The signature and the elliptic genus of π_2-finite manifolds with circle actions

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Abstract

We prove the vanishing of various characteristic numbers, such as the signature and the $\widehat{A}$-genus, on manifolds with finite second homotopy group and which admit smooth S^1-actions. More precisely, we prove the vanishing of various coefficientes of the elliptic genus on non-spin π_2-finite manifolds when the S^1-action either satisfies a "parity" condition or has isolated fixed points only.

1 Introduction

We shall call a manifold with finite second homotopy group a π_2-*finite manifold*. Our interest in the topology of these manifolds arose from geometrical considerations that can be found in [9, 10, 13, 16]. We have found that they have several features remarkably similar to those of spin manifolds and, in fact, this paper elaborates further in this direction. These features become particularly interesting when observed on *non-spin* π_2-finite manifolds on which the Dirac operator cannot be defined globally. Nevertheless, the $\widehat{A}$-genus vanishes on non-spin π_2-finite manifolds with circle actions. Note that in this case, the $\widehat{A}$-genus no longer represents the index of an elliptic operator and, in principle, is only a rational number [8].

In Section 2, we define *odd* and *even* S^1-actions on oriented manifolds, and prove that every circle action on a π_2-finite manifold is either even or odd. Given this fact, we prove various vanishing results for characteristic numbers such as the following statement (see Theorem 2.1 (i))

Let M be an oriented, connected, compact π_2-finite manifold, $|\pi_2(M)| < \infty$. Assume M admits an odd smooth S^1-action. Then the signature of M vanishes

$$\mathrm{sign}(M) = 0.$$

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The elliptic genus Φ of a compact, oriented, smooth $2n$-dimensional manifold M can be defined as the index of the signature operator d_s twisted in the following way

$$\Phi(M) = \mathrm{ind}\left(d_s \otimes \bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{q^i} T \otimes \bigotimes_{i=1}^{\infty} S_{q^i} T\right), \tag{1}$$

where $T = TM_c$ is the complexified tangent bundle, $S_a T = \sum_{j=0}^{\infty} a^j S^j T$, $\bigwedge_a T = \sum_{j=0}^{\infty} a^j \bigwedge^j T$, and $S^j T$, $\bigwedge^j T$ denote the j-th symmetric and exterior tensor powers of T respectively (c.f. [17, 18, 11]). The first few terms of the series of bundles are

$$\begin{aligned} R_0 &= 1 \\ R_1 &= 2T \\ R_2 &= 2(T^{\otimes 2} + T) \\ R_3 &= 2T^{\otimes 3} + \bigwedge\nolimits^3 T + S^3 T + 4T^{\otimes 2} + 2T, \ldots \end{aligned}$$

so that the constant term in $\Phi(M)$ is $\mathrm{sign}(M)$. Witten conjectured the rigidity of (1) under S^1-actions on spin manifolds [17, 18]. The rigidity theorem was proved by Taubes [15], Bott and Taubes [3].

In [8, 10], we proved the vanishing of the $\widehat{A}$-genus and the rigidity of the elliptic genus under S^1-actions on non-spin π_2-finite manifolds, which show that these non-spin manifolds are similar to spin manifolds with respect to S^1-actions. As mentioned above, we elaborate further on this similarity by proving the vanishing of the elliptic genus under various conditions on the action.

In Section 3 we assume that the S^1-action has isolated fixed points only and derive further vanishing theorems.

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2 Even and odd S^1-actions

Let M be an oriented, connected, compact $2n$-dimensional manifold. Assume M is endowed with a (non-trivial) smooth S^1-action. Let M^{S^1} denote the fixed point set of the circle action. At each point $p \in M^{S^1}$, the tangent space of M splits as a sum of S^1 representations

$$T_p M = T_p M^{S^1} \oplus L^{m_1} \oplus \ldots \oplus L^{m_k},$$

where L^a denotes the S^1 representation on which $\lambda \in S^1$ acts by multiplication by λ^a. The space $T_p M^{S^1}$ is a trivial representation of S^1.

The numbers $m_1, \ldots, m_k$ are called the exponents (or weights) of the S^1-action at the (connected component of M^{S^1} containing the) point p. They are not canonical and their sign can be changed in pairs. Consider the sum of the exponents $S(p) = \sum_{i=1}^k m_i$ at $p \in M^{S^1}$. The number $S(p)$ is constant on the same connected component of M^{S^1}, but may vary for different connected components.

