Continual Lie algebras and noncommutative counterparts of exactly solvable models

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Abstract

Noncommutative counterparts of exactly solvable models are introduced on the base of a generalization of Saveliev–Vershik continual Lie algebras. Examples of noncommutative Liouville and sin/sh–Gordon equations are given. The simplest soliton solution to the noncommutative sine–Gordon equation is found.

1 Introduction

Noncommutative field theories [8] attract attention due to their interesting internal structure and appearance in the noncommutative geometry, string theory, and other areas of the theoretical physics. The first step in an investigation of integrable noncommutative theories is to consider analogues of solvable models on noncommutative *two-dimensional* manifolds. So far, various approaches such as the gauged bi-complex [11, 12] or the $*$-product zero curvature approach [10, 13] (see also [14]) were applied in attempts to construct noncommutative integrable counterparts of classical integrable models. At the same time, it was shown in [5] that a general field theory defined on a noncommutative space-time is non-unitary. Nevertheless, in the two-dimensional case one can consider noncommutative euclidian models in order to avoid the non-unitarity.

The general idea is to introduce an associative $*$-product of functions depending on noncommutative coordinates. A natural candidate is the Moyal product [9]. Then one has to introduce a noncommutative counterpart of an integrable model system of equations and show that such model is integrable. A first naïve attempt is to substitute all multiplications of functions in a commutative integrable equations by a $*$-product. In the case of the Moyal deformation that way fails in general to preserve integrability. I.e., in some cases, such as the U(N) principal chiral model [10], the integrability survives, but in most cases not.

In this paper, we start with the definition of $*$-product Saveliev–Vershik continual Lie algebras, provide certain examples of $(S_*, K_{*\,0,\pm})$-mappings which determine such algebras, and then we propose a unifying approach to the construction of noncommutative counterparts of nonlinear exactly solvable models. Noncommutative models obtained in these frames appear to be generalizations of systems of integro-differential type equations constructed with the help of Saveliev–Vershik continual Lie algebras. Taking into account the powerful experience of the group-theoretical (representation theory) approach to integrable models in the commutative case, we conjecture that noncommutative $*$-product continual Lie algebra models are also exactly solvable subject to certain conditions. In particular cases of the Liouville and sin/h–Gordon models, we show how to obtain their noncommutative analogues using $*$-product continual Lie algebras. In contrast to other approaches, we managed to avoid extra constraints on functions due to an appropriate choice of $A_\pm$-pairs constructed with the help of continual Lie algebras generators. Aiming at the investigation of more general representation theory solutions to noncommutative Toda models we present a simplest soliton solution to the noncommutative sine–Gordon equation.

2 Saveliev-Vershik continual Lie algebras

The continual Lie algebras were introduced in [1] and then studied in [2, 3, 4]. Suppose E is a (noncommutative) associative algebra over the field $\mathbb{R}$ or $\mathbb{C}$, and K_0, $K_\pm : E \times E \to E$ are bilinear mappings. The local Lie part of a continual Lie algebra is defined as $\hat{g} = g_{-1} \oplus g_0 \oplus g_{+1}$, where g_i, $i = 0, \pm 1$ is isomorphic to E. The subspace g_i consists of the elements $g_i = \{X_i(\phi), \phi \in E\}, i = 0, \pm 1$. The generators $X_i(\phi)$ are subject to the commutation relations

$$[X_0(\phi), X_0(\psi)] = X_0([\phi, \psi]), \tag{2.1}$$

$$[X_0(\phi), X_{\pm 1}(\psi)] = X_{\pm 1}(K_\pm(\phi, \psi)), \tag{2.2}$$

$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(K_0(\phi, \psi)), \tag{2.3}$$

for all ϕ, $\psi \in E$. It is also assumed that the Jacobi identities are satisfied which leads to the conditions on mappings $K_{0,\pm}$

$$K_{\pm}([\phi,\psi],\chi) = K_{\pm}(\phi, K_{\pm}(\psi,\chi)) - K_{\pm}(\psi, K_{\pm}(\phi,\chi)), \tag{2.4}$$

$$[\psi, K_0(\phi,\chi)] = K_0(K_+(\psi,\phi),\chi)) + K_0(\phi, K_-(\psi,\chi)), \tag{2.5}$$

for all ϕ, ψ, $\chi \in E$. Then an infinite dimensional algebra $g(E;K) = g'(E;K)/J$ is called a continual contragredient Lie algebra where $g'(E;K)$ is a Lie algebra freely generated by $\hat{g}$, and J is the largest homogeneous ideal having trivial intersection with g_0 [2, 3].

