QUARTIC 3-FOLD: PFAFFIANS, INSTANTONS AND HALF-CANONICAL CURVES

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Abstract. A generic quartic 3-fold X admits a 7-dimensional family of representations as the Pfaffian of an 8 by 8 skew-symmetric matrix of linear forms. This provides a 7-dimensional moduli space M of rank 2 vector bundles on X. A precise geometric description of a 14-dimensional family of half-canonical curves C of genus 15 in X such that the above vector bundles are obtained by Serre's construction from C is given. It is proved that the Abel–Jacobi map of this family factors through M, and the resulting map from M to the intermediate Jacobian is quasi-finite. In particular, every component of M has non-negative Kodaira dimension. Some other constructions of rank 2 vector bundles with small Chern classes are discussed; it is proved that the smallest possible charge of an instanton on X is 4.

Introduction

This paper is a part of the study of moduli spaces of vector bundles with small Chern classes on certain Fano threefolds. It provides some non-existence results and constructions of a few moduli components of vector bundles on a quartic threefold. One of them is the component of kernel bundles, defined similarly to that of [MT], [IM] for the case of a cubic threefold. Our work received a strong pulse with the publication of the paper of Beauville [B], which allowed to simplify some arguments used in [MT] and at the same time put our results in a more general framework of Pfaffian hypersurfaces. We also prove that there are no normalized rank 2 stable vector bundles on X with $c_2 < 4$ and exhibit two constructions of stable vector bundles with $c_1 = 0$ and $c_2 = 4$. Only one of them provides instantons; we show in fact that all the instantons of charge 4 are obtained by this construction.

In [MT], it was proved that the Abel–Jacobi map of the family of normal elliptic quintics lying on a general cubic threefold V factors through a moduli component of stable rank 2 vector bundles on V with Chern numbers $c_1 = 0, c_2 = 2$, whose general point represents a

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vector bundle obtained by Serre's construction from an elliptic quintic. The elliptic quintics mapped to a point of the moduli component vary in a 5-dimensional projective space inside the Hilbert scheme of curves, and the map from the moduli component to the intermediate Jacobian is quasi-finite. Later, in [IM], this modular component was identified with the variety of representations of V as a linear section of the Pfaffian cubic in $\mathbb{P}^{14}$ and it was proved that the degree of the quasi-finite map is 1, so the moduli component is birational to the intermediate Jacobian $J^2(X)$. Beauville mentions in [B] a recent work of Druel, yet unpublished, which proves that the moduli space $M_V(2;0,2)$ is irreducible, so its unique component is the one described above.

In the present paper, we prove that a generic quartic threefold X admits a 7-dimensional family of essentially different representations as the Pfaffian of an 8×8 skew-symmetric matrix of linear forms. Thanks to [B], this provides a 7-dimensional family of arithmetically Cohen–Macaulay (ACM for short) vector bundles on X, obtained as the bundles of kernels of the 8×8 skew-symmetric matrices of rank 6 representing points of X. We show that this family is a smooth open set M_X in the moduli space of stable vector bundles $M_X(2;3,14)\simeq M_X(2;-1,6)$. The ACM property means the vanishing of the intermediate cohomology $H^i(X,\mathcal{E}(j))$ for all $i=1,2$, $j\in\mathbb{Z}$.

We give also a precise geometric characterization of the ACM curves arising as schemes of zeros of sections of the above kernel vector bundles. According to Beauville, they are half-canonical ACM curves of degree 14 in $\mathbb{P}^4$; we show that they are linear sections of the rank 4 locus $Z\subset\mathbb{P}(\wedge^2\mathbb{C}^7)$ in the projectivized space of the 7×7 skew-symmetric matrices. Linear sections of Z arose already in the literature: Rødland [R] studied the sections $\mathbb{P}^6\cap Z$, which are Calabi–Yau threefolds. We show that such curves fill out open sets of smooth points of the Hilbert schemes of X (of dimension 14) and of $\mathbb{P}^4$ (of dimension 56), and that the isomorphism classes of smooth members of this family fill out a 32-dimensional moduli component $\overset{\circ}{\mathcal{M}}{}^4_{15}$ of curves of genus 15 with a theta-characteristic linear series of dimension 4.

Next we study the Abel–Jacobi map of the ACM half-canonical curves of genus 15 in X. It factors through M_X via Serre's construction: the fibers over points of M_X are $\mathbb{P}^7$, and the resulting map from M_X to $J^2(X)$ is étale quasi-finite, hence its image is 7-dimensional. The role of the above half-canonical curves is similar to that of normal elliptic quintics in the case of the cubic threefold V, where the Abel–Jacobi map factors through the instanton moduli space with fibers $\mathbb{P}^5$ and with a 5-dimensional image; as $\dim J^2(V)=5$, the image is an open subset of $J^2(V)$. The result for a quartic threefold is somewhat weaker: here we do not know whether the degree of the quasi-finite map is 1 and whether M_X is irreducible. Moreover, as $7 = \dim M_X < 30 = \dim J^2(X)$, we cannot conclude, as in the case of a cubic threefold, that the image of M_X is an open subset of an Abelian variety; we can only state that every component of it, and hence of M_X itself, has a non-negative Kodaira dimension.

In Section 1, we gather preliminary results: a criterion for stability of rank 2 sheaves, Bogomolov's inequality, and prove the non-existence of normalized rank 2 stable vector bundles with small second Chern classes. We provide two constructions of such vector bundles for $c_1 = 0$, $c_2 = 4$ and discuss the geometry of the instanton component(s).

In Section 2, we prove that a generic quartic 3-fold is Pfaffian, in using the same method as was used by Adler in his Appendix to [AR] for a cubic threefold: take a particular quartic which is Pfaffian and prove that the differential of the Pfaffian map from the family of all the 8×8 skew-symmetric matrices of linear forms to the family of quinary quartics is of maximal rank. We prove also basic facts about M_X: stability, dimension 7, smoothness.

Section 3 treats half-canonical ACM curves of genus 15 on X and in $\mathbb{P}^4$.

Section 4 applies the technique of the Tangent Bundle Sequence of Clemens–Griffiths [CG] and Welters [W] to the calculation of the differential of the Abel–Jacobi map for the family of the above half-canonical curves C. It identifies the kernel of the differential with $H^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))^\vee$, and we prove that it has dimension 7.

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1. Generalities and the case $c_1 = 0$

Let X be a smooth quartic threefold. It is well known that $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$, generated by the class of the hyperplane section H, and the group of algebraic 1-cycles modulo topological equivalence is also isomorphic to $\mathbb{Z}$, generated by the class of a line $l \subset X$. For two integers k, α, we will denote by $M_X(2; k, \alpha)$ the moduli space of *stable* rank 2 vector bundles $\mathcal{E}$ on X with Chern classes $c_1 = k[H]$ and $c_2 = \alpha[l]$. We will often identify the Chern classes with integers in using the generators $[H], [l]$ of the corresponding groups of algebraic cycles. We have $[H]^2 = 4[l]$.

