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On Some Properties of Graph Maps: Spectral Decomposition, Misiurewicz Conjecture and Abstract Sets of Periods

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Abstract. We completely characterize sets of periods of cycles which arbitrary continuous graph maps may have. We also verify the conjecture of M.Misiurewicz and prove that for any graph X there exists a number L=L(X) such that any continuous self-mapping of X with cycles of periods 1,2,...,L has in fact cycles of all possible periods. In this studying we need the spectral decomposition for graph maps [B3] which we describe briefly in Section 1.

0. Introduction

Let us call one-dimensional branched manifolds *graphs*. We study properties of a set $P(f)$ of periods of cycles of a graph map f. One of the well-known and impressive results on this topic is Sharkovskii theorem [S1] about the co-existence of periods of cycles for maps of the real line. To formulate it let us introduce the following *Sharkovskii ordering* for positive integers:

$$(*) \qquad 3 \prec 5 \prec 7 \prec \dots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \dots \prec 8 \prec 4 \prec 2 \prec 1$$

Denote by $S(k)$ the set of all such integers m that $k \prec m$ or $k = m$ and by $S(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$.

Theorem[S1]. *Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous map. Then either $P(g) = \emptyset$ or there exists such $k \in \mathbf{N} \cup 2^\infty$ that $P(g) = S(k)$. Moreover for any such k there exists a map $g : [0,1] \to [0,1]$ with $P(g) = S(k)$ and there exists a map $g_0 : \mathbf{R} \to \mathbf{R}$ with $P(g_0) = \emptyset$.*

Other information about sets of periods of cycles for one-dimensional maps is contained in papers [AL,M] for maps of the circle, [ALM] for maps of the letter Y and [Ba] for maps of the n-od.

Sharkovskii theorem implies that if a map $f : \mathbf{R} :\to \mathbf{R}$ has a cycle of period 3 then it has cycles of all possible periods. The following conjecture, which was formulated by M.Misiurewicz at the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990, seems to be closely related to the mentioned property of maps of the real line.

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Misiurewicz Conjecture. *For a graph X there exists an integer $L = L(X)$ such that for a continuous map $f : X \to X$ inclusion $P(f) \supset \{1, 2, \dots, n\}$ implies $P(f) = \mathbb{N}$.*

We verify Misiurewicz conjecture in Section 2. Clearly it implies that sets of periods of cycles of graph maps have some general properties no matter what graph is considered. Moving in this direction we describe in Section 3 sets $A \subset \mathbb{N}$, for which there exists a graph Y and a continuous map $g : Y \to Y$ with $P(g) = A$. Namely, a set $A \subset \mathbb{N}$ is called *an abstract set of periods* ($\equiv ASP$) iff there exist a graph X and a continuous map $f : X \to X$ such that $P(f) = A$. A set B is called *an 0-abstract set of periods* ($\equiv ASP_0$) iff there exist a graph X and a continuous map $g : X \to X$ such that $h(f) = 0$, $P(f) = B$. Set $lZ \equiv \{li : i \geq 1\}$, $Q(n) \equiv \{2^i n : i \geq 0\}$. The main theorem of Section 3 is the following

Theorem 3.1. *1) A set $A \subset \mathbb{N}$ is an ASP iff it almost coincides with a finite union of some sets lZ or $Q(n)$.*

2) A set $A \subset \mathbb{N}$ is an ASP_0 iff it almost coincides with a finite union of some sets $Q(n)$.

In what follows we need *the spectral decomposition for graph maps* [B3] similar to that for maps of the interval [B1,B2]; the decomposition is briefly described in Section 1.

Notations

$\operatorname{int} Z$ is the interior of a set Z;

∂Z is the boundary of Z;

$\overline{Z}$ is the closure of Z;

f^n is the n-fold iterate of a map f;

$\operatorname{orb} x \equiv \{f^n x\}_{n=0}^{\infty}$ is the *orbit (trajectory)* of x;

$\omega(x)$ is the limit set of $\operatorname{orb} x$;

$\mathbb{N} \equiv \{1, 2, 3, \dots\}$ is the set of natural numbers;

$\operatorname{Per} f$ is the set of all periodic points of a map f;

$P(f)$ is the set of all periods of periodic points of a map f;

$h(f)$ is a topological entropy of a map f.

1. The Spectral Decomposition

In this section we briefly describe the spectral decomposition for one-dimensional maps (for the proofs see [B3]). Let us begin with some historical remarks.

A.N.Sharkovskii constructed the decomposition of the set $\omega(f) = \bigcup_{x\in I}\omega(x)$ for continuous interval maps $f : I \to I$ in [S2]. Then in [JR] Jonker and Rand constructed for unimodal maps the decomposition which is in fact close to that of Sharkovskii; however they used completely different methods based on symbolic dynamics. In [H] the decomposition for piecewise-monotone maps with discontinuities was constructed by Hofbauer and then Nitecki in [N] considered the decomposition for piecewise-monotone continuous maps from more geometrical point of view. The author's papers [B1,B2] were devoted to the case of arbitrary continuous interval maps; they contained the different approach to the problem in question which allowed us to obtain some new corollaries (e.g. describing generic properties of invariant measures for interval maps). The similar approach was used in [B3] to construct the decomposition for graph maps and now we pass to the desription of the results of the paper [B3].

Let X be a graph, $f : X \to X$ be a continuous map. We use terms *edge, vertex, endpoint* in the usual sense; the numbers of edges and endpoints of X are denoted by $\mathrm{Edg}(X)$, $\mathrm{End}(X)$. If necessary we add some "artificial" vertices to make all edges of a graph homeomorphic to an interval. We construct the decomposition of the set $\omega(f)$, which is defined similar to that for interval maps. First we need some definitions. A closed connected set $Y \subset X$ is called *subgraph.* A subgraph Y is called *periodic (of period k)* if $Y, fY, \dots, f^{k-1}Y$ are pairwise disjoint and $f^kY = Y$; the union of all iterations of Y is denoted by $\mathrm{orb}\,Y$ and called *a cycle of subgraphs.* Let $Y_0 \supset Y_1 \supset \dots$ be periodic subgraphs of periods $m_0, m_1, \dots$; then m_{i+1} is divided by m_i ($\forall i$). If $m_i \to \infty$ then the subgraphs Y_i, $i = 1, 2, \dots$ are said to be *generating.* We call any invariant closed set $S \subset Q = \bigcap(\mathrm{orb}\ Y_i)$ *a solenoidal set* and denote the solenoidal set $Q \cap \omega(f)$ by $S_\omega(Q)$ (note that $\omega(f)$ is closed for graph maps, see [B3]).