2.1 $*$-product continual Lie algebras

Suppose a noncommutative space E is endowed with an associative (in general noncommutative) $*$-product. In this paper we are interested in continual Lie algebras whose mappings $K_{0,\pm}$ act on the product $E \times E$ in such a way that the functions of E are multiplied with the help of a chosen $*$-product. We denote such mappings by K_{*0} and $K_{*\pm}$. Introduce a generalization of Saveliev–Vershik continual Lie algebras which would fit our purposes. Let S_* be a mapping $S_* : E \times E \to E$. We define a *$*$-product continual Lie algebra* $g(E; S_*, K_*)$ with the help of the commutation relations

$$[X_0(\phi), X_0(\psi)] = X_0(S_*(\phi,\psi)), \tag{2.6}$$

$$[X_0(\phi), X_{\pm 1}(\psi)] = X_{\pm 1}(K_{*\pm}(\phi,\psi)), \tag{2.7}$$

$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(K_{*0}(\phi,\psi)), \tag{2.8}$$

for all ϕ, $\psi \in E$. The Jacobi identities imposed on the generators $X_{0,\pm 1}$ imply the conditions (which generalize (2.4-2.5)) on mappings $(S_*, K_{*\,0,\pm})$

$$K_{*\,\pm}(S_*(\phi,\psi),\chi) = K_{*\,\pm}(\phi, K_{*\,\pm}(\psi,\chi)) - K_{*\,\pm}(\psi, K_{*\,\pm}(\phi,\chi)), \tag{2.9}$$

$$S_*(\psi, K_{*\,0}(\phi,\chi)) = K_{*\,0}(K_{*\,+}(\psi,\phi),\chi)) + K_{*\,0}(\phi, K_{*\,-}(\psi,\chi)), \tag{2.10}$$

$$S_*(\phi, S_*(\psi,\chi)) + S_*(\psi, S_*(\chi,\phi)) + S_*(\chi, S_*(\phi,\psi)) = 0, \tag{2.11}$$

for all ϕ, ψ, $\chi \in E$. The anti-symmetry condition in (2.6) implies for any ϕ, $\psi \in E$ that

$$S_*(\phi,\psi) = -S_*(\psi,\phi). \tag{2.12}$$

Algebras $g(E;S_*,K_*)$ as well as algebras $g(E;K)$ [2] are $\mathbb{Z}$-graded, i.e., $g = g(E;S_*,K_*)$ can be decomposed into a direct sum $g = \bigoplus_{n\in\mathbb{Z}} g_n$ of subspaces subject to the grading condition $[g_n, g_m] \subset g_{n+m}$. A nontrivial task is to find examples of such mappings $(S_*,\ K_{*0,\pm})$ that would satisfy conditions (2.9-2.12). Though some examples (with $S_*(\phi,\psi) = [\phi,\psi]$) that involve noncommutative product have been found in [2, 3], we believe that the set of such mappings is much more broader. Let us present new examples of mappings (S_*,K_*) that generalize linear action mappings used in [2, 3]. We will take advantages of these $*$-product continual Lie algebra examples in the noncommutative counterparts of classical exactly solvable models construction.

I. The simplest nontrivial example of mappings (S_*,K_*) is

$$S_*(\phi,\psi) = [\phi,\psi]_*\,, \tag{2.13}$$

$$K_{*\,0}(\phi,\psi) = \phi * \psi, \tag{2.14}$$

$$K_{*\,+}(\phi,\psi) = \phi * \psi, \tag{2.15}$$

$$K_{*\,-}(\phi,\psi) = -\psi * \phi, \tag{2.16}$$

where $[\phi,\psi]_* = \phi * \psi - \psi * \phi$. Another set of mappings (S_*,K_*) can be obtained interchanging $K_{*\,+}$ with $K_{*\,-}$. The mappings in the examples II–V are deformations of the example I mappings.