By the definition of the Chern classes and by Riemann–Roch– Hirzebruch, we have for $\mathcal{E} \in M_X(2; k, \alpha)$:

$c_1(\mathcal{E}(n)) = c_1(\mathcal{E}) + 2n[H] = (k+2n)[H]$, $c_2(\mathcal{E}(n)) = c_2(\mathcal{E}) + n[H]c_1(\mathcal{E}) + n^2[H]^2 = (\alpha + 4kn + 4n^2)[l]$, $\chi(\mathcal{E}) = \frac{2}{3}k^3 - \frac{1}{2}k\alpha + k^2 - \frac{1}{2}\alpha + \frac{7}{3}k + 2$.

A rank 2 torsion free sheaf $\mathcal{E}$ on X is *normalized*, if $c_1(\mathcal{E}) = k[H]$ with $k = 0$ or $k = -1$. We can make $\mathcal{E}$ normalized in replacing it by a suitable twist $\mathcal{E}(n)$. If $\mathcal{E}$ is locally free, we have $\mathcal{E}^\vee \cong \mathcal{E} \otimes (\det \mathcal{E})^{-1}$, so that $\mathcal{E}$ is self-dual when $k = 0$.

The following lemmas are well known:

Lemma 1.1. *Let $\mathcal{E}$ be a normalized rank 2 reflexive sheaf on a non-singular projective variety X with* $\mathrm{Pic}(X) \simeq \mathbb{Z}$. *Then it is stable if and only if $h^0(\mathcal{E}) = 0$.*

Proof. Any saturated torsion free rank 1 subsheaf of $\mathcal{E}$ is invertible of the form $\mathcal{O}_X(m)$ and gives an exact triple

$$0 \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z(k-m) \longrightarrow 0 \ , \tag{1}$$

where Z is a subscheme of X of codimension 2. The triple breaks the Gieseker stability of $\mathcal{E}$ if and only if $m \geq 0$.

If we assume that $\mathcal{E}$ has global sections, then there exists a triple (1) with $m = 0$, hence $\mathcal{E}$ is not stable. If we assume that $\mathcal{E}$ has no global sections, then in any triple (1) for $\mathcal{E}$, we have $m < 0$, because $h^0(\mathcal{E}) \geq h^0(\mathcal{O}_X(m))$. Hence $\mathcal{E}$ is stable. □

Lemma 1.2. *Let $\mathcal{F}$ be a rank r semistable reflexive sheaf on X. Then $(2rc_2(\mathcal{F}) - (r-1)c_1^2(\mathcal{E})) \cdot H \geq 0$. If $r = 2$ and $\mathcal{F}$ is stable, then the inequality is strict.*

Proof. This is Bogomolov's Theorem, proved by him for T-(semi)stable sheaves [Bo]. For another approach to the proof and for relations between different notions of (semi)stability, see e. g. [Ko]. □

Proposition 1.3. *Let X be a smooth quartic threefold. Then the following statements hold:*

(i) $M_X(2;0,\alpha) = \varnothing$ for all odd α and for $\alpha \leq 2$.

(ii) $M_X(2;-1,\alpha) = \varnothing$ for $\alpha \leq 3$.

Proof. The case of odd α in (i) follows trivially from Riemann–Roch–Hirzebruch: we have $\chi(\mathcal{E}) = 2 - \frac{1}{2}\alpha$. For the remaining cases, the proof goes exactly as in [B-MR]. The first step is to show that if $M_X(2;\epsilon,\alpha) \neq \varnothing$ $(\epsilon = 0, -1)$, then $h^0(\mathcal{E}(1)) \neq 0$ for all $\mathcal{E} \in M_X(2;\epsilon,\alpha)$. The second step is to verify that there are no curves on X that might be zero loci of sections of $\mathcal{E}(1)$.

So, let $\mathcal{E} \in M_X(2;\epsilon,\alpha)$, $\epsilon = 0, -1$, $\alpha \leq 2 - \epsilon$. Assume that $h^0(\mathcal{E}(1)) = 0$. By Serre duality, $h^3(\mathcal{E}(1)) = h^0(\mathcal{E}(\epsilon - 2)) = 0$. Hence $h^2(\mathcal{E}(1)) \geq$

$\chi(\mathcal{E}(1))$. We have $\chi(\mathcal{E}(1)) = 10 - \frac{3}{2}\alpha$ if $\epsilon = 0$ and $\chi(\mathcal{E}(1)) = 6 - \alpha$ if $\epsilon = -1$. Hence $\dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{O}(-2)) = h^1(\mathcal{E}^\vee(-2)) > 0$, and there exists a non-trivial extension of vector bundles

$$0 \longrightarrow \mathcal{O}_X(-2) \xrightarrow{\sigma} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 . \tag{2}$$

We have $\Delta(\mathcal{F}) = (c_1(\mathcal{F})^2 - 3c_2(\mathcal{F}))H = 16 - 3\alpha > 0$ if $\epsilon = 0$ and $\Delta(\mathcal{F}) = 12 - 3\alpha$ if $\epsilon = -1$, so $\mathcal{F}$ is unstable by Lemma 1.2. To fix ideas, restrict ourselves to the case $\epsilon = 0$, the other case being completely similar.

The unstability of $\mathcal{F}$ can manifest itself in two ways: either $\mathcal{F}$ contains a rank 1 saturated subsheaf $\mathcal{O}_X(n)$ with $n > c_1(\mathcal{F})/\operatorname{rk}\mathcal{F} = -2/3$, or there exists a non-trivial morphism of sheaves $\mathcal{F} \xrightarrow{\phi} \mathcal{O}_X(n)$ with $n < c_1(\mathcal{F})/\operatorname{rk}\mathcal{F}$. In the first case, $n \geq 0$, hence $h^0(\mathcal{F}) \neq 0$, hence $h^0(\mathcal{E}) \neq 0$, and this contradicts the stability of $\mathcal{E}$.

In the second case, $n \leq -1$. If $n < -2$, then $\phi\sigma = 0$ and ϕ descends to a non-trivial morphism $\mathcal{E} \longrightarrow \mathcal{O}_X(n)$, which contradicts the stability of $\mathcal{E}$. Hence $n = -1$ or -2. It cannot be -2, because otherwise the extension (2) would be split. So $n = -1$ and we obtain the exact triple

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{I}_Z(-1) \longrightarrow 0 ,$$

in which $\mathcal{F}'$ is a reflexive rank 2 sheaf. We have $c_1(\mathcal{F}') = -[H]$, $c_2(\mathcal{F}') \leq c_2(\mathcal{F}) - c_1(\mathcal{F}')c_1(\mathcal{O}_X(1)) = (\alpha - 4)[l] < 0$, hence $\mathcal{F}'$ is unstable by Bogomolov's inequality. By Lemma 1.1, $h^0(\mathcal{F}') \neq 0$, which implies $h^0(\mathcal{F}) \neq 0$ and hence $h^0(\mathcal{E}) \neq 0$. This contradicts the stability of $\mathcal{E}$.