One can use a transitive shift in an Abelian zero-dimensional infinite group as a model for the map on a solenoidal set. Namely, let $D = \{n_i\}$ be a sequence of integers, n_{i+1}

is divided by $n_i\,(\forall i)$ and $n_i \to \infty$. Let us consider a subgroup $H(D) \subset \mathbf{Z}_{n_0} \times \mathbf{Z}_{n_1} \times \dots$, defined in the following way:

$$H(D) \equiv \{(r_o, r_1, \dots) : r_{i+1} \equiv r_i \pmod{m_i}\,(\forall i)\}.$$

Denote by τ the minimal shift in $H(D)$ by the element $(1, 1, \dots)$.

Theorem 1.1[B3]. *Suppose that $\{Y_i\}$ are generating subgraphs and that they have periods $\{m_i\}$. Let $Q = \bigcap_{i\geq 0} \operatorname{orb} Y_i$. Then there exists a continuous surjective map $\varphi : Q \longrightarrow H(D)$ with the following properties:*

1) $\tau \circ \varphi = \varphi \circ f$ (i.e.φ semiconjugates $f|Q$ to τ);

2) there exists the unique set $S \subset Q \cap \overline{\operatorname{Per} f}$ such that $\omega(x) = S$ for any $x \in Q$ and if $\omega(z) \cap Q \neq \emptyset$ then $S \subset \omega(z) \subset S_\omega$;

3) for any $\bar{r} \in H(D)$ the set $J = \varphi^{-1}(\bar{r})$ is a connected component of Q and $\varphi|S_\omega$ is at most 2-to-1;

4) $h(f|Q) = 0$.

Let us turn to another type of an infinite limit set. Let $\{Y_i\}_{i=1}^{l}$ be a collection of connected graphs, $K = \bigcup_{i=1}^{l} Y_i$. A continuous map $\psi : K \longrightarrow K$ which permutes these graphs cyclically is called *non-strictly periodic* or *non-strictly l-periodic*; for example if Y is a periodic subgraph then $f|\operatorname{orb} Y$ is non-strictly periodic. In what follows we will consider monotone semiconjugations between non-strictly periodic graph maps (a continuous map $g : X \longrightarrow Y$ is *monotone* provided $g^{-1}(Y)$ is connected for any $y \in Y$). We need the following

Lemma 1.1. *Let X be a graph. Then there exists a number $r = r(X)$ such that if $M \subset X$ is a cycle of subgraphs and $g : M \longrightarrow Y$ is monotone then the following property holds for any $y \in M$: $\operatorname{card}\{\partial(g^{-1}(y))\} \leq r(X)\,(\forall y \in M)$.*

Lemma 1.1 makes natural the following definition. If $\varphi : K \longrightarrow M$ is continuous, monotone, semiconjugates a non-strictly periodic map $f : K \longrightarrow K$ to a non-strictly periodic map $g : M \longrightarrow M$ and there is a closed f-invariant set $F \subset K$ such that $\varphi(F) = M$ and $\varphi^{-1}(y) \cap F \subset \partial(\varphi^{-1}(y))\,(\forall y \in M)$ then we say that φ *almost conjugates* $f|F$ to g.

Let Y be an n-periodic subgraph, $\operatorname{orb} Y = M$. Denote by $E(M, f)$ the following set:

$$E(M,f) \equiv \{x \in M : \text{for any open } U \ni x, U \subset M \text{ we have } \overline{\operatorname{orb} U} = M\}$$

provided it is infinite. We call the set E(M,F) ***a basic set*** and denote it by $B(M, f)$ provided $\operatorname{Per}(f|M) \neq \emptyset$; otherwise we denote $E(M, f)$ by $C(M, f)$ and call it ***a circle-like set.***

Theorem 1.2[B3]. *Let Y be an n-periodic subgraph, $M = orbY$ and $E(M, f) \neq \emptyset$. Then there exist a transitive non-strictly n-periodic map $g : K \to K$ and a monotone continuous surjection $\varphi : M \to K$ which almost conjugates $f|E(M, f)$ to g. Furthermore, the following properties hold:*

1) $E(M, f)$ is a perfect set;

2) $f|E(M, f)$ is transitive;

3) if $\omega(z) \supset E(M, f)$ then $\omega(z) = E(M, f)$;

4) if $E(M, f) = C(M, f)$ is a circle-like set then K is a union of n circles, g permutes them, g^n on any of them is an irrational rotation and $h(g) = h(f|E(M, f)) = 0$;

5) if $E(M, f) = B(M, f)$ is a basic set then $h(f|B(M, f)) > 0$, $B(M, f) \subset \overline{\operatorname{Per} f}$ and there exist a number k and a closed subset $D \subset B(M, f)$ such that $\varphi(D)$ is connected, sets $f^i D \cap f^j D$ and $\varphi(f^i D) \cap \varphi(f^j D) (0 \le i < j < kn)$ are finite, $f^{kn} D = D$, $\bigcup_{i=0}^{kn-1} f^i D = B(M, f)$ and $f^{kn}|D$, $g^{kn}|\varphi D$ are topologically mixing.

A number kn from the statement 5) of Theorem 1.2 is called ***a period*** of $B(M, f)$.

In Section 3 we will need some results which can be easily deduced from Theorem 1.2 and establish the connection between a period of $B(M, f)$ and periods of cycles belonging to M. One of them is Lemma 1.2; let us formulate here another one.

Assertion 1.1. *Let M be a cycle of subgraphs, $y \in M$ be a periodic point with period l, $B(M, f)$ be the correspondent basic set of period m, $D \subset B(M, f)$ and φ have the same sense as in Theorem 1.2. Then the following statements are true:*

1) $m \le l \cdot r(X)$, where $r(X)$ was defined in Lemma 1.1;

2) if l is not divided by m then $\varphi(f^i y) \notin int(\varphi D)$ for any i.

To formulate the decomposition theorem denote by Z_f the set of all cycles maximal by inclusion among all limit sets of f.

Theorem 1.3[B3]. *Let $f : X \longrightarrow X$ be a continuous graph map. Then there exist a finite number of circle-like sets $\{C(K_i, f)\}_{i=1}^k$, an at most countable family of basic sets $\{B(L_j, f)\}$ and a family of solenoidal sets $\{S_\omega(Q_\alpha)\}$ such that*

$$\omega(f) = Z_f \bigcup (\bigcup_{i=1}^{k} C(K_i)) \bigcup (\bigcup_j B(L_j)) \bigcup (\bigcup_\alpha (S_\omega(Q_\alpha)).$$

Moreover, there exist numbers $\gamma(X)$ and $\nu(X)$ such that $k \leq \gamma(X)$, the only possible intersections in the decomposition are between basic sets and at most $\nu(X)$ basic sets can intersect.

