(X_v/W(k_v)) \quad \forall \quad i \in \mathbb{N}.$$

Let $|k_v| = p^r$, then by defining $\mathrm{Frob}_{X,\mathrm{Cris}} := \mathrm{Frob}_X^r$ we obtain a $W(k_v)$-linear (or $W(\bar{k}_v)$)-linear) endomorphism of $H^i_{\mathrm{Cris}}(X_v/W(k_v))$ (or $H^i_{\mathrm{Cris}}(X_v/W(\bar{k}_v))$).

Let F_v^0 be the fraction field of $W(k_v)$. It corresponds to the maximal unramified extension of $\mathbb{Q}_p$ in F_v. Let $H^i_{\text{Cris}}(X_v)$ denote the F_v^0-vector space $H^i_{\text{Cris}}(X_v/W(k_v)) \otimes F_v^0$. The *Crsytalline Frobenius* is the endomorphism

$$\Psi_{X,\text{Cris}} : H^i_{\text{Cris}}(X_v) \to H^i_{\text{Cris}}(X_v),$$

obtained by base extension of $\text{Frob}_{X,\text{Cris}}$ to F_v^0.

Suppose now that the motive $h(X) \in \mathcal{M}^{ab}_{AH}(F)$. We fix a $i \in \mathbb{N}$ and denote $M := h^i(X)$. Henceforth, we denote the crystalline Frobenius morphism $H^i_{\text{Cris}}(X_v) \to H^i_{\text{Cris}}(X_v)$ by $\Psi_{M,\text{Cris}}$.

By [17, 3.6] and [15, eqn.4.3] one can show that there exists a finite extension F' of F and an abelian variety A/F' such that

$$M_{F'} := h(X_{/F'}) \in \langle h^1(A), \mathbb{Q}(1)\rangle.$$

and

$$G_A^{ad} \cong G_{M_{F'}}^{ad},$$

where G_A and $G_{M_{F'}}$denotes the motivic Galois group associated to the motives $h(A)$ and $M_{F'}$, respectively. After another finite base extension we may suppose that G_A and $G_{M_{F'}}$ are connected. Let v' be an extension of v to F'. By properties of Tannakian categories [10, II,2.9], we get a map $\theta : G_A \to G_{M_F'}$. As in the proof of theorem 2.3 one shows that the map θ is compatible with the action of the Galois group $\Gamma_{F'}$.

Let us prove that under the given conditions A has potential good reduction at v'. As X is smooth and has good reduction at v, the action of the inertia group on $H^i_{et}(X_{/\bar{F}}, \mathbb{Q}_\ell)$ is trivial. As $G_A^{ad} \cong G_{M_{F'}}^{ad}$, thus by the compatibilty of θ with the action of the Galois group $\Gamma_{F'}$, it follows that $I := \rho_{A,\ell}(I_{v'}) \subseteq \mathcal{Z}_A(\mathbb{Q}_\ell)$, where $\mathcal{Z}_A$ denotes the centre of G_A. Now G_A is a subgroup of the general linear algebraic group $\text{GL}(W)$, where $W := H^1_\tau(A(\mathbb{C}), \mathbb{Q})$ is the betti cohomlogy of A. Again by the ℓ-adic monodromy theorem of Grothendieck we may suppose that, by passing over to a finite base extension if necessary, $I_{v'}$ acts unipotently on

$$W \otimes \mathbb{Q}_\ell = H^1_{et}(A_{/\bar{F}}, \mathbb{Q}_\ell).$$

Let Ω be a fixed algebraic closure of $\mathbb{Q}_\ell$. We have a canonical injection

$$\mathcal{Z}_A(\mathbb{Q}_\ell) \hookrightarrow \mathcal{Z}_{A/\Omega}(\Omega).$$

For any $\sigma \in I \subseteq \mathcal{Z}_A(\mathbb{Q}_\ell)$ denote $\bar{\sigma}$ the image of σ in $\mathcal{Z}_{A/\Omega}(\Omega)$. As G_A is reductive thus $\mathcal{Z}_{A/\Omega}$ is a diagonalizable algebraic group.

Since $\mathcal{Z}_{A/\Omega} \subset \text{GL}(W_{/\Omega})$ is diagonalizable thus there exists an invertible matrix $P \in \text{Mat}_m(\Omega)$ such that $P \cdot \bar{\sigma} \cdot P^{-1}$ is a diagonal matrix

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$$

Where $m := \dim W$. As $\bar{\sigma}$ is unipotent thus $\lambda_1 = \cdots = \lambda_m = 1$. So, we get that $\bar{\sigma}$ is the identity matrix. Since $\sigma \in I$ was an arbitrary element thus I is trivial. Now using Néron-Ogg-Shfarevich condition [22, Thm. 1] we conclude that the abelian variety A has potential good reduction.