IIa.

$$S_*(\phi,\psi) = K\phi * \psi - K\psi * \phi, \tag{2.17}$$

$$K_{*\,0}(\phi,\psi) = K\phi * \psi, \tag{2.18}$$

$$K_{*\,+}(\phi,\psi) = K\phi * \psi, \tag{2.19}$$

$$K_{*\,-}(\phi,\psi) = -K\psi * \phi, \tag{2.20}$$

$$K^2 = K, \tag{2.21}$$

i.e., K is a projector. In the examples II–V we assume that all K-mappings act on a $*$-product of functions as $K(\phi * \psi) = K\phi * K\psi$. It is easy to notice that the example IIa can be obtained from the example I by the transform $M_1(K) = m_* \circ K_1 \circ m_*^{-1}$, where m_* is a mapping $m_* : E \otimes E \to E * E$, m_*^{-1} is its inverse, and K_1 is the standard notation for $K \otimes 1$. Thus, $(S_*^{IIa}, K_{*\,0,\pm}^{IIa}) = M_1(K) \circ (S_*^I, K_{*\,0,\pm}^I)$.

IIb.

$$S_*(\phi,\psi) = \phi * K\psi - \psi * K\phi, \tag{2.22}$$

$$K_{*\,0}(\phi,\psi) = \phi * K\psi, \tag{2.23}$$

$$K_{*\,+}(\phi,\psi) = \phi * K\psi, \tag{2.24}$$

$$K_{*\,-}(\phi,\psi) = -\psi * K\phi, \tag{2.25}$$

$$K^2 = K. \tag{2.26}$$

The example IIb is connected with the example I though the transform $M_2(K) = m_* \circ K_2 \circ m_*^{-1}$, where $K_2 = 1 \otimes K$.

IIIa.

$$S_*(\phi,\psi) = [\phi,\psi]_*\,, \tag{2.27}$$

$$K_{*\,0}(\phi,\psi) = \pm K\phi * \psi, \tag{2.28}$$

$$K_{*\,+}(\phi,\psi) = K\phi * \psi, \tag{2.29}$$

$$K_{*\,-}(\phi,\psi) = -\psi * \phi, \tag{2.30}$$

$$K^2 = 1, \tag{2.31}$$

and $(K_{*\,0}^{IIIa}, K_{*\,+}^{IIIa}) = M_1(K) \circ (\pm K_{*\,0}^{I}, K_{*\,+}^{I})$, while the mappings S_* and $K_{*\,-}$ remain the same.

IIIb.

$$S_*(\phi,\psi) = [\phi,\psi]_*\,, \tag{2.32}$$

$$K_{*\,0}(\phi,\psi) = \pm \phi * K\psi, \tag{2.33}$$

$$K_{*\,+}(\phi,\psi) = \phi * \psi, \tag{2.34}$$

$$K_{*\,-}(\phi,\psi) = -\psi * K\phi, \tag{2.35}$$

$$K^2 = 1, \tag{2.36}$$

and $(K_{*\,0}^{IIIb}, K_{*}^{IIIb}) = M_2(K) \circ (\pm K_{*\,0}^{I}, K_{*\,-}^{I})$, while the mappings S_* and $K_{*\,+}$ remain the same.

IV.

$$S_*(\phi,\psi) = [\phi,\psi]_*\,, \tag{2.37}$$

$$K_{*\,0}(\phi,\psi) = K_0\phi * K_0\psi, \tag{2.38}$$

$$K_{*\,+}(\phi,\psi) = K_+\phi * \psi, \tag{2.39}$$

$$K_{*\,-}(\phi,\psi) = -\phi * K_-\psi, \tag{2.40}$$

$$K_0 \circ K_+ = K_0 \circ K_- = 1, \tag{2.41}$$

where $K_{0,\,\pm}$ are some mappings. This example is given by $K^{IV}_{*\,0} = K_0 \circ K^I_{*\,0}$, $K^{IV}_{*\,+,-} = M_{1,2}(K_{+,-}) \circ K^I_{*\,+,-}$,

V.

$$S_*(\phi,\psi) = [\phi,\psi]_*\,, \tag{2.42}$$

$$K_{*\,0}(\phi,\psi) = \phi * \psi, \quad \text{or} \tag{2.43}$$

$$K_{*\,0}(\phi,\psi) = \psi * \phi, \tag{2.44}$$

$$K_{*\,\pm}(\phi,\psi) = [\phi,\psi]_*\,, \tag{2.45}$$

i.e., $K^V_{*\,\pm} = K^I_{*\,+} + K^I_{*\,-}$, $K^V_{*\,0} = K^I_{*\,0}$, or $K^V_{*\,0} = m_* \circ P_{12} \circ m_*^{-1} \circ K^I_{*\,0}$, where $P_{12}(A \otimes B) = (B \otimes A)$. One can verify that the sets of mappings in the examples I–V satisfy conditions (2.9-2.11) following from the Jacobi identities as well as the anti-symmetry condition (2.12). These sets of mappings (with a linear form of a (S_*, K_*)-action on functions of E) represent a small and a simplest subset of $*$-product continual Lie algebras.