Theorem 1.3 shows that one can consider mixing graph maps as models for graph maps on basic sets. The following theorem seems to be important in this connection; to formulate it we need the definition of maps with the specification property (see, for example, [DGS]).

Theorem 1.4[B3]. *Let $f : X \longrightarrow X$ be a continuous mixing graph map. Then f has the specification property.*

It is well-known [DGS] that maps with the specification have nice properties concerning the set of invariant measures. Using them and Theorems 1.1 - 1.4 we can describe generic properties of invariant measures for graph maps. First we need some definitions. Let $T : X \longrightarrow X$ be a map of a compact metric space into itself. The set of all T-invariant Borel normalized measures is denoted by D_T. A measure $\mu \in D_T$ with $supp\,\mu$ containing in one cycle is said to be a $CO - measure$. The set of all CO-measures concentrated on cycles with minimal period p is denoted by $P_T(p)$. Let $V(x)$ be the set of accumulation points of time-averages of iterations of the point x. A point $x \in X$ is said to have *maximal oscillation* if $V_T(x) = D_T$.

Theorem 1.5[B3]. *Let B be a basic set. Then:*

1) for any l the set $\bigcup_{p \geq l} P_{f|B}(p)$ is dense in $D_{f|B}$;

2) the set of all ergodic non-atomic invariant measures μ with $supp\,\mu = B$ is a residual subset of $D_{f|B}$;

3) if $V \subset D_{f|B}$ is a non-empty closed connected set then the set of all such points x that $V(x) = V$ is dense in X (in particular every measure $\mu \in D_{f|B}$ has a generic point);

4) points with maximal oscillation are residual in B.

Theorem 1.6[B3]. *Let μ be an invariant measure. Then the following properties of μ are equivalent:*

1) there exists such a point x that $\operatorname{supp}\mu \subset \omega(x)$;

2) μ has generic points;

3) μ is concentrated on a circle-like set or can be approximated by CO-measures.

In particular, CO-measures are dense in all ergodic neasures which are not concentrated on circle-like sets.

Let us introduce two notions. If $n \geq 1$ then set $nZ \equiv \{in : i \geq 1\}$, $Q(n) \equiv \{2^i n : i \geq 0\}$. Then if A and B are such sets that $A \setminus B$ and $B \setminus A$ are finite then say that A *almost coincides* with B and if $B \setminus A$ is finite then say that A *almost contains* B and denote it by $A \underset{a}{\supset} B$. In Lemma 1.2 we need the following easy property of maps with the specification.

Property 1.1. *If T is a map with the specification then $P(T)$ almost coincides with $\mathbf{N}$.*

Assertion 1.1, Property 1.1 and Theorem 1.2 easily imply the following

Lemma 1.2. *Let $f : X \to X$ be a graph map, B be a basic set of f, m be a period of B. Then $P(f|B)$ almost coincides with mZ.*

Lemma 1.2 show how sets mZ appear in abstract sets of periods; at the same time sets $Q(n)$ correspond roughly speaking to the invariant subgraphs on which a map has zero entropy. Let us call a subset of a graph *an interval* if it is homeomorphic to the interval [0,1] (we use for intervals standart notations $[a, b], [a, b), (a, b], (a, b)$). To conclude this section let us formulate the following

Lemma 1.3[B3]. *Suppose that $y_n \to y$, $y_n \in Per\, f$ and there exists an interval I with an endpoint y such that $y_n \in I\, (\forall n)$. Let*

$$F = F(\{y_i\}) \equiv \{z : orb\, y_n \cap U \neq \emptyset \text{ for any open } U \ni z \text{ and infinitely many } n\}.$$

Then $fF = F$, F is a cycle or an infinite set and there exists such x that $\omega(x) \supset F$ and $\omega(x)$ is not a circle-like set.

2. Misiurewicz Conjecture

During the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990 M.Misiurewicz formulated the following

Conjecture. *For a graph X there exists an integer $L = L(X)$ such that for a continuous map $f : X \to X$ inclusion $P(f) \supset \{1, 2, \ldots, n\}$ implies $P(f) = \mathbb{N}$.*

We verify this conjecrure and give a sketch of the proof. First let us formulate the following

Lemma 2.1. *Let R be a positive integer. Then one can find such $N = N(R) > R$ that for any $M \geq N$ there exist positive integers $0 = a_0 < a_1 < a_2 < \cdots < a_l = M$ with the following properties:*

1) $a_{i+1} - a_i \geq R\,(0 \leq i < l)$;

2) for any proper divisor s of M there exists $j, 1 \leq j < l$ such that a_j is divided by s.

PROOF: Let $M = p_1^{b_1} \cdot \ldots p_k^{b_k}$, where $p_1, \ldots, p_k$ are prime integers. Set $m_i = \dfrac{M}{p_i}$, $1 \leq i \leq k$. Clearly numbers $\{m_i\}$ have the required property 2). So it is sufficient to find numbers $a_0 = 1 < a_1 < \cdots < a_l = M$ such that $a_{i+1} - a_i \geq R$, $0 \leq i < l$ and for any j there exists such i that a_i is divided by m_j. To this end suppose that $\{q_1 < q_2 < \cdots < q_r\}$ is the set of all prime integers less that $R + 1$ and set $\alpha = \min(\dfrac{1}{q_{i+1}} - \dfrac{1}{q_i})_{i=1}^{r-1}$, $N = \max(\dfrac{R}{\alpha}, 3q_r)$.

Now if $\dfrac{M}{p_k p_{k-1}} \geq R$ then $\dfrac{M}{p_i} - \dfrac{M}{p_{i+1}} \geq \dfrac{M}{p_k p_{k-1}} \geq R$. If $\dfrac{M}{p_k p_{k-1}} < R$ then $p_1, p_2, \cdots < p_{k-2} \leq R$ and so

$$m_i - m_{i+1} = \frac{M}{p_i} - \frac{M}{p_{\,i+1}} \geq \alpha M \geq \alpha N \geq R\,(1 \leq i \leq k-2).$$

Thus it remains to consider the differences $\dfrac{M}{p_{k-1}} - \dfrac{M}{p_{k-2}}, \dfrac{M}{p_k} - \dfrac{M}{p_{k-1}}$ which is left to the reader. Clearly we may assume that $N(R)$ increases with R. ∎

Now let us fix for the rest of this section a graph X and a continuous map $f : X \to X$.

