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HIGHER ORDER DERIVED FUNCTORS AND THE ADAMS SPECTRAL SEQUENCE

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Abstract. Classical homological algebra considers chain complexes, resolutions, and derived functors in additive categories. We describe "track algerrbas in dimension n", which generalize additive categories, and we define *higher order* chain complexes, resolutions, and dervied functors. We show that higher order resolutions exist in higher track categories, and that they determine higher order Ext-groups. In particular, the E_m-term of the Adams spectral sequence $(m \leq n+2)$ is a higher order Ext-group, which is determined by the track algebra of higher cohomology operations.

Introduction

Topologists have been working on the problem of calculating the homotopy groups of spheres for around 80 years, and many methods have been developed for this purpose. One of the most useful is the Adams spectral sequence $E_2, E_3, E_4, \ldots,$, converging to the p-completed stable homotopy groups of the sphere. Adams computed the E_2-term of the spectral sequence, and showed that it is algebraically determined:

$$E_2^{s,t} \;\cong\; \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p)\ ,$$

where the derived functor Ext is taken for modules over the mod p Steenrod algebra $\mathcal{A}$ of primary mod p cohomology operations (cf. [A1]). Since the work of Adams in [A2], it has been generally believed that higher order cohomology operations can be used to compute the higher terms of the Adams spectral sequence.

However, it remained unclear what kind of algebra $\mathcal{B}_{(n)}$ would be formed by cohomology operations of order $(n+1)$. For $n=0$, the algebra $\mathcal{B}_{(0)} = \mathcal{A}$ is the Steenrod algebra, which determines E_2. It is shown in [B2] that the algebra of secondary cohomology operations, $\mathcal{B}_{(1)}$, can be described by a differential algebra $\mathcal{B}$, which was computed in [*loc. cit.*], leading to the calculation of E_3 as a "secondary Ext-group" over $\mathcal{B}$. For this, the notion of secondary derived functors was developed in [BJ2] in the context of track categories – that is, categories enriched in groupoids.

It is the purpose of this paper to exhibit higher order derived functors in track algebras – in particular, higher order Ext-groups – which allow the calculation of the higher terms $E_n^{*,*}$ $(n \geq 2)$ in the Adams spectral sequence. This generalizes Adams' original result for $n=2$, and the results in [B2, BJ2] for $n=3$.

The elements of the Steenrod algebra $\mathcal{A}$ are (stable) homotopy classes of maps between mod p Eilenberg-Mac Lane spaces. Here we consider the space of all such maps, which together constitute the *Eilenberg-Mac Lane mapping algebra* (see Section 7 below). We associate with each mapping algebra a *track algebra of dimension* n $(n \geq 0)$ (see Section 11 below), and prove as our main result:

Theorem A. *Higher order resolutions exist in a track algebra of dimension* n*, and such resolutions determine higher order* Ext*-groups* E_m *for* $m \leq n+2$*. If the track algebra is the one determined by the Eilenberg-Mac Lane mapping algebra, these higher order* Ext*-groups compute the* E_m*-terms of the Admas spectral sequence for* $m \leq n+2$*.*

The track algebra of dimension n associated to the Eilenberg-Mac Lane mapping algebra constitutes the algebra $\mathcal{B}_{(n)}$ of $(n+1)$*-st order mod* p *cohomology operations.* It is conjectured in [B3] that $\mathcal{B}_{(n)}$ can be computed in terms of a suitable differential algebra for all $n \geq 0$, as is the case for $n=0$ and $n=1$.

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1. Left cubical sets

We first recall some properties of cubical sets, and introduce the notion of *left* cubical sets, which are used to describe higher nullhomotopies.

Let $I = [0,1]$ be the unit interval and let $I^n = I \times \cdots \times I$ be the n-dimensional cube. We have inclusions $d^i_\varepsilon : I^{n-1} = I^{i-1} \times \{\varepsilon\} \times I^{n-i} \subset I^n$ for $1 \le i \le n$ and $\varepsilon \in \{0,1\}$. Here I^0 is a single point.

Let $\square$ denote the category whose objects are cubes I^n $(n \ge 0)$, and whose morphisms are generated by d^i_ε and the projections $s^i : I^n \longrightarrow I^{n-1}$.

A *pointed cubical set* is a functor $K : \square^{\mathrm{op}} \longrightarrow \mathbf{Set}^*$, where $\mathbf{Set}^*$ is the category of pointed sets. As usual, $K(I^n)$ is denoted K_n and $* \in K_n$ is the base point. We write $\dim(a) = n$ if $a \in K_n$. See [C], [J], or [I] for further details on category of cubical sets.

1.1. **Definition.** Let $\overline{\square}$ be the subcategory of $\square$ consisting of objects I^n $(n \ge 0)$ and morphisms generated by d^i_0. A *left cubical set* is a functor $\overline{\square}^{\mathrm{op}} \to \mathbf{Set}^*$. We write ∂^i for $(d^i_0)^* : K_n \to K_{n-1}$ $(1 \le i \le n)$. We also consider the full subcategories $\overline{\square}_n \subset \overline{\square}$ consisting of objects I^m $(0 \le m \le n)$. A functor $\overline{\square}_n^{\mathrm{op}} \longrightarrow \mathbf{Set}^*$ is called a *left n-cubical set.*

1.2. *Remark.* Given a pointed cubical set K, one obtains a left cubical set $\mathrm{nul}(K)$ by setting

$$\mathrm{nul}(K)_m \ := \ \{a \in K_m \mid \quad (d^i_1)^* a = * \quad \text{for} \quad 1 \le i \le m\}.$$

Accordingly, one gets the left n-cubical set $\mathrm{nul}_n(K)$ as a restriction of $\mathrm{nul}(K)$ to $\overline{\square}_n^{\mathrm{op}}$.

Note that nul is a functor from pointed cubical sets to left cubical sets. Its left adjoint $\mathcal{U} : (\mathbf{Set}^*)^{\overline{\square}^{\mathrm{op}}} \to (\mathbf{Set}^*)^{\square^{\mathrm{op}}}$ may be thought of as a "universal enveloping cubical set" functor, described as follows: given a left cubical set M, the pointed cubical set $\mathcal{U}(M)$ has one n-cube I^n_a for each left n-cube $a \in M$, with $(d^i_1)^* I^n_a = *$ (the base point) for each $1 \le i \le n$. In addition, there is a degenerate $(n+k)$-cube:

(1.3) $\quad (s^{j_1})^* \dots (s^{j_k})^* I^n_a$ in $\mathcal{U}(M)$ for each iterated projection $s^{j_k} \dots s^{j_1} : I^{n+k} \to I^n$ in $\square$

(with identifications according to the cubical identities).

It is readily verified that $\mathcal{U}(M)$ is indeed a pointed cubical set, with a natural isomorphism:

$$\mathrm{Hom}_{(\mathbf{Set}^*)^{\overline{\square}^{\mathrm{op}}}}(M, \mathrm{nul}(K)) \ \xrightarrow{\cong} \ \mathrm{Hom}_{(\mathbf{Set}^*)^{\square^{\mathrm{op}}}}(\mathcal{U}(M), K) \tag{1.4}$$

for $K \in (\mathbf{Set}^*)^{\square^{\mathrm{op}}}$ and $M \in (\mathbf{Set}^*)^{\overline{\square}^{\mathrm{op}}}$. Moreover, both functors preserve dimensions of all cubes, so they commute with the n-skeleton functor, yielding a left adjoint $\mathcal{U}_n$ to nul_n.

For any cubical set K, let $\mathcal{C}_K$ be the partially ordered set of all k-cubes $(k \ge 0)$ of K, ordered under inclusion. We have $K \cong \mathrm{colim}_{I^k \in \mathcal{C}_K} \ I^k$, where each I^k is thought of as a cubical set. We use this to define a monoidal structure on $\mathbf{Set}^{\square^{\mathrm{op}}}$, given by:

$$K \otimes L \ := \ \mathrm{colim}_{I^j \in \mathcal{C}_K,\ I^k \in \mathcal{C}_L} \ I^{j+k} \tag{1.5}$$

(see [J, §3]). If K and L are pointed, there is a cubical *smash* functor

$$K \overline{\otimes} L \ := \ (K \otimes L) \ / \ (\{*\} \otimes L \amalg K \otimes \{*\}) \tag{1.6}$$

on $(\mathbf{Set}^*)^{\square^{\mathrm{op}}}$, which also is also defined on $(\mathbf{Set}^*)^{\overline{\square}^{\mathrm{op}}}$. Moreover, nul and $\mathcal{U}$ are monoidal with respect to $\overline{\otimes}$ on $(\mathbf{Set}^*)^{\square^{\mathrm{op}}}$ and $(\mathbf{Set}^*)^{\overline{\square}^{\mathrm{op}}}$, respectively.

Now, let $(X, *)$ be a pointed space and let $\mathcal{S}_\square X$ be the *singular* pointed cubical set: thus $(\mathcal{S}_\square X)_n$ is the set of all maps $I^n \longrightarrow X$, with base point $o : I^n \longrightarrow \{*\} \subset X$.

Then, $\mathrm{nul}(X) = \mathrm{nul}(\mathcal{S}_\square X)$ is given by all maps $a : I^n \to X$ with $a d^i_1 = o$ for $1 \le i \le n$. Accordingly, we let $\mathrm{nul}_n(X) := \mathrm{nul}_n(\mathcal{S}_\square X)$

1.7. **Definition.** The *left n-cubical set* $\mathrm{Nul}_n(X)$ is defined by

$$\mathrm{Nul}_n(X)_m := \begin{cases} \mathrm{nul}(X)_m & \text{for} \quad m < n, \\ \mathrm{nul}(X)_n / \simeq & \text{for} \quad n = m. \end{cases}$$

Here, we set $a \simeq b$ for $a, b \in \mathrm{nul}(X)_n$ if the maps $a, b : I^n \longrightarrow X$ are homotopic relative to the boundary ∂I^n of the cube I^n. Let $\{a\}$ be the equivalence class of a; we call $\{a\}$ an *n-track* in X.

There is a surjective map of left n-cubical sets:

$$\mathrm{nul}_n(X) \twoheadrightarrow \mathrm{Nul}_n(X)\ , \tag{1.8}$$

which is the identity in dimension $< n$ and which carries a with $\dim(a) = n$ to the n-track $\{a\}$. We point out that the left n-cubical set $\mathrm{Nul}_n(X)$ is not the restriction of a cubical set.

1.9. *Remark.* Let $\overline{\Delta}$ be the category with sets $\{1, 2, \dots, n\}$ $(n \geq 0)$ as objects, and order preserving injective maps as morphisms. There is an isomorphism of categories $\overline{\Delta} \cong \overline{\square}$ which carries $\{1, 2, \dots, n\}$ to I^n and carries $\{1, \dots, \widehat{i}, \dots, n\} \subset \{1, \dots, n\}$ to d_0^i. Here, $\widehat{i}$ indicates that we omit i.

2. n-graded categories enriched in left cubical sets

Cubical sets, and left cubical sets, have a natural grading by the dimension of the cubes. Thus categories enriched in (left) cubical sets are in particular graded categories, described as follows:

A *graded set* K is a sequence of sets K_n $(n \geq 0)$. We write $\dim(x) = n$ if $x \in K_n$. An n-*set* L is a finite sequence $L_0, \dots, L_n$ of sets. For example, the n-*skeleton* $K_0, \dots, K_n$ of a graded set is an n-set. A *graded category* $\mathbf{G}$ is a category in which each morphism f has a dimension $\dim(f) \geq 0$ such that the composition fg satisfies

$$\dim(fg) = \dim(f) + \dim(g).$$

Thus, all morphism sets $\mathrm{Mor}_{\mathbf{G}}(X, Y)$ are graded sets.

An n-*graded category* consists of morphism sets which are n-sets and composition fg is defined if $\dim(f) + \dim(g) \leq n$. For example, the n-skeleton of a graded category is an n-graded category.

An n-*graded category enriched in left cubical sets* is a n-graded category such that morphism sets $\mathrm{Mor}(X, Y)$ are left n-cubical sets with operators $(d_0^i)^* = \partial^i$ satisfying

$$\partial^i(fg) = \begin{cases} (\partial^i f)g & \text{for } \ i \leq \dim(f) \\ f(\partial^{i-\dim(f)} g) & \text{for } \ i > \dim(f) \end{cases} \tag{2.1}$$

Moreover, the zero morphisms $o^n \in \mathrm{Mor}(X, Y)_n$ $(n \geq 0)$ satisfy

$$o^n g = o^{n+\dim(g)} \quad \text{and} \quad f o^m = o^{\dim(f)+m}.$$

For example, let $\mathbf{C}$ be a category enriched in $(\mathbf{Top}^*, \wedge)$, where $\wedge$ is the smash product of pointed topological spaces. Thus for every $X, Y \in \mathrm{Obj}\,(\mathbf{C})$, there is a *zero morphism* $o \in \mathrm{Mor}_{\mathbf{C}}(X, Y)$, satisfying $og = o$ and $fo = o$ for any $f, g \in \mathrm{Mor}_{\mathbf{C}}$. Then $\mathrm{nul}(\mathbf{C})$ is given by the left cubical set $\mathrm{nul}(\mathrm{Mor}_{\mathbf{C}}(X, Y))$. The composition $f \otimes g$ defined by

$$f \otimes g : I^n \times I^m \xrightarrow{f \times g} \mathrm{Mor}_{\mathbf{C}}(Y, X) \times \mathrm{Mor}_{\mathbf{C}}(Z, Y) \xrightarrow{\mu} \mathrm{Mor}_{\mathbf{C}}(Z, X)\ ,$$

where μ is the composition in $\mathbf{C}$. Thus $\mathrm{nul}(\mathbf{C})$ is a category enriched in left cubical sets as above.

The n-skeleton of $\mathrm{nul}(\mathbf{C})$, denoted by $\mathrm{nul}_n \mathbf{C}$, is given by the n-cubical sets $\mathrm{nul}_n \mathrm{Mor}_{\mathbf{C}}(X, Y)$. One has the quotient functor

$$\mathrm{nul}_n \mathbf{C} \twoheadrightarrow \mathrm{Nul}_n \mathbf{C}$$

given by the quotient maps:

$$\mathrm{nul}_n \mathrm{Mor}_{\mathbf{C}}(X, Y) \longrightarrow \mathrm{Nul}_n \mathrm{Mor}_{\mathbf{C}}(X, Y)$$

(see (1.8)). Here, $\mathrm{Nul}_n \mathbf{C}$ is an n-graded category with the composition defined by the equivalence class $\{f \otimes g\}$ for $\dim(f) + \dim(g) = n$. The n-graded categories $\mathrm{nul}_n \mathbf{C}$ and $\mathrm{Nul}_n \mathbf{C}$ are enriched in left n-cubical sets.

For $n = 0$, the (0-graded) category

$$\mathrm{Nul}_0 \mathbf{C} = \pi_0 \mathbf{C}$$

has morphisms $X \longrightarrow Y$ given by the path components of $\mathrm{Mor}_{\mathbf{C}}(X, Y)$.

3. The chain category $\mathbb{Z}_\otimes$

A chain complex in any pointed category $\mathcal{M}$ may be defined as a pointed functor from a suitable indexing category. To define higher order chain complexes, we require a more elaborate indexing category, which we now describe.

Let $\star$ and J be elements which generate the free monoid

$$N \ := \ \mathrm{Mon}(\star, J) \ .$$

Let $\deg, \dim : N \longrightarrow (\mathbb{N}_0, +)$ be monoid homomorphisms defined by

$$\begin{array}{ll} \deg(\star) = 1 & \qquad \deg(J) = 1 \\ \dim(\star) = 0 & \qquad \dim(J) = 1. \end{array}$$

Elements in N are words which consist of letters $\star$ and J. For example, $V = \star\star J \star JJ$ is such a word, with $\deg(V) = 6$ the length of the word V, and $\dim(V) = 3$ the number of letters J in V. Let $\emptyset$ be the empty word, which is the unit in the monoid N.