Thus we have proved $h^0(\mathcal{E}(1)) \neq 0$. As $\operatorname{Pic} X \cong \mathbb{Z}$, the scheme $C = C_s$ of zeros of a non-trivial section s of $\mathcal{E}(1)$ is a l. c. i. of pure codimension 2. Hence $\mathcal{E}(1)$ fits into the following exact triple

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{E}(1) \longrightarrow \mathcal{I}_C(2) \longrightarrow 0 . \tag{3}$$

We have $[C] = (4 + \alpha)[l]$, $\omega_C = \mathcal{O}_C(1)$, $2p_a(C) - 2 = 4 + \alpha$. It remains only to verify the case of $\alpha = 2$. We have $p_a(C) = 4$ and C is embedded into $\mathbb{P}^4$ by a subsystem of the canonical system. The exact triple (3), twisted by $\mathcal{O}_X(-1)$, implies that C is not contained in a hyperplane. Hence C is not connected. It cannot be a union of more than one connected components either, because at least one of them should be of degree ≤ 3 and hence ω_C cannot be ample.

The proof is completed in a similar way in the case $k = -1$. $\square$

For $k = 0$, we have proved that there are no stable rank 2 vector bundles with $\alpha < 4$. However, they do exist for $\alpha = 4$. Indeed, assuming that $h^0(\mathcal{E}(1)) \neq 0$, we find only two possibilities for the zero locus C af a non-trivial section of $\mathcal{E}(1)$: either C is a canonical curve of degree 8, or the union of two canonical curves of degree 4 which spans $\mathbb{P}^4$ (including the non-reduced limits of such curves). In the first case, standard calculations show that the generic quartic contains an 8-dimensional family of such curves C. Only in this case the vector bundle $\mathcal{E}$ has *natural cohomology*, that is, for every t, $h^i(\mathcal{E}(t)) \neq 0$ for at most one value of i. It is reasonable to call *instantons* the stable vector bundles with natural cohomology, such that $c_1(\mathcal{E}) = 0$ and the instanton condition $h^1(\mathcal{E}(-2)) = 0$ is verified. Thus we have the following statement:

Proposition 1.4. *(i) On any smooth quartic threefold X, there is an irreducible component M_X^0 of $M_X(2;0,4)$ which parametrizes the vector bundles obtained by Serre's construction from the curves $C = C_1^4 \sqcup C_2^4$, where C_i^4 are plane sections of X. These vector bundles satisfy $h^1(\mathcal{E}(1)) = 1$, hence they are not instantons.*

(ii) Let C be a smooth complete intersection of 3 quadrics in $\mathbb{P}^4$. Then there exists a smooth quartic threefold X containing C, and the vector bundles on X obtained by Serre's construction (3) from the curve C and from its generic deformations in X sweep out a component $M_X^1(C)$ of $M_X(2;0,4)$, different from M_X^0. The vector bundles $\mathcal{E} \in M_X^1(C)$ are instantons.

(iii) Let X be a smooth quartic threefold. Then any component M of $M_X(2;0,4)$, such that $h^0(\mathcal{E}(1)) \neq 0$ for some $\mathcal{E} \in M$, is one of the above components M_X^0, $M_X^1(C)$.

Remark 1.5. Let $\mathcal{C}_8^5(X)$ be the 8-dimensional family of curves C on the general $X = X_4$ as in Proposition 1.4, (ii), and let $v_2 : X \to \mathbb{P}^{14}$ be the Veronese map. For $C \in \mathcal{C}_8^5(X)$, one can see that the 11-space $\mathbb{P}^{11}(C) =< v_2(C) >\subset \mathbb{P}^{14}$ lies in a unique rank 6 quadric $Q = Q_C \supset v_2(X)$. Indeeed, if C is the intersection of three quadrics $q_i = 0$ $(i = 1,2,3)$, then the equation of X can be written in the form $q_1(x)\tilde{q}_1(x) + q_2(x)\tilde{q}_2(x) + q_3(x)\tilde{q}_3(x) = 0$, which provides the rank 6 quadric $Q = \{l_1\tilde{l}_1 + l_2\tilde{l}_2 + l_3\tilde{l}_3 = 0\}$, where $l_i, \tilde{l}_i$ are the linear forms in the Veronese embedding corresponding to the quadratic forms $q_i, \tilde{q}_i$. It is a degenerate cone whose ridge is $\mathbb{P}^8$, the kernel of the quadratic form, and whose base is G, a non-degenerate 4-dimensional quadric in $\mathbb{P}^5$. The curves C_s of zeros of sections $s \in H^0(\mathcal{E}(1))$ form a projective space $\mathbb{P}^3$, naturaly identified with one of the two $\mathbb{P}^3$'s parametrizing projective 2-planes $\{l_1' = l_2' = l_3' = 0\}$ in G, the equations of C_s being of the form $q_1' = q_2' = q_3' = 0$, where q_i' correspond to l_i' under the Veronese map. The above $\mathbb{P}^{11}$ is the cone over this $\mathbb{P}^3$ with "vertex" $\mathbb{P}^8$.

Let $\Lambda_C = \{\mathbb{P}^{11}_t : t \in \mathbb{P}^3\}$ be the ruling of Q_C defined by $\mathbf{P}^{11}(C) \in \Lambda$, $\mathbf{P}^{11}(C) = \mathbf{P}^{11}_o$. Then the sections of $E_C(1)$ are exactly the curves $C_t = \mathbf{P}^{11}_t \cap v_2(X) \subset v_2(X) \cong X$, $t \in \mathbf{P}^3$, $C = C_o$.

Thus we can represent M^1_X, the union of the 5-dimensional components $M^1_X(C)$ of Proposition 1.4, as the variety $\tilde{D}_6(v_2(X)) := \{$ the $\mathbb{P}^{11}$'s contained in some rank 6 quadric $Q \supset v_2(X)$ $\}$, which is, in its turn, the double cover of the family $D_6(v_2(X)) := \{$the rank 6 quadrics $Q \supset v_2(X)$ $\}$.

Remark 1.6. According to [Tyu], (3.1.45), the virtual dimension of $M_X(2;0,2)$ is 1, so one could expect a curve of isomorphism classes of stable vector bundles with $c_1 = 0, c_2 = 2$. But we have proved that $M_X(2;0,2)$ is empty, providing thus one more example of a situation when dimension is different from the virtual one.

