ON ARTHUR'S Φ-FUNCTION

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Abstract. Write Θ^E for the stable character associated to an irreducible finite-dimensional representation E of a connected real reductive group G. Let M be the centralizer of a maximal torus T, and denote by $\Phi_M(\gamma, \Theta^E)$ Arthur's extension of $|D_M^G(\gamma)|^{\frac{1}{2}}\Theta^E(\gamma)$ to $T(\mathbb{R})$. In this paper we give a simple explicit expression for $\Phi_M(\gamma, \Theta^E)$, when γ is elliptic in G.

1. Introduction

Let G be a connected reductive group over $\mathbb{R}$, and T a maximal torus in G. Assume that G has a discrete series of representations. Let A be the split part of T, and M the centralizer of A in G. It is a Levi subgroup of G containing T. Let E be an irreducible finite-dimensional representation of $G(\mathbb{C})$, and consider the packet Π_E of discrete series representations π of $G(\mathbb{R})$ which have the same infinitesimal and central characters as E. Write Θ_π for the character of π, and put

$$\Theta^E = (-1)^{q(G)} \sum_{\pi \in \Pi_E} \Theta_\pi.$$

Here $q(G)$ is half the dimension of the symmetric space associated with G. Note that $\Theta^E(\gamma)$ will not extend to all elements $\gamma \in T(\mathbb{R})$, in particular to $\gamma = 1$. Define the number $D_M^G(\gamma)$ by

$$D_M^G(\gamma) = \det(1 - \mathrm{Ad}(\gamma); \mathrm{Lie}(G)/\mathrm{Lie}(M)).$$

Then a result of Arthur and Shelstad [1] states that the function

$$\gamma \mapsto |D_M^G(\gamma)|^{\frac{1}{2}}\Theta^E(\gamma),$$

defined on the set of regular elements $T_{\mathrm{reg}}(\mathbb{R})$ extends continuously to $T(\mathbb{R})$. We denote this extension by $\Phi_M(\gamma, \Theta^E)$. These quantities give the contribution from the real place to the L^2-Lefschetz numbers of Hecke operators in [1] and [2]. An expression for $\Phi_M(\gamma, \Theta^E)$ as essentially a sum over elements in the Weyl group W of T in G appears in the proof of Lemma 4.1 in [2]. Although this expression suffices

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to prove the lemma, it can be considerably refined when γ is in the maximal elliptic subtorus $T_e(\mathbb{R})$ of $T(\mathbb{R})$.

The following theorem is proved in section 4.

Theorem 1. *If $\gamma \in T_e(\mathbb{R})$, then*

$$\Phi_M(\gamma, \Theta^E) = (-1)^{q(L)} \cdot |W_L| \cdot \sum_{\omega \in W^{LM}} \varepsilon(\omega) \cdot \mathrm{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}).$$

Here we write L for the centralizer of T_c in G, where T_c is the maximal compact subtorus of T. Also write W_L and W_M for the Weyl groups of T in L and M. The latter are subgroups of W which commute and have trivial intersection. Here W^{LM} is a certain set of representatives for the cosets $(W_L \times W_M)\backslash W$. It is defined explicitly in section 5. We write ε for the sign character of W. Finally by $V^M_{\omega(\lambda_B+\rho_B)-\rho_B}$ we denote the irreducible finite-dimensional representation of M, with highest weight $\omega(\lambda_B + \rho_B) - \rho_B$, where λ_B is the B-dominant highest weight of E.

In particular, we obtain the extremely simple expression,

$$\Phi_A(1, \Theta^E) = (-1)^{q(G)} \cdot |W|,$$

in the case of a split torus $T = A$.

We now describe the organization of this paper.

In section 2, we spell out the relationship between the root systems of G, L, and M. There are two distinct systems of chambers in $X_*(A)\otimes_{\mathbb{Z}}\mathbb{R}$ obtained from these root systems which are important to understand.

In section 3, we take the aforementioned lemma a step further to express $\Phi_M(\gamma, \Theta^E)$ explicitly as a linear combination of characters. (Actually we do the computation for any stable virtual character Θ, as it is no more difficult.) The sum over W simplifies to a sum over Kostant representatives W^M.

In section 4, in which we deal specifically with $\Phi_M(\gamma, \Theta^E)$, we distill out the action of W_L. A sum over W^{LM} remains. At a key step we use a result of section 5, the computation of an alternating sum of stable discrete series constants.