Example. Consider the S^1 action on $\mathbb{CP}^2$ given by

$$\lambda \cdot [x_0 : x_1 : x_2] \longrightarrow [x_0 : \lambda x_1 : \lambda^2 x_2],$$

where $\lambda \in S^1$. The (isolated) fixed points for this action are $l_0 = [1 : 0 : 0]$, $l_1 = [0 : 1 : 0]$ and $l_2 = [0 : 0 : 1]$. The exponents of the S^1-action at each point are:

$$\begin{array}{lrrl} l_0 & 1 & 2 & S(l_0) = 3 \\ l_1 & -1 & 1 & S(l_1) = 0 \\ l_2 & -2 & -1 & S(l_2) = -3 \end{array}$$

Notice that the numbers $S(l_i)$ do not even have the same parity.

On the other hand, there are classes of manifolds with S^1-actions whose numbers $S(p)$ have the same parity for every fixed point $p \in M^{S^1}$. For example, spin manifolds satisfy such a condition [1], as well as π_2-finite manifolds (c.f. [8, 10], see below). We may assume that M^{S^1} is non-empty since otherwise M is rationally zero-bordant and $\Phi(M) = 0$.

Definition 2.1 *A circle action on an oriented $2n$-dimensional manifold M with non-empty fixed point set will be called* even *if the sum $S(p) \equiv 0 \pmod 2$ for all $p \in M^{S^1}$, and* odd *if $S(p) \equiv 1 \pmod 2$ for all $p \in M^{S^1}$.*

In the next few paragraphs we verify that S^1-actions on π_2-finite manifolds are either odd or even, and that the codimension of the fixed point submanifolds behave as in the spin case [1].

Lemma 2.1 *Let M be an oriented, connected, compact $2n$-dimensional manifold with finite second homotopy group, $|\pi_2(M)| < \infty$. Assume M admits a smooth S^1 action. Then $S(p_1) = S(p_2)$ for all $p_1, p_2 \in M^{S^1}$. In particular, the S^1-action is either even or odd.*

Proof. Consider the S^1-decompositions of the tangent space at the two points in terms of generator L of the representation ring $R(S^1) = \mathbb{Z}\{L\}$ (the ring of finite Laurent series in L)

$$\begin{aligned} T_{p_1}M &= L^{m_1} \oplus \ldots \oplus L^{m_n}, \\ T_{p_2}M &= L^{m'_1} \oplus \ldots \oplus L^{m'_n}, \end{aligned}$$

where the m_i and m'_i are the exponents of the S^1 action at each point. By [4], the virtual representation $T_{p_1}M - T_{p_2}M$ can be factored as follows

$$T_{p_1}M - T_{p_2}M = (1-L)^2 \otimes \left(\bigoplus\nolimits_j b_j L^j \right), \tag{2}$$

where the set $\{b_j \in \mathbb{Z}\}$ is finite. Clearly

$$S(p_1) - S(p_2) = \sum_j b_j \cdot j - 2\sum_j b_j \cdot (j+1) + \sum_j b_j \cdot (j+2) = 0.$$

□

Lemma 2.1 allows us to talk about even and odd actions on π_2-finite manifolds in a similar fashion to that of the spin case.

Proposition 2.1 *Let M be an oriented, connected, compact $2n$-dimensional manifold with finite second homotopy group, admitting a smooth S^1-action. Let $\mathbb{Z}_2 = \{\pm 1\}$ be the subgroup of S^1 generated by the involution.*

Let X be a connected component of the $\mathbb{Z}_2$-fixed point set $M^{\mathbb{Z}_2}$ such that $X \cap M^{S^1} \neq \emptyset$. Then

$$\begin{aligned} \operatorname{codim}(X) &\equiv 0 \pmod 4 \quad \textit{if the action is even,} \\ \operatorname{codim}(X) &\equiv 2 \pmod 4 \quad \textit{if the action is odd.} \end{aligned}$$

Proof. Let $p \in X \cap M^{S^1}$ and consider the S^1 decomposition of

$$T_p(M) = L^{m_1} \oplus \ldots \oplus L^{m_n},$$

where $m_i \in \mathbb{Z}$. This (complex) decomposition reduces to twice the (real) decomposition under the subgroup $\mathbb{Z}_2$. The summands with even m_i reduce to twice the trivial real representation of $\mathbb{Z}_2$, and the ones with m_i odd reduce to twice the non-trivial representation. Therefore, half the codimension of X is

$$\begin{aligned} \frac{1}{2}\operatorname{codim}(X) &= \sum_{m_i \,\mathrm{odd}} 1 \equiv \sum_{m_i \,\mathrm{odd}} m_i \pmod 2 \\ &\equiv \sum_{i=1}^{2n} m_i \pmod 2 \\ &\equiv \begin{cases} 0 \pmod 2 & \text{if the action is even} \\ 1 \pmod 2 & \text{if the action is odd} \end{cases} \end{aligned}$$

□

From now on, we shall assume that $\dim(M) \equiv 0 \pmod 4$.