3 Noncommutative counterparts of exactly solvable models

In commutative two-dimensional case, an exactly integrable system may be introduced [6] through the zero curvature condition on holomorphic and anti-holomorphic elements of a flat connection in the trivial holomorphic principal fibre bundle over a manifold $\mathcal{M}$ endowed with commutative coordinates $z^{\pm}$. Suppose g is a (infinite dimensional) Lie algebra endowed with a $\mathbb{Z}$-grading. Then the general form of holomorphic and antiholomorphic elements $A_\pm$ of a connection is the following

$$A_\pm = e^0_\pm + \sum_{a=1}^{m_\pm} e^a_\pm, \tag{3.1}$$

where $g_{m_\pm}$ is the highest (lowest) subspace in the grading decomposition of g used in $A_\pm$, $e^0_\pm = \sum_{i=1}^{n_0} u_{\pm i}\ x^0_i$, $n^0 = dim\ g_0$; $e^a_\pm = \sum_{i=1}^{n^a_\pm} f^a_{\pm i}\ x^a_{\pm i}$, $n^a_\pm = dim\ g_{\pm a}$, $x^0_i \in g_0$, $x^a_{\pm i} \in g_{\pm a}$ are elements of the subspaces $g_{0,\pm a}$ of a chosen $\mathbb{Z}$-grading, and $u_{\pm i}(z^+, z^-)$, $f_{\pm i}(z^+, z^-)$ are *arbitrary* differentiable functions of $z^{\pm}$. The zero curvature condition $[\partial_+ + A_+, \partial_- + A_-] = 0$, leads to the system of equations

$$\partial_+ e^0_- - \partial_- e^0_+ + [e^0_+, e^0_-] + \sum_{a=1}^{min(m_+,m_-)} [e^a_+, e^a_-] = 0, \tag{3.2}$$

$$\sum_{a=1}^{m_\pm} (\partial_\mp e^a_\pm + [e^0_\mp, e^a_\pm]) - \sum_{1\leq b<a\leq m_\pm} [e^a_\mp, e^b_\pm] = 0. \tag{3.3}$$

It turns out that the system (3.2-3.3) is exactly integrable. The exact integrability of a model means that it is possible to find solutions depending on a sufficient number of arbitrary functions of $z^{\pm}$ in the form of a series which is finite in the case of a finite dimensional Lie algebra, and infinite (though absolutely convergent) in a finite growth infinite dimensional Lie algebra g case [6]. Exactly solvable models defined on multidimensional manifolds were introduced in [7].

On the base of $*$-product continual Lie algebras defined in the previous section, it is possible to construct noncommutative counterparts of exactly integrable models considered in [1, 2, 3]. Suppose E is an associative noncommutative algebra of functions depending on noncommutative variables $z^{\pm} = (z^{\pm}_\mu)$, which satisfy the commutation relations $\left[z^+_\mu, z^-_\nu\right] = i\theta^{\mu\nu}$, where $\mu, \nu = 1, ..., N$, and $\theta^{\mu\nu}$ are real constants. In this paper we discuss models defined on a noncommutative *two-dimensional* space $\mathcal{M}$. Further generalizations to noncommutative multidimensional case will be presented in a separate paper. Introduce an associative and, in general, noncommutative $*$-product on E. Consider the elements of the form

$$A_\pm = X_0(u_\pm) + \sum_{a=\pm 1}^{m_\pm} \sum_{i=1}^{n^a_\pm} X^a_{\pm i}(f^a_{\pm i}), \tag{3.4}$$

where $X_0 \in g_0$, $X^a_{\pm i} \in g_{\pm a}$, $i = 1, ..., n^a_\pm = dim\ g_{\pm a}$, and $u_{\pm_i}(z^+, z^-)$, $f^a_{\pm_i}(z^+, z^-)$ are arbitrary differentiable functions of noncommutative coordinates $z^{\pm}$. Let us also substitute the usual bracket in the zero curvature condition with the $*$-product deformed bracket $[A, B]_* = A * B - B * A$. Then the $*$-product zero curvature condition

$$[\partial_+ + A_-, \partial_- + A_+]_* = 0, \tag{3.5}$$

applied to the elements (3.4) generates the system of nonlinear equations

$$\partial_+ X_0(u_-) - \partial_- X_0(u_+) + X_0(S_*(u_+,u_-)) + \sum_{a=1}^{min(m_+,m_-)} \left[e_+^a, e_-^a\right]_* = 0, \quad (3.6)$$