Remark 1.7. For $k = -1$, we leave open the cases of $\alpha = 4, 5$. We construct in what follows a 7-dimensional component, or a union of 7-dimensional components of $M_X(2;-1,6)$ (we do not approach the question on the number of these components).

2. Generic quartic 3-fold is Pfaffian

Let E be an 8-dimensional vector space over $\mathbb{C}$. Fix a basis $e_0, \dots, e_7$ for E, then $e_{ij} = e_i \wedge e_j$ for $0 \le i < j \le 7$ form a basis for the Plücker space $\wedge^2 E$. Let x_{ij} be the corresponding (Plücker) coordinates. The embedding of the Grassmannian $G = G(2,E)$ in $\mathbb{P}^{27} = \mathbb{P}(\wedge^2 E)$ is precisely the locus of rank 2 skew symmetric 8×8 matrices M with elements x_{ij} above the diagonal. Let $G \subset \Omega \subset \Xi \subset \mathbb{P}^{27}$ be the filtration of $\mathbb{P}^{27}$ by the rank of M, that is $\Omega = \{M \mid \mathrm{rk}\, M \le 4\}$, $\Xi = \{M \mid \mathrm{rk}\, M \le 6\}$. Then G, $\Omega \setminus G$, $\Xi \setminus \Omega$ and $\mathbb{P}^{27} \setminus \Xi$ are orbits of $PGL(8)$, acting via $\wedge^2$ of its standard representation (see e. g. [SK]), and we have $G = \mathrm{Sing}\, \Omega$, $\dim G = 12$, $\Omega = \mathrm{Sing}\, \Xi$, $\dim \Omega = 21$. Ξ is defined by the quartic equation $\mathrm{Pf}(M) = 0$, where Pf stands for the Pfaffian of a skew-symmetric matrix. We will call Ξ the *Pfaffian hypersurface* of $\mathbb{P}^{27}$.

Let $H \subset \mathbb{P}^{27}$ be a 4-dimensional linear subspace which is not contained in Ξ. Then the intersection $H \cap \Xi$ will be called a Pfaffian quartic 3-fold. As $\mathrm{codim}_{\Xi}\, \Omega = 5$, the linear section $H \cap \Xi$ is nonsingular for general H. Suppose that a quartic 3-fold $X \subset \mathbb{P}^4$ has two different representations $\phi_1 : X \xrightarrow{\sim} H_1 \cap \Xi$, $\phi_2 : X \xrightarrow{\sim} H_2 \cap \Xi$ as linear sections of Ξ. We will call them equivalent if $\phi_2 \circ \phi_1^{-1}$ is the restriction of a transformation from $PSL(8)$.

Proposition 2.1. *A generic quartic 3-fold admits a 7-parameter family of non-equivalent representations as linear sections of the Pfaffian hypersurface in* $\mathbb{P}^{27}$.

Proof. We are using the same argument as that of [AR], Theorem (47.3). The family of quartic 3-folds in $\mathbb{P}^4$ is parametrized by $\mathbb{P}^{69}$, and that of the Pfaffian representations of quartic 3-folds by an open set in the variety $\mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27})$ of linear morphisms between the two projective spaces. So, we are going to specify one particular quartic 3-fold $X_0 = \{F_0 = 0\}$ which admits a Pfaffian representation $F_0 = \mathrm{Pf}(M_0)$, then we will show that the differential of the map $f : \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) \dashrightarrow \mathbb{P}^{69}$ at M_0 is surjective, and this will imply that f is dominant.

Let

$$M_0 = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_1 & 0 \\ -x_1 & 0 & 0 & x_5 & 0 & 0 & -x_3 & -x_1 \\ -x_2 & 0 & 0 & x_1 & x_1 & 0 & 0 & -x_4 \\ -x_3 & -x_5 & -x_1 & 0 & x_2 & 0 & 0 & 0 \\ -x_4 & 0 & -x_1 & -x_2 & 0 & x_3 & x_1 & 0 \\ -x_5 & 0 & 0 & 0 & -x_3 & 0 & x_4 & x_2 \\ -x_1 & x_3 & 0 & 0 & -x_1 & -x_4 & 0 & x_5 \\ 0 & x_1 & x_4 & 0 & 0 & -x_2 & -x_5 & 0 \end{bmatrix},$$

$F_0 = \mathrm{Pf}(M_0) = x_1^3x_2 - x_1^3x_3 + x_2^3x_3 - x_1x_2x_3^2 - x_1x_2^2x_4 + x_1^2x_3x_4 + x_1x_2x_3x_4 + x_3^3x_4 - x_1^2x_4^2 + x_1x_2x_4^2 + x_1^3x_5 - x_1^2x_2x_5 - x_1x_2^2x_5 - x_1^2x_3x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_4^3x_5 + x_2x_3x_5^2 - x_1x_4x_5^2 + x_1x_5^3$.

A point $M \in \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27})$ is the proportionality class of an 8×8 skew-symmetric matrix of linear forms l_{ij} and is given by its $5 \cdot 28 = 140$ homogeneous coordinates (a_{ijk}) such that $l_{ij} = \sum_k a_{ijk}x_k$ $(0 \leq i < j \leq 8, 1 \leq k \leq 5)$. We have $\partial f(M)/\partial a_{ijk} = x_k \,\mathrm{Pf}_{ij}(M)$, where $\mathrm{Pf}_{ij}(M)$ denotes the Pfaffian of the 6×6 matrix obtained by deleting the i-th and the j-th rows and the i-th and the j-th columns of M.

Computation by the Macaulay 2 program [GS] shows that, for the above matrix M_0, the 140 quartic forms $x_k \,\mathrm{Pf}_{ij}(M_0)$ generate the whole 70-dimensional space of quinary quartic forms, hence f is of maximal rank at M_0. One can also easily make Macaulay 2 to verify that X_0 is in fact nonsingular, though this is not essential for the above proof.

It remains to verify that the generic fiber of the induced map $\bar{f} : PGL(5)\backslash \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27})/PGL(8) \dashrightarrow PGL(5)\backslash \mathbb{P}^{69}$ is 7-dimensional. By counting dimensions, one sees that this is equivalent to the fact that the stabilizer of a generic point of the Grassmannian $G(5, 28) = PGL(5)\backslash \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{27})$ in $PGL(8)$ is 0-dimensional.