Lemma 2.2. *There exists a number $m = m(X)$ such that if $a \in X$ and $[a, b_1], [a, b_2], \ldots,$ $[a, b_{m+1}]$ are intervals then one of them contains some of others.*

PROOF: Left to the reader. ∎

Suppose that there exist an edge $I = [a, b] \subset X$ and two periodic points, $P \in I$ of prime period $p > m(X)$ and $Q \in X$ of prime period $q > m(x)$, $p \neq q$ such that if $[P, Q] \subset I$ then $(P, Q) \cap (\operatorname{orb} Q \cup \operatorname{orb} P) = \emptyset$; fix them for Lemmas 2.3 - 2.7.

Lemma 2.3. *We have $f^{p(q-1)m(X)}[P, Q] \supset \operatorname{orb} Q$, $f^{q(p-1)m(X)}[P, Q] \supset \operatorname{orb} P$ and so $f^t[P, Q] \supset \operatorname{orb} Q \cup \operatorname{orb} P$ for $t \geq pqm(X) - \min(p, q) \cdot m(X)$.*

PROOF: Consider all the intervals of type $\{T_i = [P, c_i]\}_{i=1}^{k}$, where $c_i \in \operatorname{orb} Q$, containing no points of $\operatorname{orb} Q$ but c_i (some of points $\{c_i\}$ may coincide with each other). Then $k \leq m(X)$ and we may assume $Q = c_1, [P, Q] = T_1$. On the other hand for any i there exists $j = j(i)$ such that $f^p T_i \supset T_j$. Hence there exist such numbers l and n that $l + n \leq k$ and, say, $f^{pl} T_1 \supset T_2, f^{pn} T_2 \supset T_2$ which implies that $f^{pnj} T_2 \supset \{f^{pni} c_2\}_{i=0}^{j}$. But p, q are prime numbers and $n \leq m(X) < q$; thus $\{f^{ipn} c_2\}_{i=0}^{q-1} = \operatorname{orb} Q$ and $f^{pn(q-1)+lp}[P, Q] \supset$ $\operatorname{orb} Q$ (recall that $T_1 = [P, Q]$). It implies that $f^{p(q-1)m(X)}[P, Q] \supset \operatorname{orb} Q$. Similarly $f^{q(p-1)m(X)}[P, Q] \supset \operatorname{orb} P$ and we are done. ∎

Let us call subintervals of I with endpoints from $\operatorname{orb} Q$ or $\operatorname{orb} P$ *basical intervals* provided their interiors contain no points from $\operatorname{orb} P$ or $\operatorname{orb} Q$. In what follows basical interval will be called *P-interval, Q-interval* or *PQ-interval* depending on periodic orbits containing its endpoints. Furthermore, suppose that there are two intervals $G \subset X$ and $H \subset X$ and a continuous map $\varphi : X \to X$ such that $\varphi(G) \supset H$ and there is a subinterval $K \subset G$ such that $\varphi(K) = H$; then say that G $\varphi - covers$ H. Note the following property: if G φ-covers H and H ψ-covers M then G $\psi \circ \varphi$-covers M.

Lemma 2.4. *Let $Z \subset X$ be an interval, $Y = [\alpha, \beta] \subset X$ be an edge and $g : X \to X$ be a continuous map; suppose that $\alpha, \beta \in g(Z)$. Then there are points $\gamma, \delta \in Y$ such that $g(Z) \cap Y = [\alpha, \gamma] \cup [\delta, \beta]$ and Z g-covers $[\alpha, \gamma]$ and $[\delta, \beta]$.*

PROOF: Left to the reader. ∎

Lemma 2.5. *Let A be a PQ-interval. Then for any $i \geq pqm(X)$ this interval f^i-covers all basical intervals except at most one.*

PROOF: Follows from Lemmas 2.3 and 2.4. ∎

Lemma 2.6. *Suppose that $card(orb\, P \cap I) \geq 4$, $card(orb\, Q \cap I) \geq 4$. Then the following statements are true.*

1) Either for any P-interval M there exists $i < p^2$ such that $f^i M$ contains a PQ-interval or there exist two P-intervals Y and Z such that each of them f^i-covers both of them for $i \geq (p-1)^2$.

2) Either for any Q-interval N there exists $i < q^2$ such that $f^i N$ contains a PQ-interval or there exist two Q-intervals Y' and Z' such that each of them f^i-covers both of them for $i \geq (p-1)^2$.

PROOF: We will prove only statement 1). Consider a P-interval $[c, d]$ which has a neighbouring PQ-interval, say, $[d, e]$. Let the point c be closer to the point a than the point d (recall that $I = [a, b] \supset [c, d] \cup [d, e]$). Divide the proof by steps.

Step 1. *If $f^i[c, d]$ contains a PQ-interval then for any P-interval M there exists such $j \leq p - 1 + i$ that $f^j M$ contains a PQ-interval.*

Indeed, for any P-interval M one can find such $m < p$ that either $f^m M \supset [c, d]$ or $f^m M \supset [d, e]$ which implies the required.

Step 2. *Suppose there exists such $i < (p-1)^2$ that $f^i[c, d]$ contains a PQ-interval. Then for any P-interval M there exists an integer $j < (p-1)^2 + p$ such that $f^j M$ contains a PQ-interval.*

Step 2 easily follows from Step 1.

Denote by x the closest to e point from orb P lying to the other side of e than d; clearly x may not exist.

Step 3. *Suppose that $f^i[c,d]$ does not contain PQ-intervals for $i < (p-1)^2$. Then for $i \geq (p-1)(p-2)$ the interval $[c,d]$ f^i-covers $[a,d]$ (and $[x,b]$ provided x exists).*

Let $l < p$ be such that $f^l c = d$. Then $f^l[c,d] \supset [c,d]$ and moreover $[c,d]$ f^l-covers $[c,d]$. But p is a prime integer which as in Lemma 2.3 implies that $f^i[c,d] \supset \operatorname{orb} P$ for every $i \geq l(p-2)$. Since $f^i[c,d]$ does not contain $[d,e]$ for $l(p-2) \leq i < l(p-1)$ we have by Lemma 2.4 that $[c,d]$ f^i-covers $[a,d]$ (and $[x,b]$ provided x exists). But $[c,d]$ f^i-covers $[c,d]$ which easily implies that for any $i \geq l(p-2)$ the interval $[c,d]$ f^l-covers $[a,d]$ (and $[x,b]$ provided x exists).