Theorem 2.1 *Let M be an oriented, connected, compact $4n$-dimensional manifold with finite second homotopy group, $|\pi_2(M)| < \infty$. Assume M admits a (non-trivial) smooth S^1-action, and let $\mathbb{Z}_2 = \{\pm 1\}$ the subgroup generated by the involution $-1 \in S^1$.*

(i) *If the action is odd, then the elliptic genus and the signature of M vanish*

$$\Phi(M) = 0 \qquad \textit{and} \qquad \operatorname{sign}(M) = 0.$$

(ii) *If the action is even and* $\operatorname{codim}(Y) \geq 4r$ *for all the connected components* Y *of* $M^{\mathbb{Z}_2}$ *that contain* S^1*-fixed points, then the characteristic numbers* $\widehat{A}(M, R'_j)$ *vanish for* $1 \leq j \leq r-1$*. If* $r \geq n/2$ *then* $\Phi(M)$ *does not depend on the variable* q *and* $\Phi(M) = \operatorname{sign}(M)$*. If* $r > n/2$*, then the elliptic genus and the signature of* M *vanish*

$$\Phi(M) = 0 \qquad \textit{and} \qquad \operatorname{sign}(M) = 0.$$

(iii) *In particular* [8], *the* $\widehat{A}$*-genus of* M *vanishes regardless of the parity of the action*

$$\widehat{A}(M) = 0.$$

Remark. Note that the vanishing of characteristic numbers such as $\widehat{A}(M)$ and $\widehat{A}(M, T)$ is new and highly non-trivial on non-spin π_2-finite manifolds, since they do not represent the index of an elliptic operator.

Proof. The main tool for the proof is the rigidity theorem of the elliptic genus for π_2-finite manifolds with S^1-actions [8, 10]. Namely, let G be a Lie group acting on an oriented, smooth manifold by diffeomorphisms that preserve the orientation. The coefficients of q^k in $\Phi(M)$ are of the form $\operatorname{sign}(M, R_k)$ where the bundle R_k is such that the G-action lifts to it: $R_0 = 1$, $R_1 = 2T$, $R_2 = 2(T \oplus T \otimes T)$, etc. Thus, for $g \in G$, the equivariant signature $\operatorname{sign}(M, R_k)_g$ is defined for all k, and we get an equivariant elliptic genus

$$\Phi(M)_g = \sum_{k=0}^{\infty} \operatorname{sign}(M, R_k)_g \cdot q^k,$$

which is also a power series in q. The rigidity theorem of the elliptic genus for π_2-finite manifolds states that

$$\Phi(M) = \Phi(M)_g \tag{3}$$

for all $g \in G$. Notice that this is certainly not the case for manifolds with infinite second homotopy group, such as the complex projective plane $\mathbb{CP}^2$.

Recall that $\Phi(M)$ has the following expansion at the other cusp [11]

$$\tilde{\Phi}(M) = \frac{1}{q^{n/2}} \sum_{j=0}^{\infty} \widehat{A}(M, R'_j),$$

where R'_j is a sequence of representations in terms of T starting with $R'_0 = 1$, $R'_1 = -T$, $R'_2 = \bigwedge^2 T + T$, etc., and $\widehat{A}(M, R'_j)$ denotes the $\widehat{A}$-genus twisted by the bundle R'_j. This (rational) characteristic number is defined for all oriented compact manifolds. It constitutes the index of the Dirac operator with coefficients in R'_j only when the manifold is spin, in which case it is an integer.

By the detailed analysis of [7], which is also valid in our case due to the rigidity theorem [8, 10], we can consider only those components of M^g of $M^{\mathbb{Z}_2}$ that contain

S^1-fixed points. Let $M^g \circ M^g$ denote its (transversal) self-intersection. $M^g \circ M^g$ is oriented and smooth. By [12]

$$\Phi(M)_g = \Phi(M^g \circ M^g).$$

Hence, by the rigidity (3)

$$\Phi(M) = \Phi(M^g \circ M^g),$$

which implies

$$\tilde{\Phi}(M) = \tilde{\Phi}(M^g \circ M^g). \tag{4}$$

The codimension of M^g is positive and even, therefore the right hand side of (4) has a pole of order less than $n/2$, which implies that the coefficient of $q^{n/2}$ in the left hand side of (4) must vanish, proving (iii).

If the action is odd, $\mathrm{codim}(M^g \circ M^g) \equiv 4 \pmod 8$. The order of the pole for $\tilde{\Phi}(M)$ in q is both integral and half integral in (4). Hence, $\Phi(M) = 0$, proving (i).