Take a generic 4-dimensional linear subspace $H \subset \mathbb{P}^{27}$. Then the quartic 3-fold $X = H \cap \Xi$ is generic, and hence $\mathrm{Aut}(X)$ is trivial. Thus the stabilizer G_H of H in $PGL(8)$ acts trivially an X, and hence on H. This implies the triviality of G_H by (5.3) of [B]. $\square$

Let now $\mathcal{K}$ be the kernel bundle on Ξ whose fiber at $x \in \Xi$ is $\ker x$. Thus $\mathcal{K}$ is a rank 2 vector subbundle of the trivial rank 8 vector bundle $E_\Xi = E \otimes_{\mathbb{C}} \mathcal{O}_\Xi$ over $\Xi_0 = \Xi \setminus \Omega$. Let $\phi : X \longrightarrow H \cap \Xi$ be a representation of a nonsingular quartic 3-fold $X \subset \mathbb{P}^4$ as a linear section of Ξ. Giving ϕ is equivalent to specifying a skew-symmetric 8×8 matrix M of linear forms such that the equation of X is $\mathrm{Pf}(M) = 0$. Such a representation yields a rank 2 vector bundle $\mathcal{E} = \mathcal{E}_\phi$ on X, defined by $\mathcal{E} = \phi^* \mathcal{K}$. According to [B], Proposition 8.2, the scheme of zeros of any section $s \neq 0$ of $\mathcal{E}$ is an arithmetically Cohen– Macaulay (ACM) 1-dimensional scheme C of degree 14, not contained in any quadric hypersurface and such that its canonical bundle $\omega_C \simeq \mathcal{O}_C(2)$. Varieties satisfying the last condition are usually called half-canonical. Moreover, $\mathcal{E}$ is also ACM and has a resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^8 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}^8 \longrightarrow \mathcal{E} \longrightarrow 0 \qquad (4)$$

This implies in particular that two Pfaffian representations ϕ_1, ϕ_2 are equivalent if and only if the corresponding vector bundles $\mathcal{E}_1, \mathcal{E}_2$ are isomorphic. By (8.1) ibid., $\mathcal{E}$ can be given also by Serre's construction as the middle term of the extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{C,X}(3) \longrightarrow 0 \ , \qquad (5)$$

where $\mathcal{I}_{C,X}$ denotes the ideal sheaf of C in X. Thus, the following assertion holds.

Corollary 2.2. *A generic quartic 3-fold $X \subset \mathbb{P}^4$ has a 7-dimensional family of isomorphism classes of rank 2 ACM vector bundles $\mathcal{E}$ with $\det \mathcal{E} \simeq \mathcal{O}(3)$ and $h^0(\mathcal{E}) = 8$ which are characterized by either one of the following equivalent properties:*

(i) $\mathcal{E}$ as a sheaf on $\mathbb{P}^4$ possesses a resolution of the form (4) with a skew-symmetric matrix of linear forms M;

(ii) the scheme of zeros of any section $s \neq 0$ of $\mathcal{E}$ is an ACM half-canonical curve C of degree 14 and arithmetic genus 15, not contained in any quadric hypersurface in $\mathbb{P}^4$;

(iii) $\mathcal{E}$ can be obtained by Serre's construction from a curve $C \subset X$ as in (ii).

In fact the vector bundles under consideration are stable, so the above 7-parameter family is a part of the moduli space of vector bundles.

Theorem 2.3. *Let X be a generic quartic 3-fold and $M_X(2;-1,6)$ the moduli space of stable rank 2 vector bundles $\mathcal{G}$ on X with Chern classes $c_1 = -[H], c_2 = 6[l]$, where $[l] \in H^2(X,\mathbb{Z})$ is the class of a line. Then the isomorphism classes of the ACM vector bundles of the form $\mathcal{G} = \mathcal{E}(-2)$, where $\mathcal{E}$ are vector bundles introduced in Corollary 2.2, form an irreducible open subset M_X of dimension 7 in the nonsingular locus of $M_X(2;-1,6)$.*

Proof. Stability. If $\mathcal{E}$ is given by the extension (5), then twisting by $\mathcal{O}_X(-2)$ and using $h^0(\mathcal{I}_{C,X}(k)) = 0$ for $k \leq 2$ ((ii) of Lemma 2.2), we see that $h^0(\mathcal{E}(-2)) = 0$. The stability follows from Lemma 1.1.

Smoothness and dimension. The stability implies that $\mathcal{E}$ is simple, that is $h^0(\mathcal{E}^\vee \otimes \mathcal{E}) = 1$. Hence the tangent space $T_{[\mathcal{E}]}M_X(2;0,2[l])$ at $[\mathcal{E}]$ is identified with $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) = H^1(X, \mathcal{E}^\vee \otimes \mathcal{E})$, and if $H^2(X, \mathcal{E}^\vee \otimes \mathcal{E}) = 0$, then $M_X(2;0,2[l])$ is smooth at $[\mathcal{E}]$ of local dimension $\dim_{[\mathcal{E}]} M_X(2;0,2[l]) = h^1(\mathcal{E}^\vee \otimes \mathcal{E})$.

As $\mathrm{rk}\,\mathcal{E} = 2$, we have $\mathcal{E}^\vee \simeq \mathcal{E} \otimes (\det \mathcal{E})^{-1} \simeq \mathcal{E}(-3)$. By Serre duality, $h^3(\mathcal{E}^\vee \otimes \mathcal{E}) = h^0(\mathcal{E}^\vee \otimes \mathcal{E}(-1)) = 0$. By (4), $h^0(\mathcal{E}(-3)) = \chi(\mathcal{E}(-3)) = 0$. Together with the ACM property for $\mathcal{E}$ this gives $h^i(\mathcal{E}(-3)) = 0$ for all $i \in \mathbb{Z}$. Now, from (5) tensored by $\mathcal{E}(-3)$, we obtain the isomorphisms

$$H^i(\mathcal{E}^\vee \otimes \mathcal{E}) = H^i(\mathcal{E} \otimes \mathcal{E}(-3)) = H^i(\mathcal{E} \otimes \mathcal{I}_C) \quad \forall\, i \in \mathbb{Z}\,. \tag{6}$$

Further, the restriction sequence

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{I}_C \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_C \longrightarrow 0 \tag{7}$$

yields $\chi(\mathcal{E} \otimes \mathcal{I}_C) = \chi(\mathcal{E}) - \chi(\mathcal{E}|_C) = 8 - 14 = -6$, so to finish the proof, it remains to prove the vanishing of $h^2(\mathcal{E} \otimes \mathcal{I}_C)$. By [B], (8.9), the vanishing of $h^2(\mathcal{E}nd_0(\mathcal{E}))$ follows from the fact that the map f, introduced in the proof of Proposition 2.1, is dominant. As $h^2(\mathcal{O}_X) = 0$, we have $0 = h^2(\mathcal{E}nd_0(\mathcal{E})) = h^2(\mathcal{E}nd(\mathcal{E})) = h^2(\mathcal{E} \otimes \mathcal{I}_C)$, and we are done. □

3. Curves of degree 14 and genus 15 in $\mathbb{P}^4$

Let $X = \{F = 0\}$ be a generic quartic 3-fold in $\mathbb{P}^4$, and $X = H \cap \Xi$ (so, the $\mathbb{P}^4$ is identified with H) a Pfaffian representation for X. For the sake of functoriality, we should have defined Ξ as embedded in $\mathbb{P}(\wedge^2(E^\vee))$, so that the points $x \in X$ be interpreted as alternating bilinear forms of rank 6 on E, whilst $G = G(2,8) \subset \mathbb{P}(\wedge^2 E)$; to avoid this dichotomy we will work in coordinates, identifying E with $E^\vee$. Let $\mathcal{E}$ be the corresponding rank 2 vector bundle and C the scheme of zeros of a section $s \neq 0$ of $\mathcal{E}$. Let $H_{14,15}$, resp. $H^X_{14,15}$ denote the union of the components of the Hibert scheme of curves in $\mathbb{P}^4$, resp. X whose generic points represent a curve C as above. For generic s, the curve C is nonsingular.