We associate with J the unit interval $I = [0,1]$ and with $\star$ the one point space $\{0\}$. For any word V, let $\overline{V}$ be the space defined by

$$\overline{V} = \begin{cases} I & \text{if} \quad V = J \\ \{0\} & \text{if} \quad V = \star \\ \overline{V}_1 \times \overline{V}_2 & \text{if} \quad V = V_1V_2 \ . \end{cases}$$

We say that V is *in the boundary of* W with $V, W \in N$ if there is an inclusion $\overline{V} \subset \overline{W}$. This implies $\deg V = \deg W$ and $\dim V \leq \dim W$. By projecting the spaces $\{0\}$, one gets the homeomorphism

$$\overline{V} \cong I^{\dim V} \ .$$

If V is in the boundary of W, there is a unique inclusion $d_{V,W}$ of cubes in the category $\boxdot$ (see Definition 1.1) such that

$$\begin{tikzcd} \overline{V} \arrow[r, "\cong"] \arrow[d] & I^{\dim V} \arrow[d, "d_{V,W}"] \\ \overline{W} \arrow[r, "\cong"] & I^{\dim W} \end{tikzcd}$$

commutes.

Now, consider elements $\star$ and $I_n \quad (n \geq 1)$, which generate the monoid

$$M \ := \ \mathrm{Mon}(\star, I_n, n \geq 1)/I_n \circ I_m = I_{n+m} \ .$$

The multiplication in M is denoted by $\circ$. Here, $\mathrm{Mon}(\star, I_n \ : \ n \geq 1)$ denotes the free monoid. In M, we divide out the relation $I_n \circ I_m = I_{n+m}$ for $n, m \geq 1$.

There is a canonical isomorphism of monoids

$$M \xrightarrow{\cong} N$$

which carries $\star$ to $\star$ and I_n to the n-fold product $J^n = J \cdots J$. Using this isomorphism, we obtain the functions $\deg$ and $\dim$ on M.

We introduce on M a further multiplication $\otimes$ defined by

$$V \otimes W \ = \ V \circ \star \circ W \qquad \text{for} \quad V, W \in M. \tag{3.1}$$

Here $V \times W$ is the product of elements $V, \star, W$ in the monoid M. The operation $\otimes$ is associative, but it has no unit. For the empty word $\emptyset \in M$, we get

$$\emptyset \otimes \emptyset \ = \ \emptyset \circ \star \circ \emptyset \ = \ \star.$$

3.2. **Definition.** We define the *chain category* $\mathbb{Z}_\otimes$ to be the following graded category: the objects in $\mathbb{Z}_\otimes$ are the integers $i, j, \ldots \in \mathbb{Z}$. In addition to the identities 1_i, with $\dim(1_i) = 0$, the morphisms in $\mathbb{Z}_\otimes$ consist of

$$(i, V) : i \ \xrightarrow{V} \ i - \deg V - 1 = j$$

for all $V \in M$. The composition of $V : i \longrightarrow j$ and $W : j \longrightarrow j - \deg W - 1 = k$ $(W \in M)$, is defined

$$(i, W \otimes V) : i \xrightarrow{W \otimes V} i - \deg(W \otimes V) - 1 = k \ .$$

Here we have $\deg(W \otimes V) = \deg W + \deg V + 1$, so that the composition is well defined. We also omit $\otimes$ in the notation of the composite.

More precisely, morphisms in $\mathbb{Z}_\otimes$ are pairs (i, V), where $i \in \mathbb{Z}$, $V \in M$, and i is the source of the morphism (i, V) (aslow written $V : i \longrightarrow j$). The target j satisfies $j = i - \deg V - 1$.

The category $\mathbb{Z}_\otimes$ is graded by dimension of elements in M. In fact, we have $\dim(W \otimes V) = \dim(W) + \dim(V)$. The *n-skeleton* $\mathbb{Z}^n_\otimes$ of $\mathbb{Z}_\otimes$ $(n \geq 0)$ is an n-graded category. The 0-skeleton $\mathbb{Z}^0_\otimes$ consists only of identities and of the morphisms $V : i \to i - \deg V - 1$, where V is a power of the element $\star$ in M.

If $V = \emptyset$, then $\emptyset : i \longrightarrow i - 1$ is in $\mathbb{Z}^0_\otimes$. The composition is $\emptyset \otimes \emptyset = \star : i \longrightarrow i - 2$, and so on. We observe:

3.3. **Lemma.** *The category* $\mathbb{Z}_\otimes$ *is freely generated by the morphisms* $(i, \emptyset) : i \to i - 1$ *and* $(i, I_k) : i \longrightarrow i - k - 1$ *for* $i \in \mathbb{Z}$. $k \geq 1$.

4. Higher order chain complexes

We are now in a position to define the notion of a higher order chain complex:

Given an n-graded category $\mathbf{T}$ enriched in left n-cubical sets (for example, $\mathbf{T} = \mathrm{Nul}_n \, \mathbf{C}$), we consider a functor of n-graded categories

$$K : \mathbb{Z}^n_\otimes \longrightarrow \mathbf{T}$$

which carries an object $i \in \mathbb{Z}$ to the object $K_i := K(i)$ in $\mathbf{T}$. We say that K satisfies the *inclusion property* if the following holds:

Given morphisms $V, W : i \longrightarrow j$ in $\mathbb{Z}^n_\otimes$ such that V is in the boundary of W, then the induced morphisms $K(V)$ and $K(W)$ in $\mathbf{T}$ satisfy the equation

$$K(V) = d^*_{V,W} K(W) \quad \text{in} \quad \mathrm{Mor}_{\mathbf{T}}(K_i, K_j) \ . \tag{4.1}$$

Here $d^*_{V,W}$ is defined by the structure of $\mathrm{Mor}_{\mathbf{T}}(K_i, K_j)$ as a left cubical set.

4.2. **Definition.** A functor K satisfying the inclusion property (4.1) is called an *n-th order pre-chain complex* in $\mathbf{T}$.

Let $N > M$ and $\mathbb{Z}(N, M) = \{k \in \mathbb{Z}, N \geq k \geq M\}$. Then we obtain the full subcategory

$$\mathbb{Z}(N, M)_\otimes \subset \mathbb{Z}_\otimes$$

consisting of objects $k \in \mathbb{Z}(N, M)$. We say that K is *concentrated* in $\mathbb{Z}(N, M)$ if $K : \mathbb{Z}(N, M)^n_\otimes \longrightarrow \mathbf{T}$ is a functor of n-graded categories.

Assume a quotient functor $\mathbf{T}^0 \longrightarrow \mathbf{A}$ is given, which yields the induced morphisms

$$\delta_i = K(i, \emptyset)_* : K_i \longrightarrow K_{i-1} \qquad \text{in } \mathbf{A}$$

for each $i \in \mathbb{Z}$. We then say that K is *based on the diagram*

$$K_N \longrightarrow \ldots \longrightarrow K_i \xrightarrow{\delta_i} K_{i-1} \xrightarrow{\delta_{i-1}} K_{i-2} \longrightarrow \ldots \longrightarrow K_M \tag{4.3}$$

in the category $\mathbf{A}$.

Now let $\mathbf{C}$ be a category enriched in pointed spaces with zero morphisms. For $\mathbf{T} = \mathrm{Nul}_n \, \mathbf{C}$, we consider a functor K with the inclusion property,

$$K : \mathbb{Z}^n_\otimes \longrightarrow \mathrm{Nul}_n \, \mathbf{C} \ .$$

We have in $\mathbb{Z}^n_\otimes$ the $(n+1)$-tuple of morphisms $i \longrightarrow i - n - 2$:

$$(i, \partial I_{n+1}) = \begin{cases} (i, \emptyset \otimes I_n), \\ (i, I_n \otimes \emptyset), \\ (i, I_r \otimes I_s), \quad r + s = n, r \geq 1, s \geq 1 \end{cases} \tag{4.4}$$

which yields the $(n+1)$-tuple of n-tracks

$$K(i, \partial I_{n+1}) \ = \ (K(i, \emptyset \otimes I_n), \ K(i, I_1 \otimes I_{n-1}), \ \ldots, \ K(i, I_{n-1} \otimes I_1), \ K(i, I_n \otimes \emptyset)).$$

These tracks are represented by maps $I^n \longrightarrow \mathrm{Mor}_{\mathbf{C}}(K_i, K_{i-n-2})$. In fact, these n-tracks yield up to homotopy a well defined map

$$\alpha : S^n \approx \partial(I^{n+1}) \longrightarrow \mathrm{Mor}_{\mathbf{C}}(K_i, K_{i-n-2})$$

on the boundary of the $(n+1)$-cube. Hence, the map α yields an *obstruction element*

$$\mathcal{O}K(i, \partial I_{n+1}) \in D_n(K_i, K_{i-n-2}) = \pi_n \mathrm{Mor}_{\mathbf{C}}(K_i, K_{i-n-2}). \tag{4.5}$$

4.6. **Definition.** We say that K is an *n-th order chain complex* in $\mathrm{Nul}_n \mathbf{C}$ if the obstruction elements (4.5) vanish for all i. This is the *obstruction property* of K.

Below, we study the properties of obstruction elements.

4.7. **Definition.** Let $\mathbf{C}$ be as above and let

$$K_0 \xleftarrow{\delta_1} K_1 \xleftarrow{\delta_2} K_2 \longleftarrow \dots \xleftarrow{\delta_{n+2}} K_{n+2}, \qquad n \geq 1 \tag{4.8}$$

be a diagram in $\mathbf{A} = \pi_0(\mathbf{C})$. Consider all functors

$$K : \mathbb{Z}(0, n+2)^n_{\otimes} \longrightarrow \mathbf{T} = \mathrm{Nul}_n \mathbf{C}$$

satisfying the inclusion property, which are based on the diagram (4.8). Each such functor yields an obstruction element

$$\mathcal{O}K(n+2, \partial I_{n+1}) \in D_n(K_{n+2}, K_0) = \pi_n \mathrm{Mor}_{\mathbf{C}}(K_{n+2}, K_0).$$

The set of all these elements is the classical *higher order Toda bracket*

$$\langle \delta_1, \dots, \delta_{n+2} \rangle \subset D_n(K_{n+2}, K_0)$$

(see [W]).

The set can be empty. If there exists a n-th order chain complex K based on the diagram (4.8), then of course $0 \in \langle \delta_1, \dots, \delta_{n+2} \rangle$ by the obstruction property of K.

5. The W-construction

An alternative description of higher order chain complexes can be given using the bar construction $W\mathcal{K}$, going back to Boardman-Vogt (see [BV, §3] and [Bo, §6]). This construction is a topologically-enriched "cofibrant replacement" for any small category $\mathcal{K}$, which serves as the indexing category for lax versions of functors $\mathcal{K} \to \mathbf{Top}$. A cubically enriched variant of $W\mathcal{K}$ was defined in [BJT, §3.1] and [BB, §3.4]; we shall require the following pointed setting:

5.1. **Definition.** Let $\mathcal{K}$ be a small category enriched in $(\mathbf{Set}^*, \wedge)$ (so zero morphisms o are defined). The *pointed W-construction* on $\mathcal{K}$, denoted by $W_*\mathcal{K}$, is the category enriched in $((\mathbf{Set}^*)^{\square^{\mathrm{op}}}, \overline{\otimes})$ with object set $\mathrm{Obj}\,\mathcal{K}$ defined as follows:

First, for every $a, b \in \mathrm{Obj}\,\mathcal{K}$, the underlying graded pointed category of $W_*\mathcal{K}$ has an (indecomposable) morphism (n-cube) $I^n_{f_\bullet}$ in $W_*\mathcal{K}(a,b)_n$ associated to each composable sequence

$$f_\bullet = (a = a_{n+1} \xrightarrow{f_{n+1}} a_n \xrightarrow{f_n} a_{n-1} \dots a_1 \xrightarrow{f_1} a_0 = b) \tag{5.2}$$

of length $n+1$ in $\mathcal{K}$. In addition, $W_*\mathcal{K}(a,b)$ has a degenerate $(n+k)$-cube $(s^{j_1})^* \dots (s^{j_k})^* I^n_{f_\bullet}$ for each iterated projection $s^{j_k} \dots s^{j_1} : I^{n+k} \to I^n$ in $\square$ (with identifications according to the cubical identities). The zero morphism in degree k is $I^k_o := (s_k)^* \dots (s_1)^* I^0_o$, and we identify $I^n_{f_\bullet}$ with I^n_o whenever at least one of the maps $f_1, \dots, f_{n+1}$ is o. Then $W_*\mathcal{K}$ is freely generated as a graded category with zero morphisms by these cubes. Composition in the category $W_*\mathcal{K}$ is denoted by $\overline{\otimes}$.

The cubical structure is determined by the face maps of the non-degenerate indecomposable cubes $I^n_{f_\bullet}$ and the cubical identities, as follows:

(a) The i-th 1-face of $I^n_{f_\bullet}$ is $I^{n-1}_{f_1 \circ \dots \circ (f_i \cdot f_{i+1}) \circ \dots f_{n+1}}$ – that is, we carry out (in the category $\mathcal{K}$) the i-th composition in $f_\bullet$.

(b) The i-th 0-face of $I^n_{f_\bullet}$ is the composite $I^i_{f_0 \circ \dots \circ f_i} \overline{\otimes} I^{n-i-1}_{f_{i+1} \circ \dots \circ f_{n+1}}$.

(c) The cubical structure on the composites $I^j_{f_\bullet} \overline{\otimes} I^k_{g_\bullet}$ is defined by (1.5) (or (2.1)).

5.3. **Definition.** Let Γ be the category enriched in $(\mathbf{Set}^*, \wedge)$ with object set $\mathbb{Z}$ and a single non-zero arrow $d_{k+1} : k+1 \to k$ for each $k \in \mathbb{Z}$, satisfying $d_k \circ d_{k+1} = o$ for all k.

5.4. **Proposition.** *Let $\mathcal{M}$ be a category enriched in cubical sets with zero morphisms. There is a one-to-one correspondence between pointed cubical functors $W_*\Gamma \to \mathcal{M}$ and pre-chain complexes in* $\operatorname{nul}\mathcal{M}$*, which restricts to a one-to-one correspondence between pointed cubical functors $\operatorname{sk}_n W_*\Gamma \to \mathcal{M}$ and n-th order pre-chain complexes in $\operatorname{nul}_n \mathcal{M}$.*

Proof. Since $\mathbb{Z}_\otimes$ is a free graded category, by Lemma 3.3, we can define a one-to-one functor of graded categories $\Phi : \mathbb{Z}_\otimes \to W_*\Gamma$ which is the identity on objects by setting $\Phi(i, \emptyset) := I^0_{d_i}$ and $\Phi(i, I_k) := I^k_{f_\bullet}$, for $f_\bullet := (i \xrightarrow{d_i} i-1 \to \dots i-k \xrightarrow{d_{i-k}} i-k-1)$.

We can endow $\mathbb{Z}_\otimes$ with the structure of a category $\widehat{\mathbb{Z}}_\otimes$ enriched in $((\mathbf{Set}^*)^{\square^{\mathrm{op}}}, \overline{\otimes})$ by setting $d^*_{V,W}(W) = V$ if $V \subseteq W$, and adding zero morphisms. Note that a functor $K : \mathbb{Z}_\otimes \to \operatorname{nul}\mathcal{M}$ is a pre-chain complex if and only if it induces a pointed cubical functor $\hat{K} : \widehat{\mathbb{Z}}_\otimes \to \operatorname{nul}\mathcal{M}$.

The universal enveloping functor $\mathcal{U} : (\mathbf{Set}^*)^{\square^{\mathrm{op}}} \to (\mathbf{Set}^*)^{\square^{\mathrm{op}}}$ of Remark 1.2 is monoidal with respect to $\overline{\otimes}$, so the adjunction (1.4) extends to categories of enriched functors. Moreover, Φ induces a natural isomorphism of pointed cubical categories

$$\mathcal{U}(\widehat{\mathbb{Z}}_\otimes) \;\cong\; W_*\Gamma \;, \tag{5.5}$$

so left cubical functors $\widehat{\mathbb{Z}}_\otimes \to \operatorname{nul}\mathcal{M}$ indeed correspond to pointed cubical functors $W_*\Gamma \to \mathcal{M}$. Since this correspondence preserves the grading, the same is true for n-th order pre-chain complexes. $\square$

6. Resolutions and derived functors

We now recall some basic definitions of resolutions and derived functors in the context of additive categories:

Let $\mathbf{A}$ be a category enriched in abelian groups, i.e., a preadditive category. Then we denote the morphism sets in $\mathbf{A}$ by

$$\operatorname{Hom}_{\mathbf{A}}(X, Y) = \operatorname{Mor}_{\mathbf{A}}(X, Y)$$

for objects X, Y in $\mathbf{A}$. This is an abelian group, and morphisms $f : X' \longrightarrow X$ and $g : Y' \longrightarrow Y$ in $\mathbf{A}$ induce homomorphisms $\operatorname{Hom}(f, Y)$ and $\operatorname{Hom}(X, g)$. Let $\mathbf{a}$ be a full subcategory of $\mathbf{A}$.