$$\sum_{a=1}^{m_\pm} \left(\partial_\mp e_\pm^a + \left[X_0(u_\mp), e_\pm^a\right]_*\right) - \sum_{1\le b<a\le m_\pm} \left[e_\mp^a, e_\pm^b\right]_* = 0, \quad (3.7)$$

where $e_\pm^a = \sum_{i=1}^{n_\pm^a} X_{\pm i}^a(f_{\pm i}^a)$. The system (3.6-3.7) generalizes (3.2-3.3) as well as the system of equations introduced in [2] for the case of Saveliev–Vershik continual Lie algebras. Note that (3.6-3.7) is much more nontrivial with respect to (3.2-3.3) since it allows to generate noncommutative nonlinear systems of *integro-differential* equations. As it was shown in [10], an associative $*$-product deformed zero curvature condition (3.5) considered as a compatibility condition for a pair of $*$-product deformed linear differential equations implies the existence of an infinite number of conserved charges (the invertibility of certain operators was assumed). In this paper we consider the simplest case of the elements (3.4) when $m_- = m_+ = 1$. The higher grading case, i.e., when $m_\pm > 1$, will be described elsewhere. Further in the paper a general associative noncommutative $*$-product is assumed unless we specify it precisely. One can take for instance the Moyal product [9]

$$f(z^+,z^-) * g(z^+,z^-) = m \circ e^P \left[f(z^+,z^-) \otimes g(\xi^+,\xi^-)\right]\Big|_{z^\pm=\xi^\pm}, \quad (3.8)$$

where $P = 1/2\ \theta^{\mu\nu}\ \partial_{+\mu}^{(z)} \otimes \partial_{-\nu}^{(\xi)},$, $\mu,\ \nu = 1,...,N$, and m is the mapping $m : f\otimes g \to f\cdot g$, where the dot denotes the usual multiplication of functions. Finally, we formulate the following conjecture which will be proven in a separate paper [17].

Conjecture. *The system of equations (3.6–3.7) on functions of noncommutative variables $z^\pm$ associated with the $*$-(Moyal) product continual Lie algebra of the constant growth ($g_n \simeq g_1 \simeq E$) is exactly solvable. Its solutions depend on a set of functions of noncommutative variables subject to certain conditions.*

4 Examples

Exactly integrable models arising from Saveliev–Vershik continual Lie algebras were considered in [2, 3]. In particular, a class of equations constructed on the base of continual Lie algebras with a linear form of $K_{0,\pm}$-mappings action (for example, $K_{\pm}(\phi,\psi) = (K_{\pm}\phi)\cdot\psi$, $K_0(\phi,\psi) = K_0(\phi\cdot\psi)$) was discussed. We believe that the usage of $*$-product continual Lie algebras defined with the help of mappings (S_*, K_*) opens a way to more sophisticated exactly solvable models. In this section we give examples of noncommutative models that correspond to continual Lie algebras with $*$-deformed mappings.

A. *Noncommutative Liouville equation.*

Consider a $*$-product continual Lie algebra defined by (2.6-2.8) with the mappings (2.13-2.16) (example I). Let the elements $A_\pm$ be

$$A_+ = X_0(u) + X_+(1), \tag{4.1}$$

$$A_- = X_-(f), \tag{4.2}$$

where $u(z^+, z^-)$ and $f(z^+, z^-)$ are arbitrary functions depending on noncommutative coordinates $z^\pm$. The $*$-product deformed zero curvature condition (3.5) applied to (4.1-4.2) gives

$$-\partial_-((f)^{-1}_{*\,L} * \partial_+ f) + f = 0. \tag{4.3}$$

We assume that there exists the left inverse function f_*^{-1} with respect to the $*$-product, i.e., $(f)^{-1}_{*L} * f = 1$. Now take $f = e_*^{\beta\phi}$, β is a constant (we assume also that the $*$-multiplication by a constant coincides with the usual multiplication) with the $*$-exponential defined by $e_*^x = \sum_{n=0}^{\infty}\frac{1}{n!}x_*^n$, where $x_*^n = x * ... * x$ (n-times $*$-product). Then we arrive at the noncommutative Liouville equation

$$\partial_-\left(\left(e_*^{\beta\phi}\right)^{-1}_{*\,L} * \partial_+ e_*^{\beta\phi}\right) = e_*^{\beta\phi}. \tag{4.4}$$

In the commutative limit, when the $*$-product is substituted with the usual product, we get the Liouville equation.

