Remarks on a Conjecture of S. Rusheweyh

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REMARKS ON A CONJECTURE OF RUSCHEWEYH

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Introduction. The subject of this note is a conjecture formulated by S. Ruscheweyh at the Conference on Complex Analysis, held at Bar-Ilan University, November, 1994.

Conjecture. *(S. Ruscheweyh* [R1]*). Let f be a function, holomorphic in a neighborhood of the closed unit disk and $f(0) = 0$. Assume that for some $\gamma > 0$ the following identity holds on the unit circle:*

$$|f'(z)| = \gamma(1 - |f(z)|^2) \quad , \quad |z| = 1. \tag{1}$$

Then, $f(z) = cz^N$ for some $N \in \mathbb{N} \cup \{0\}$ and $c \in \mathbb{C}$.

The conjecture is confirmed in the case when $f'(z) \neq 0$ for $z \neq 0$ (S. Ruscheweyh [R2], S.L. Krushkal [K]). The aim of this note is to prove the conjecture for other classes of function f, namely for multi-valent and entire functions. Another purpose (maybe more important) is to attract attention of the readers to this very nice and intriguing problem. We also give a geometric interpretation of the fact for conformal mappings. In that case it can be understood as a rigidity property of hyperbolic metrics.

We will use the following notations:

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$\mathbb{C}$ - the complex plane; $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$, $\overline{\Delta} = \{z \in \mathbb{C}\colon |z| \leq 1\}$, $\partial\Delta = \overline{\Delta} \setminus \Delta$, $\Delta^* = \Delta \setminus \{0\}$,

$H(\overline{\Delta})$ - the space of all functions holomorphic in a neighborhood of the closed disk $\overline{\Delta}$. Our main result is:

Theorem 1. *Let $f \in H(\overline{\Delta}), f(0) = 0$ belong to one of two following classes:*

(a) *f is a multi-valent function in Δ, i.e the multiplicity $\#f^{-1}(f(z)), z \in \Delta$ is finite and constant.*

(b) *f extends to $\mathbb{C}$ as an entire function.*

Then the condition (1) implies $f(z) = cz^N$, for some $c \in BC$ and $N \in \mathbb{N} \cup 0$[1]

1. Reduction of the problem. The following lemma reduces Theorem 1 to a geometric characterization of the image $\Omega = f(\Delta)$.

Lemma 1.1. *If $f \in H(\Delta) \cap C^1(\overline{\Delta})$ and the functions $|f|$ and $|f'|$ are constant on $\partial\Delta$ then $f(z) = cz^N$ for some $N \in \mathbb{N} \cup \{0\}$, $c \in \mathbb{C}$. Here $f'(z), |z| = 1$, is understood as a continuous extension of f' to $\overline{\Delta}$.*

Proof. Suppose $f(z)\overline{f(z)} = \alpha^2$ and $f'(z)\overline{f'(z)} = \beta^2$ for $|z| = 1$. The differentiation of both sides of the first identity in φ, $z = e^{i\varphi}$ yields

$$f'_\varphi \overline{f} + f(\overline{f})'_\varphi = 0.$$

Multiply both sides by $f'_\varphi f$:

$$\left(f'_\varphi\right)^2 \alpha^2 + f^2 \mid f'_\varphi \mid^2 = 0.$$

Note that $f'_\varphi(z) = izf'(z)$ and, therefore, $\mid f'_\varphi(z) \mid^2 = \mid f'(z) \mid^2 = \beta^2, |z| = 1$. Then,

$$-\alpha^2 z^2 (f')^2 + \beta^2 f^2 = 0.$$

[1] After our preprint was written we were informed by S. Ruscheweyh that he too independently had found the prove of the Conjecture for class (b).

The left hand side is holomorphic in Δ, hence the identity holds in Δ and we arrive at two equations in Δ:

$$\alpha z f' - \beta f = 0 \qquad \text{or} \qquad \alpha z f' + \beta f = 0.$$

Excluding the trivial case $\alpha = 0$, we obtain $f(z) = cz^{\pm\beta/\alpha}$ and the holomorphic solution is $f(z) = cz^N$, $N = \beta/\alpha \in \mathbb{C}$. $\square$

The next lemma shows that in Theorem 1 the restriction $f\colon \partial\Delta \to f(\partial\Delta)$ is a local diffeomorphism.