6.1. **Definition.** Let X be an object in $\mathbf{A}$. An $\mathbf{a}$-*resolution* of X is a diagram

$$A_\bullet = (\dots \xrightarrow{\delta_2} A_1 \xrightarrow{\delta_1} A_0 \xrightarrow{\delta_0} A_{-1})$$

in $\mathbf{A}$ with $A_{-1} = X$ and $A_i \in \mathbf{a}$ for $i \geq 0$, such that, for all objects B in $\mathbf{a}$, the induced diagram $\operatorname{Hom}(B, A_\bullet)$ is an exact sequence of abelian groups; in particular, $\operatorname{Hom}(B, \delta_0)$ is surjective.

An $\mathbf{a}$-*coresolution* of Y is a diagram

$$A^\bullet = (A_1 \xrightarrow{\delta_1} A_0 \xrightarrow{\delta_0} A_{-1} \xrightarrow{\delta_{-1}} \dots)$$

in $\mathbf{A}$ with $A_1 = Y$ and $A_i \in \mathbf{a}$ for $i \geq 0$, such that for all objects B in $\mathbf{a}$ the induced diagram $\operatorname{Hom}(A^\bullet, B)$ is an exact sequence of abelian groups. Here $\operatorname{Hom}(\delta_1, B)$ is surjective.

The next result is proved in [BJ2, 1.3]:

6.2. **Lemma.** *Suppose*

(1) the coproduct of any family of objects of $\mathbf{a}$ *exists in* $\mathbf{A}$ *and belongs to* $\mathbf{a}$ *again,*
(2) there is a small subcategory $\mathbf{g}$ *of* $\mathbf{a}$ *such that every object of* $\mathbf{a}$ *is a retract of a coproduct of a family of objects from* $\mathbf{g}$*,*

then every object of $\mathbf{A}$ *has an* $\mathbf{a}$*-resolution.*

The dual statement also holds: suppose

(3) the product of any family of objects of $\mathbf{a}$ *exists in* $\mathbf{A}$ *and belongs to* $\mathbf{a}$ *again,*
(4) there is a small subcategory $\mathbf{g}$ *of* $\mathbf{a}$ *such that every object of* $\mathbf{a}$ *is a retract of a product of a family of objects from* $\mathbf{g}$*,*

then every object of $\mathbf{A}$ *has an* $\mathbf{a}$*-coresolution.*

One obtains (3) and (4) by replacing the categories $\mathbf{A}$ and $\mathbf{a}$, respectively, in (1) and (2) by the opposite categories $\mathbf{A}^{\mathrm{op}}$ and $\mathbf{a}^{\mathrm{op}}$. Given a functor $F : \mathbf{A} \longrightarrow \mathcal{A}$, where $\mathcal{A}$ is an abelian category and F is linear (i.e., enriched in the category of abelian groups), then *derived functors* are defined by the homology (respectively, cohomology)

$$(L_n F)(X) = \mathrm{H}_n F(A_\bullet),$$
$$(R^n F)(Y) = \mathrm{H}^n F(A^\bullet).$$

Here $A_\bullet$ (respectively, $A^\bullet$) is a resolution of X (respectively, a coresolution of Y).

We need the following concept of a Σ-algebra which allows the definition of a bigraded Ext -group.

6.3. **Definition.** A Σ-*algebra* $\mathbf{A} = (\mathbf{A}, \mathbf{a}, \Sigma)$ is an additive category $\mathbf{A}$ together with an additive subcategory $\mathbf{a}$ and an additive endofunctor $\Sigma : \mathbf{A} \to \mathbf{A}$ of $\mathbf{A}$ which carries $\mathbf{a}$ to $\mathbf{a}$ and which carries an $\mathbf{a}$-resolution $A_\bullet$ of X in $\mathbf{A}$ to an $\mathbf{a}$-resolution $\Sigma A_\bullet$ of ΣX in $\mathbf{A}$. Dually, we define an Ω-*algebra* $\mathbf{A} = (\mathbf{A}, \mathbf{a}, \Omega)$ where Ω carries an $\mathbf{a}$-coresolution of X in $\mathbf{A}$ to an $\mathbf{a}$-coresolution of ΩX in $\mathbf{A}$.

Given a Σ-algebra $\mathbf{A}$ and objects X, Y in $\mathbf{A}$, we define the *bigraded* Ext-*group* by the cohomology

$$\begin{aligned} E_2^{r,s} &= \mathrm{Ext}^r_{\mathbf{A}}(\Sigma^s X, Y), \\ &= \mathrm{H}^r \operatorname{Hom}_{\mathbf{A}}(\Sigma^s A_\bullet, Y), \\ &= \operatorname{kernel} \operatorname{Hom}_{\mathbf{A}}(\Sigma^s \delta_{r+1}, Y) / \operatorname{image} \operatorname{Hom}_{\mathbf{A}}(\Sigma^s \delta_r, Y). \end{aligned} \tag{6.1}$$

Here $\Sigma^s = \Sigma \circ \ldots \circ \Sigma$ is the s-fold composite of Σ. Such groups appear in the E_2-term of the Adams spectral sequence.

7. Mapping algebras

In this section we consider topological analogues of Σ-algebras and Ω-algebras of Definition 6.3, in order to provide a setting for defining higher order resolutions, and thus higher order derived functors.

7.1. **Definition.** Let $\mathbf{C}$ be a category enriched in pointed spaces with zero morphisms. Then $\mathbf{C}$ is a Σ-*mapping algebra* if the category $\mathbf{A} = \pi_0 \mathbf{C}$ is a Σ-algebra and the bifunctor $(n \geq 1)$

$$D_n : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \longrightarrow \mathbf{Ab}$$

$$D_n(X, Y) = \pi_n \operatorname{Mor}_{\mathbf{C}}(X, Y)$$

satisfies

$$\tau_\Sigma : D_n(X, Y) = \operatorname{Hom}_{\mathbf{A}}(\Sigma^n X, Y)$$

for X in $\mathbf{a}$ and Y in $\mathbf{A}$. Here $\Sigma^n = \Sigma \circ \ldots \circ \Sigma$ is the n-fold composite of the endofunctor Σ of $\mathbf{a}$.

Dually $\mathbf{C}$ is the Ω-*mapping algebra* if the category $\mathbf{A} = \pi_0 \mathbf{C}$ is an Ω-algebra and $(n \geq 1)$

$$\tau_\Omega : D_n(X, Y) = \operatorname{Hom}_{\mathbf{A}}(X, \Omega^n Y)$$

for X in $\mathbf{A}$ and Y in $\mathbf{a}$.

7.2. **Definition.** A Σ-mapping algebra $\mathbf{C}$ is *complete* if the endofunctor Σ of $\mathbf{A} = \pi_0 \mathbf{C}$ is induced by an endofunctor Σ of $\mathbf{C}$ and if a binatural transformation

$$\tau_\Sigma : \operatorname{Mor}_{\mathbf{C}}(\Sigma A, Y) \longrightarrow \Omega \operatorname{Mor}_{\mathbf{C}}(A, Y)$$

is given, where we use the topological loop space functor on pointed spaces. Moreover, the functor $\Sigma : \mathbf{C} \longrightarrow \mathbf{C}$ preserves zero morphisms and coproducts in $\mathbf{C}$.

An Ω-mapping algebra $\mathbf{C}$ is *complete* if the endofunctor Ω of $\mathbf{A} = \pi_0 \mathbf{C}$ is induced by an endofunctor Ω of $\mathbf{C}$ and if a binatural transformation

$$\tau_\Omega : \operatorname{Mor}_{\mathbf{C}}(Y, \Omega A) \longrightarrow \Omega \operatorname{Mor}_{\mathbf{C}}(Y, A)$$

is given. Moreover, the functor $\Omega : \mathbf{C} \to \mathbf{C}$ preserves zero morphisms and products in $\mathbf{C}$. Iteration of τ_Σ (respectively, τ_Ω) induces the isomorphisms τ_Σ (respectively, τ_Ω) in Definition 7.1.

7.3. **Example.** There are a number of different simplicial model categories of spectra, including the Γ-spaces of [BF], the S-modules of [EKMM], and the symmetric spectra of [HSS]. All of these have pointed versions (cf. [Hov, Prop. 1.1.8]). In this and later sections, we let $\mathbf{Spec}^*$ be any category of pointed spectra which is enriched in pointed topological spaces (or simplicial sets), with function spaces of pointed maps

$$\mathrm{Mor}(X,Y) = \mathrm{Map}^*(X,Y) \quad \text{for } X \text{ and } Y \text{ in } \mathbf{Spec}^* .$$

We always assume that X and Y are both fibrant and cofibrant in our chosen model category.

Clearly zero morphisms $o : X \to * \to Y$ are defined in $\mathbf{Spec}^*$. Let $\mathcal{X}$ be a class of objects in $\mathbf{Spec}^*$ such that $\mathcal{X}$ is closed under coproducts and suspension Σ:,, that is, for $A, A' \in \mathcal{X}$ we have $A \vee A', \Sigma A \in \mathcal{X}$. Then we have

$$\{\mathcal{X}\} \subset \mathbf{Spec}^* ,$$

where $\{\mathcal{X}\}$ is the full subcategory in $\mathbf{Spec}^*$ with objects in $\mathcal{X}$. Then $\mathbf{C} = \mathbf{Spec}^*$ with $\mathbf{a} = \pi_0\{\mathcal{X}\} \subset \mathbf{A} = \pi_0\mathbf{C}$ is a complete Σ-mapping algebra.

Dually, let $\mathcal{Y}$ be a class of objects in $\mathbf{Spec}^*$ such that $\mathcal{Y}$ is closed under products and loop functor Ω, that is, for $B, B' \in \mathcal{Y}$, we have $B \times B'$ $(\Omega B \in \mathcal{Y})$. Then we have

$$\{\mathcal{Y}\} \subset \mathbf{Spec}^* ,$$

where $\{\mathcal{Y}\}$ is the full subcategory in $\mathbf{Spec}^*$ with objects in $\mathcal{Y}$. Then $\mathbf{C} = \mathbf{Spec}^*$ with $\mathbf{a} = \pi_0\{\mathcal{Y}\} \subset \mathbf{A} = \pi_0\mathbf{C}$ is a complete Ω-mapping algebra.

7.4. **Example.** Let p be a prime and let $H = \mathrm{H}(\mathbb{Z}/p)$ be the Eilenberg-Mac Lane spectrum. Let $\mathcal{Y}$ be given by all products

$$\Omega^{n_1}H \times \Omega^{n_2}H \times \ldots \times \Omega^{n_k}H$$

with $k \geq 0$, $n_i \geq 0$ for $i = 1, \ldots, k$. Then $\mathbf{C} = \mathbf{Spec}^*$ with $\mathbf{a} = \pi_0\{\mathcal{Y}\}$ is a complete Ω-mapping algebra, which we call the Eilenberg-Mac Lane mapping algebra. This is used in the Adams spectral sequence.

7.5. *Remark.* In the examples of mapping algebras above the category $\mathbf{C} = \mathbf{Spec}^*$ is very large. For computations, however, we consider only the mapping algebras $\mathbf{C}'$ which are generated by $\{\mathcal{X}\}$ (respectively, $\{\mathcal{Y}\}$) and two further objects X and Y in $\mathbf{Spec}^*$.

8. Existence of higher order resolutions

We can use the definitions of Section 7 to state our main results on resolutions, which will be proved subsequently.

Let $\mathbf{C}$ be a Σ-mapping algebra with $\mathbf{a} \subset \mathbf{A} = \pi_0\mathbf{C}$. If a n-th order chain complex

$$K : \mathbb{Z}(\infty, -1)^n_{\otimes} \longrightarrow \mathrm{Nul}_n\,\mathbf{C}$$

is based on an $\mathbf{a}$-resolution in $\mathbf{A}$,

$$A_\bullet = (\ \ldots \xrightarrow{\delta_2} A_1 \xrightarrow{\delta_1} A_0 \xrightarrow{\delta_0} A_{-1}\,) ,$$

of $X = A_{-1}$, we say that K is an *n-th order resolution* of X in $\mathrm{Nul}_n\,\mathbf{C}$.

8.1. **Resolution Theorem.** *If there exists an $\mathbf{a}$-resolution $A_\bullet$ of X in $\mathbf{A}$, then there exists an n-th order resolution K of X in $\mathrm{Nul}_n\,\mathbf{C}$ $(n \geq 1)$. In fact, given an $\mathbf{a}$-resolution $A_\bullet$ of X in $\mathbf{A}$, an n-th order resolution K of X exists which is based on $A_\bullet$.*

8.2. *Remark.* The Theorem shows that, if 'minimal' $\mathbf{a}$-resolutions exist (as in the case of the Adams spectral sequence), then also an n-th order minimal resolution exists which is based on a minimal resolution in $\mathbf{A}$. This is of high importance for computations.

Dually, let $\mathbf{C}$ be a Ω-mapping algebra with $\mathbf{a} \subset \mathbf{A} = \pi_0\mathbf{C}$. If an n-th order chain complex

$$L : \mathbb{Z}(+1, -\infty)^n_{\otimes} \longrightarrow \mathrm{Nul}_n\,\mathbf{C}$$

is based on an $\mathbf{a}$-coresolution in $\mathbf{A}$

$$A^\bullet = (A_1 \xrightarrow{\delta_1} A_0 \xrightarrow{\delta_0} A_{-1} \xrightarrow{\delta_{-1}} \ldots) ,$$

with $A_1 = Y$, we say that L is an *n-th order coresolution* of X in $\mathrm{Nul}_n\,\mathbf{C}$.

8.3. **Dual Resolution Theorem.** *If there exists an* $\mathbf{a}$*-coresolution* $A^\bullet$ *of* Y *in* $\mathbf{A}$*, then there exists an* n*-th order coresolution* L *of* Y *in* $\mathrm{Nul}_n\,\mathbf{C}$ $(n \geq 1)$*. In fact, given an* $\mathbf{a}$*-coresolution* $A^\bullet$ *of* Y *in* $\mathbf{A}$*, an* n*-th order coresolution* L *of* Y *exists which is based on* $A^\bullet$*.*

8.4. *Remark.* In view of Lemma 3.6 (a) in [BJ2], a 1-order resolution in $\mathrm{Nul}_1\,\mathbf{C}$ is a secondary resolution in the sense of [BJ2].

9. Left cubical balls

For the proof of the Resolution Theorems 8.1 and 8.3, we require the notion of a *left cubical ball*, which serves as a book-keeping device to describe the combinatorics of higher tracks, and allows us to define the associated obstructions.

A *ball* of dimension n is a finite regular CW-complex B with a subcomplex ∂B and a homeomorphism of pairs

$$(E^n, S^{n-1}) \approx (B, \partial B)$$

where E^n is the Euclidean ball. Two balls B, B' are equivalent if there is a cellular isomorphism $B \approx B'$. A ball B is a union

$$B = B_1 \cup \ldots \cup B_k$$

of closed n-cells B_i in B. We say that A is a *sub-ball* of B if $A = B_{i_1} \cup \ldots \cup B_{i_t}$ for $1 \leq i_1 < \ldots < i_t \leq k$ is a ball and if for $t < k$, the closure of the complement $B - A$ in B is also a ball, denoted by A_B, so that $B = A \cup A_B$.

If A is also a sub-ball of a ball C with $S = A \cap A_B = A \cap A_C$, then we obtain the *union of complements*

$$A_B \cup A_C \;=\; A_B \cup_s A_C \ ,$$

which is also a ball.