B. *Noncommutative K-Liouville equation.*

Consider a $*$-product continual Lie algebra with the mappings (2.37-2.41) (example IV). The same $A_\pm$-pair (4.1-4.2) subject to the $*$-product curvature condition (3.5), leads to the model which we call the noncommutative K-Liouville equation

$$\partial_- \left(\partial_+ e_*^{\beta\phi} * \left(K_- e_*^{\beta\phi} \right)_{*\,R}^{-1} \right) = K_0 e_*^{\beta\phi}, \tag{4.5}$$

where the subscript R denotes the right inverse function with respect to the $*$-product.

C. *Noncommutative sin/h–Gordon equation.*

Consider the pair of operators

$$A_+ = X_0(u) + X_+(1/2) + X_-(1/2), \tag{4.6}$$

$$A_- = X_-(f) + X_+(f_{*\,L}^{-1}), \tag{4.7}$$

where the generators $X_{0,\pm}(\phi)$ belong to a continual Lie algebra of the example I. Then the $*$-deformed zero curvature condition (3.5) applied to (4.6-4.7) leads to the equations

$$-\partial_- \left(\partial_+ \left(e_*^{i\beta\phi} \right)_{*\,L}^{-1} * e_*^{i\beta\phi} \right) = \frac{1}{2} \left(e_*^{i\beta\phi} - \left(e_*^{i\beta\phi} \right)_{*\,L}^{-1} \right), \tag{4.8}$$

$$\partial_- \left(\left(e_*^{i\beta\phi} \right)_{*\,L}^{-1} * \partial_+ e_*^{i\beta\phi} \right) = \frac{1}{2} \left(e_*^{i\beta\phi} - \left(e_*^{i\beta\phi} \right)_{*\,L}^{-1} \right), \tag{4.9}$$

where we have taken $f = e_*^{i\beta\phi}$. We call (4.8-4.9) the noncommutative counterpart of the sin/h–Gordon model. In a similar way, one can obtain a noncommutative Bullough-Dodd equation. Note that if the derivatives $\partial_\pm$ are with respect to the chosen $*$-product (i.e., the Leibnitz rule works) then $\partial_\pm \left(\left(e_*^{i\beta\phi} \right)_{*\,L}^{-1} * e_*^{i\beta\phi} \right) = 0$, and the equations (4.8) and (4.9) are in fact equivalent. In the commutative limit, (4.8–4.9) collapse to one sin/h–Gordon equation.

Let a $*$-product in (4.8-4.9) be the Moyal product (3.8). Then (4.8) and (4.9) are equivalent to

$$\partial_- \left(e_*^{-i\beta\phi} * \partial_+ e_*^{i\beta\phi} \right) = \frac{1}{2} \left(e_*^{i\beta\phi} - e_*^{-i\beta\phi} \right). \tag{4.10}$$

It is easy to prove that the noncommutative sine–Gordon equation (4.10) possesses a soliton solution. Under a soliton solution to a noncommutative equation we mean a function of noncommutative coordinates $z^{\pm}$ which has a classical commutative soliton solution as the limit when $\theta = 0$. The soliton solution to (4.10) has the form

$$e_*^{-i\beta\phi(z^+,z^-)} = (1 + e_*^{\gamma}) * (1 - e_*^{\gamma})_*^{-1}\,, \tag{4.11}$$

where $\gamma = z^- - z^+$. Here $(1 - e_*^{\gamma})_*^{-1}$ denotes the left inverse to the function $1 + e_*^{\gamma}$ with respect to the Moyal product. As in the commutative limit, the noncommutative sine–Gordon equation possesses also a more general soliton solution $e_*^{-i\beta\phi(z^+,z^-)} = (1 + Qe_*^{\gamma}) * (1 - Qe_*^{\gamma})_*^{-1}$, where $\gamma = \omega_+ z^- - \omega_- z^+$, Q, $\omega_{\pm}$ are some constants, and $1/2$ in (4.10) is rescaled to $Q/2\omega_+\omega_-$.

The proof is rather direct. Note that due to the properties of the Moyal product and the linear dependence of γ on $z^{\pm}$, the noncommutative exponential e_*^{γ} coincides with the ordinary exponential e^{γ}. Also, the differentiations $\partial_{\pm}$ are with respect to the Moyal product and, in contrast to the case of $e_*^{\phi(z^+,z^-)}$, one can make use of usual formulae for when differentiating e^{γ} with respect to $z^{\pm}$. In order to prove (4.11) one has to substitute it into (4.10), and take into account the properties of e^{γ}. The solution (4.11) coincides with the classical soliton solution in the commutative limit. It can also be proven that a classical *multisoliton* solution to the sine–Gordon equation written in a similar form as (4.11) does satisfy the noncommutative sine–Gordon equation (4.10).