Step 4. *Suppose that $f^i[c,d]$ does not contain PQ-intervals for $i < (p-1)^2 + p$. Then for any P-interval M and $i \geq (p-1)^2$ we have that M f^i-covers $[a,d]$ (and $[x,b]$ provided x exists).*

Clearly there exists $l < p$ such that either M f^l-covers $[c,d]$ or M f^l-covers $[d,e]$. Now by Step 3 $f^{(p-1)(p-2)}[c,d] \supset M$; so if M f^l-covers $[d,e]$ then $f^{(p-1)(p-2)+l}[c,d] \supset [d,e]$ which is a contradiction. Thus M f^l-covers $[c,d]$ and by Step 3 we get the required.

Now suppose there exists a P-interval M such that $f^i M$ contains no PQ-intervals for $i < p^2$. Then by Step 1 $f^i[c,d]$ contains no PQ-intervals for $i < p^2 - (p-1) = (p-1)^2 + p$. Applying Step 4 and using simple geometrical arguments we may assert that there exist two P-intervals Y and Z such that $Y \cap Z = \emptyset$ and for any $i \geq (p-1)^2$ the interval Y f^i-covers intervals Y, Z and the interval Z f^i-covers intervals Y, Z which completes the proof of Lemma 2.6. ▮

Lemma 2.7. *Suppose that $card(orb\, P \cap I) \geq 4$, $card(orb\, Q \cap I) \geq 4$. Let*

$$T \equiv T(p,q) \equiv N(pqm(X) - \min(p,q) \cdot m(X) + [\max(p,q)]^2)$$

(recall that function $N(x)$ was defined in Lemma 2.1). Then $P(f) \supset \{i : i \geq T\}$ and $h(f) > 0$.

PROOF: Let us make use of Lemmas 2.1 and 2.6 and consider all possible cases.

Case A. *There exist such P-intervals Y and Z that each of them f^i-covers both of them for $i \geq (p-1)^2$.*

Let $k \geq N((p-1)^2)$ be an integer. By Lemma 2.1 one can easily see that there exist integers $1 = a_0 < a_1 < \cdots < a_l = k$, $a_{i+1} - a_i \geq (p-1)^2$ such that for any proper divisor s of k there exists a_i which is divided by s. Properties of f^i-covering imply that there exists an interval $K \subset Y$ such that $f^{a_i}K \subset Z$ for any $1 < i < l$ and $f^k K = Y$. Hence there exists a point $\zeta \in Y$ such that $f^{a_i}\zeta \in Z$ for $0 < i < l$ and $f^k\zeta = \zeta$; by the properties of the numbers $\{a_i\}$ it implies that k is the minimal period of the point ζ and so $P(f) \supset \{i : i \geq N((p-1)^2)\} \supset \{i : i \geq T\}$. Standart one-dimensional arguments show also that $h(f) > 0$ (see, for example, [BGMY]).

Case B. *There are such Q-intervals Y' and Z' that each of them f^i-covers both of them for $i \geq (q-1)^2$.*

Similarly to Case A we have $P(f) \supset \{i : i \geq N((q-1)^2)\} \supset \{i : i \geq T\}$ and $h(f) > 0$.

Case C. *For any basical interval M there exists a number $s = s(M) < [\max(p,q)]^2$ such that $f^s M$ contains a PQ-interval.*

Let for definitness $p > q$. Then similarly to Lemma 2.5 we can conclude by Lemmas 2.3 and 2.4 that any basical interval M f^i-covers all basical intervals except at most one of them for $i \geq H = pqm(X) - qm(X) + p^2$. Choose four basical intervals $\{M_j\}_{j=1}^4$ which are pairwise disjoint and show that for any $k \geq N(H)$ there exists a periodic point ζ of minimal period k.

Let $k \geq N(H)$. As in Case A choose integers $1 = a_0 < a_1 < \cdots < a_l = k$ with the properties from Lemma 2.1. Let $u = a_l - a_{l-1}$. Then it is easy to see that there exists such basical interval, say, M_1, that at least two other basical intervals, say M_2 and M_3, f^u-cover M_1. On the other hand one can easily show that there are two numbers $i, j \in \{2,3,4\}$ and two intervals $K_i \subset M_1$ and $K_j \subset M_1$ such that for any $1 \leq v \leq l-2$ we have $f^{a_v}(K_i) \subset M_{r(v)}$ and $f^{a_v}(K_j) \subset M_{t(v)}$ where $r(v), t(v) \in \{2,3,4\}$ are appropriate integers and moreover $f^{a_{l-1}}K_i = M_i$, $f^{a_{l-1}}K_j = M_j$. Clearly one of the numbers i, j belongs to the set $\{2,3\}$; let, say, $i = 2$. Then choosing correspondent subintervals and using simple properties of f-coverings one can easily find an interval $K \subset M_1$ such that $f^{a_v} K \cap M_1 = \emptyset$, $1 \leq v \leq l-1$, and $f^k K = M_1$. Thus f has a periodic point of minimal period k. Moreover, it is clear that $h(f) > 0$ which completes the proof. ∎

Theorem 2.1. *Let X be a graph, $s = Edg(X)+1$ and $\{p_i\}_{i=1}^{s}$ be s ordered prime integers greater than $4Edg(X)$. Set $L = L(X) = T(p_s, p_{s-1})$. If a continuous map $f : X \longrightarrow X$ is such that $P(f) \supset \{1, 2, \dots, L\}$ then $P(f) = \mathbb{N}$ and $h(f) > 0$.*

PROOF: Clearly in the situation of Theorem 2.1 one can find two periodic points with properties from Lemma 2.7. It completes the proof. ∎

Remark 1[B4]. *If X is a tree then one may set $L(X) = 2(p-1)End(X)$ where p is the least prime integer greater than $End(X)$.*

The preliminary version of Sections 1,2 was a subject of the author's talk at the Conference on Dynamical Systems and Ergodic Theory in the memory of Dr. Prof. H.Michel in Güstrow, October 1990 (that version will probably appear in the volume of Proceedings of the Conference in Güstrow in Lecture Notes in Mathematics).

3. Abstract Sets of Periods for Graph Maps

One of the well-known results about periods of cycles of graph maps is the famous Sharkovskii theorem on the co-existence of periods of cycles for maps of the real line. To formulate it let us introduce the following *Sharkovskii ordering* for positive integers:

$$(*) \qquad 3 \prec 5 \prec 7 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \cdots \prec 8 \prec 4 \prec 2 \prec 1$$

Denote by $S(k)$ the set of all such integers m that $k \prec m$ or $k = m$ and by $S(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$.

Theorem[S1]. *Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous map. Then either $P(g) = \emptyset$ or there exists such $k \in \mathbb{N} \cup 2^\infty$ that $P(g) = S(k)$. Moreover for any such k there exists a map $g : [0,1] \longrightarrow [0,1]$ with $P(g) = S(k)$ and there exists a map $g_0 : \mathbb{R} \longrightarrow \mathbb{R}$ with $P(g_0) = \emptyset$.*

In what follows we need the following corollary of Sharkovskii theorem.