9.1. **Example.** Let T^n_0 be the union of all cells $I^{i-1} \times \{0\} \times I^{n+i-1}$ in I^{n+1}, and let T^n_1 be the union of all cells $I^{i-1} \times \{1\} \times I^{n+i-1}$ $(i = 1, \ldots, n+1)$. Then T^n_0 and T^n_1 are balls of dimension n, with $n+1$ closed n-cells.

9.2. **Definition.** A *left cubical ball* is a ball B with a 0-vertex $0 \in B - \partial B$ with the following properties. Each closed n-cell B_i is equivalent to I^n, and each closed $(n-1)$–cell $\overline{e}$ is equivalent to I^{n-1}, such that for $\overline{e} \subset B_i \cap B_j$ the diagram

$$\begin{tikzcd}
B_j & \overline{e} \arrow[l, phantom, "\supset"] \arrow[r, phantom, "\subset"] & B_i \\
I^n \arrow[u, "h_j", "\approx"'] & I^{n-1} \arrow[l, "d_{e,j}"'] \arrow[r, "d_{e,i}"] \arrow[u, "\approx"'] & I^n \arrow[u, "\approx", "h_i"']
\end{tikzcd}$$

commutes. Here $d_{e,j}$ and $d_{e,i}$ are morphisms in the left cubical category $\overline{\square}$. The vertex 0 is also a vertex of each B_i and the equivalence $h_i : I^n \approx B^i$ carries 0 to 0. Moreover, the union

$$h_1(T^{n-1}_1) \cup \ldots \cup h_k(T^{n-1}_1) = \partial B$$

is the boundary of B.

Examples of left cubical balls of dimension 2 appear in in Figures 1 and 2.

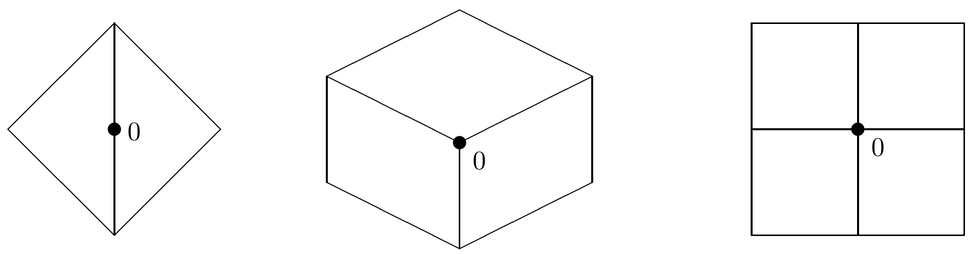

FIGURE 1. Some left cubical balls of dimension 2

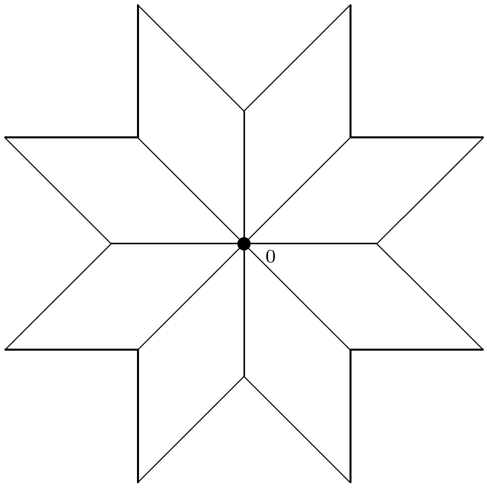

Figure 2. A 2-dimensional left cubical ball

9.3. **Example.** The push out of $I^n \longleftarrow T_0^{n-1} \longrightarrow I^n$, called the *double* of I^n, is a left cubical ball. Moreover, T_0^n is a left cubical ball.

9.4. **Lemma.** *Let A be a sub-ball of B and C, where B and C are left cubical; then the union of complements $A_B \cup A_C$ is left cubical.*

9.5. *Remark.* Let B be a left cubical ball of dimension n with k closed n-cells. Then B is equivalent to the double of I^n for $k = 2$ and B is equivalent to T_0^n for $k = n+1$. For $2 < k < n+1$, such a ball does not exist. For $k \geq n+1$ there is a 1-1-correspondence between left cubical balls (up to equivalence) and simplicial complexes homeomorphic to the $(n-1)$-sphere S^{n-1}. The correspondence carries B to the boundary of a small neighbourhood of 0 in B.

10. Obstructions

Let X be a pointed space with $o \in X$ the base point. Let B be a ball and let $a : B \longrightarrow X$ be a map with $a(\partial B) = o$. We obtain the map

(10.1)
$$\overline{a} : S^n \approx E^n/S^{n-1} \approx B/\partial B \xrightarrow{a} X ,$$

which represents an element $\mathcal{O}(a) \in \pi_n(X)$ in the n-th homotopy group of X. Now let $B = B_1 \cup \ldots \cup B_k$ be a left cubical ball. Then

$$I^n \xrightarrow{h_i} B_i \subset B \xrightarrow{a} X$$

is a left n-cube representing an n-track $a_i \in \mathrm{Nul}_n(X)_n$.

Then for $\overline{e} \subset B_i \cap B_j$ we have the *gluing condition in* B (see Definition 9.2).

(10.2)
$$d^*_{e,i} a_i = d^*_{e,j} a_j .$$

10.3. **Lemma.** *Each k-tuple $(a_1, \ldots, a_k)$ of n-tracks a_i in* $\mathrm{Nul}_n(X)_n$ *satisfying* (10.2) *yields (up to homotopy relative to the boundary) a well defined map $a : B \to X$ with $a(\partial B) = o$. This defines the* obstruction $\mathcal{O}_B(a_1, \ldots, a_k) = \mathcal{O}(a)$ *in $\pi_n(X)$ as above.*

Now let $B = T_0^n = B_1 \cup \ldots \cup B_{n+1}$ and let $a_1, \ldots, a_{n+1} \in \mathrm{Nul}_n(X)_n$ be n-tracks satisfying (10.2). Then we get the *boundary property*:

10.4. **Lemma.** $\mathcal{O}_{T_0^n}(a_1, \ldots, a_{n+1}) = 0$ *if and only if there exist* $\overline{a} \in \mathrm{Nul}_{n+1}(X)_{n+1}$ *with $\partial^i \overline{a}$ representing a_i.*

Proof. We choose representatives a_i' of a_i which define a map

$$\overline{\overline{a}} : \partial I^{n+1} \longrightarrow X$$

with $\overline{\overline{a}}(T_1^n) = 0$ and $\overline{\overline{a}}|_{T_0^n} = a'_1 \cup \ldots \cup a'_{n+1}$. Here $\overline{\overline{a}}$ extends to I^{n+1} if and only if $\mathcal{O}(a'_1 \cup \ldots \cup a'_{n+1}) = 0$. □

The next result is the *Complement Rule.*

10.5. **Lemma.** *Let* $B = A_1 \cup \ldots \cup A_r \cup B_1 \cup \ldots \cup B_t$ *and* $C = A_1 \cup \ldots \cup A_r \cup C_1 \cup \ldots \cup C_s$ *be left cubical balls with the sub-ball* $A = A_1 \cup \ldots \cup A_r$. *Then*

$$\mathcal{O}_C(a_1, \ldots, a_r, c_1, \ldots c_s) = 0$$

implies that for $D = A_B \cup A_C$

$$\mathcal{O}_B(a_1, \ldots, a_r, b_1, \ldots, b_t) = \mathcal{O}_D(b_1, \ldots, b_t, c_1, \ldots, c_s).$$

Of course, there is the following *Double Rule*:

10.6. **Lemma.** *If* $B = B_1 \cup B_2$ *is the double of* I^n *then for* $a_1 = a_2$ *we have:*

$$\mathcal{O}_B(a_1, a_2) = 0 \ .$$

10.7. **Definition.** Let $B = B_1 \cup \ldots \cup B_k$ be a left cubical ball. Then for each $1 \le i \le k$ we have a map

$$\overline{\varepsilon}_i : I^n \approx B_i \subset B \approx E^n \ ,$$

where I^n and E^n are oriented by the inclusions of I^n and E^n in $\mathbb{R}^n$. We set $\varepsilon_i = +1$ if the map $\overline{\varepsilon}_i$ is orientation preserving, otherwise $\varepsilon_i = -1$. We call ε_i the *orientation sign* of B_i.

Let $B = B_1 \cup B_2$ be the double of I^n. Then $\varepsilon_1 = -\varepsilon_2$, and we can choose B_1 so that $\varepsilon_1 = 1$, In this case we define the *action* + of $\alpha \in \pi_n(X)$ on an n-track $a \in \mathrm{Nul}_n(X)_n$ by the n-track $a + \alpha$ which satisfies $\mathcal{O}(a + \alpha, a) = \alpha$ $(n \ge 1)$.

10.8. **Lemma.** *The action + yields a well defined effective and transitive action of the group* $\pi_n(X)$ *on the set of all n-tracks* $a \in \mathrm{Nul}_n(X)_n$ *which coincide on the boundary (that is,* $\partial^i a = b_i$, *where* $(b_1, \ldots, b_n)$ *is fixed).*

10.9. **Lemma.** *Let* $B = B_1 \cup \ldots \cup B_k$ *be a left cubical ball and let* $\mathcal{O}_B(a_1, \ldots, a_k)$, $\mathcal{O}_B(a'_1, \ldots, a'_k)$ *be defined, where*

$$\begin{cases} a'_i = a_i & \textit{for } i \neq j \\ a'_j = a_j + \alpha & \textit{for } i = j, \ \alpha \in \pi_n(X). \end{cases}$$

Then we have the Action Formula*:*

$$\mathcal{O}(a'_1, \ldots, a'_k) = \mathcal{O}(a_1, \ldots, a_k) + \varepsilon_j \alpha \ .$$

11. n-TRACK CATEGORIES

We now define the concept of an n-track category, which encompasses the properties needed for the construction of higher order resolutions.

Let $\mathbf{C}$ be a category enriched in pointed spaces with zero morphisms. Let $n \ge 1$ and let

$$\begin{aligned} &\mathbf{T} \ = \ \mathrm{Nul}_n \, \mathbf{C} \ , \\ &\mathbf{A} \ = \ \pi_0 \mathbf{C} \ , \\ &D : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \longrightarrow \mathbf{Ab} \ , \quad D(X,Y) = \pi_n \, \mathrm{Mor}_{\mathbf{C}}(X,Y) \quad , \\ &\mathcal{O}_B(a_1, \ldots, a_k) \qquad\qquad \text{is defined in} \quad \mathrm{Nul}_n \, \mathrm{Mor}_{\mathbf{C}}(X,Y) \quad (\text{see } (10.1)). \end{aligned}$$

Then $(\mathbf{T}, \mathbf{A}, D, \mathcal{O}_B)$ has the following properties of an n-track category. Here we assume for $n = 1$ that $\pi_1 \, \mathrm{Mor}_{\mathbf{C}}(X,Y)$ is abelian for all objects X, Y in $\mathbf{C}$.

11.1. **Definition.** An *n-track category* $(n \ge 1)$

$$\mathbf{T} = (\mathbf{T}, \mathbf{A}, D, \mathcal{O})$$

is given by an n-graded category $\mathbf{T}$, a quotient functor $\mathbf{T}^0 \longrightarrow \mathbf{A}$; a bifunctor $D : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \longrightarrow \mathbf{Ab}$ and an obstruction operator $\mathcal{O}$. The following properties hold:

(1) $\mathbf{T}$ is enriched in left n-cubical sets and has zero morphisms, that is, for all objects X, Y in $\mathbf{T}$, we have the n-cubical set $\mathrm{Mor}_{\mathbf{T}}(X,Y)$ with operators $(d_o^i)^* = \partial^i$ and zero elements $o^t \in \mathrm{Mor}_{\mathbf{T}}(X,Y)_t$ such that

$$\begin{aligned} \partial^i(fg) &= (\partial^i f)g \quad \text{for} \quad i \leq \dim(f) \\ \partial^i(fg) &= f(\partial^{i-\dim(f)}g) \quad \text{for} \quad i > \dim(f) \\ o^t g &= o^{t+\dim(g)} \\ f o^h &= o^{\dim(f)+h} . \end{aligned}$$

Here fg is the composite in the n-graded category $\mathbf{T}$, which is defined if $\dim(f)+\dim(g) \leq n$.

(2) The 0-skeleton $\mathbf{T}^0$ is the subcategory of $\mathbf{T}$ consisting of morphisms f with $\dim(f) = 0$, this is a category together with a functor $q : \mathbf{T}^0 \longrightarrow \mathbf{A}$ which is the identity on objects and full (quotient functor). Moreover, D is a bifunctor

$$D : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \longrightarrow \mathbf{Ab}$$

into the category of abelian groups. Here D defines via q a bifunctor on $\mathbf{T}^0$ which satisfies $(o^0)^* = o$ and $(o^0)_* = o$. For a zero morphism $o^0 : X \longrightarrow Y$ in $\mathbf{T}^0$ we obtain the zero morphism $o_{X,Y} = q(o^0)$ in $\mathbf{A}$.

For $f : X \longrightarrow Y$ in $\mathbf{T}^0$, we have $q(f) = o_{X,Y}$ if and only if there is $F : X \longrightarrow Y$ in $\mathbf{T}$ with $\dim(F) = 1$ and $\partial^1 F = f$. This is the boundary property in dimension 1.

(3) The obstruction operator $\mathcal{O}$ yields for each left cubical ball B an element

$$\mathcal{O}_B(a_1, \ldots, a_k) \in D(X,Y)$$

where $a_1, \ldots, a_k \in \mathrm{Mor}_{\mathbf{T}}(X,Y)_n$ is a k-tuple satisfying the gluing condition in B, see (10.2).

This obstruction operator satisfies the complement rule, the double rule, and the action formula as in Section 10. Here the *action* $+$ of $D(X,Y)$ on the set $\mathrm{Mor}_{\mathbf{T}}(X,Y)_n$ is defined by:

$$\text{if} \quad \mathcal{O}_B(a_1, a) = \alpha, \quad \text{then} \quad a_1 = a + \alpha .$$

Here B is the double of I^n with $\varepsilon_1 = +1$.

The action $+$ is transitive and effective on the set of all elements a in $\mathrm{Mor}_{\mathbf{T}}(X,Y)_n$ which coincide on the boundary (that is, $\partial^i a = b_i$, where $(b_1, \ldots, b_n)$ is fixed).

(4) The obstruction operator satisfies for

$$f \in \mathrm{Mor}_{\mathbf{T}}(X', X)_0 \quad \text{and} \quad g \in \mathrm{Mor}_{\mathbf{T}}(Y, Y')_0$$

the *naturality rule*

$$\begin{aligned} \mathcal{O}_B(ga_1, \ldots, ga_k) &= g_* \mathcal{O}_B(a_1, \ldots, a_k) \\ \mathcal{O}_B(a_1 f, \ldots, a_k f) &= f^* \mathcal{O}_B(a_1, \ldots, a_k) . \end{aligned}$$

Here f^* AND g_* denote the induced maps on D. This implies $g(a+\alpha) = ga + g_*\alpha$ and $(a+\alpha)f = af + f^*\alpha$.

(5) The obstruction operator satisfies the following *triviality rule*: For morphisms

$$Z \xleftarrow{f} Y \xleftarrow{g} X$$

in $\mathbf{T}$ with $\dim(f), \dim(g) \leq n$ and

$$\dim(f) + \dim(g) = n+1$$

we have the $(n+1)$-tuple $(a_1, \ldots, a_{n+1})$ in $\mathrm{Mor}_{\mathbf{T}}(X,Z)_n$ given by

$$a_t = \begin{cases} (\partial^t f)g & \text{for} \quad 1 \leq t \leq \dim(f), \\ f(\partial^{t-\dim(f)} g) & \text{for} \quad \dim(f) < t \leq n+1. \end{cases}$$

This $(n+1)$-tuple satisfies the gluing condition in $B = \mathbf{T}_0^n$. The associated obstruction

$$\mathcal{O}_B(a_1, \ldots a_{n+1}) = 0$$

is trivial.

We now are able to define n-th order chain complexes in an n-track category, for this we replace $\mathrm{Nul}_n\,\mathbf{C}$ by $\mathbf{T}$ as follows, see Section 4.