In the commutative limit, a multisoliton solution to the sine–Gordon equation can be obtained from the general solution though the vertex operator construction for the $\widehat{sl_2}$ algebra [16] (see also [15]). The construction of more general solutions and a proof of the exact integrability of the Liouville and sin/h–Gordon (and, in general, of conformal affine Toda models) noncommutative counterparts require a representation theory development for corresponding $*$-product continual Lie algebras. That point as well as a noncommutative vertex operator construction will be discussed in a forthcoming paper [17].

Remark: noncommutative sl_2 and bi-complex sin/h–Gordon systems. In [10] some other version of the Moyal product noncommutative sin/h–Gordon model was constructed. The main equation and two additional constraints have the form

$$\partial_+\partial_-\varphi + m^2/\beta\ \sinh_*(\beta\varphi) = 0, \tag{4.12}$$

$$\partial_-(e_*^{-\beta\varphi}) + \beta/2\,(e_*^{-\beta\varphi} * \partial_-\varphi + \partial_-\varphi * e_*^{-\beta\varphi}) = 0, \tag{4.13}$$

$$\partial_-(e_*^{\beta\varphi}) - \beta/2\,(e_*^{\beta\varphi} * \partial_-\varphi + \partial_-\varphi * e_*^{\beta\varphi}) = 0, \tag{4.14}$$

where m, β are constants, and $sin_*(\beta\varphi) = \frac{1}{2i}(e_*^{i\beta\varphi} - e_*^{-i\beta\varphi})$, $(sinh_*(\beta\varphi) = i\, sin_*(\beta\varphi)|_{\beta\to -i\beta})$. In [10] the system (4.13-4.14) was obtained though the $*$-product zero curvature condition (3.5) applied to the $A_\pm$-elements constructed on the base of the Pauli matrices $\sigma_{\pm,3}$, $\sigma_\pm = 1/2\,(\sigma_x \pm i\sigma_-)$ (i.e., a spin 1/2 matrix representation of the Lie algebra sl_2, which is a discrete limit of a continual Lie algebra), $A_+ = -m\lambda/2\left(e_*^{\beta\varphi}\sigma_- + e_*^{-\beta\varphi}\sigma_+\right)$, $A_- = m\lambda/2\,(\sigma_- + \sigma_+) - \beta/2\,\partial_-\varphi\,\sigma_3$. Using an expansion in the noncommutativity parameter θ, it was shown in [10] that a one-soliton solution to the classical sine–Gordon equation satisfies the equations (4.13-4.14).

In [12] another noncommutative version of the Moyal product sine–Gordon equation was obtained though the bi-complex approach [11] to (noncommutative) integrable models. The main equation and the constraint are

$$\partial_-(e_*^{-i\varphi} * \partial_+ e_*^{i\varphi} - e_*^{i\varphi} * \partial_+ e_*^{-i\varphi}) = i\gamma \sin_* \varphi, \tag{4.15}$$

$$\partial_-(e_*^{i\varphi} * \partial_+ e_*^{-i\varphi} + e_*^{-i\varphi} * \partial_+ e_*^{i\varphi}) = 0, \tag{4.16}$$

where γ is a constant.

As it was mentioned in [10] that various noncommutative versions of an integrable model come from an ambiguity in a noncommutative counterpart of derivatives containing expressions. We argue that all such versions can be obtained in the frames of our unifying $*$-product continual Lie algebra approach. Note that a *matrix representation* for the sl_2 Lie algebra generators was used in order to construct the $A_\pm$-pair for the equations (4.12-4.14). The extra constraints come from the off-diagonal terms due to the choice of an *arbitrary* function be equal to $\partial_-\varphi$. In contrast to that, in the example C we chose a suitable set of a continual Lie algebra mappings (S_*, K_*) in order to avoid any extra constraints. At the same time, it is possible to obtain the equations of noncommutative sin/h–Gordon systems constructed though the sl_2-matrix representation and the bi-complex approach by means of $A_\pm$ elements containing generators of certain degenerate $*$-product continual Lie algebras.