In section 5, we prove the result mentioned above, in the context of abstract root systems. It is independent from the rest of the paper.

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2. L-chambers and $\mathcal{P}$-chambers

Let G be a connected reductive group over $\mathbb{R}$, and T a maximal torus of G. Assume that G has a discrete series, or equivalently, that G has an elliptic maximal torus.

Write T_c, respectively A, for the maximal compact, resp. split, subtori of T with centralizers L, resp. M, in G. Write R for the root system of T in G, and R_L, resp. R_M, for the set of roots of T in L, resp. M. Then R_L is the subset of R consisting of real roots, and R_M is the subset of imaginary roots. Write W_L and W_M for the respective Weyl groups. They are commuting subgroups of W with trivial intersection. Note that W_L fixes each root in R_M.

A is contained as a split maximal torus in L_{der}, the derived group of L, and we may identify R_L with the set of roots of A in L_{der}.

Write $\mathfrak{a}_M$ for $X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. For any $\alpha \in R \backslash R_M$ the root hyperplane H_α of $X^*(T)_{\mathbb{R}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ gives a hyperplane in $\mathfrak{a}_M$. Let us consider two kinds of chambers in $\mathfrak{a}_M$ obtained from these. Define $\mathcal{P}$-chambers to be those obtained by deleting from $\mathfrak{a}_M$ all the hyperplanes H_α, with $\alpha \in R \backslash R_M$. Define L-chambers to be those obtained by deleting all the H_α with $\alpha \in R_L$. The latter are the Weyl chambers for A in L_{der}; therefore W_L acts simply transitively on them.

Observe that $R_L \subset (R \backslash R_M)$. Any additional hyperplanes coming from roots in $R \backslash (R_L \cup R_M)$ divide the L-chambers into $\mathcal{P}$-chambers. Thus every $\mathcal{P}$-chamber is contained in a unique L-chamber.

Write $\mathcal{P}(M)$ for the set of parabolic subgroups of G admitting M as a Levi component. There is a one-to-one correspondence between $\mathcal{P}(M)$ and the set of $\mathcal{P}$-chambers in $\mathfrak{a}_M$, obtained as follows: for $P = MN \in \mathcal{P}(M)$, the corresponding $\mathcal{P}$-chamber is

$$\mathfrak{a}_P^+ = \{x \in \mathfrak{a}_M : \langle \alpha, x \rangle > 0, \text{ for all } \alpha \in R_N\},$$

where R_N denotes the set of roots of T in $\mathrm{Lie}(N)$.

Recall that the set of L-chambers is in bijection with the set of Borel subgroups of L containing T, or equivalently the set of positive root systems R_L^+ in the root system R_L.

Now let C_P be a $\mathcal{P}$-chamber, and let $P = MN$ be the corresponding element of $\mathcal{P}(M)$. It is easy to see that $R_N \cap R_L$ is a positive system in R_L, and this corresponds to an L-chamber C_L. Thus we have defined a map $C_P \mapsto C_L$ from the set of $\mathcal{P}$-chambers to the set of L-chambers. It is the obvious one which associates to C_P the unique L-chamber containing C_P.

3. A Linear Combination of Characters

A stable virtual character is a finite $\mathbb{Z}$-linear combination Θ of characters Θ_π so that

$$\Theta(\gamma) = \Theta(\gamma')$$

whenever γ and γ' are regular, stably conjugate elements of $G(\mathbb{R})$.

In Lemma 4.1 of [2], it is proved that for a stable virtual character Θ on $G(\mathbb{R})$, the function

$$\gamma \mapsto |D_M^G(\gamma)|^{\frac{1}{2}}\Theta(\gamma)$$

on $T_{\mathrm{reg}}(\mathbb{R})$ extends continuously to $T(\mathbb{R})$. A key ingredient of the proof is the fact that the expression at the bottom of page 497 is a linear combination of irreducible finite-dimensional representations of M. In this section we will compute explicitly the coefficients and the representations involved, in the case where the element a appearing in the proof is equal to 1.