11.2. **Definition.** Let $(\mathbf{T},\mathbf{A},D,\mathcal{O})$ be an n-track category. A functor of n-graded categories

$$K:\mathbb{Z}(N,M)^n_\otimes \longrightarrow \mathbf{T}$$

satisfying the inclusion property (4.1) is an *n-th order pre-chain complex* in $\mathbf{T}$. This is an *n-th order chain complex* in $\mathbf{T}$ if for $i, i-n-2\in\mathbb{Z}(N,M)$, the obstructions

$$\mathcal{O}K(i,\partial I_{n+1})=\mathcal{O}_B(b_1,\dots,b_{n+1})=0$$

vanish. Here B is the left cubical ball $B=\mathbf{T}^n_0$, and

$$K(i,\partial I_{n+1})=\begin{cases} b_1=K(i,\emptyset\otimes I_n)\\ b_{r+1}=K(i,I_r\otimes I_{n-r}) & \text{for } 1\le r\le n-1\\ b_{n+1}=K(i,I_n\otimes\emptyset)\end{cases}$$

(see (4.5)). Since K is a functor we have

$$\begin{array}{lclcl} K(i,\emptyset\otimes I_n) &=& K(i-n-1,\emptyset)K(i,I_n) &=& \delta_{i-n-1}K(i,I_n)\\ K(i,I_r\otimes I_s) &=& K(i-s-1,I_r)K(i,I_s) && \\ K(i,I_n\otimes\emptyset) &=& K(i-1,I_n)K(i,\emptyset) &=& K(i-1,I_n)\delta_i \end{array}$$

where the right hand side denotes composition in $\mathbf{T}$. We define higher order Toda brackets in $\mathbf{T}$ in the same way as in Definition 4.6

$$\langle\delta_1,\dots,\delta_{n+2}\rangle\subset D(K_{n+2},K_0)\ .$$

12. Track categories and 1-track categories

We show that each abelian track category with zero morphisms has the structure of a 1-track category. This shows that n-track categories are n-dimensional analogues of track categories for every $n\ge 1$.

A *track category* is a category $\mathbf{C}$ enriched in groupoids. For objects X, Y in $\mathbf{C}$ we have the groupoid $\mathrm{Mor}_{\mathbf{C}}(X,Y)$ with objects f, g and morphisms $F:f\longrightarrow g$.

The morphisms $F:f\longrightarrow f$ form the automorphism group $\mathrm{Aut}_{\mathbf{C}}(f)$, and we write $f\simeq g$ if there is $F:f\longrightarrow g$. Let $\dim(f)=0$, $\dim(F)=1$, $(d^1_0)^*F=f$, and $(d^1_1)^*F=g$. Morphisms of dimension 0 form the category $\mathbf{C}_0$, and the homotopy relation $\simeq$ defines the homotopy category

$$\mathbf{A}=\pi_0\mathbf{C}=\mathbf{C}_0/\simeq\ .$$

Let $\mathbf{C}$ be *abelian*, i.e., all automorphism groups $\mathrm{Aut}_{\mathbf{C}}(f)$ are abelian groups. We assume that $\mathbf{C}$ has zero morphisms $o_{X,Y}\in\mathrm{Mor}_{\mathbf{C}}(X,Y)_0$. Then we get a bifunctor

$$D:\mathbf{A}^{\mathrm{op}}\times\mathbf{A}\longrightarrow\mathbf{Ab},$$

$$D(X,Y)=\mathrm{Aut}_{\mathbf{C}}(o_{X,Y}).$$

We define the 1-category $\mathbf{T}$ associated to $\mathbf{C}$ by

$$\begin{cases}\mathrm{Mor}_{\mathbf{T}}(X,Y)_0=\mathrm{Mor}_{\mathbf{C}}(X,Y)_0\\ \mathrm{Mor}_{\mathbf{T}}(X,Y)_1=\{(F,f),F:f\longrightarrow o_{X,Y}\}\subset\mathrm{Mor}_{\mathbf{C}}(X,Y)_1.\end{cases}$$

Let ∂^1 be defined by $\partial^1(F,f)=f$, and let the zero elements be given by $o^0=o_{X,Y}$, $o^1=$ identity of $o_{X,Y}$.

12.1. **Proposition.** *Let $\mathbf{C}$ be an abelian track category with zero morphisms. Then $\mathbf{C}$ yields the 1-category $\mathrm{Nul}_1\,\mathbf{C}=(\mathbf{T},\mathbf{A},D,\mathcal{O})$ with $\mathbf{T}$, $\mathbf{A}$ and D as above and with the following obstruction operator $\mathcal{O}$.*

Up to equivalence there is only one left cubical ball B of dimension 1: this is the double of I, which is equivalent to T^1_0. Given $a_1=(F,f)$ and $a_2=(G,g)$ with gluing condition $\partial^1a_1=f=g=\partial^1a_2$, let

$$\mathcal{O}_B(a_1,a_2)\ :=\ FG^{-1}\in\mathrm{Aut}_{\mathbf{C}}(o_{X,Y})$$

be the *obstruction*. The action for $\alpha \in \mathrm{Aut}_{\mathbf{C}}(o_{X,Y})$ and $a = (F, f)$ is given by $a + \alpha = (\alpha F, f)$, with $\mathcal{O}_B(a + \alpha, a) = \alpha F F^{-1} = \alpha$. The triviality rule of $\mathcal{O}$ is satisfied, since for a diagram

$$\begin{tikzcd} Z & Y \arrow[l, "g"'] \arrow[l, bend right, phantom, "\uparrow G"] \arrow[l, bend right=60, "o"'] & X \arrow[l, "f"'] \arrow[l, bend right, phantom, "\uparrow F"] \arrow[l, bend right=60, "o"'] \end{tikzcd},$$

in $\mathbf{C}$ we have the formula $Gf = gF$, so that

$$\mathcal{O}_B(a_1, a_2) = o \quad \text{for} hsma_1 = (\partial^1 F)G \quad \text{and} \quad a_2 = F(\partial^1 G)\ .$$

12.2. **Example.** Let $\mathbf{C}$ be a category enriched in groupoids with zero morphisms and let $\mathbf{C}$ be abelian. Then the 1-track category $\mathrm{Nul}_1(\mathbf{C})$ is defined and a *triple Toda bracket*

$$\langle \delta_1, \delta_2, \delta_3 \rangle \quad \text{in} \quad \mathrm{Nul}_1(\mathbf{C})$$

coincides with the classical triple Toda bracket in $\mathbf{C}$. Moreover, a 1-st order chain complex in $\mathrm{Nul}_1(\mathbf{C})$ as defined in 11.2 coincides with a *secondary chain complex* in $\mathbf{C}$ as studied in [BJ2].

12.3. *Remark.* Abelian track categories are classified by cohomology, see [BW], [BD], [P], [B1], [BJ1]. It would be interesting to classify accordingly 1-track categories and n-track categories for $n \geq 1$.

13. The inductive step of the resolution theorem

An n-track category $\mathbf{T} = (\mathbf{T}, \mathbf{A}, D, \mathcal{O})$ is *a Σ-track algebra in dimension n* if $\mathbf{A} = (\mathbf{A}, \mathbf{a}, \Sigma)$ is a Σ-algebra and

$$D(X, Y) = \mathrm{Hom}_{\mathbf{A}}(\Sigma^n X, Y)$$

for X in $\mathbf{a}$ and Y in $\mathbf{A}$. See Definition 6.3. We say that $\mathbf{T}$ is a *Ω-track algebra in dimension n* if $\mathbf{A} = (\mathbf{A}, \mathbf{a}, \Omega)$ is an Ω-algebra and

$$D(X, Y) = \mathrm{Hom}_{\mathbf{A}}(X, \Omega^n Y)$$

for Y in $\mathbf{a}$ and X in $\mathbf{A}$.

13.1. **Theorem.** *Let $\mathbf{T}$ be a Σ-track algebra in dimension n and consider a functor of n-graded categories*

$$K : \mathbb{Z}(\infty, -1)^n_{\otimes} \longrightarrow \mathbf{T}$$

which is a pre-chain complex and which is based on an $\mathbf{a}$-resolution $A_\bullet$ of X in $\mathbf{A}$. Then there exists a functor

$$K' : \mathbb{Z}(\infty, -1)^n_{\otimes} \longrightarrow \mathbf{T}$$

which coincides with K in dimension $\leq n-1$ and which is an n-th order chain complex in $\mathbf{T}$ (and is based on $A_\bullet$).

The dual also holds.

13.2. **Theorem.** *Let $\mathbf{T}$ be an Ω-track algebra in dimension n and consider a functor of n-graded categories*

$$L : \mathbb{Z}(+1, -\infty)^n_{\otimes} \longrightarrow \mathbf{T}$$

which is a pre-chain complex and which is based on an $\mathbf{a}$-coresolution $A^\bullet$ of Y in $\mathbf{A}$. Then there exists a functor

$$L' : \mathbb{Z}(+1, -\infty)^n_{\otimes} \longrightarrow \mathbf{T}$$

which coincides with L in dimension $\leq n-1$ and which is an n-th order chain complex in $\mathbf{T}$ (and is based on $A^\bullet$).

Proof. The functor K' is determined by K in dimension $\leq n-1$ and by

$$K'(i, I_n) = K(i, I_n) + \alpha_i, \quad i \geq n-1 \tag{13.3}$$

in dimension n. See Lemma 3.3. Here the elements α_i are obtained inductively as follows. We have to choose α_i, $i \geq n$, in such a way that the obstruction

$$\xi(\alpha_{i-1}, \alpha_i) = \mathcal{O}_B(\delta_{i-n-1} K'(i, I_n), b_2, \ldots, b_n, K'(i-1, I_n)\delta_i) \tag{13.4}$$

vanishes with $b_{r+1} = K(i, I_r \otimes I_{n-r})$ for $1 \leq r \leq n-1$, see 11.2. We start with $i = n+1$. In this case $(\delta_0)_*$ is surjective since $A_\bullet$ is a resolution with $\delta_0 : A_0 \longrightarrow A_{-1}$, $A_{-1} = X$. The action rule shows

$$\xi(\alpha_{i-1}, \alpha_i) = \xi(0,0) + \varepsilon_1(\delta_{i-n-1})_*\alpha_i + \varepsilon_{n+1}(\delta_i)^*\alpha_{i-1}. \tag{13.5}$$

Here $\varepsilon_1, \ldots, \varepsilon_{n+1}$ are the orientation signs for the left cubical ball $B = T_0^n$. For $i = n+1$ we get for $\alpha_n = 0$ the equation

$$\xi(0, \alpha_{n+1}) = \xi(0,0) + \varepsilon_1(\delta_0)_*\alpha_{n+1}.$$

Since $(\delta_0)_*$ is surjective there is α_{n+1} with $\xi(0, \alpha_{n+1}) = 0$. We now consider (13.5) for $i = n+2$. Then we show that

$$(\delta_0)_*\xi(\alpha_{n+1}, \alpha_{n+2}) = 0. \tag{13.6}$$

Since $A_\bullet$ is a resolution this shows that

$$\xi(\alpha_{n+1}, \alpha_{n+2}) \in \mathrm{image}(\delta_1)_*. \tag{13.7}$$

Since by (13.5) we have

$$\xi(\alpha_{n+1}, \alpha_{n+2}) = \xi(0,0) + \varepsilon_1(\delta_1)_*\alpha_{n+2} + \varepsilon_{n+1}\delta_{n+2}^*\alpha_{n+1}, \tag{13.8}$$

we can choose α_{n+2} with $\xi(\alpha_{n+1}, \alpha_{n+2}) = 0$. This way we get inductively α_i, $i \geq n$, such that $\xi(\alpha_{i-1}, \alpha_i) = 0$. Hence K' defined by (13.3) satisfies the obstruction property and hence is an n-th order chain complex as in the Theorem. In the next lemma we show that (13.6) holds. □

We introduce the following notation on the 'boundary' of I_{n+1}, $n \geq 0$. Let

$$\partial I_1 = \emptyset \otimes \emptyset$$

and for $n \geq 1$ let

$$\partial I_{n+1} = (\emptyset \otimes I_n, I_1 \otimes I_{n-1}, I_2 \otimes I_{n-2}, \ldots, I_{n-1} \otimes I_1, I_n \otimes \emptyset). \tag{13.9}$$

(see (4.5)). We also write

$$\langle n \rangle = (I_1 \otimes I_{n-1}, I_2 \otimes I_{n-2}, \ldots, I_{n-1} \otimes I_1)\ ,$$

so that

$$\partial I_{n+1} = (\emptyset \otimes I_n, \langle n \rangle, I_n \otimes \emptyset).$$

Given a functor $K' : \mathbb{Z}(\infty, -1)^n_\otimes \longrightarrow \mathbf{T}$ which is a pre-chain complex, we obtain for $i \geq n \geq 1$ the obstruction element

$$\mathcal{O}_B K'(i, \partial I_{n+1})$$

where $B = T_0^n$. This corresponds to (13.4) in the proof above.

13.10. **Hauptlemma.** *Let $n \geq 1$, $i \geq n+2$, and assume*

$$\mathcal{O}_B K'(i-1, \partial I_{n+1}) = 0.$$

Then we also have

$$\mathcal{O}_B K'(i, \emptyset \otimes \partial I_{n+1}) = 0.$$

For the proof of Hauptlemma 13.10, we use the following equation given by the triviality rule with $r + s = n+1$, $r \geq 1$, $s \geq 1$, $i \geq n$.

$$\mathcal{O}_B K'(i, \partial_{r,s}) = 0, \tag{13.11}$$

where

$$\partial_{r,s} = ((\partial I_r) \otimes I_s, I_r \otimes (\partial I_s)).$$

The assumption implies

$$\mathcal{O}_B K'(i, (\partial I_{n+1}) \otimes \emptyset) = 0 \tag{13.12}$$

by the naturality rule.

Proof of Hauptlemma 13.10 for $n = 1$. In this case we have the triviality rule (13.11) which we write as

$$(\emptyset \otimes \emptyset \otimes I_1, I_1 \otimes \emptyset \otimes \emptyset) \sim 0 \ . \tag{13.13}$$

The assumption implies (13.12):

$$Y = (\emptyset \otimes I_1, I_1 \otimes \emptyset) \otimes \emptyset \sim 0. \tag{13.14}$$

We have to show

$$X = \emptyset \otimes (\emptyset \otimes I_1, I_1 \otimes \emptyset) \sim 0 \ . \tag{13.15}$$

In fact, by the complement rule and (13.13), we get

$$X \sim (I_1 \otimes \emptyset \otimes \emptyset, \emptyset \otimes I_1 \otimes \emptyset)$$

so that $X \sim 0$ by (13.14). □

Proof of Hauptlemma 13.10 for $n = 2$. We omit $\otimes$ in the notation and write VW for $V \otimes W$. By (13.11), we find:

$$((\emptyset I_1, I_1\emptyset)I_1, I_2\emptyset\emptyset) \sim 0 \tag{13.16}$$

and

$$(\emptyset\emptyset I_2, I_1(\emptyset I_1, I_1\emptyset)) \sim 0. \tag{13.17}$$

By the assumption (13.12) we have

$$Y = (\emptyset I_2, I_1 I_1, I_2\emptyset)\emptyset \sim 0. \tag{13.18}$$

We have to show

$$X = \emptyset(\emptyset I_2, I_1 I_1, I_2\emptyset) \sim 0. \tag{13.19}$$

By the complement rule and (13.17) (replacing $\emptyset\emptyset I_2$) we get:

$$X \sim (I_1(\emptyset I_1, I_1\emptyset), \emptyset I_1 I_1, \emptyset I_2\emptyset) = X' \ .$$

By the complement rule and (13.16) (replacing $I_2\emptyset\emptyset$) we get

$$Y \sim (\emptyset I_2\emptyset, I_1 I_1\emptyset, (\emptyset I_1, I_1\emptyset)I_1) = Y' \ .$$

Here we have $X' = Y'$, so that $X \sim X' = Y' \sim Y \sim 0$. □

Proof of Hauptlemma 13.10. By (13.11) we have the relations

$$\partial_{r,s} \sim 0 \qquad \text{for } r + s = n + 1,\, r \geq 1. \tag{13.20}$$

By (13.12), the assumption implies that

$$Y = (\partial I_{n+1}) \otimes \emptyset \ \sim \ 0 \ . \tag{13.21}$$

We have to show that

$$X = \emptyset \otimes (\partial I_{n+1}) \sim 0. \tag{13.22}$$

We now apply the complement rule inductively to Y by use of $\partial_{r,s}$ for $s = 1, \ldots, [n/2]$. This yields the equivalence $Y \sim Y'$. Similarly, we apply the complement rule inductively to X by use of $\partial_{r,s}$ for $r = 1, \ldots, [n/2]$. This yields $X \sim X'$. If n is even, we have $Y' = X'$, so that $0 \sim Y \sim Y' = X' \sim X$, by (13.21). If $n = 2n' + 1$ is odd, we can use $\partial_{n'+1,n'+1}$ to show that $X' \sim Y'$. By (13.21), this implies that $0 \sim Y \sim Y' \sim X' \sim X$. □

The proof of Hauptlemma 13.10 involves left cubical balls with the number of cells $\leq [n/2](n - 1) + n + 1$.