Lemma 1.2. *If $f \in H(\overline{\Delta})$ satisfies* (1), *then f' does not vanish on $\partial\Delta$.*

Proof. Assuming, on the contrary, that $f'(z_0) = 0$ for some $z_0 \in \partial\Delta$, we obtain, due to (1), $|f(z_0)| = 1$. Equation (1) is invariant under rotation, so we can assume $z_0 = -1$, $f(z_0) = 1$. The Taylor series at $z_0 = 1$ has the form

$$f(z) = 1 + a_k(z+1)^k + a_{k+1}(z+1)^{k+1} + \cdots, a_k \neq 0,\ k \geq 2.$$

Take $z_\varepsilon = -1 + \varepsilon e^{i\varphi}$ where $\varepsilon > 0$ and $\varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})$ is chosen so that $\operatorname{Re} a_k e^{ik\varphi} > 0$. Then

$$\operatorname{Re} f(z_\varepsilon) = 1 + \varepsilon^k \ \operatorname{Re} a_k e^{ik\varphi} + 0(\varepsilon^k),$$

and we have $z_\varepsilon \in \Delta$ and $\operatorname{Re} f(z_\varepsilon) > 1$ when ε is sufficiently small. Therefore, $|f(z_\varepsilon)| > 1$, which contradicts the Maximum Modulo Principle because (1) implies $|f(z)| \leq 1$ for $z \in \partial\Delta$. $\square$

Remark. Lemma 1.2 together with the equation (1) shows that $f(\Delta)$ is compactly contained on Δ. In light of Lemma 1.2 the problem geometrically looks now as follows. Given is a function $f \in H(\overline{\Delta})$. Consider the restriction $ds\ \big|_\gamma$ of the hyperbolic metric in Δ on the curve $\gamma = f(\partial\Delta)$. Suppose that $f^*(ds\ \big|_\gamma) = c d\varphi$, $\partial\Delta = \{\varepsilon^{i\varphi} : \varphi \in [0, 2\pi)\}$. Must γ be a circle?

2. Proof of Theorem 1(a).

Lemma 2.1. *Let Ω be a domain in $\mathbb{C}$ with a connected piece-wise smooth boundary $\partial\Omega$, and $0 \in \Omega$. Suppose that for any $k \in \mathbb{N}$*

$$\int\limits_{\partial\Omega} w^k \frac{|dw|}{1-|w|^2} = 0. \tag{2}$$

Then Ω is a disk, $\Omega = \Delta_r = \{|z| < r\}$.

Proof. Denote the measure $\dfrac{|dw|}{1-|w|^2}$ on $\partial\Omega$ by $d\mu$. Condition (2) implies that the measure $d\mu$ is orthogonal on $\partial\Omega$ to any function $g \in H(\Delta) \cap C(\overline{\Delta})$ with $g(0) = 0$. Since μ is real-valued, the same is true for any harmonic function g and, moreover,

$$g(0) = \frac{1}{M} \int\limits_{\partial\Omega} g(w) d\mu(w),$$

where $M = \int\limits_{\partial\Omega} d\mu$ and g is any harmonic function in Δ and continuous in $\overline{\Delta}$. Thus the measure $\mu_1 = \dfrac{1}{M}\,\mu$ is the normalized harmonic measure for the domain Ω at the point $w = 0$. Substituting $w = \psi(z)$, where $\psi\colon \Delta \to \Omega$ is a Riemann mapping, $\psi(0) = 0$, gives the normalized harmonic measure for the unit disk Δ at $z = 0$:

$$\psi^*(d\mu_1)(z) = \frac{1}{M} \frac{|\psi'(z)|\,|dz|}{1-|\psi(z)|^2} = \frac{1}{2\pi}\,|dz|.$$

Thus we obtain for ψ the same equation

$$|\psi'(z)| = \gamma(1-|\psi(z)|^2)\;, \quad \gamma = \frac{M}{2\pi}.$$

Since the Ruscheweyh conjecture is true for the conformal mapping ψ ([R2], [K]), it follows that $\psi(z) = rz$, $r = \text{const}$, and, consequently, $\Omega = \psi(\Omega) = \Delta_r$. $\square$

We proceed the proof of Theorem 1(a). First, if f is multi-valent, then

$$f(\partial\Delta) = \partial f(\Delta) = \partial\Omega.$$

According to Lemma 1.2, the boundary $\partial\Omega$ is a piece-wise real-analytic curve. For $k \in \mathbb{N}$ consider integrals the J_k, $\quad k = 1, 2, \dots$:

$$\int\limits_{\partial\Omega} w^k \frac{|dw|}{1-|w|^2} = N \int\limits_{\partial\Delta} [f(z)]^k \frac{|f'(z)|}{1-|f(z)|^2}\,|dz|$$

$$= N\gamma \int\limits_{\partial\Delta} [f(z)]^k\,|dz| = 2N\pi[f(0)]^k = 0.$$

Here N is the multiplicity of f. It remains to apply Lemma 2.1 and Lemma 1.1.