We translate the set-up of the proof in [2] as follows. We take Γ to be the identity component of $T(\mathbb{R})$. The root system R_Γ is then simply R_L. Fix a positive root system R_L^+ in R_L, and let C be the corresponding L-chamber in $\mathfrak{a}_M$. We then choose a parabolic subgroup $P = MN$ so that $R_L \cap R_N \subseteq R_L^+$. Note that $R_L \cap R_N$ is also a system of positive roots, so this condition is equivalent to having $R_L \cap R_N = R_L^+$. Thus we simply require that the $\mathcal{P}$-chamber corresponding to P be contained in C.

Although at the end of our computations we will allow γ to be non-regular, we choose now γ to be a regular element of $\Gamma = T_c(\mathbb{R}) \cdot \exp(\bar{C})$.

The expression is

$$\sum_B m(B) \frac{\Delta_P(\gamma) \cdot \lambda_B(\gamma)}{\Delta_B(\gamma)}. \tag{1}$$

The sum ranges over Borels containing T, which correspond to elements of W.

Here λ_B is the B-dominant highest weight of E,

$$\Delta_B = \prod_{\alpha > 0} (1 - \alpha^{-1}), \text{ and } \Delta_P = \prod_{\alpha \in R_N} (1 - \alpha^{-1}).$$

Fix now a Borel B of G with $T \subseteq B \subseteq P$, for the rest of this paper.

Recall the set of Kostant representatives W^M for the Weyl group W_M of M, relative to B. It is the set $\{w \in W | w^{-1} R_M^+ \subset R^+\}$.

If $w \in W$, write $w * B$ for wBw^{-1}.

We will use the observation that for $\omega \in W^M, (\omega * B)_M = B_M$. Indeed, if $\alpha \in R^+ \cap R_M$, then $\omega^{-1}\alpha \in R^+$, which implies that $\alpha \in \omega R^+ \cap R_M$.

Our sum (1) breaks up as

$$\sum_{\omega \in W^M} m(\omega * B) \cdot \Delta_P(\gamma) \cdot \sum_{w_M \in W_M} \frac{w_M(\omega \lambda_B)(\gamma)}{\Delta_{w_M \omega * B}(\gamma)}. \tag{2}$$

We would prefer the denominator inside the sum to be $\Delta_{w_M * B_M}(\gamma)$. (Recall that $B_M = B \cap M$.) Note that $\Delta_P \cdot \Delta_{B_M} = \Delta_B$, since R^+ is the disjoint union of R_M^+ and R_N.

So we consider the quantity

$$\frac{\Delta_P \cdot \Delta_{w_M * B_M}}{\Delta_{w_M \omega * B}} = \frac{\Delta_B \cdot \Delta_{w_M * B_M}}{\Delta_{B_M} \cdot \Delta_{w_M \omega * B}}. \tag{3}$$

Observe that if $\mathcal{B}$ is a Borel, $\Delta_{\mathcal{B}} = \delta_{\mathcal{B}} \cdot \rho_{\mathcal{B}}^{-1}$, where $\delta_{\mathcal{B}} = \prod_{\alpha > 0} (\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}})$ and $\rho_{\mathcal{B}}$ is the usual half sum of positive roots. Since $\delta_{w * \mathcal{B}} = \varepsilon(w) \delta_{\mathcal{B}}$, we compute that

$$\frac{\Delta_{w * \mathcal{B}}}{\Delta_{\mathcal{B}}} = \varepsilon(w) \cdot (\rho_{\mathcal{B}} - w \rho_{\mathcal{B}}).$$

Thus (3) becomes

$$\varepsilon(\omega)(w_M(\omega \rho_B - \rho_{B_M}) - \rho_B + \rho_{B_M}).$$

Next observe that for $w_M \in W_M$,

$$w_M(\rho_B - \rho_{B_M}) = \rho_B - \rho_{B_M}.$$

Indeed, the roots of R^+ not in R^+_M are in R_N, and are thus normalized by W_M. So the above expression simplifies to

$$\varepsilon(\omega) \cdot w_M(\omega\rho_B - \rho_B).$$

We can therefore rewrite (2) as

$$\sum_{\omega \in W^M} m(\omega * B) \cdot \varepsilon(\omega) \cdot \sum_{w_M \in W_M} \frac{w_M(\omega(\lambda_B + \rho_B) - \rho_B)(\gamma)}{\Delta_{w_M * B_M}(\gamma)}. \tag{4}$$

Since ω is a Kostant representative, the weight $\omega(\lambda_B + \rho_B) - \rho_B$ is positive for B_M, and we may use the Weyl character formula to rewrite this as

$$\sum_{\omega \in W^M} m(\omega * B) \cdot \varepsilon(\omega) \cdot \mathrm{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}). \tag{5}$$

Here $V^M_{\omega(\lambda_B+\rho_B)-\rho_B}$ denotes the irreducible finite-dimensional representation of M with highest weight $\omega(\lambda_B + \rho_B) - \rho_B$.