14. Track algebras and proof of the Resolution Theorem

In order to prove Resolution Theorem 8.1, we need to relate track categories of different dimensions, as follows:

A *total n-track category* $\mathbf{T}(\leq n)$ is a sequence of m-track categories

$$\mathbf{T}(m) = (\mathbf{T}(m), \mathbf{A}, D_m, \mathcal{O}^m) \qquad \text{for} \quad m = 1, 2, \ldots, n$$

together with quotient functors

$$q : \mathbf{T}(m+1)^m \longrightarrow \mathbf{T}(m)$$

which is the identity on objects and is full and is the identity functor on $(m-1)$-skeleta

$$q : \mathbf{T}(m+1)^{m-1} = \mathbf{T}(m)^{m-1} .$$

Moreover, the *boundary property* of Lemma (10.4) holds – that is, for $B = T_0^m$, we have

$$\mathcal{O}_B^m(a_1, \ldots, a_{m+1}) = 0$$

if and only if there exists $\overline{a} \in \mathbf{T}(m+1)_{m+1}$ with $q(\partial^i \overline{a})$ representing a_i for $i = 1, \ldots, m+1$.

14.1. **Example.** Let $\mathbf{C}$ be a category enriched in pointed spaces with zero morphisms. Then

$$\mathrm{Nul}_{\leq n}\,\mathbf{C} \;:=\; (\mathrm{Nul}_n\,\mathbf{C}, \mathrm{Nul}_{n-1}\,\mathbf{C}, \ldots, \mathrm{Nul}_1\,\mathbf{C})$$

is a total n-track category.

We say that $\mathbf{T}(\leq n)$ is a *Σ-track algebra* if $\mathbf{A} = (\mathbf{A}, \mathbf{a}, \Sigma)$ is a Σ-algebra as in Definition 6.3 and

$$D_m(X, Y) = \mathrm{Hom}_{\mathbf{A}}(\Sigma^m X, Y)$$

for $m = 1, \ldots, n$ and X in $\mathbf{a}$ and Y in $\mathbf{A}$.

Dually we say $\mathbf{T}(\leq n)$ is an *Ω-track algebra* if $\mathbf{A} = (\mathbf{A}, \mathbf{a}, \Omega)$ is an Ω-algebra as in (6.1) and

$$D_m(X, Y) = \mathrm{Hom}_{\mathbf{A}}(X, \Omega^m Y)$$

for $m = 1, \ldots, n$ and X in $\mathbf{A}$ and Y in $\mathbf{a}$.

14.2. **Example.** Let $\mathbf{C}$ be a Σ-mapping algebra then $\mathrm{Nul}_{\leq n}\,\mathbf{C}$ is a Σ-track algebra. If $\mathbf{C}$ is an Ω-mapping algebra then $\mathrm{Nul}_{\leq n}\,\mathbf{C}$ is an Ω-track algebra.

We now obtain the following Resolution Theorems, which generalize those of Section 8.

14.3. **Theorem.** *Let $\mathbf{T}(\leq n)$ be a Σ-track algebra and let $A_\bullet$ be an $\mathbf{a}$-resolution of X in $\mathbf{A}$. Then there exists an n-th order chain complex*

$$K : \mathbb{Z}(\infty, -1)^n_\otimes \longrightarrow \mathbf{T}(n)$$

*which is based on $A_\bullet$. We call K an n-*th order resolution *of X in $\mathbf{T}(n)$.*

14.4. **Theorem.** *Let $\mathbf{T}(\leq n)$ be an Ω-track algebra and let $A^\bullet$ be an $\mathbf{a}$-coresolution of Y in $\mathbf{A}$. Then there exists an n-th order chain complex*

$$L : \mathbb{Z}(+1, -\infty)^n_\otimes \longrightarrow \mathbf{T}(n)$$

*which is based on $A^\bullet$. We call L an n-*th order coresolution *of Y in $\mathbf{T}(n)$.*

Proof. The boundary property shows that there exists a functor

$$K'(1) : \mathbb{Z}(\infty, -1)^1_\otimes \longrightarrow \mathbf{T}(1)$$

which satisfies the inclusion property and which is based on $A_\bullet$. Hence by Theorem 13.1 we find a 1-order chain complex $K(1)$ which is based on $A_\bullet$. Now the boundary property shows that there is a functor

$$K'(2) : \mathbb{Z}(\infty, -1)^1_\otimes \longrightarrow \mathbf{T}(2)$$

which satisfies the inclusion property and which based on $A_\bullet$. Again the boundary property shows there exists $K'(3)$, so that by Theorem 13.1 one obtains $K(3)$. Inductively, we thus have $K = K(n)$. □

14.5. **Example.** Let $\mathbf{C}$ be the Eilenberg-Mac Lane Ω-mapping algebra. Then *minimal* coresolutions $A^\bullet$ of Y are defined in $\mathbf{A}$ and hence we can find an n-th order coresolution of Y in $\mathrm{Nul}_{\leq n}\,\mathbf{C}$ based on $A^\bullet$. We call $\mathrm{Nul}_{\leq n}\,\mathbf{C}$ the *algebra of cohomology operations of order* $\leq n+1$. This is an Ω-track algebra. It is convenient to consider the dual of $\mathrm{Nul}_{\leq n}\,\mathbf{C}$, which is a Σ-track algebra and for which $\mathbf{a}$ is the category of finitely generated free modules over the Stenrod algebra.

14.6. *Remark.* The main result of [B2] computes the algebra of cohomology operations of order ≤ 2 in terms of a bigraded differential algebra $\mathcal{B}$ over the ring $\mathbb{Z}/p^2$. This leads to the conjecture that also the algebra of cohomology operations of order $\leq n$ $(n \geq 1)$, can be described up to equivalence by a bigraded differential algebra over $\mathbb{Z}/p^2$.

15. Higher order Ext-groups

In this section we deduce from higher order resolutions the associated higher order derived functors, which are higher order Ext -groups. We shall see that the E_n-term in the Adams spectral sequence is such a higher order Ext-group for $n \geq 2$.

It is classical that the E_2 of the Adams spectral sequence is given by the 'primary' Ext-groups of homological algebra, see (6.1). In [BJ2] we studied the secondary Ext-groups which determine E_3.

Let $\mathbf{T}(\leq n)$ $(n \geq 1)$ be a Σ-track algebra so that for $m = 1, \ldots, n$ we have the m-track category

(15.1)
$$\mathbf{T}(m) = (\mathbf{T}(m), \mathbf{A}, D_m, \mathcal{O}^m)$$

with $\mathbf{a} \subset \mathbf{A}$ and $D_m(A,X) = \mathrm{Hom}_{\mathbf{A}}(\Sigma^m A, X)$ for objects A in $\mathbf{a}$ and X in $\mathbf{A}$. Let $A_\bullet$ be an $\mathbf{a}$-resolution of X in $\mathbf{A}$ and let

(15.2)
$$K : \mathbb{Z}(\infty, -1)^n_\otimes \longrightarrow \mathbf{T}(n)$$

be a n-th order resolution of X based on $A_\bullet$ (see Theorem 14.3). Furthermore, let Y be another object of $\mathbf{A}$, and consider the diagram in $\mathbf{A}$:

(15.3)
$$\begin{tikzcd} \cdots \arrow[r] & A_{r+m+1} \arrow[r] & A_{r+m} \arrow[r] & \cdots \arrow[r] & A_r \arrow[r, "\delta_r"] \arrow[d, "\beta"] & \cdots \arrow[r] & A_0 \arrow[r] & X \\ & & & & Y & & & \end{tikzcd}$$

The row of the diagram is the $\mathbf{a}$-resolution $A_\bullet$ of X. We assume that β is a cocycle, that is,

(15.4)
$$\beta\delta_{r+1} = 0.$$

Then β represents an element $\{\beta\}$ in the Ext-group

(15.1)
$$\begin{aligned} E_2^{r,0} &= \mathrm{Ext}^r_{\mathbf{A}}(X,Y) \\ &= \mathrm{H}^r\,\mathrm{Hom}_{\mathbf{A}}(A_\bullet, Y) \\ &= \mathrm{kernel}\,\delta^*_{r+1}/\,\mathrm{image}\,\delta^*_r, \end{aligned}$$

where

$$\delta^*_r : \mathrm{Hom}_{\mathbf{A}}(A_{r-1}, Y) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(A_r, Y)\ .$$

Using the $\mathbf{a}$-resolution $\Sigma^s A_\bullet$ of $\Sigma^s X$, we get accordingly for $s \geq 0$ the bigraded Ext-group (see (6.1)),

$$E_2^{r,s} = \mathrm{Ext}^r_{\mathbf{A}}(\Sigma^s X, Y).$$

We shall define a differential

(15.5)
$$d_2 = d_2^{r,s} : E_2^{r,s} \longrightarrow E_2^{r+2,s+1}.$$

Moreover, inductively for $m \geq 2$ we consider subquotients $E_m^{r,s}$ of $E_2^{r,s}$, together with differentials

(15.6)
$$d_m = d_m^{r,s} : E_m^{r,s} \longrightarrow E_m^{r+m,s+m-1}$$

satisfying $d_m d_m = 0$, and

$$E_{m+1}^{r,s} = \mathrm{kernel}(d_m^{r,s})/\,\mathrm{image}(d_m^{r-m,s-m+1}).$$

We call $E_m^{r,0}$ for $m = 2, \ldots, n+1$ the *higher order* Ext-*groups* associated to the n-th order resolution K of X above. replacing X by $\Sigma^s X$, we obtain the groups $E_m^{r,s}$, accordingly.

15.7. **Definition.** Let $\overline{\beta} \in E_{m+1}^{r,0}$ be represented by $\{\beta\} \in E_2^{r,0}$ $(1 \leq m \leq n)$, and let L be a $(m-)$–order chain complex

$$L : \mathbb{Z}(\infty, r-1)_{\otimes}^{m-1} \longrightarrow \mathbf{T}(m-1)$$

based on the diagram

$$\longrightarrow A_{r+m+1} \longrightarrow \ldots \longrightarrow A_{r+1} \longrightarrow A_r \xrightarrow{\beta} Y\,,$$

in $\mathbf{A}$. We assume also that L restricted to $\mathbb{Z}(\infty, r)$ coincides with the $(m-1)$-skeleton of K in $\mathbf{T}(m-1)$. The boundary property in $\mathbf{T}(m)$ shows that there is a functor

$$\hat{L} : \mathbb{Z}(\infty, r-1)_{\otimes}^{m} \longrightarrow \mathbf{T}(m)$$

which is a pre-chain complex such that $\hat{L}$ restricted to $\mathbb{Z}(\infty, r)$ coincides with the m-skeleton of K, and such that the $(m-1)$-skeleton $\hat{L}(m-1)$ of $\hat{L}$ satisfies $q\hat{L}(m-1) = L$ in $\mathbf{T}(m-1)$. We then obtain the obstruction

$$\mathcal{O}\hat{L}(r+m+1, \partial I_{m+1}) = \mathcal{O}_B(b_1, \ldots, b_{m+1}) \in \mathrm{Hom}_{\mathbf{A}}(\Sigma^m A_{r+m+1}, Y)\ ,$$

where $B = T_0^m$ and

$$\hat{L}(r+m+1, \partial I_{m+1}) = \begin{cases} b_1 = \hat{L}(r+m+1, \emptyset \otimes I_m), \\ b_{k+1} = \hat{L}(r+m+1, I_k \otimes I_{m-1}), \quad 1 \leq k \leq m-1, \\ b_{m+1} = \hat{L}(r+m+1, I_m \otimes \emptyset) \end{cases}$$

(see (4.5)). Now the element $\mathcal{O}\hat{L}(r+m+1, \partial I_{m+1})$ represents the differential

$$d_{m+1}^{r,0}(\overline{\beta}) \in E_{m+1}^{r+m+1,m}\ .$$

15.8. **Theorem.** *Let* $\mathbf{T}(\leq n) = \mathrm{Nul}_{\leq n}\,\mathbf{C}$ *be the* Σ*-track algebra given by the complete* Σ*-mapping algebra* $\mathbf{C}$ *of Example 7.3. Then Definition 15.7 yields a well defined sequence of* Ext*-groups* $E_m^{r,s}$ *for* $m = 2, \ldots, n+2$. *These groups depend on the weak equivalence class of the* Σ*-track algebra* $\mathbf{T}(\leq n)$, *and not on the choice of the* n*-th order resolution of* X.

15.9. **Theorem.** *Let* $\mathbf{T}(\leq n) = \mathrm{Nul}_{\leq n}\,\mathbf{C}$ *be the* Σ*-track algebra given by the dual* $\mathbf{C}$ *of the Eilenberg-Mac Lane* Ω*-mapping algebra. Then the* Ext*-groups* $E_m^{r,s}$ $(2 \leq m \leq n+2)$, *yield the* m*-term* E_m *of the Adams spectral sequence which converges to the stable homotopy set* $\{Y, X\}$ *for finite spectra* X *and* Y.

For $n = 1$ this result is proved in [BJ2, Section 7].

16. Strictification of higher order resolutions

In this section we use the strictification of higher order resolutions to prove Theorems 15.8 and 15.9.

Let $\mathbf{C}$ be a Σ-mapping algebra as in Example 7.3. Hence $\mathbf{C}$ is given by an underlying model category and cubes

$$I^n \longrightarrow \mathrm{Mor}_{\mathbf{C}}(X, Y)$$

having an adjoint

$$(16.1) \qquad (I^n \times X)/(I^n \times *) \longrightarrow Y\ .$$

Here $(I \times X)/(I \times *)$ is the pointed *cylinder* of X. We then have the additive category $\mathbf{A} = \pi_0 \mathbf{C}$, and the full additive subcategory $\mathbf{a} = \pi_0\{\mathcal{X}\}$ given by the class of spectra $\mathcal{X}$ in 7.3. We define the full subcategory $\hat{\mathbf{a}}$, with

$$(16.2) \qquad \mathbf{a} \subseteq \hat{\mathbf{a}} \subseteq \mathbf{A}\ .$$

Here $\hat{\mathbf{a}}$ consists of all abjects A in $\mathbf{A}$ which are isomorphic in $\mathbf{A}$ to an object in $\mathbf{a}$.

16.3. **Definition.** Let $\mathbf{T}$ be an n-graded category (such as $\mathrm{Nul}_n\,\mathbf{C}$ or $\mathrm{nul}_n\,\mathbf{C}$) with a quotient functor $q : \mathbf{T}^0 \to \mathbf{A}$. Let $K, L : \mathbb{Z}(\infty,-1)^n_\otimes \to \mathbf{T}$ be functors of n-graded categories. A *weak equivalence* $\tau : K \to L$ over X is a natural transformation τ which for objects i in $\mathbb{Z}_\otimes$ consists of a map

$$\tau_i : K_i \to L_i \quad \text{in} \quad \mathbf{T}^0$$

which induces an isomorphism $q\tau_i$ in $\mathbf{A}$. For $i=-1$, the map $\tau_{-1} : K_{-1} = X = L_{-1}$ is the identity of X. For a morphism $V : i \to j$ in $\mathbb{Z}_\otimes$, we have the commutative diagram in $\mathbf{T}$:

$$\begin{tikzcd} K_i \arrow[r, "\tau_i"] \arrow[d, "K(V)"'] & L_i \arrow[d, "L(V)"] \\ K_j \arrow[r, "\tau_j"] & L_j \end{tikzcd}$$

or equivalently, $\tau_j K(V) = L(V)\tau_i$. Let $\sim$ be the equivalence relation generated by weak equivalences over X.