5 Conclusions

We have introduce $*$-product continual Lie algebras which are generalizations of Saveliev–Vershik continual Lie algebras. Some examples of (S_*, K_*)-

mappings that determine $*$-product continual Lie algebras are found. Making use of advantages of such algebras properties we have formulated a new unifying approach to the construction of noncommutative counterparts of exactly solvable models. Then we conjecture the exact solvability of such counterparts. The examples of noncommutative Liouville and sine/sh–Gordon equations are presented, and the simplest soliton solution to the noncommutative sine–Gordon equation is found. There is no doubt that there exist more complicated (of non-linear type of mappings action) examples of $*$-product continual Lie algebras. Such examples should lead to noncommutative models of integro–differential type. Among the first aims is a development of a representation theory based procedure of solving nonlinear systems of equation introduced in this paper. Some progress has already been made [17] in the direction to noncommutative soliton vertex operators which, by analogy with the commutative case, should generate soliton solutions to noncommutative affine Toda models. On the other hand, it is possible to incorporate known examples of noncommutative analogues of completely integrable models [11, 13, 14] to the constructions discussed in the paper. Finally, we mention a possibility for super-generalizations of continual Lie algebras, and construction of corresponding solvable models.

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References

[1] M. V. Saveliev. Integro–differential non–linear equations and continual Lie algebras. Commun. Math. Phys. 121, 283–290, 1989;

[2] M. V. Saveliev, A. M. Vershik. Continuum analogues of contragredient Lie algebras. Commun. Math. Phys. 126, 367, 1989;

[3] M. V. Saveliev, A. M. Vershik. New examples of continuum graded Lie algebras. Phys. Lett. A, 143, 121, 1990;

[4] A. M. Vershik,; B. B. Shoikhet. Graded Lie algebras whose Cartan subalgebra is the algebra of polynomials of one variable. (Russian) Teoret. Mat. Fiz. 123 (2000), no. 2, 345–352; translation in Theoret. and Math. Phys. 123, no. 2, 701–707, 2000;

[5] J. Gomis, T. Mehen. Space-Time Noncommutative Field Theories and Unitarity, Nucl. Phys. B591, 265-276, 2000, hep-th/0005129;

[6] A. N. Leznov, M. V. Saveliev, Group-Theoretical Methods for Integration of Non–Linear Dynamical Systems. Progress in Physics Series, v.15, Birkhauser–Verlag, Basel, 1992;

[7] A. V. Razumov, M. V. Saveliev. Multidimensional Toda type systems, Teor. Mat. Fiz. 112, 254-282, 1997; Theor. Math. Phys. 112, 999-1022, 1997, hep-th/9609031;

[8] N. Seiberg, E. Witten. String Theory and Noncommutative Geometry. JHEP 9909, 032, 1999, hep-th/9908142; M. R. Douglas, N. A. Nekrasov. Noncommutative Field Theory. Rev. Mod. Phys. 73, 977–1029, 2001, hep-th/0106048;

[9] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer. Deformation theory and quantization. I. Deformation of symplectic structures; II. Physical applications. Ann. Phys. 110–111, 61–110, 111–151 (1978);

[10] I. Cabrera-Carnero, M. Moriconi. Noncommutative Integrable Field Theories in 2d, hep-th/0211193;

[11] A. Dimakis, F. Müller-Hoissen. Noncommutative Korteweg-de-Vries Equation, hep-th/0007074; A Noncommutative Version of the Nonlinear Schrodinger Equation. hep-th/0007015; Bicomplexes and Integrable Models. J. Phys. A33, 6579-6592, 2000, nlin.SI/0006029;

[12] M. T. Grisaru, S. Penati. An integrable noncommutative version of the sine–Gordon system, hep-th/0112246;

[13] L.D. Paniak. Exact Noncommutative KP and KdV Multi-solitons. Preprint: PUPT-1982, hep-th/0105185;

[14] M. Hamanaka, K. Toda. Towards Noncommutative Integrable Systems. Preprint UT-02-54, hep-th/0211148; Noncommutative Burgers Equation. Preprint UT-02-56, hep-th/0301213;

[15] M. V. Saveliev, A. B. Zuevsky. Quantum vertex operators for the sine–Gordon model. *International Journal of Modern Physics A*, Vol. 15, No. 24, p. 3877–3897, 2000;

[16] D.I. Olive, N. Turok, J.W.R. Underwood. Affine Toda solitons and vertex operators. Nucl. Phys. B409, 509-546, 1993;

[17] A. Zuevsky. Noncommutative affine Toda models. In preparation.