3. Proof of Theorem 1(b) (The case of entire functions).

3.1. Riccati equation Define $Y(z) = f^*(z)$; where $f^*(z) = \overline{f\left(\frac{1}{\bar{z}}\right)}$. Then Y is a holomorphic function in $\mathbb{C} \setminus \{0\}$ and on the unit circle $Y(z) = \overline{f(z)}$. For $|z| = 1$ we have

$$\overline{f'(z)} = -z^2 Y'(z)$$

and, therefore, equation (1) can now be rewritten as the Riccati equation in $\mathbb{C}$:

$$-z^2 f' Y' = \gamma^2 (1 - fY)^2. \tag{3}$$

The function $Y = f^*$ is a holomorphic solution of (3) in $\mathbb{C} \setminus \{0\}$ while f and f' are considered as the coefficients of this equation. In the next item we want analyze the character of the singularities of it's solutions at the origin. In fact, the information is non-explicitly contained in the theory of Riccati equations (for example,[B]. p.155). We include the analysis here in order to make reading more transparent and independent.

3.2 A particular meromorphic solution of (3) in a neighborhood of $z = 0$.

Denote by N the order of zero of the function f at $z = 0$. Then $f(z) = z^N \varphi(z)$, where φ is holomorphic and $\varphi(0) \neq 0$. For f' we have a representation $f'(z) = z^{N-1}\psi(z)$ with ψ holomorphic and $\psi(0) = N\varphi(0)$. Thus, (3) can be rewritten as

$$Y' = -\frac{\gamma}{z^{N+1}\psi} (1 - z^N \varphi Y)^2. \tag{4}$$

Now we are looking for a local solution Y_1 near the point $z = 0$ having, at this point, a pole of order N:

$$Y_1(z) = \frac{a + v(z)}{z^N} \tag{5}$$

where v is holomorphic at $z = 0$, $v(0) = 0$ and $a = \text{const}$. Inserting (5) into (4)and comparing the Laurent series in both sides of (4), we obtain the following relation:

$$N^2 a\varphi(0) = \gamma^2 (1 - a\varphi(0))^2 \tag{6}$$

Therefore, the coefficient a can take two values:

$$a^{\pm} = \frac{1}{\varphi(0)} \left[1 + \frac{N^2}{2\gamma^2} \pm \frac{N}{2\gamma^2} \sqrt{N^2 + 4\gamma^2} \right]. \tag{7}$$

Choosing $a = a^+$ in (7), from (4)and (5) obtain:

$$zv' = A(z) + B(z)v + C(z)v^2 \tag{8}$$

where coefficients on the right hand side are:

$$A(z) = -\frac{\gamma^2}{\psi}\,(1 - a\varphi)^2 + Na, \quad B(z) = N + \frac{2\gamma^2}{\psi}\,(1 - a\varphi)\varphi, \quad C(z) = -\frac{\gamma^2}{\psi}\,\varphi^2.$$

Note that $A(0) = \dfrac{\gamma^2}{N\varphi(0)}\,(1 - a\varphi(0))^2 + aN = 0$ due to (6). The function $B(z)$ has a series expansion

$$B(z) = B_0 + B_1 z + \cdots$$

with $B_0 = B(0) = N + \dfrac{2\gamma}{N\varphi(o)}\,(1 - a\varphi(0))\,\varphi(0)$. Formula (7) implies

$$B_0 = -\sqrt{N^2 + 4\gamma^2}.$$

Now we can rewrite (8) as

$$zv' - B_0 v = A(z) + (B(z) - B_0)v + C(z)v^2 = R(z, v). \tag{9}$$

where $R(z, v)$ is holomorphic in both variables z and v at the point $z = v = 0$ and satisfies $R(0,0) = R'_v(0,0) = 0$. Equation (9) is known as the Briot-Bouquet equation, and if $B_0 \notin \mathbb{N}$, which holds in our case, then there exists a unique holomorphic solution $v = v(z), v(0) = 0$ defined near the point $z = 0$ [I, p. 295]. Thus,

$$Y_1(z) = \frac{a}{z^N} + \frac{v(z)}{z^N}$$

is a holomorphic solution of (4) in a punctured neighborhood of $z = 0$, having at $z = 0$ a pole of order N.