Let $k_{v'}$ be the corresponding extension of the residue field k_v. If $n := [k_{v'} : k_v]$ then we denote by $\Phi_{v'} := \Phi_v^n \in \Gamma_{F'}$, the corresponding arithmetic Frobenius at v'. Denote the fraction field of the ring of Witt vectors $W(k_{v'})$ by $F^0_{v'}$. The theorem of Blasius [5, Theorem 5.3] allows us to construct a fibre functor

$$H_{\text{Cris}} : \mathcal{M}_A \to \text{Vect}_{F^0_{v'}}$$

and it also says that $\Psi_{A,\text{Cris}}$ induces an automorphism of this functor. i.e if $G_{A,\text{Cris}}$ is the automorphism group of the fibre functor H_{Cris}, then $\Psi_{A,\text{Cris}} \in G_{A,\text{Cris}}(F^0_{v'})$. Let $G_{M_{F'},\text{Cris}}$ be the automorphism group of the fibre functor H_{Cris} restricted to the Tannakian subcategory generated by $M_{F'}$.

Let $\Psi'_{M,\text{Cris}}$ be the crystalline Frobenius morphism $H^i_{\text{Cris}}(X_{v'}) \to H^i_{\text{Cris}}(X_{v'})$, for the motive $M_{F'}$. Then, we have $\Psi'_{M,\text{Cris}} \in G_{M_{F'},\text{Cris}}(F^0_{v'})$. The algebraic group $G_{M_{F'},\text{Cris}}$ is an inner form of $G_{M_{F'}}$. Thus, we have an isomorphism

$$\text{Conj}^*(G_{M_{F'},\text{Cris}}) \cong \text{Conj}^*(G_{M_{F'}})_{/F^0_v},$$

where $*$ denotes anyone one of the algebraic varieties $\text{Conj}(-)$ or $\text{Conj}'(-)$. Let $\ell \neq p$ be a prime number and $\rho_{M,\ell} : \Gamma_F \to G_M(\mathbb{Q}_\ell)$ be the canonical ℓ-adic Galois representation associated to the motive M.

For $* = M_{F'}, A$, let $\text{Cl}_{\text{Cris}} : G_{*/\text{Cris}} \to \text{Conj}'G_{*/\text{Cris}}$ be the quotient maps obtained by the conjugation action of $G^{\natural ad}_{*/\text{Cris}}$ on $G_{*/\text{Cris}}$.

As X has good reduction at v, therefore we may apply [15, Theorem 4.2.1] to conclude that the conjugacy class $\text{Cl}(\rho_{M,\ell}(\Phi_{v'})) \in \text{Conj}'G_{M_{F'}}(\mathbb{Q})$ for all $\ell \neq p$ and is independent of ℓ. We denote this common element of $\text{Conj}'G_{M_{F'}}(\mathbb{Q})$ by $\text{Cl}_M\text{Fr}_{v'}$. Then we have the following:

Theorem 4.1. *The element* $\text{Cl}_{\text{Cris}}(\Psi'_{M,\text{Cris}}) \in \text{Conj}'(G_{M_{F'},\text{Cris}})(F^0_{v'})$ *coincides with the image of* $\text{Cl}_M\text{Fr}_{v'}$ *in* $\text{Conj}'(G_{M_{F'},\text{Cris}})(F^0_{v'})$.

Remark 4.2. The statement makes sense because we have

$$\text{Conj}'(G_{M_{F'}})(\mathbb{Q}) \subset \text{Conj}'(G_{M_{F'}})(F^0_{v'}) \cong \text{Conj}'(G_{M_{F'},\text{Cris}})(F^0_{v'})$$

Proof of 4.1. After making a finite base extension if necessary, we may suppose that $\Psi_{A,\mathrm{Cris}}$ is weakly neat. As A has good reduction at v', thus by [18, Theorem 4.2] we see that $\mathrm{Cl}_{\mathrm{Cris}}(\Psi_{A,\mathrm{Cris}}) \in \mathrm{Conj}'(G_{A,\mathrm{Cris}})(F^0_{v'})$ coincides with the image of $\mathrm{Cl}_A\mathrm{Fr}_{v'}$ in $\mathrm{Conj}'(G_{A,\mathrm{Cris}})(F^0_{v'})$.