4. A Formula for $\Phi_M(\gamma, \Theta^E)$

To identify (5) with $\Phi_M(\gamma, \Theta^E)$, we replace $m(\omega * B)$ with $n(\gamma, \omega * B)$ as on page 500 of [2], and multiply it by the factor $\delta_P^{\frac{1}{2}}(\gamma)$:

$$\delta_P^{\frac{1}{2}}(\gamma) \cdot \sum_{\omega \in W^M} n(\gamma, \omega * B) \cdot \varepsilon(\omega) \cdot \mathrm{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B}). \tag{6}$$

Here δ_P is the modulus character of P. (We are still only considering regular γ.)

Write A_G for the split component of the center of G. Let $\lambda_0 \in X^*(A_G)$ denote the character by which A_G acts on E. It extends to $X^*(T)_{\mathbb{R}}$ in the usual way, and is W-invariant.

Let T_e denote the subtorus of T generated by T_c and A_G. It is the maximal subtorus of T which is elliptic in G.

Write p_M for the projection from $X^*(T)_{\mathbb{R}}$ to $X^*(A)_{\mathbb{R}}$, and note that it is W_L-invariant. The group W_L fixes each root of M, thus it acts on W^M. For every orbit of this action, there is a unique member ω so that $p_M(\omega(\lambda_B + \rho_B - \lambda_0))$ is dominant with respect to C. We denote the set of these elements by W^{LM}, one element for each orbit of W_L on W_M.

If $\lambda \in X^*(T)$ and $w_L \in W_L$, then plainly $w_L\lambda - \lambda \in \mathfrak{a}_M^*$. Write $(\chi_{w_L,\omega,B}, \mathbb{C}_{w_L,\omega,B})$ for the one-dimensional representation of M, acting through A, with weight $w_L\omega(\lambda_B+\rho_B) - \omega(\lambda_B+\rho_B)$. Note that T_c and A_G act trivially on $\mathbb{C}_{w_L,\omega,B}$, thus so does T_e.

Thus we have

$$V^M_{w_L\omega(\lambda_B+\rho_B)-\rho_B} \cong V^M_{\omega(\lambda_B+\rho_B)-\rho_B} \otimes \mathbb{C}_{w_L,\omega,B}.$$

Our formula (6) is now (replacing $\omega \in W^M$ with $w_L\omega$, where ω is now in W^{LM}):

(7)
$$\delta_P^{\frac{1}{2}}(\gamma)\cdot \sum_{\omega\in W^{LM}} \varepsilon(\omega)\cdot \mathrm{tr}(\gamma; V^M_{\omega(\lambda_B+\rho_B)-\rho_B})\cdot \sum_{w_L\in W_L} \varepsilon(w_L)\cdot \chi_{w_L,\omega,B}(\gamma)\cdot n(\gamma, w_L\omega * B),$$

Of course we now wish to simplify the inner sum. Recall from page 500 of [2] that

$$n(\gamma, w_L\omega * B) = \bar{c}(x, p_M(w_L\omega\lambda_B + w_L\omega\rho_B - \lambda_0)),$$

where x is in the interior of C. Here $\bar{c}(x,\lambda)$ is the integer-valued "stable discrete series constant" on

$$(X_*(A/A_G)_{\mathbb{R}})_{\mathrm{reg}} \times (X^*(A/A_G)_{\mathbb{R}})_{\mathrm{reg}},$$

as defined, for instance, on page 493 of [2]. Recall that $\lambda_0 \in X^*(T)_{\mathbb{R}}$ is obtained from the character $\lambda_0 \in X^*(A_G)$ by which A_G acts on E, and is thus W-invariant.

As p_M commutes with w_L, the inner sum of (7) is now

$$\sum_{w_L\in W_L} \varepsilon(w_L)\cdot \bar{c}(x, w_L\Lambda)\cdot \chi_{w_L,\omega,B}(\gamma), \tag{8}$$

where $\Lambda = p_M(\omega\lambda_B + \omega\rho_B - \lambda_0)$.