Corollary S. *Suppose that $I \subset J \subset \mathbf{R}$ are closed intervals, $f : I \to J$ is a continuous map onto and either $I = J$ or there is a periodic point y such that $orb\, y \subset I$ and endpoints of I belong to $orb\, y$. Consider the set $Per\, f$ of all periodic points of f with orbits belonging to I. Then the set $P(f)$ of their periods is $S(k)$ for some $k \in \mathbf{N} \cup 2^\infty$.*

PROOF: It is sufficient to consider the case when $I = [a, b] \subset [c, d] = J$, $a, b \in \mathrm{orb}_f y$. Define $g : J \to J$ as follows: $g|[a, b] = f$, $g|[c, a] = f(a)$, $g|[b, d] = f(b)$. Then $P(g) = S(k)$ for some $k \in \mathbf{N} \cup 2^\infty$. At the same time the only possible g-periodic but not f-periodic points are those with orbits entering $[c, a] \cup [b, d]$; clearly it means that these points belong to f-orbit of y and so Per $f =$ Per g and $P(f) = P(g) = S(k)$. ∎

Other information about sets of periods of cycles for one-dimensional maps is contained in papers [AL,M] for maps of the circle, [ALM] for maps of the letter Y and [Ba] for maps of the n-od.

We describe in Section 3 possible sets of periods of cycles for graph maps with no restrictions on a graph. Namely, a set $A \subset \mathbf{N}$ is called ***an abstract set of periods*** ($\equiv ASP$) iff there exist a graph X and a continuous map $f : X \to X$ such that $P(f) = A$. A set B is called ***an 0-abstract set of periods*** ($\equiv ASP_0$) iff there exist a graph X and a continuous map $g : X \to X$ such that $h(f) = 0$, $P(f) = B$. Recall also that $lZ \equiv \{li : i \geq 1\}$, !!$Q(n) \equiv \{2^i n : i \geq 0\}$. The main theorem of Section 3 is the following

Theorem 3.1. *1) A set $A \subset \mathbf{N}$ is an ASP iff it almost coincides with a finite union of some sets lZ or $Q(n)$.*

2) A set $A \subset \mathbf{N}$ is an ASP_0 iff it almost coincides with a finite union of some sets $Q(n)$.

A key role will play the following

Lemma 3.1. *Let $f : X \to X$ be a continuous graph map, y_i be f-periodic points of periods n_i. Then taking a subsequence we may assume that one of the following possibilities A) and B) holds.*

A) There are a sequence of cycles of subgraphs $M_i \supset orb\, y_i$ and a number p such that for any i there exists a basic set $B(M_i, f)$ of period p and n_i is divided by p.

B) There are a sequence of pairs of intervals $J_i \supset I_i$ and a number p such that $f^p I_i = J_i$, intervals $fI_i, \ldots, f^{p-1}I_i, J_i$ are pairwise disjoint, $\operatorname{orb} y_i \subset \bigcup_{j=0}^{p-1} f^j I_i$ $(\forall i)$ and either $f^p I_i = J_i$ or endpoints of I_i belong to $\operatorname{orb} y_i$.

PROOF: We may assume that $n_i \nearrow \infty, y_i \to y$ and there is an interval $[a, y]$ such that $[a, y)$ contains no vertices of X and $y_i \in [a, y)\,(\forall i)$. Consider the set $F = F(\{y_i\})$ (see Lemma 1.3) using Theorems 1.1 - 1.3 and Lemma 1.3.

Case 1. *The point y belongs to a circle-like set.*

This possibility is excluded (see Lemma 1.3).

Case 2. *The set F is not a cycle.*

According to the spectral decomposition and Lemma 1.3 we need to consider two subcases.

Subcase 2a. *The set F belongs to a solenoidal set.*

In this case there exists a p-periodic interval I containing no vertices of X and such that $\operatorname{orb} y_i \subset \operatorname{orb} I$ for all sufficiently large i. Clearly it is enough to set $I_i = J_i = I$; then the possibility B) of Lemma 3.1 holds.

Subcase 2b. *The set F belongs to a basic set $B = B(M, f)$.*

Let p be a period of B. Furthermore, let $g : K \to K$ be a transitive non-strictly periodic graph map and $\varphi : M \to K$ be a monotone continuous surjection which almost conjugates $f|B$ to g (such a map φ exists by Theorem 1.2). Finally let the set D be the same as in Theorem 1.2.5).

The fact that F is not a cycle and Lemma 1.3 imply that F is infinite and so (taking if necessary a subsequence) we may assume that $\varphi(f^r y_i) \in \operatorname{int}(\varphi D)$ for any i and some $r = r(i)$. By Assertion 1.1 it implies that n_i is divided by p for any i, i.e. the possibility A) of Lemma 3.1 holds.

Case 3. *The set F is a cycle (i.e. $y \in Per\, f, orb\, y_i \to orb\, y = F$).*

Let a period of y be k. Consider a map $\psi = f^k$. Clearly we may assume that there are small intervals $[y, z_1] = T_1, \dots, [y, z_l] = T_l$ such that $n_j > l\,(\forall j)$ and the following holds:

i) $(y, z_s) \cap (y, z_t) = \emptyset\,(s \neq t)$;

ii) the set $U = \bigcup_{r=1}^{l} [y, z_r)$ is a neighbourhood of y;

iii) $y_i \in T_1$, card(orb $y_i \cap T_1) > 1$ and $\mathrm{orb}_\psi\, y_i \subset U\,(\forall i)$;

iv) there exists a neighbourhood $V = \bigcup_{i=1}^{l} R_i$ of the point y such that for any i we have $R_i = [y, \zeta_i) \supset [y, z_i]$, $V \setminus y$ contains no vertices of X, $\psi^j U \subset V$ for $0 \le j \le l$ and also $f^e V \cap f^d V = \emptyset\,(0 \le e < d < k)$.

Denote by $Y_r^{(i)}$ the smallest subinterval of T_r containing $\{orb_\psi\, y_i \cap T_r\}$; if $Y_r^{(i)} \neq \emptyset$ then set $Y_r^{(i)} = [\alpha_r^{(i)}, \beta_r^{(i)}]$ where $\beta_r^{(i)}$ is closer to the point y than $\alpha_r^{(i)}$. Consider some subcases.