Now, we have a canonical commutative diagram:

$$\begin{array}{ccc} G_A & \xrightarrow{\mathrm{Cl}} & \mathrm{Conj}'G_A \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\pi} \\ G_{M_{F'}} & \xrightarrow{\mathrm{Cl}} & \mathrm{Conj}'G_{M_{F'}} \end{array}$$

By base extension to $F^0_{v'}$ we obtain the following diagram :

$$\begin{array}{ccc} G_{A,\mathrm{Cris}} & \xrightarrow{\mathrm{Cl}_{\mathrm{Cris}}} & \mathrm{Conj}'G_{A,\mathrm{Cris}} \\ \downarrow{\scriptstyle\theta\otimes 1} & & \downarrow{\scriptstyle\pi_{\mathrm{Cris}}} \\ G_{M_{F'},\mathrm{Cris}} & \xrightarrow{\mathrm{Cl}_{\mathrm{Cris}}} & \mathrm{Conj}'G_{M_{F'},\mathrm{Cris}} \end{array} \tag{10}$$

By definition we have $\theta \otimes 1(\Psi_{A,\mathrm{Cris}}) = \Psi'_{M,\mathrm{Cris}}$. Thus, by commutativity of (10) we obtain $\mathrm{Cl}_{\mathrm{Cris}}(\Psi'_{M,\mathrm{Cris}}) = \pi_{\mathrm{Cris}}(\mathrm{Cl}_{\mathrm{Cris}}(\Psi_{A,\mathrm{Cris}})) = \pi(\mathrm{Cl}_A\mathrm{Fr}_{v'})$. But, by [15, Theorem 4.2.1] we know that $\mathrm{Cl}_M\mathrm{Fr}_{v'} = \pi(\mathrm{Cl}_A\mathrm{Fr}_{v'})$. Thus,

$$\mathrm{Cl}_{\mathrm{Cris}}(\Psi'_{M,\mathrm{Cris}}) = \mathrm{Cl}_M\mathrm{Fr}_{v'}.$$

□

From now on we suppose that G_M is connected. This implies that $G_{M,\mathrm{Cris}}$ is also connected. As X has good reduction at v, thus by the results of Deligne on Weil conjectures [9] and by using smooth specialization (proper-smooth base change)we conclude that the $\mathbb{Q}_\ell$-linear map $\rho_{M,\ell}(\Phi_v)$ has coefficients in $\mathbb{Q}$ and is independent of ℓ. By a result of Katz and Messing [13, Theorem 1], we conclude that the F^0_v-linear map $\Psi_{M,\mathrm{Cris}}$ has the same characteristic polynomial as $\rho_{M,\ell}(\Phi_v)$. This implies in particular that if $\Psi_{M,\mathrm{Cris}}$ is weakly neat then $\rho_{M,\ell}(\Phi_v)$ is also weakly neat. In this situation, we may apply [15, Theorem 4.2.6] to conclude that $\mathrm{Cl}(\rho_{M,\ell}(\Phi_v)) \in \mathrm{Conj}'(G_M)(\mathbb{Q})$ for all $\ell \neq p$ and is independent of ℓ. We denote this common element by $\mathrm{Cl}_M\mathrm{Fr}_v$. This leads us to the following theorem:

Theorem 4.3. *Assume that $\Psi_{M,\mathrm{Cris}}$ is weakly neat. Then, the element*

$$\mathrm{Cl}_{\mathrm{Cris}}(\Psi_{M,\mathrm{Cris}}) \in \mathrm{Conj}'(G_{M,\mathrm{Cris}})(F^0_v)$$

coincides with the image of $\mathrm{Cl}_M\mathrm{Fr}_v$ *in* $\mathrm{Conj}'(G_{M,\mathrm{Cris}})(F^0_v)$.

Proof of 4.3. By theorem 4.1 there exists a finite extension F' of F and an extension v' of v, such that $\mathrm{Cl}_{\mathrm{Cris}}(\Psi'_{M,\mathrm{Cris}}) = \mathrm{Cl}_M\mathrm{Fr}_{v'}$. Let $n = [k_{v'} : k_v]$. As we assumed that G_M is connected, therefore $G_{M,\mathrm{Cris}} \cong G_{M_{F'},\mathrm{Cris}}$. We also have $\mathrm{Conj}'G_{M,\mathrm{Cris}}(F_v^0) \subseteq \mathrm{Conj}'G_{M,\mathrm{Cris}}(F_{v'}^0)$.