Similarly to the previous section, introduce the rank filtration on the 7×7 skew-symmetric matrices: $G' = G(2,7) \subset Z \subset \mathbb{P}^{20} = \mathbb{P}(\wedge^2(\mathbb{C}^7))$. According to [R], we have $\dim G' = 10$, $\deg G' = 42$, $\omega_{G'} = \mathcal{O}_{G'}(-7)$, $\dim Z = 17$, $\deg Z = 14$, $\omega_Z = \mathcal{O}_Z(-14)$. G' will be identified with a subvariety of G for the standard inclusion $\mathbb{C}^7 \subset \mathbb{C}^8$.

Proposition 3.1. *The following assertions hold:*

(i) $h^0(\mathcal{N}_{C/X}) = 14$, $h^1(\mathcal{N}_{C/X}) = 0$. *Hence* $H^X_{14,15}$ *is smooth at* $[C]$ *of local dimension* 14.

(ii) $h^0(\mathcal{N}_{C/\mathbb{P}^4}) = 56$, $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = 0$. *Hence* $H_{14,15}$ *is smooth at* $[C]$ *of local dimension* 56.

(iii) C *can be identified with a section of the rank* 4 *locus* Z *of* 7×7 *skew-symmetric matrices by a* 4*-dimensional linear subspace* $L \subset \mathbb{P}^{20}$.

Proof. (i) The restriction sequence (7) yields $h^2(\mathcal{E} \otimes \mathcal{I}_C) = h^1(\mathcal{E}|_C)$. We proved in Theorem 2.3 the vanishing of $h^2(\mathcal{E} \otimes \mathcal{I}_C)$. As C is the scheme of zeros of a section of $\mathcal{E}$, we have $\mathcal{E}|_C \simeq \mathcal{N}_{C/X}$. So, we obtain $h^1(\mathcal{N}_{C/X}) = 0$. By Riemann–Roch, $h^0(\mathcal{N}_{C/X}) = 14$ and we are done.

(ii) We have $h^1(\mathcal{N}_{C/X}) = 0$. We are going to show that this implies $h^1(\mathcal{N}_{C/\mathbb{P}^4}) = 0$. First, by Serre duality, $0 = h^1(\mathcal{N}_{C/X}) = h^0(\mathcal{N}^\vee_{C/X}(2))$. From the restriction sequence

$$0 \longrightarrow \mathcal{I}_{C,X}(2) \longrightarrow \mathcal{O}_X(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0$$

and from the fact that $\omega_C = \mathcal{O}_C(2)$, we deduce that $h^1(\mathcal{I}_{C,X}(2)) = 0$. Now, the exact triple

$$0 \longrightarrow \mathcal{I}^2_{C,X}(2) \longrightarrow \mathcal{I}_{C,X}(2) \longrightarrow \mathcal{N}^\vee_{C/X}(2) \longrightarrow 0$$

yields $h^1(\mathcal{I}^2_{C,X}(2)) = 0$. The ACM property for C and

$$0 \longrightarrow \mathcal{I}_{C,\mathbb{P}^4}(-2) \xrightarrow{F} \mathcal{I}^2_{C,\mathbb{P}^4}(2) \longrightarrow \mathcal{I}^2_{C,X}(2) \longrightarrow 0$$

imply $h^1(\mathcal{I}^2_{C/\mathbb{P}^4}(2)) = 0$. Now, the triple

$$0 \longrightarrow \mathcal{I}^2_{C,\mathbb{P}^4}(2) \longrightarrow \mathcal{I}_{C,\mathbb{P}^4}(2) \longrightarrow \mathcal{N}^\vee_{C,\mathbb{P}^4}(2) \longrightarrow 0$$

and the Serre duality give $h^0(\mathcal{N}^\vee_{C/\mathbb{P}^4}(2)) = h^1(\mathcal{N}_{C/\mathbb{P}^4}) = 0$. By Riemann–Roch, $h^0(\mathcal{N}_{C/\mathbb{P}^4}) = 56$.

(iii) The sections of $\mathcal{E}$ are naturally identified with elements of $E^\vee$ via the embedding of $\mathcal{E}$ into the trivial rank 8 vector bundle $E_X = E \otimes \mathcal{O}_X$. Let $\mathrm{Cl} : \Xi \setminus \Omega \longrightarrow G = G(2,8)$ be the classifying map, sending each $x \in \Xi \setminus \Omega$ to the projectivized kernel of x, considered as a point of G, and Cl_X the restriction of Cl to X. We can choose the coordinates in E in such a way that $s = x_7$. Hence $C = \mathrm{Cl}_X^{-1}(\sigma_{11}(\mathbb{P}^6))$, where $\mathbb{P}^6$ is the hyperplane $\{x_7 = 0\}$ in $\mathbb{P}^7 = \mathbb{P}(E)$, and $\sigma_{11}(\mathbb{P}^6) = G' \subset G$ is the Schubert subvariety of all the lines contained in the hyperplane. We can also write $C = \mathrm{Cl}^{-1}(G') \cap H$. The closure of the 24-fold $\mathrm{Cl}^{-1}(G')$ in Ξ is defined by the 7 cubic Pfaffians $\mathrm{Pf}_{r7}(x), 0 \leq r \leq 6$.