Subcase 3a. *There is an infinite set C of such integers i that for any $j \le l$, $r \le l$ we have $y \notin \psi^j(Y_r^{(i)})$.*

Let $i \in C$ and $x_1, x_2 \in \mathrm{orb}\, y_i$ belong to the same interval, say, T_r. Then by iv) and the hypothesis of Subcase 3a we may conclude that for any $0 \le m \le l$ the set $\psi^m[x_1, x_2]$ belongs to one of the intervals from the family $\{R_j\}$. So for every $i \in C$ there exists a number $0 < s_i \le l$ such that $\psi^{s_i} Y_1^{(i)} \subset (y, \zeta_1]$ and moreover $f^d Y_1^{(i)} \cap f^e Y_1^{(i)} = \emptyset$ for $0 < d < e \le s_i k$. Taking a subsequence $E \subset C$ we may assume that $s_i = s \le l\,(\forall i \in E)$, so the number $p = ks$, the intervals $Y_1^{(i)} \equiv I_i$ and $\psi^s Y_1^{(i)} = f^{ks} Y_1^{(i)} \equiv J_i$ are those required in possibility B) of Lemma 3.1.

Subcase 3b. *For any sufficiently large i there exist such $j = j(i) \le l$ and $r = r(i) \le l$ that $y \in \psi^j(Y_r^{(i)})$.*

To consider Subcase 3b we need the following Assertion 3.1 which is more or less easy and traditional for one-dimensional dynamics (cf. [BGMY]) so that we leave the proof to the reader.

Assertion 3.1. *In the situation of Subcase 3b there exist intervals*

$$L_i \subset Y^{(i)}_{r(i)} = [\alpha^{(i)}_{r(i)}, \beta^{(i)}_{r(i)}], \quad N_i \subset [\beta^{(i)}_{r(i)}, y]$$

and a number $t_i = t$ such that $\psi^t N_i = \psi^t L_i = [\alpha^{(i)}_{r(i)}, y]$. Moreover, there exists a ψ^t-invariant set Σ_i with the following properties:

1) $\psi^t|\Sigma_i$ is at most 2-to-1 semiconjugated to the full Bernoulli shift with two states;

2) for every $\zeta \in \Sigma_i$, every small open interval W such that $\zeta \in W$ and every integer d there exist an open interval $\mathcal{U}, \zeta \in \mathcal{U} \subset W$ and such integer s that $\psi^{std}\mathcal{U} = [\alpha^{(i)}_{r(i)}, y]$ and $\psi^{tm}\mathcal{U} \subset N_i \cup L_i \, (0 \leq m < sd)$;

3) there exists a point x such that $\Sigma_i = \omega_\psi(x) \subset \omega_f(x)$.

Now consider a basic set $B_i = B(M_i, f) \supset \omega_f(x) \supset \Sigma_i$. Then by the definition we have $M_i \supset [\alpha^{(i)}_{r(i)}, y]$. By Assertion 1.1.1) we may assume that all B_i have the same period, say, p. Moreover, we may assume that there is a number $r \leq l$ such that $r(i) = r \, (\forall i)$.

Let g, φ, $D_i \subset B_i$ have the same meaning as in Theorem 1.2.5) and be chosen so that $(\Sigma_i \cap D_i)$ is infinite. We will prove that $\varphi[\alpha^{(i)}_r, y] \subset \varphi(D_i)$. Indeed, take such a point $z \in \Sigma_i \cap D_i$ that $\varphi(z) \in \mathrm{int}(\varphi D_i)$, then (using Assertion 3.1) take a small neghbourhood $W, z \in W$ and a number s such that $\varphi W \subset \mathrm{int}(\varphi D_i)$, $\psi^{stp} W = [\alpha^{(i)}_r, y]$.Clearly by properties of φ we have $\varphi\psi^{stp} \subset \varphi D_i$, so $\varphi[\alpha^{(i)}_r, y] \subset \varphi D_i$. But $\beta^{(i)}_r$ lies in $[\alpha^{(i)}_r]$ between sets $N_i \cap \Sigma_i$ and $L_i \cap \Sigma_i$ belonging to B_i; together with the properties of φ it implies that $\varphi\beta^{(i)}_r \in \mathrm{int}(\varphi D_i)$.

Now by Assertion 1.1.2) the fact that $\varphi\beta^{(i)}_r \in \mathrm{int}(\varphi D_i)$ implies that f-period n_i of $\beta^{(i)}_r$ (which is equal to that of y_i) is divided by p. So we get to the possibility A) of Lemma 3.1 which concludes the proof. ∎

Theorem 3.1. *1) A set $A \subset \mathbb{N}$ is an ASP iff it almost coincides with a finite union of some sets lZ or $Q(n)$.*

2) A set $A \subset \mathbb{N}$ is an ASP_0 iff it almost coincides with a finite union of some sets $Q(n)$.

Proof: 1.i) Let us prove first that if $f : X \to X$ is a continuous graph map then $P(f)$ has the required form. To this end let us introduce some notions.

Consider the family $\mathcal{A}$ of all sets $T(d,n) \equiv \{di : i \geq n\}$ belonging to $P(f)$. Suppose that for some number d there exists such number n that $T(d,n) \in \mathcal{A}$; then d is called *a difference.* Denote for any difference d by $n(d)$ the minimal such integer that $T(d,n(d) \in \mathcal{A}$; denote also the family of all sets $T(d,n(d)) \in \mathcal{A}$ by $\mathcal{R}$. Clearly, '$\underset{a}{\supset}$' is a partial ordering in $\mathcal{R}$ and if $T(d,n)$ is a maximal element of $\mathcal{R}$ then d is not divided by any other difference.

Denote the family of all $\underset{a}{\supset}$-maximal elements of $\mathcal{R}$ by $\mathcal{R}_{max}$ and call *minimal differences* all those d that $T(d,n(d)) \in \mathcal{R}_{max}$. By the definition for any $T(d,n) \in \mathcal{A}$ there exists $T(d',n') \in \mathcal{R}_{max}$ such that $T(d',n') \underset{a}{\supset} T(d,n)$. For any minimal difference d denote by $m(d)$ a prime integer greater than $n(d)$; moreover, choose $m(d)$ so that if $d_1 \neq d_2$ then $m(d_1) \neq m(d_2)$. Let us also call *starting periods* numbers $d \cdot m(d)$ where d is a minimal difference.