We would like to consider the limit of (8) as x approaches 0. Recall we can write $\gamma = \gamma_c \cdot \exp(x)$, with $\gamma_c \in T_c(\mathbb{R})$ and x in $\bar{C}$. Also recall that γ is still regular (not for long!). Consider the above formula with γ_c fixed and x going to 0 along regular elements of $\bar{C}$. Fix some element x_0 in the interior of C. The value

$$\bar{c}(x, w_L\Lambda) = \bar{c}(x_0, w_L\Lambda)$$

is unchanged, but $\chi_{w_L,\omega,B}(\gamma)$ approaches $\chi_{w_L,\omega,B}(\gamma_c)=1$.

Thus (8) converges to

$$\sum_{w_L\in W_L}\varepsilon(w_L)\cdot\bar{c}(x_0,w_L\Lambda)$$

for some $x_0\in C$.

But this is simply $(-1)^{q(L)}|W_L|$, by Proposition 1(ii) in Section 5 below. Here we use that $\omega\in W^{LM}$. Note that -1 is in the Weyl group of the root system by the argument on page 499 of [2].

It is easy to modify this argument to get the same limit as x approaches an element of $X_*(A_G)_{\mathbb{R}}$.

Finally note that δ_P is a positive character and therefore trivial on the compact group $T_c(\mathbb{R})$. It is thus trivial on $T_e(\mathbb{R})$.

Now consider irregular γ. We take the limit in (7) and obtain our theorem:

Theorem 1. *If $\gamma\in T_e(\mathbb{R})$, then*

$$\Phi_M(\gamma,\Theta^E)=(-1)^{q(L)}\cdot|W_L|\cdot\sum_{w\in W^{LM}}\varepsilon(w)\cdot\operatorname{tr}(\gamma;V^M_{w(\lambda_B+\rho_B)-\rho_B}).\tag{9}$$

For the reader's convenience, we review the definition of W^{LM}.

The definition depends on the choice of a parabolic $P=MN$ and a Borel subgroup B with $T\subseteq B\subseteq P$. The choice of B gives a set of positive roots R^+ for R and a set of positive roots R^+_M for R_M. It also gives B-dominant elements λ_B and ρ_B of $X^*(T)_{\mathbb{R}}$. The choice of P determines an L-chamber C as in Section 2. Recall the character λ_0 determined by A_G on E and the projection p_M from $X^*(T)_{\mathbb{R}}$ to $X^*(A)_{\mathbb{R}}$. Then

$$W^{LM}=\{w\in W|w^{-1}R^+_M\subseteq R^+\text{ and }p_M(w(\lambda_B+\rho_B-\lambda_0))\text{ is dominant w.r.t. }C\}.$$

We now evaluate (9) for Φ_M on the extreme cases for T. If $T=A$ is split, then $M=A$, $L=G$, W^{LM} is trivial, but so is T_c. We conclude that for $z\in A_G(\mathbb{R})$,

$$\Phi_A(z,\Theta^E)=(-1)^{q(G)}\cdot|W|\cdot\lambda_0(z).$$

If T is elliptic, then $M=G$, $L=T$, W^{LM} is again trivial, and so for $\gamma\in T$,

$$\Phi_G(\gamma,\Theta^E)=\operatorname{tr}(\gamma;E).$$

Note that this agrees with the results of Theorems 5.1 and 5.2 of [2], since

$$\mathrm{tr}(\gamma^{-1}; E^*) = \mathrm{tr}(\gamma; E).$$

5. The Sum of the Stable Discrete Series Constants

Let $(X, X^*, R, \check{R})$ be a root system. Write W for the Weyl group of the root system, and ε for its sign character. Assume that R generates the real vector space X and that $-1 \in W$. Write $q(R)$ for $(|R^+| + \dim(X))/2$, as in [2]. Let x_0 be a regular element of X, and λ a regular element of X^*. Write C_0 for the chamber of X containing x_0, and $C_0^\vee$ for its dual chamber in X^*. Recall the stable discrete series constants $\bar{c}_R(x_0, \lambda)$ from section 3 of [2].

Proposition 1. *We have the following formulas for sums of discrete series constants:*

(i) *For all such λ, $\sum_{w \in W} \bar{c}_R(wx_0, \lambda) = |W|$.*
(ii) *For $\lambda = \lambda_0 \in C_0^\vee$, we have $\sum_{w \in W} \varepsilon(w) \cdot \bar{c}_R(wx_0, \lambda_0) = (-1)^{q(R)}|W|$.*

The same formulas hold if we sum over the W-orbit of λ rather than that of x_0.