3.3 A general solution of (3) near the point $z = 0$.

Having at hands a particular solution Y_1 constructed above, we can apply the standard method of reducing the Riccati equation to a linear differential equation. Let Y be a general solution and $u = Y - Y_1 \neq 0$. Then u satisfies the Bernoulli equation

$$u' = \frac{2\gamma^2(1 - fY_1)f}{z^2 f'}\; u - \frac{\gamma^2 f^2}{z^2 f'}\; u^2. \tag{10}$$

Introducing

$$\eta = \frac{1}{u}$$

and dividing both sides in (10) by u^2 we arrive at the linear equation

$$-\eta' = \frac{2\gamma^2(1 - fY_1)f}{z^2 f'}\; \eta - \frac{\gamma^2 f^2}{z^2 f'}.$$

Solving it, we get:

$$u(z) = \frac{\exp\left(\int \frac{2\gamma^2(1 - fY_1)f}{z^2 f'}\, dz\right)}{C + \int \frac{\gamma^2 f^2}{z^2 f'}\, \exp\left(\int \frac{2\gamma^2(1 - fY_1)f}{z^2 f'}\, dz\right)}$$

where C is an arbitrary constant. Using the Riccati equation (3), we can replace the integrand $\frac{2\gamma^2(1 - fY_1)}{z^2 f'}$ by $-\frac{2Y_1}{1 - fY}$ and, finally, write

$$Y = Y_1 + u = Y_1 + \frac{\exp\left(-\int \frac{2Y_1' f}{1 - fY_1}\, dz\right)}{C + \int \frac{\gamma^2 f^2}{z^2 f'}\, \exp\left(-\int \frac{2Y_1' f}{1 - fY_1}\, dz\right)}. \tag{11}$$

The function $1 - fY_1$ is holomorphic and $(fY_1)(0) = \varphi(0)a = 1 + \frac{N^2}{2\gamma^2} + \frac{N}{2\gamma}\sqrt{N^2 + 4\gamma^2} > 1$. We have also

$$Y_1'(z) = \frac{-Na}{z^{N+1}} + \cdots \quad , \quad f(z) = \varphi(0) z^N + \cdots$$

and, therefore, we have the decomposition near $z = 0$:

$$\frac{-2Y_1'(z) f(z)}{1 - f(z)Y_1(z)} = \frac{\alpha}{z} + c_0 + c_1 z + \cdots$$

where the residue

$$\alpha = \frac{2N\varphi(0)a}{1-\varphi(0)a}.$$

Therefore,

$$\exp(\int \frac{-2Y_1'f}{1-fY_1}) = \exp(\alpha\ln(z) + c_0 z + \cdots) = z^\alpha e^\Psi,$$

Ψ is holomorphic, $\psi(0) = 0$. Also,

$$\frac{\gamma^2 f^2}{z^2 f'} = \frac{\gamma^2(\varphi(0)z^N + \cdots)^2}{z^2(N\varphi(0)z^{N-1} + \cdots)} = \frac{\gamma^2\varphi(0)}{N} z^{N-1} + \cdots .$$

So, equation (11) may be rewritten in the following form:

$$Y(z) = Y_1(z) + \frac{z^\alpha e^\Psi}{C + \int\left(\frac{\gamma^2\varphi(0)}{N} z^{N-1} + \cdots\right) z^\alpha e^\Psi dz} \tag{12}$$

$$= \frac{z^\alpha e^\psi}{C + z^{\alpha+N}(c_0 + c_1 z + \cdots)}$$

where $c_0 = \dfrac{\gamma^2\varphi(0)}{N^2} \neq 0$. Thus the following Lemma is obtained:

Lemma 3.1. *Any local holomorphic solution Y of the Riccati equation* (3) - (4) *in a punctured neighborhood of $z=0$ has at $z=0$ singularity which is either a branch point, when $C \neq 0$ and α is integer, or a pole of order N, where N is the order of zero of the function f at $z=0$.*

Note that for $C = 0$, we obtain the second solution with a pole at $z = 0$. This solution corresponds to the second possible value for a in (5), $a = a^-$.