Now consider the family $\mathcal{B}$ of all sets $Q(m) = \{2^i m : i \geq 0\} \subset P(f)$ for which there is no set $T(d,n) \underset{a}{\supset} Q(m)$. Sets from $\mathcal{B}$ are partially ordered by inclusion; let us denote by $\mathcal{B}_{max}$ the family of all maximal elements of $\mathcal{B}$ and call *roots* all those m that $Q(m) \in \mathcal{B}_{max}$. Finally let us call a number $l \in P(f)$ *a period of finite type* if it does not belong to sets from either $\mathcal{R}_{max}$ or $\mathcal{B}_{max}$; the set of all periods of finite type is denoted by $\mathcal{F}$.

To prove Theorem 3.1 it is enough to show that $\mathcal{R}_{max}, \mathcal{B}_{max}, \mathcal{F}$ are finite sets. Suppose this is not the case; it means that the set of all minimal differences, roots and periods of finite type is infinite. Let us show that then the set of all starting periods, roots and periods of finite type is infinite. Indeed, none of roots are equal to each other or to some starting periods. At the same time one starting period may correspond to no more than finite number of minimal differences. So the set in question is infinite. Take for every starting period, root and period of finite type the correspondent periodic point. This way we get an infinite sequence $\{y_i\}$ of periodic points of periods n_i and we may assume that $n_i \nearrow \infty$. Let us apply Lemma 3.1 and consider some cases.

Case A. *There is a sequence of cycles of subgraphs M_i and a number p such that for every i $y_i \in M_i$, there exists a basic set $B_i = B(M_i, f)$ of period p and n_i is divided by p.*

By Lemma 1.2 $P(f|B_i)$ almost coincides with pZ; so there exists a set $T(d,n) \in \mathcal{R}_{max}$ such that $T(d,n) \underset{a}{\supset} P(f|B_i) \underset{a}{\supset} pZ$. At the same time $n_i \in pZ\,(\forall i)$. Hence $n_i \in T(d,n)$ and n_i is divided by d for all sufficiently large i. If $d = 1$ then we are done because $P(f)$ almost coincides with $\mathbf{N}$. On the other hand if $d > 1$ then the choice of starting periods shows that if $e \cdot m(e)$ is a starting period divided by d then e is divided by d or $m(e)$ is divided by d (because $m(e)$ is a prime integer); the same argument proves that there are no more than one integer of type $m(e)$ divided by d (namely, in this case $d = m(e)$ must be a prime number). But the properties of minimal differences show that a minimal difference e may be divided by d only if $e = d$. So there are only finitely many starting periods among numbers $\{n_i\}$. Moreover, it is easy to see that there are only finitely many roots and periods of finite type among numbers $\{n_i\}$ which is a contradiction. Note that in fact we have proved that there is no such d that $T(d,n) \underset{a}{\supset} \{n_i\}$ where $T(d,n) \in \mathcal{A}$.

Case B. *There is a sequence of pairs of intervals $J_i \supset I_i \ni y_i$ and a number p such that for any i we have $f^p I_i = J_i$, intervals $fI_i, \dots, f^p I_i = J_i$ are pairwise disjoint, $\mathrm{orb}\, y_i \subset \bigcup_{j=0}^{p-1} I_i$ and either $f^p I_i = I_i = J_i$ or endpoints of I_i belong to $\mathrm{orb}\, y_i$.*

Let us apply Corollary S to $f^p I_i$. Consider the set R of periods of all periodic points ζ for which there exists such i that $\mathrm{orb}\, \zeta \subset \bigcup_{j=0}^{p-1} f^j I_i$. Then $\{n_i\} \subset R$ and by Corollary S there exists such k that $R = pS(k)$ (here either $k \in \mathbf{N}$ or $k = 2^\infty$). Consider two subcases.

Cubcase B1. $k \in \mathbf{N}$

Clearly the property $n_i \to \infty$ implies that $k = 2^l(2m+1), m \geq 1$. Then we see that $T(2^l p, 2^l p(2m+1)) \in \mathcal{A}$ and at the same time $T(2^l p, 2^l p(2m+1)) \underset{a}{\supset} R \supset \{n_i\}$; so we are done by what has been proved in Case A.

Subcase B2. $k = 2^\infty$

If there is a set $T(d,n) \in \mathcal{A}$ such that $T(d,n) \underset{a}{\supset} R$ then we get to the same contradiction as earlier. Suppose there is no such set $T(d,n)$. Then $R \in \mathcal{B}$ and there is a set $Q(m) \in \mathcal{B}_{max}$ such that $R \subset Q(m)$. Hence $\{n_i\} \subset Q(m)$. But it is easy to see that there are only finitely many starting periods, roots and periods of finite type belonging to $Q(m)$ which is a contradiction. It concludes the proof of the first part of statement 1) of Theorem 3.1.

1.ii) Now suppose there is a set A which almost coincides with the finite union of some sets lZ and $Q(m)$. To construct a graph map $f : X \to X$ such that $P(f) = A$ let us first note that we do not suppose X to be connected. So it is enough to show that the following two statements are true.

Statement 1. *For any $m \geq 0$ there exists a graph map $g : Y \to Y$ such that we have $P(g) = \{i : i \geq m\} = T(1, m)$*

By the results of [AL,M] it is easy to see that there exists a map $g_m : S^1 \to S^1$ with $P(g_m) = T(1, m)$.

Statement 2. *There is a map $\psi : [0,1] \to [0,1]$ such that $P(\psi) = \{1, 2, 4, 8, \dots\} = Q(1)$.*

This fact is well-known.

Taking into account the existence of graph maps g with $Per\, g = \emptyset$ (e.g. irrational rotation) one can easily construct the required graph map, so the rest of construction is left to the reader. It completes the proof of the first statement of Theorem 3.1.

2.i) To prove that every graph map g with zero entropy has a set of periods $P(g)$ which almost coincides with a finite union of some sets $Q(l)$ one could repeat the same arguments as in the proof of the first statement of Theorem 3.1 taking into account that graph maps with zero entropy have no basic set (since a map on a basic set has a positive entropy, Theorem 1.2.5). An alternative proof follows from Theorem 2.1 which implies that a graph map g with zero entropy cannot contain a set of type nZ in its set of periods $P(g)$.

2.ii) The construction is similar to that in the proof of the first statement of Theorem 3.1 and is left to the reader. ∎

Corollary 3.1. *There are no graph maps f, g with $h(f) = 0, h(g) > 0, P(f) = P(g)$.* ∎

Corolloary 3.2. *Suppose that $f : X \to X$ is a graph map. Then the following properties are equivalent:*

1) $h(f) > 0$;

2) there is a number n such that $P(f) \underset{a}{\supset} nZ$;

3) there are numbers d, r such that $P(f) \underset{a}{\supset} \{r, r + d, r + 2d, \dots\}$;

4) a density of $P(f)$ is positive;

5) an upper density of $P(f)$ is positive. ∎

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