We make a few comments before beginning the proof. The proof begins by using the "inductive" property (4) of the discrete series constants from page 493 of [2], to change the sum over chambers into a sum over certain facets of X. In fact we consider those facets which separate the chambers of X, i.e., those which span the root hyperplanes Y of X.

In the course of the proof, we (mis)use the term "facet" only in reference to these particular facets, of codimension 1. So a facet in this sense will be the common face of two adjacent chambers.

The hyperplanes Y have their own chambers, and we examine the relationship between the facets and these smaller chambers. Not every facet is equal to such a chamber, as in the case of B_3 when Y is the root hyperplane of a long root. The facets in Y give a B_2 system, but the chambers of R_Y give an $A_1 \times A_1$ system.

Finally induction on the rank of the root system gives the calculation.

Proof. The second formula follows from the first by applying Theorem 3.2(2) on page 494 of [2].

We induce on $r = \dim X$. The proposition is clear when $r = 0$.

We associate these discrete series constants with the various chambers and facets of X, and introduce some appropriate notation.

Write $c(\mathcal{C})$ for $\bar{c}_R(x,\lambda)$, when x is in the interior of a chamber $\mathcal{C}$.

Suppose F is a facet in X, y is in the interior of F, and $\bar{F} := \operatorname{span}(F) = Y$. Then write $c(F) = \bar{c}_{R_Y}(y, \lambda_Y)$, notation as on page 493 of [2].

Thus if F is the common face of distinct chambers $\mathcal{C}$ and $\mathcal{C}'$, then

$$2c(F) = c(\mathcal{C}) + c(\mathcal{C}').$$

Each chamber has r faces, and it follows that

$$r \cdot \sum_{\mathcal{C}} c(\mathcal{C}) = 2 \sum_F c(F), \tag{10}$$

where we are summing over all chambers and then all facets.

We show the right hand side of (10) is equal to $r \cdot |W|$ to prove the proposition.

Now every facet is on some root hyperplane $X_\alpha = X_{-\alpha}$, so we have

$$2 \sum_F c(F) = \sum_{\alpha \in R} \sum_{\bar{F} = X_\alpha} c(F).$$

We now work with the inner sum. There is a root system on X_α whose set of coroots is $\check{R} \cap X_\alpha$, which defines chambers $\mathcal{C}_\alpha$ in X_α and constants $c_\alpha(\mathcal{C}_\alpha)$. Write W_α for the Weyl group of X_α. We have

$$\sum_{\bar{F} = X_\alpha} c(F) = \sum_{\mathcal{C}_\alpha} \sum_{F \subset \mathcal{C}_\alpha} c_\alpha(\mathcal{C}_\alpha) = \sum_{\mathcal{C}_\alpha} \sum_{W_\alpha \backslash \{F \subset X_\alpha\}} c_\alpha(\mathcal{C}_\alpha) = \sum_{W_\alpha \backslash \{F \subset X_\alpha\}} \sum_{\mathcal{C}_\alpha} c_\alpha(\mathcal{C}_\alpha).$$

For the first equality, note that every facet F with $\bar{F} = X_\alpha$ is contained in a some chamber $\mathcal{C}_\alpha$.

The second equality follows because W_α acts transitively on the chambers C_α.

Write $n(\alpha)$ for the order of $W_\alpha \backslash \{F \subset X_\alpha\}$. It is equal to the number of facets in a given chamber $\mathcal{C}_\alpha$. Then by induction the above is merely

$$n(\alpha) \cdot |W_\alpha|,$$

which is exactly the number of facets in X_α. It follows that (10) is simply equal to twice the total number of facets in X.

Since W has r orbits on the set of facets in X, and the stabilizer in W of any facet has order 2, we conclude that the total number of facets is half of $r \cdot |W|$, as desired. □

References

[1] J. Arthur, The L^2-Lefschetz numbers of Hecke operators, Invent. Math. **97** (1989), 257-290.

[2] M. Goresky, R. Kottwitz, and R. Macpherson, Discrete Series Characters and the Lefschetz Formula for Hecke operators, Duke Math. Journal, Vol. **89**, No. 3 (1997),477-554.

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