16.4. **Lemma.** *$K, L : \mathbb{Z}(\infty,-1)^n_\otimes \to \mathrm{Nul}_n\,\mathbf{C}$ be n-th order resolutions of X. If $K \sim L$ are weakly equivalent over X, then the higher* Ext*-groups defined by K and L are isomorphic.*

We shall show that the higher Ext-groups actually do not depend on the choice of resolution of X. For this, we use the strictification of resolutions.

16.5. **Definition.** Let $\mathbf{T}$ be an n-graded category (like $\mathrm{Nul}_n\,\mathbf{C}$ or $\mathrm{nul}_n\,\mathbf{C}$) and let $K : \mathbb{Z}(\infty,-1)^n_\otimes \to \mathbf{T}$ be a functor of n-graded categories. Then we say that K is *N-strict* with $N \geq 0$ if for all $i \leq N$ and $k = 1,\dots,n$ we have $K(i, I_k) = o$. This shows that $\delta_i = K(i,\emptyset) : K_i \to K_{i-1}$ yields a sequence of maps in $\mathbf{T}^0$

$$K_N \xrightarrow{\delta} K_{N-1} \xrightarrow{\delta} \dots \xrightarrow{\delta} K_0 \xrightarrow{\delta} K_{-1}$$

with $K_{-1} = X$ and $\delta\delta = o$ This is a *strict chain complex* in $\mathbf{T}^0$.

We say that K is *N-fibrant* if there are fiber sequences

$$Z_i \xrightarrow{j_i} K_i \overset{p}{\twoheadrightarrow} Z_{i-1} \qquad p \circ j_i = o\ ,$$

in the model category with $\delta_i = j_{i-1}p$ for $i < N$, and δ_N admits a factorization

$$K_N \xrightarrow{\overline{p}} Z_{N-1} \xrightarrow{j_{N-1}} K_{N-1}\ .$$

Moreover, K is *N-exact* if for $i < N$ and A in $\mathbf{a}$ the induced sequence

$$\mathrm{Hom}_{\mathbf{A}}(A, Z_i) \to \mathrm{Hom}_{\mathbf{A}}(A, K_i) \to \mathrm{Hom}_{\mathbf{A}}(A, Z_{i-1})$$

is a short exact sequence of abelian groups.

16.6. **Theorem.** *Let $n \geq 1$ and $N \geq 0$, and let $K : \mathbb{Z}(\infty,-1)^n_\otimes \to \mathrm{Nul}_n\,\mathbf{C}$ be an n-th order resolution of X based on the $\mathbf{a}$-resolution $A_\bullet$ of X in $\mathbf{A}$. Then there exists an N-(strict, fibrant, exact) n-th order resolution L of X based on an $\hat{\mathbf{a}}$-resolution $\hat{A}_\bullet$ of X in $\mathbf{A}$ such that $L \sim K$ are weakly equivalent over X.*

Here we use the large category $\hat{\mathbf{a}}$ in (16.2). The resolutions $A_\bullet$ and $\hat{A}_\bullet$ yield by the weak equivalence $L \sim K$ over X the commutative diagram in $\mathbf{A}$:

$$\begin{tikzcd} \cdots \arrow[r] & A_1 \arrow[r, "\delta"] \arrow[d, "\cong"'] & A_0 \arrow[r, "\delta"] \arrow[d, "\cong"] & X \arrow[d, "="] \\ \cdots \arrow[r] & \hat{A}_1 \arrow[r, "\delta"] & \hat{A}_0 \arrow[r, "\delta"] & X \end{tikzcd}$$

Here the vertical arrows are isomorphisms in $\mathbf{A}$ and we have $A_i = K_i$ and $\hat{A}_i = L_i$ for $i \geq -1$.

16.7. *Remark.* The dual of Theorem 16.6 holds for coresolutions.

Proof of Theorem 16.6 for $n = 1$. We use induction over N. Assume the result is true for $n = 1$ and $N \geq 0$. Then the map $\overline{p}$ with $\delta_{N-1} = j\overline{p}$ admits a factorization

(16.8)
$$\overline{p} \;:\; K_N \overset{\sim}{\rightarrowtail} L_n \overset{p}{\twoheadrightarrow} Z_{N-1}$$

which defines L_n. Hence we get the diagram

(16.9)
$$\begin{tikzcd}
 & K_{N+1} \arrow[d, "\delta"] \arrow[ddd, bend left=60, "o"] \\
L_N \arrow[d] & K_N \arrow[l, "\sim"'] \arrow[d, "\overline{p}"] \arrow[r, phantom, "\overset{H}{\Longrightarrow}"] & {} \\
Z_{N-1} \arrow[d] \arrow[r, equal] & Z_{N-1} \arrow[d, "j"] \\
L_{N-1} \arrow[r, equal] & K_{N-1}
\end{tikzcd}$$

Here $H = K(N+1, I_1)$ satisfies $pH : o \Rightarrow o$, so that pH is a map $\alpha : \Sigma K_{N+1} \to Z_{N-2}$ which is nullhomotopic, since $j_{N-2}\alpha \simeq o$ as follows from the obstruction property of K and N-exactness. The lift of $j_{N-2}\alpha \sim o$ through p_{N-1} shows that the track H can be represented by a homotopy $\overline{H} : \overline{p}\delta \simeq o$; that is, the 1-track $\{j_{N-1}\overline{H}\}$ coincides with H. We then get the following diagram with the cylinder IK_{N+1} and inclusions i_0, i_1 of the cylinder. We set $L_{N+1} := K_{N+1}$.

(16.10)
$$\begin{tikzcd}
L_{N+1} \arrow[r, tail, "i_0"] \arrow[d, dotted, "\delta"] \arrow[dd, bend right=60, "o"'] & IK_{N+1} \arrow[dl, "\hat{H}"'] & K_{N+1} \arrow[l, tail, "i_1"'] \arrow[d] \arrow[dd, bend left=60] \\
L_N \arrow[d] & & K_N \arrow[ll, tail, "\sim"] \arrow[d] \arrow[r, phantom, "\overset{\overline{H}}{\Longrightarrow} o"] & {} \\
Z_{N-1} \arrow[rr, equal] & & Z_{N-1}
\end{tikzcd}$$

Here $\hat{H}$ is a lift of $\overline{H}$ through p_N, so that the diagram commutes with $p_N\hat{H}i_0 = o$. Hence for $\delta = \hat{H}i_0$, the left hand side is $(N+1)$-strict and $(N+1)$-fibrant. Moreover, the left hand side is $(N+1)$-exact, that is:

(16.11)
$$(p_N)_* \;=\; \mathrm{Hom}_{\mathbf{A}}(A, p_N) \;:\; \mathrm{Hom}_{\mathbf{A}}(A, L_N) \;\to\; \mathrm{Hom}_{\mathbf{A}}(A, Z_{N-1})$$

is surjective for all A in $\mathbf{a}$. In fact, we have for $\alpha \in \mathrm{Hom}_{\mathbf{A}}(A, Z_{N-1})$ the equation $\delta j_{N-1}\alpha = 0$, so that by exactness of $A_\bullet$ we have $j_{N-1}\alpha = \delta\beta = j_{N-1}p_N\beta$, so that $\alpha = p_N\beta$ by injectivity of $(j_{N-1})_*$. Moreover, $(j_N)_*$ is injective since we have the fiber sequence where $\mathrm{Hom}_{\mathbf{A}}(A, \Omega p_N) = \mathrm{Hom}_{\mathbf{A}}(\Sigma A, p_N)$ is surjective, since $\Sigma A \in \mathbf{a}$.

We now construct weak equivalences

(16.12)
$$L \overset{i_0}{\longrightarrow} R \overset{i_1}{\longleftarrow} K$$

where i_0 and i_1 are the identity in degrees $< N$. In degree N the resolution L is given by the diagram above. In dimension 0, diagram (16.12) is given by the commutative diagram:

(16.13)
$$\begin{tikzcd}
\cdots \arrow[r] & K_{N+2} \arrow[r] \arrow[d] & K_{N+1} \arrow[r] \arrow[d] & K_N \arrow[r] \arrow[d, "\sim"] & K_{N-1} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "I\delta"] & IK_{N+2} \arrow[r, "I\delta"] & IK_{N+1} \arrow[r, "\hat{H}"] & L_N \arrow[r] & K_{N-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & K_{N+2} \arrow[r] \arrow[u] & K_{N+1} \arrow[r] \arrow[u] & L_N \arrow[r] \arrow[u, equal] & L_{N-1} \arrow[r] \arrow[u, equal] & \cdots
\end{tikzcd}$$

It is easy to find appropriate $R(N+1, I_1)$, $R(N+2, I_1)$, and $R(M, I_1) = IK(M, I_1)$ for $M \geq N+3$, so that (16.12) is well defined. Here we use the adjoint maps in (16.1). This completes the proof of Theorem 16.6 for $n = 1$. □

16.14. **Transport Lemma.** *Let K be an n-th order resolution of X in $\mathrm{Nul}_n\,\mathbf{C}$ with $(n-1)$-skeleton $K^{(n-1)}$. Let*

$$L^{(n-1)} \xrightarrow{f} K^{(n-1)} \xrightarrow{g} R^{(n-1)} \qquad \text{in } \mathrm{Nul}_n\,\mathbf{C}$$

be weak equivalences over X. Then there exist unique n-th order resolutions L and R in $\mathrm{Nul}_n\,\mathbf{C}$ together with weak equivalences

$$L \xrightarrow{\tilde{f}} K \xrightarrow{\tilde{g}} R$$

which, restricted to $(n-1)$-skeleta, coincide with f and g respectively.

Proof. We use Proposition II.2.11 in [B1] for sets of n-tracks. □

16.15. **Lemma.** *Let $q : \mathrm{nul}_n\,\mathbf{C} \to \mathrm{Nul}_n\,\mathbf{C}$ be the quotient map of Section 2 and let qK, qL be n-th order resolutions of X in $\mathrm{Nul}_n\,\mathbf{C}$. If $qK \sim qL$ are weakly equivalent, then also $K \sim L$ are weakly equivalent over X in $\mathrm{nul}_n\,\mathbf{C}$.*

16.16. **Lemma.** *Let $n \geq 2$ and $N \geq 1$, and let K be an n-th order resolution of X in $\mathrm{Nul}_n\,\mathbf{C}$. Assume that the $(n-1)$-skeleton $K^{(n-1)}$ in $\mathrm{nul}_n\,\mathbf{C}$ is $(N-1)$-(strict, fibrant, exact). Then there is an n-th order resolution L of X which is is N-(strict, fibrant, exact), such that $L \sim K$ are weakly equivalent.*

Proof. Let $i \leq N$. Since the $(n-1)$-skeleton is strict, the n-track $K(i, I_n)$ is given by an element

$$\alpha_i \in \mathrm{Hom}_{\mathbf{A}}(\Sigma^n K_i, K_{i-n-1}) \qquad i-n-1 \geq -1 \ .$$

By the obstruction property of K we have $\delta\alpha_i \pm \alpha_{i-1}\delta = 0$.

We now use the induction over i and assume $\alpha_j = 0$ for $j < i$. Then $\delta\alpha_i = 0$, and the exactness yields β with $\alpha_i = \beta\delta$. We construct weak equivalences $L \to R \leftarrow K$ in $\mathrm{Nul}_n\,\mathbf{C}$ which in dimension 0 are given by the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i+1} \arrow[r] \arrow[d] & K_i \arrow[r] \arrow[d] & K_{i-1} \arrow[r] \arrow[d, equal] & \cdots , & K \arrow[d, "i_0"] \\
\cdots \arrow[r, "I\delta"] & IK_{i+1} \arrow[r] & IK_i \arrow[r] & K_{i-1} \arrow[r] & \cdots , & R \\
\cdots \arrow[r] & K_{i+1} \arrow[r] \arrow[u] & K_i \arrow[r] \arrow[u] & K_{i-1} \arrow[r] \arrow[u, equal] & \cdots , & L \arrow[u, "i_1"]
\end{tikzcd}$$

The $(n-2))$-skeleton of R is strict. We define $R(i, I_{n-1})$ by β. Then we can choose $R(i, I_n)$ such that i_0 is a well-defined map and L is i-strict in $\mathrm{Nul}_n\,\mathbf{C}$. □

Proof of Theorem 16.6 for $n \geq 2$. By induction on n, we assume that the Theorem holds for $n-1$. Let K be a resolution of X in $\mathrm{Nul}_n\,\mathbf{C}$, and let $K^{(n-1)}$ be the $(n-1)$-skeleton of K in $\mathrm{nul}_n\,\mathbf{C}$. For $qK^{(n-1)}$ we get by assumption a weak equivalence $qK^{(n-1)} \sim qL^{(n-1)}$, where L is N-strict. Hence by Lemma 16.15 we have $K^{(n-1)} \sim L^{(n-1)}$, and by the Transport Lemma 16.14 we get $K \sim L$ in $\mathrm{Nul}_n\,\mathbf{C}$, where $L^{(n-1)}$ is strict. Now Lemma 16.16 yields $L \sim L'$ in $\mathrm{Nul}_n\,\mathbf{C}$, where L' is N-strict. □

Proof of Theorem 15.8. Let K and L be two resolutions of X in $\mathrm{Nul}_n\,\mathbf{C}$. By Theorem 16.6 we have $L \sim L'$ and $K \sim K'$, where L' and K' are N-(strict, fibrant, exact) for large N. This yields a map of spectral sequences $E_K \to E_L$ which induces an isomorphism on the E_2-term. Hence $E_K \to E_L$ is also an isomorphism. □

Proof of Theorem 15.9. An N-(strict, fibrant, exact) coresolution of X for large N, as in Remark 16.7, corresponds to the X-coaugmented sequence in [BJ2, §6.7] given by the Adams fiber tower [BJ2, 7.1]. □

16.17. *Remark.* Strictification results for ∞-homotopy commutative diagrams appear in [BV, Theorem IV.4.37] and [DKS, Theorem 2.4], inter alia. However, these do not yield the precise case needed for Theorem 16.6. The explicit construction given in this context may be of independent interest.

17. The differential d_2

The first interesting higher order Ext_3-group involves the d_2-differential of the spectral sequence, which we now describe:

Let $\mathbf{T}(\leq 1)$ be a Σ-track algebra and let K be a resolution in $\mathbf{T}(1)$ of X, based on $A_\bullet$ in $\mathbf{A}$, see (15.2). Then we define

$$d_2 : \mathrm{Ext}^r_{\mathbf{A}}(X,Y) \longrightarrow \mathrm{Ext}^{r+1}_{\mathbf{A}}(\Sigma X, Y)$$

as follows. For $\{\beta\} \in \mathrm{Ext}^r_{\mathbf{A}}(X,Y)$ with $\beta : A_r \to Y$ in $\mathbf{A}$, we have $\beta\delta_{r+1} = 0$, so that there is a 1-track H with $\partial^1 H = \beta\delta_{r+1}$. On the other hand K yields a 1-track G with $\partial^1 G = \delta_{r+1}\delta_{r+2}$. Then the obstruction of Definition 15.7 is

(17.1) $$\omega = \mathcal{O}\hat{L}(r+2, \partial I_2) = \mathcal{O}(H\delta_{r+2}, \beta G) \in [\Sigma A_{r+2}, Y]$$

and this element represents $d_2\{\beta\} = \{\omega\}$.