Lemma 4.4. $(\mathrm{Cl}_{\mathrm{Cris}}(\Psi_{M,\mathrm{Cris}}))^n = \mathrm{Cl}_{\mathrm{Cris}}(\Psi'_{M,\mathrm{Cris}}) \in \mathrm{Conj}'G_{M,\mathrm{Cris}}(F_{v'}^0)$.

Proof. It suffices to show that $(\Psi_{M,\mathrm{Cris}})^n = \Psi'_{M,\mathrm{Cris}} \in G_{M,\mathrm{Cris}}(F_{v'}^0)$. The inclusion $G_{M,\mathrm{Cris}}(F_v^0) \hookrightarrow G_{M,\mathrm{Cris}}(F_{v'}^0)$ maps $\Psi_{M,\mathrm{Cris}}$ to its base extension to $F_{v'}^0$ i.e the endomorphism $\Psi_{M,\mathrm{Cris}} \otimes \mathrm{id}_{v'} : H^i_{\mathrm{Cris}}(M) \otimes_{F_v^0} F_{v'}^0 \to H^i_{\mathrm{Cris}}(M) \otimes_{F_v^0} F_{v'}^0$.

Now we have

$$\Psi'_{M,\mathrm{Cris}} = (\mathrm{Frob}_{v'})^{rn} \otimes_{W(k_{v'})} \mathrm{id}_{v'} \cong (\mathrm{Frob}_{v'} \otimes_{W(k_{v'})} \mathrm{id}_{v'})^{rn} \cong$$

$$((\mathrm{Frob}_v \otimes_{W(k_v)} W(k_{v'})) \otimes_{W(k_{v'})} \mathrm{id}_{v'})^{rn} \cong (\mathrm{Frob}_v \otimes_{W(k_v)} \mathrm{id}_{v'})^{rn}$$

$$\cong ((\mathrm{Frob}_v)^r \otimes_{W(k_v)} \mathrm{id}_{v'})^n \cong (\Psi_{M,\mathrm{Cris}})^n$$

□

As $\Phi_{v'} = {\Phi_v}^n$, therefore $(\mathrm{Cl}_M\mathrm{Fr}_v)^n = \mathrm{Cl}_M\mathrm{Fr}_{v'}$. By the previous theorem 4.1, we have $\mathrm{Cl}_M\mathrm{Fr}_{v'} = \mathrm{Cl}_{\mathrm{Cris}}(\Psi'_{M,\mathrm{Cris}})$. Thus, using the previous lemma we conclude that $(\mathrm{Cl}_M\mathrm{Fr}_v)^n = (\mathrm{Cl}_{\mathrm{Cris}}(\Psi_{M,\mathrm{Cris}}))^n$.

Fix a prime number ℓ and let $\Omega \supset \bar{\mathbb{Q}}_p, \bar{\mathbb{Q}}_\ell$ be an algebraically closed field. We have two weakly neat elements $\rho_{M,\ell}(\Phi_v), \Psi_{M,\mathrm{Cris}} \in G_M(\Omega) \cong G_{M,\mathrm{Cris}}(\Omega)$ which have same characteristic polynomial. Therefore, we may apply the following lemma

Lemma 4.5. *Let Ω be an algebraically closed field containing K. Let V be a faithful K-linear representation of G and $\alpha, \beta \in G(\Omega)$ be two weakly neat elements having same characteristic polynomial in the representation V. If for some positive integer n we have $\bar{\phi}_n(\mathrm{Cl}(\alpha)) = \bar{\phi}_n(\mathrm{Cl}(\beta))$, then* $\mathrm{Cl}(\alpha) = \mathrm{Cl}(\beta)$*.*

By taking α and β to be $\mathrm{Cl}_M\mathrm{Fr}_v$ and $\mathrm{Cl}_{\mathrm{Cris}}(\Psi_{M,\mathrm{Cris}})$, respectively, we get $\mathrm{Cl}_M\mathrm{Fr}_v = \mathrm{Cl}_{\mathrm{Cris}}(\Psi_{M,\mathrm{Cris}})$. This concludes the proof of the theorem.

□

Remark 4.6. The same arguments may be applied to any algebraic variety X/F such that $h(X_{F'}) \in \mathrm{Ob}(\mathcal{M}^{ab}_{AH}(F'))$. Thus in particular the theorem holds for hyperkähler varieties of $K3$ type.

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