As cubic forms, the Pfaffians $\mathrm{Pf}_{r7}(x)$, $0 \leq r \leq 6$ do not depend on the variables x_{p7}, $0 \leq p \leq 7$. Therefore $\mathrm{Cl}^{-1}(G')$ is isomorphic to the cone $C(Z)$ with vertex (or ridge) $\overline{\mathbb{P}}^6 = < e_{07}, ..., e_{67} >$ and base

$Z = \{x' : \mathrm{Pf}_{07}\, x' = \ldots = \mathrm{Pf}_{67}\, x' = 0\} \subset \mathbb{P}(\wedge^2 \ < e_0, ..., e_6 >)$;

here $x' = (x_{pq})_{0 \leq p,q \leq 6}$ is the 8-th principal adjoint matrix of the matrix x, i.e. x' is obtained from x by deleting its last column and row. It is well known that the vanishing of the principal minors of order $2n$ of a skew-symmetric $(2n+1) \times (2n+1)$ matrix is equivalent to the vanishing of all its minors of order $2n$, so Z is the locus of 7×7 skew-symmetric matrices of rank 4. The projection $\pi : \mathbb{P}^{27} \dashrightarrow \mathbb{P}^{20}$ with center $\overline{\mathbb{P}}^6$ maps isomorphically (for generic H) the intersection $H \cap C(Z)$ to $L \cap Z$, where $L = \pi(H)$. This ends the proof. $\square$

Let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus g and $\mathcal{M}_g^r$ the subvariety of $\mathcal{M}_g$ parametrizing half-canonical curves with a theta-characteristic D such that $\dim |D| = r$.

Corollary 3.2. *The following assertions hold:*

(i) $H_{14,15}$ *is irreducible of dimension* 56.

(ii) For generic $\mathcal{L} \in \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$*, the stabilizer of* $\mathcal{L}$ *in* $PGL(7)$*, acting on the right, is finite, and the natural map* $\mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})/PGL(7) \dashrightarrow H_{14,15}$ *is generically finite.*

(iii) The natural map $g : PGL(5)\backslash \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20}) / PGL(7) \dashrightarrow \mathcal{M}^4_{15}$ *is generically finite and its image is a* 32*-dimensional irreducible component* $\mathring{\mathcal{M}}^4_{15}$ *of* $\mathcal{M}^4_{15}$.

Proof. (i) Indeed, $H_{14,15}$ is the image of $\mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$.

(ii) This follows from the count of dimensions: $\dim \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20}) - \dim PGL(7) = (5 \cdot 21 - 1) - (7^2 - 1) = 56 = \dim H_{14,15}$.

(iii) According to Harris [H], if $r \leq \frac{1}{2}(g-1)$, then the codimension of any component of $\mathcal{M}_g^r$ in $\mathcal{M}_g$ is at most $\frac{1}{2}r(r+1)$. Applying this to our case, we see that the dimension of every component of $\mathcal{M}^4_{15}$ is at least 32. Hence the component $\mathring{\mathcal{M}}^4_{15}$, containing the image of $H_{14,15}$, is of dimension ≥ 32. The dimension of $PGL(5)\backslash \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20}) / PGL(7)$ is 32, so it remains to show that g is dominant over $\mathring{\mathcal{M}}^4_{15}$.

Take a generic C from the image of g. C is a smooth ACM curve in $\mathbb{P}^4$. By the definition of $\mathcal{M}_g^r$, every small (analytic or étale) deformation of C is accompanied by a deformation of the theta-characteristic D

embedding C into $\mathbb{P}^4$. The ACM property being generic, any generic small deformation of C is again in the image of g, and we are done. $\square$

Remark 3.3. In (ii) of the lemma, the stabilizer $G_{\mathcal{L}}$ of $\mathcal{L}$ might act by non-trivial automorphisms of C. As $\mathrm{Aut}(C)$ is finite, the subgroup $H_{\mathcal{L}}$ fixing pointwise C, and hence $L = \mathcal{L}(\mathbb{P}^4)$, is of finite index in $G_{\mathcal{L}}$. So, the first assertion of (ii) is equivalent to saying that $H_{\mathcal{L}}$ is finite. One can strengthen this assertion: the subgroup of $PGL(2n+1)$ fixing pointwise a generic linear $\mathbb{P}^2 \subset \mathbb{P}(\wedge^2\mathbb{C}^{2n+1})$ for $n \geq 2$ is finite. This is easily reduced to the $2n$-dimensional case, stated in [B], (5.3).

Proposition 3.4. *Let $\mathring{H}^X_{14,15} \subset H^X_{14,15}$ be the locus of ACM half-canonical curves $C \subset X$ of degree 14 and arithmetic genus 15, not contained in any quadric hypersurface in $\mathbb{P}^4$, and $M_X \subset M_X(2;-1,6)$ the open set defined in Theorem 2.3. Then the Serre construction defines a morphism $\phi : \mathring{H}^X_{14,15} \longrightarrow M_X$ with fiber $\mathbb{P}^7$. Moreover, $\mathring{H}^X_{14,15}$ is isomorphic locally in the étale topology over M_X to a projectivized rank 8 vector bundle on M_X.*

Proof. It is easily seen that $\dim \mathrm{Ext}^1(\mathcal{I}_C(3), \mathcal{O}_X) = 1$, so, given C, the Serre construction determines $\mathcal{E}$ uniquely. This yields ϕ as a set theoretic map. An obvious relativization of the Serre construction shows that it is indeed a morphism.

Further, we have $h^0(\mathcal{E} \otimes \mathcal{I}_C) = 1$ by stability of $\mathcal{E}$ and (6), so the projective space $\mathbb{P}^7 = \mathbb{P}(H^0(\mathcal{E}))$ is injected into $H^X_{14,15}$ by sending each section $s \neq 0$ of $\mathcal{E}$ to its scheme of zeros. Hence the fibers of ϕ are set-theoretically 7-dimensional projective spaces. The proof of the last assertion of the proposition is completely similar to that of Lemma 5.3 in [MT]. $\square$

4. Abel–Jacobi map

We are going to remind briefly the Clemens–Griffiths technique for the calculation of the differential of the Abel–Jacobi map, following Welters [W], Sect. 2. Let X be a nonsingular projective 3-fold with $h^{03} = 0$, and $X \subset W$ an embedding in a nonsingular possibly non-complete 4-fold. Let $\Phi : B \longrightarrow J^2(X)$ be the Abel–Jacobi map, where B is the base of a certain family of curves on X. The differential $d\Phi_{[Z]}$ at a point $[Z] \in B$, representing a curve Z, factors into the composition of the infinitesimal classifying map $T_{B,b} \longrightarrow H^0(Z, \mathcal{N}_{Z/X})$ and of the universal "infinitesimal Abel–Jacobi map" $\psi_Z : H^0(Z, \mathcal{N}_{Z/X}) \longrightarrow H^1(X, \Omega^2_X)^\vee = T_{J_1(X),0}$. The adjoint $\psi_Z^\vee$ is identified by the following commutative square:

$$
\begin{array}{ccc}
H^0(X, \mathcal{N}_{X/W} \otimes \omega_X) & \xrightarrow{R} & H^1(X, \Omega^2_X) \\
{\scriptstyle r_Z}\downarrow & & \downarrow{\scriptstyle \psi_Z^\vee} \\
H^0(Z, \mathcal{N}_{X/W} \otimes \omega_X|_Z) & \xrightarrow{\beta_Z} & H^0(Z, \mathcal{N}_{Z/X})^\vee .
\end{array} \tag{8}
$$