17.2. **Lemma.** *The differential d_2 is well defined.*

Proof. We first check that ω is a cocycle. In fact,

$$\omega(\Sigma\delta_{r+3}) = \mathcal{O}(H\delta_{r+2}, \beta G)(\Sigma\delta_{r+3})$$

(1) $$= \mathcal{O}(H\delta_{r+2}\delta_{r+3}, \beta G\delta_{r+3})$$

(2) $$= \mathcal{O}(H\delta_{r+2}\delta_{r+3}, \beta\delta_{r+1}G')$$

(3) $$= 0.$$

Here (1) holds by na,turality of $\mathcal{O}$. Moreover, G' in (2) is the 1-track with $\partial^1 G' = \delta_{r+2}\delta_{r+3}$ given by the resolution K so that $\mathcal{O}(G\delta_{r+3}, \delta_{r+1}G') = 0$. Hence by naturality also $\mathcal{O}(\beta G\delta_{r+3}, \beta\delta_{r+1}G') = 0$, so that (2) holds by the complement rule in the Appendix below. Finally (3) holds by the triviality rule.

Next we show that $\{\omega\}$ does not depend on the choice of H. If we choose H' instead, there is an α with $H' = H + \alpha$, and we get

$$\omega' = \mathcal{O}((H+\alpha)\delta_{r+2}, \beta G) = \omega \pm \alpha\delta_{r+2}$$

by the action rule. Hence $\omega - \omega'$ is a coboundary, so that $\{\omega'\} = \{\omega\}$.

Finally, we check that $d_2\{\beta\}$ is trivial if β is a coboundary – that is, $\beta = \beta'\delta_r$. In fact, we can then choose H to be the 1-track $\beta' G''$, where G'' with $\delta^1 G'' = \delta_r\delta_{r+1}$ is given by K, so that $\mathcal{O}(G''\delta_{r+2}, \delta_r G) = 0$. Hence also $\mathcal{O}(\beta' G''\delta_{r+2}, \beta'\delta_r G) = 0$, so that $\mathcal{O}(H\delta_{r+2}, \beta G) = 0$. □

The Lemma is proved in [BJ2] in the context of track categories, above we use only 1-track categories. The proof that $d_2 d_2 = 0$ requires the product rule below.

Next we prove that the assumption on L in Definition 15.7 is satisfied for $m = 2$. This leads to the definition of the differential d_3.

17.3. **Lemma.** *Let $d_2\{\beta\} = 0$. Then for $m = 2$ here is a chain complex L as in Definition 15.7.*

Proof. The assumption $d_2\{\beta\} = 0$ shows that $\omega = \mathcal{O}(H\delta_{r+2}, \beta G) = \alpha\delta_{r+2}$ is a coboundary. Hence we get by the action rule $H' = H \pm \alpha$, so that $\mathcal{O}(H'\delta_{r+2}, \beta G) = 0$. Hence we define the chain complex L by H' and by K. □

In the context of a Σ-track algebra $\mathbf{T}(\leq n)$ $(n \geq 1)$, the following result can be proved which is the higher dimensional analogue of Lemma 17.2.

17.4. **Proposition.** *Given L, $\hat{L}$ and*

$$\omega = \mathcal{O}\hat{L}(r+m+1, \partial I_{m+1})$$

as in Definition 15.7, then ω is a cocycle, that is:

$$\omega(\Sigma^m\delta_{r+m+2}) = 0\ .$$

Moreover, if β is a coboundary, then L and $\hat{L}$ can be chosen such that $\omega = 0$. Let L be given and let $\hat{L}$, $\overline{L}$ be two choices as in Definition 15.7. Then $\overline{\omega} = \mathcal{O}\overline{L}(r+m+1, \partial I_{m+1})$ and ω differ by a coboundary; that is: $\omega - \overline{\omega} = \alpha(\Sigma^m\delta_{r+m+1})$.

Appendix: Complete track algebras

Definition 15.7 of the differential d_{m+1} makes sense in any Σ-track algebra $\mathbf{T}(\leq n)$, but in general one cannot show that it has the properties needed to define the higher Ext-groups, such as $d_{m+1}d_{m+1} = 0$. The strictification process described in Section 16 shows that the higher Ext-groups are well-defined in the Example 7.3, and that the differential in the Adams spectral sequence is indeed given by Definition 15.7.

We now introduce the notion of a *complete* Σ-track algebra, to collect together the assumptions needed to show that the differential described above yields well-defined higher Ext-groups. These assumptions are satisfied in particular for the Σ-track category $\mathrm{Nul}_{\leq n}\,\mathbf{C}$, where $\mathbf{C}$ is a complete mapping algebra as in Definition 7.2.

Let $\mathbf{C}$ be a complete Σ-mapping algebra. Then the endofunctor $\Sigma : \mathbf{C} \longrightarrow \mathbf{C}$ induces an endofunctor

(18.1) $$\Sigma : \mathrm{Nul}_{\leq n}\,\mathbf{C} \longrightarrow \mathrm{Nul}_{\leq n}\,\mathbf{C}$$

of Σ-track algebras satisfying

(18.2) $$\Sigma\mathcal{O}_B(b_1,\dots,b_k) = \mathcal{O}_B(\Sigma b_1,\dots,\Sigma b_k)$$

for each left cubical ball of dimension $\leq n$, see (11.1) (3).

18.3. **Definition.** Let $\mathbf{T}(\leq n)$ be a Σ-track algebra and let

$$\Sigma : \mathbf{T}(\leq n) \longrightarrow \mathbf{T}(\leq n)$$

be an endofunctor of $\mathbf{T}(\leq n)$, similarly to (18.1), satisfying (18.2), such that Σ induces the endofunctor $\Sigma : \mathbf{A} \longrightarrow \mathbf{A}$ of the Σ-algebra $\mathbf{A}$. Then $\mathbf{T}(\leq n)$ is a *complete Σ-track algebra* if the sum rule and the product rule below are satisfied.

18.4. **Definition** (sum rule)**.** Let $m \leq n$. Given a pre-chain complex L in $\mathbf{T}(m)$ based on

(1) $$Y \xleftarrow{\alpha} A_0 \xleftarrow{\delta_1} A_1 \longleftarrow \dots \longleftarrow A_{m+1}$$

and given a pre-chain complex L' in $\mathbf{T}(m-1)$ based on

(2) $$Y \xleftarrow{\beta} \Sigma A_1 \xleftarrow{\Sigma\delta_2} \Sigma A_2 \longleftarrow \dots \longleftarrow \Sigma A_{m+1}$$

such that L' restricted to $\mathbb{Z}(m+1,1)$ coincides with ΣL, there exists a pre-chain complex L'' in $\mathbf{T}(m)$ based on (1) such that L'' restricted to $\mathbb{Z}(m+1,0)$ coincides with L and

(3) $$\mathcal{O}L''(m+1,\partial I_{m+1}) = \mathcal{O}L(m+1,\partial I_{m+1}) + \mathcal{O}L'(m+1,\partial I_m)`.$$

This is the *sum rule* in dimension m.

18.5. **Proposition.** *The sum rule is satisfied in* $\mathrm{Nul}_{\leq n}\,\mathbf{C}$ *in* (18.1).

Proof. Let $I \approx [0,2] = I \cup I$ be the homeomorphism of intervals carrying $t \in I$ to $2t$. Then we have:

(1) $$I^{k+1} = I \times I^k \approx (I \cup I) \times I^k = I^{k+1} \cup I^{k+1}\ .$$

for each $k \geq 0$.

For each $j = 1,\dots,m+1$, L' yields the left $(j-1)$-cube $a_j = L'(j+1, I_{j-1})$ in $\mathrm{Mor}_{\mathbf{C}}(\Sigma A_{j+1}, Y)$, which by τ_Σ in Definition 7.2 yields the j-cube $\overline{a}_j$ in $\mathrm{Mor}_{\mathbf{C}}(A_{j+1}, Y)$ adjoined to $\tau_\Sigma a_j$. Using (1), we define the j cube

$$L''(j+1, I_j) = L(j+1, I_j) \cup \overline{a}_j\ .$$

This defines L'' completely, since L'' restricted to $\mathbb{Z}(m+1,0)$ coincides with L. One can now check that the sum formula (18.4) (3) holds. $\square$

Let $\mathbf{T}$ be an $(n+k)$-category enriched in left cubical $(n+k)$-sets and let $\mathbf{T}^n$ be the n-skeleton of $\mathbf{T}$. Then $\mathbf{T}^n$ is an n-graded category enriched in n-cubical sets. We consider a pre-chain complex

$$R : \mathbb{Z}(\infty,0)^n_{\otimes} \longrightarrow \mathbf{T}^n\ .$$

18.6. **Definition.** A *chain module with operators in* R is a functor L which carries a morphism $V : i \longrightarrow -1 \quad (i \geq 0)$ in $\mathbb{Z}(\infty,-1)^n_\otimes$ to an element

$$L(V) \in \mathrm{Mor}_{\mathbf{T}}(R,Y)_{\dim(V)+k}$$

such that the inclusion property

$$L(V) = (d_{V,W} \otimes I^k)^* L(W)$$

holds if V is in the boundary of W and such that for a composite $V \otimes V'$ of morphisms in $\mathbb{Z}(\infty,-1)^n_\otimes$ the equation

$$L(V \otimes V') = L(V)R(V')$$

holds, where the right hand side denotes the composite in the $(n+k)$-category $\mathbf{T}$.

18.7. **Lemma.** *A chain module L with operators in R is determined by the elements*

$$L(m, I_m) \in \mathrm{Mor}_{\mathbf{T}}(R_m, Y)_{m+k}$$

where $I_0 = \emptyset$ and $m = 0, \ldots, n$.

Now let $B = B_1 \cup \ldots \cup B_s$ be a left cubical ball of dimension k with cells B_i and gluing maps $d_{e,i}$ as in Definition 9.2). An s-tuple $(L_1, \ldots, L_s)$ of chain modules L_i with operators in R satisfies the *gluing condition* in B if for $V : m \to -1$ we have

(18.8) $$(I^{\dim(V)} \times d_{e,i})^* L_i(V) = (I^{\dim(V)} \times d_{e,j})^* L_j(V)\ .$$

The left cubical ball $C = T^n_0$ has cells $C_1, \ldots, C_{n+1}$. The product $B \times C$ is a left cubical ball with cells $B_i \times C_j$. Let $(c_1, \ldots, c_{n+2}) = \partial I_{n+1}$, see (4.4).

18.9. **Lemma.** *Given $(L_1, \ldots L_s)$ as in* (18.8) *we obtain the tuple of $(m+k)$-cubes $(r = n+1)$ $L_i(r, c_j)$ satisfying the gluing condition in $B \times C$, so that the obstruction*

$$\mathcal{O}_{B\times C}(L_i(r,c_j),\quad i = 1,\ldots,s \quad \textit{and} \quad j = 1,\ldots,n+1) \quad \textit{in} \quad \mathrm{Hom}_{\mathbf{A}}(\Sigma^{n+k} R_r, Y)$$

is defined in the Σ-track algebra $\mathbf{T} = \mathbf{T}(n+k)$. Also the tuple $L_i(0,\emptyset)$ satisfies the gluing condition, so that the obstruction

$$\mathcal{O}_B(L_1(0,\emptyset),\ldots,L_s(0,\emptyset)) \in \mathrm{Hom}_{\mathbf{A}}(\Sigma^k R_0, Y)$$

is defined in $\mathbf{T}(k)$.

18.10. **Definition** (product rule)**.** Let R and $L_1, \ldots, L_s$ be given as above where R is based on

$$\ldots \longrightarrow R_{n+1} \longrightarrow R_n \longrightarrow \ldots \xrightarrow{\delta_1} R_0 .$$

Let $\alpha \in \mathrm{Hom}_{\mathbf{A}}(\Sigma^k R_0, Y)$ be given by $\mathcal{O}_B(L_i(0,\emptyset))$. Then there exists a pre-chain complex L' based on

$$\Sigma^k R_{n+1} \longrightarrow \Sigma^k R_n \longrightarrow \ldots \xrightarrow{\Sigma^k \delta_1} \Sigma^k R_0 \xrightarrow{\alpha} Y$$

such that the equation

$$\mathcal{O}_C(L'(n+1, \partial I_{n+1})) = \mathcal{O}_{B\times C}(L_i(0,c_j))$$

holds in $\mathrm{Hom}_{\mathbf{A}}(\Sigma^{k+n} R_{n+1}, Y)$ and L' restricted to $\mathbb{Z}(n+1,0)$ coincides with $\Sigma^k R$. This is the *product rule* in $\mathbf{T}(n+k)$.

18.11. **Proposition.** *Let $\mathbf{C}$ be a complete Σ-mapping algebra. Then $\mathrm{Nul}_{\leq(n+k)}(\mathbf{C})$ satisfies the product rule.*

Proof. We have the homeomorphism $S^k = B/\partial B$, so that we can replace $\Sigma^k R_i$ by $(B/\partial B) \wedge R_i$. Then gluing the various L_i yields L'. □

18.12. *Remark.* If $\mathbf{T}(\leq 2n)$ is a complete Σ-track algebra (Definition 18.3) the higher Ext-groups $E^{r,s}_m$ are well-defined by Definition15.7 for $m = 2, \ldots, n+2$. A proof can be given along the lones of the argument given below to show that $d_2 d_2 = 0$.

Since a complete Σ-mapping algebra $\mathbf{C}$ yields a complete Σ-track algebra $\mathrm{Nul}_{\leq 2n}(\mathbf{C})$, the higher Ext-groups are well defined in this case.

18.13. **Example.** We prove that $d_2 d_2 = 0$, as an application of the product rule. By (15.3) we get the diagram

$$
\begin{tikzcd}
A_{r+4} \arrow[r, "\delta_4"] \arrow[d, equal] & A_{r+3} \arrow[r, "\delta_3"] \arrow[d, equal] & A_{r+2} \arrow[r, "\delta_2"] \arrow[d, equal] \arrow[rr, bend left=40, "o"] \arrow[rr, phantom, shift left=3, "\uparrow G"] & A_{r+1} \arrow[r, "\delta_1"] \arrow[rr, bend right=40, "o"'] \arrow[rr, phantom, shift right=3, "\downarrow F"'] & A_r \arrow[r, "\beta"] & Y \\
R_2 & R_1 & R_0 & & &
\end{tikzcd}
$$

For $B = T_0^1 = B_1 \cup B_2$, we choose $L_1(0, \emptyset) = F\delta_2$ and $L_2(0, \emptyset) = \beta G$, where F and G are 1-cubes with $G = K(r+2, I_1)$ given by the resolution K. Then

(1) $$\alpha = d_2\beta = \mathcal{O}_B(L_1(0, \emptyset), L_2(0, \emptyset)).$$

Now let $C = T_0^1 = C_1 \cup C_2$ and $(c_1, c_2) = (I_1 \otimes \emptyset, \emptyset \otimes I_1)$. Then $L_i(2, c_j)$ is defined as follows:

$$
\begin{array}{rclcl}
L_1(2, c_1) & = & L_1(2, I_1 \otimes \emptyset) & = & FK(r+3, I_1)\delta_4 \\
L_1(2, c_2) & = & L_1(2, \emptyset \otimes I_1) & = & F\delta_2 K(r+4, I_1) \\
L_2(2, c_1) & = & L_2(2, I_1 \otimes \emptyset) & = & \beta K(r+3, I_2)\delta_4 \\
L_2(2, c_2) & = & L_2(2, \emptyset \otimes I_1) & = & \beta K(r+2, I_1) K(r+4, I_1)
\end{array}
$$

Now the product rule shows that

(2) $$d_2 d_2 \beta = d_2 \alpha = \mathcal{O}_{B \times C}(L_i(0, c_j)) = 0$$

and the rules in a 2-track category show that this obstruction is trivial. In fact, we have

(3) $$\mathcal{O}(I_1 I_1 \emptyset, \emptyset I_2 \emptyset, I_1 \emptyset I_1, \emptyset I_1 I_1)$$

(4) $$= \mathcal{O}(I_1 I_1 \emptyset, \emptyset \emptyset I_2, I_1 \emptyset I_1)$$

(5) $$= 0$$

Here (3) is the obstruction (2) and (4) is a consequence of the complement rule and

$$\mathcal{O}(\emptyset I_2, I_1 I_1, I_2 \emptyset) = 0$$

as follows from the fact that K is a resolution, the naturality yields

$$\mathcal{O}(\emptyset \emptyset I_2, \emptyset I_1 I_1, \emptyset I_2 \emptyset) = 0.$$

Moreover (5) follows from the triviality rule.

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