3.4. Proof of Theorem 1(b) We know that the inversion $Y = f^*$ satisfies the Riccati equation (3) - (4). According to Lemma 4.1, the function Y has a pole at $z = 0$ of order N, since the branch point at $z = 0$ is impossible for entire f. This implies that $f(z) = Y^*(z) = \overline{Y\left(\frac{1}{\bar{z}}\right)}$ has a pole of order N at $z = \infty$. But f is entire and has at $z = 0$ zero of the same order N. The Liouville Theorem immediately implies $f(z) = \text{const } z^N$. $\square$

Remark. Since, as it is shown above, any solution of the Riccati equation has not essential singularity at the origin, any meromorphic solution of equation (1) has to be rational.

4. Geometric interpretation.

The following Theorem is a geometric interpretation of the Ruscheweyh Conjecture for conformal mappings.

Theorem 2 (on three hyperbolic disks). *Let $D \subset G_1 \subset G_2$ be three simply-connected domains in $\mathbb{C}$. Denote by ds_1, ds_2 the hyperbolic metrics in G_1, G_2 respectively. Suppose that D is a disk in both hyperbolic spaces (G_1, ds_1) and (G_2, ds_2). Then G_1 is a disk in (G_2, ds_2).*

In other words , if a hyperbolic subspace is not a disk then all disks in it are distorted with respect to the metric of the ambient hyperbolic space.

Proof. Let G_1, G_2, D be as in Theorem 2. Let $\varphi : G_2 \to \Delta$ be a conformal equivalence according to the Riemann theorem. Denote $G_1' = \varphi(G_1)$, $D' = \varphi(D)$ and let also $\psi : G_1' \to \Delta$ be a conformal isomorphism. Denote $\Omega = \psi[\varphi(D)]$. The construction is illustrated by the following diagram:

$$\begin{array}{ccccc} G_2 & \xrightarrow{\varphi} & \Delta & & \\ \cup & & \cup & & \\ G_1 & \xrightarrow{\varphi} & G_1' & \xrightarrow{\psi} & \Delta \\ \cup & & \cup & & \cup \\ D & \xrightarrow{\varphi} & \varphi(D) & \xrightarrow{\psi} & \Omega \end{array}$$

We know that D is a hyperbolic disk in G_2 and hence $\varphi(D)$ is a hyperbolic disk in Δ. We can choose φ so that $\varphi(D)$ is a disk centered at the origin,

$$\varphi(D) = \Delta_R = \{z \in \mathbb{C} : |z| < R\}, \quad 0 < R < 1.$$

The mapping ψ can be also normalized by $\psi(0) = 0$. By the condition the disk Δ_R is also a disk with respect to the hyperbolic metric

$$ds_1' = \frac{|d\psi|}{1 - |\psi|^2}$$

in G_1'. Therefore, the restriction of ds_1' on the circle $\partial\Delta_R$ is proportional to the arc length on $\partial\Delta_R$, i.e.,

$$\frac{|\psi'(z)|}{1-|\psi(z)|^2} = \gamma_1 = \text{const}, \quad |z| = R.$$

We are in the situation of Theorem 1 with $f(z) = \psi(Rz)$ which a univalent holomorphic function in a neighborhood of the unit disk $\overline{\Delta}$, $f(0) = \psi(0) = 0$ and

$$\frac{|f'(z)|}{1-|f(z)|^2} = \gamma = R\gamma_1, \quad |z| = 1.$$

We conclude that $f(z) = cz$ and therefore $\psi(z) = \dfrac{c}{R}z$. Thus, G_1' is a disk in Δ and, respectively, $G_1 = \varphi^{-1}(G_1')$ is a hyperbolic disk in $G_2 = \varphi^{-1}(\Delta)$ with respect to the metric ds_2. $\square$

5. Concluding remarks and acknowledgments.

After this work has been completed the first author visited the University of Wuerzburg and discussed the subject with Prof. S. Ruscheweyh. He made us aware of publication [RF], where a background of the problem (related to generalizations of the Bieberbach conjecture (now theorem)) ,is given. S. Ruscheweyh presented also a short and elegant proof of his Conjecture (actually, of the result, even stronger) under an assumption that f' is vanishing only at the zeros of f.

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