Here r_Z is the map of restriction to Z, and the whole square (upon natural identifications) is the $H^0 \to H^1$ part of the commutative diagram of the long exact cohomology sequences associated to the following commutative diagram of sheaves:

$$
\begin{array}{ccccccccc}
0 & \to & \Omega^2_X & \to & \Omega^3_W \otimes \mathcal{N}_{X/W} & \to & \Omega^3_X \otimes \mathcal{N}_{X/W} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^3_X \otimes \mathcal{N}_{Z/X} & \to & \Omega^3_X \otimes \mathcal{N}_{Z/W} & \to & \Omega^3_X \otimes \mathcal{N}_{X/W} \otimes \mathcal{O}_Z & \to & 0
\end{array} \tag{9}
$$

Specifying all this to the case when X is a generic quartic 3-fold, $Z = C \subset X$ a generic curve from $H^X_{14,15}$, $W = \mathbb{P}^4$, we see that the dimensions in (8) form the array $\begin{pmatrix} 35 & 30 \\ 28 & 14 \end{pmatrix}$, that R, r_C are surjective and that $\operatorname{corank} \beta_C = \operatorname{corank} \psi_C^\vee = h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$. Dualizing, we obtain:

Lemma 4.1. *For C, X as above,* $\dim \ker \psi_C = h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$, $\dim \operatorname{im} \psi_C = 14 - h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$.

We have $\chi(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 14$, hence $h^0(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 14 + h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1))$.

Lemma 4.2. $h^0(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 21$.

Proof. Twisting the 4 exact triples in the proof of Proposition 3.1 by $\mathcal{O}(1)$, one can see that the assertion is equivalent to

$$h^2(\mathcal{I}^2_{C,\mathbb{P}^4}(3)) = 21 \ , \quad h^i(\mathcal{I}^2_{C,\mathbb{P}^4}(3)) = 0 \ \ \forall \ \ i \neq 2.$$

The last equalities follow immediately from the resolution for $\mathcal{I}^2_{C,\mathbb{P}^4}(3)$, obtained from (4) of [R] by restriction to $L = \mathbb{P}^4 \subset \mathbb{P}^6$ and twisting by $\mathcal{O}(3)$:

$$0 \to 21\mathcal{O}_{\mathbb{P}^4}(-5) \to 48\mathcal{O}_{\mathbb{P}^4}(-4) \to 28\mathcal{O}_{\mathbb{P}^4}(-3) \to \mathcal{I}^2_{C,\mathbb{P}^4}(3) \to 0.$$

$\square$

Remark 4.3. One can interprete the elements of $H^0(\mathcal{N}_{C/\mathbb{P}^4}(-1))$ as infinitesimal deformations of C preserving 14 points of some hyperplane section $S = C \cap h$ of C. It is easy to understand this value geometrically in constructing explicitly a 21-dimensional family of global, non-infinitesimal deformations of C which preserve S; one can show that every 1-parameter infinitesimal deformation lifts to a global one at list for generic C, h.

Indeed, lift C to an element $A \in \mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$, $C = A^{-1}(L \cap Z)$, $L = A(\mathbb{P}^4)$. The ACM property of C implies that S spans h, so the set U of global deformations A' of A such that $S \subset C' = A'^{-1}(A'(\mathbb{P}^4) \cap Z)$ are exactly the elements A' with the property $A|_h = A'|h$. Identify $\mathrm{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$ with the open subset of the Grassmannian $G(5, 26)$ parametrizing the graphs of the linear injective maps from $\mathbb{C}^5$ to $\mathbb{C}^{21}$. The graphs of the above elements A' correspond to those 4-dimensional planes in $\mathbb{P}^{25}$ which contain a fixed $\mathbb{P}^3$, the graph of $A|_h$. Thus U is identified with an open subset of $\mathbb{P}^{21} \subset G(5, 26)$.

We can assume A, C, h generic, so that $A(h)$ is a generic linear $\mathbb{P}^3$ in $\mathbb{P}^{20}$. The ACM property for C implies that the 14 points S are in a sufficiently general position, so that the stabilizer of S in $PGL(4) = \mathrm{Aut}(h)$ is finite, and the subgroup fixing S pointwise is trivial. This observation and Remark 3.3 imply that the stabilizer of $A(h)$ in $PGL(7)$ is finite. Hence the orbits of $PGL(7)$ have only finite intersections with U. Hence the map of U to the quotient by $PGL(7)$ is quasi-finite, as well as that to $H_{14,15}$, and its differential is injective at (a generic) A.

We have obtained a 21-dimensional family of global deformations of C preserving S. Now we want to show that any 1-parameter infinitesimal deformation of C can be lifted to a global 1-parameter one in the image of U. Indeed, the injectivity of the differential allows to lift the infinitesimal deformation to U. An element of U is a proportionality class of a 5×21 matrix, and U is an open subset in a linear $\mathbb{P}^{21}$ inside the projective space of the proportionality classes of 5×21 matrices, so any infinitesimal deformation in U is obviously lifted to a linear pencil.

Lemmas 4.1, 4.2 imply that the Abel–Jacobi map Φ has a 7-dimensional image in the 30-dimensional intermediate Jacobian $J^2(X)$ and 7-dimensional fibers. We can easily identify the irreducible components of the fiber. Indeed, by Proposition 3.4, each C is contained in a $\mathbb{P}^7 = \mathbb{P}(H^0(\mathcal{E})) \subset H^X_{14,15}$. Any rationally connected variety is contracted by the Abel–Jacobi map, so each one of its fibers is a union of these $\mathbb{P}^7$'s. As the dimension of the fiber is 7, the $\mathbb{P}^7$'s are irreducible components of the fiber. Being fibers of ϕ, the irreducible components do not meet each other, so they are in fact connected components. Thus we have proved the following theorem.

Theorem 4.4. *Let X be a generic quartic 3-fold. Let $\mathring{H}^X_{14,15} \subset H^X_{14,15}$ be defined as in Proposition 3.4, and $\Phi : \mathring{H}^X_{14,15} \longrightarrow J^2(X)$ the Abel–Jacobi map. Then the dimension of any component of $\Phi(\mathring{H}^X_{14,15})$ is equal to 7 and the fibers of Φ are the unions of finitely many disjoint*

7-dimensional projective spaces. The natural map $\psi : M_X \longrightarrow J^2(X)$, defined by $\Phi = \psi \circ \phi$, is quasi-finite and étale on M_X.

We get immediately the following obvious corollary:

Corollary 4.5. *Every component of M_X has non-negative Kodaira dimension.*

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