If the action is even the hypotheses on the codimension indicates only that the order of the pole in $\tilde{\Phi}(M^g \circ M^g)$ is smaller than the one in $\tilde{\Phi}(M)$ according to the codimension Hence the vanishing of the coefficients in (ii) □

3 Isolated fixed points

Theorems 2.1 shows that both the type of the action and the codimension of fixed point sets of the involution $-1 \in S^1$ can determine the elliptic genus and the signature of a π_2-finite manifold. In this section, inspired by [6, Section 6.1], we shall study the case when the S^1 action has isolated fixed points only. Again, the main tool will be the rigidity theorem of the elliptic genus on π_2-finite manifolds.

Let M be an oriented, compact, connected, π_2-finite $2n$-dimensional manifold admitting a smooth S^1-action with isolated fixed points only. Let $F = M^{S^1}$ be the fixed point set and $\{m_{p,i} : i = 1...n\}$ the set of exponents of the action at each $p \in F$. Notice that since p is isolated, all the exponents are non-zero.

By [2], the equivariant elliptic genus is

$$\Phi(M)_{S^1}(\lambda) = \sum_{p\in F} \mu_p(q, \lambda)$$

where λ is a topological generator of S^1. The local contributions of the fixed points are

$$\mu_p(q, \lambda) = \prod_{i=1}^{n} \frac{1 + \lambda^{-m_{p,i}}}{1 - \lambda^{-m_{p,i}}} \cdot \prod_{l=1}^{\infty} \frac{(1 + q^l\lambda^{m_{p,i}})(1 + q^l\lambda^{-m_{p,i}})}{(1 - q^l\lambda^{m_{p,i}})(1 - q^l\lambda^{-m_{p,i}})}. \tag{5}$$

Since the elliptic genus is rigid on π_2-finite manifolds with S^1-actions, $\Phi(M)_{S^1}(\lambda)$ does not depend on λ.

Let

$$\psi(\tau, z) = \frac{1 + \lambda^{-1}}{1 - \lambda^{-1}} \cdot \prod_{l=1}^{\infty} \frac{(1 + q^l\lambda)(1 + q^l\lambda^{-1})}{(1 - q^l\lambda)(1 - q^l\lambda^{-1})}$$

where $q = e^{2\pi i\tau}$ and $\lambda = e^{2\pi i z}$. The function $\psi(\tau, z)$ is meromorphic on $\mathcal{H} \times \mathbb{C}$ with zeroes of order one at $\mathbb{Z} \cdot \tau + \mathbb{Z} - 1/2$ and poles of order one at $\mathbb{Z} \cdot \tau + \mathbb{Z}$. Define

$$\varphi(\tau, z) = \sum_{p \in F} \prod_{i=1}^{n} \psi(\tau, m_{p,i} \cdot z). \tag{6}$$

Thus,

$$\Phi(M) = \varphi(\tau, z),$$

by the aforementioned rigidity, and it does not depend on z.

In order to derive more detailed information on the signature, we shall borrow some observations from A. Dessai [6, Section 6.1]. The function ψ is related to the Weierstrass Φ-function [11] and has the following properties:

- $$\psi(\tau, -z) = -\psi(\tau, z), \qquad \psi(\tau, z+1) = \psi(\tau, z),$$

  $$\psi(\tau, z+1/2) = \psi(\tau, z)^{-1}, \qquad \psi(\tau, z+\tau) = -\psi(\tau, z),$$

  $$\psi(\tau, 1/2) = 0, \qquad \psi(\tau, 1/4) = \frac{1-i}{1+i}, \qquad \psi(\tau, 3/4) = \frac{1+i}{1-i},$$

- If $a \in \mathbb{Z}$ is odd, $\psi(\tau, a/4)$ is constant in τ.
- If $a \in \mathbb{Z}$ is such that $a \not\equiv 0 \pmod 3$,

  $$\psi(\tau, (a\tau)/3) = \pm\psi(\tau, \tau/3)$$

- If $s, t \in \mathbb{Z}$, such that $s \equiv 1 \pmod 2$ and $t \equiv 0 \pmod 2$ then

  $$(\psi(\tau, s \cdot z) \cdot \psi(\tau, t \cdot z))|_{z=1/2} = \frac{s}{t}$$

  is a constant function in τ.
- If $s, t \in \mathbb{Z}$, such that $s \equiv 2 \pmod 4$ and $t \equiv 0 \pmod 4$ then

  $$(\psi(\tau, s \cdot z) \cdot \psi(\tau, t \cdot z))|_{z=1/4} = \frac{s}{t}$$

  is a constant function in τ.

For each $p \in F$ let

$$m(p)_e = \#\{m_{p,i} \equiv 0 \pmod 2\}, \qquad m(p)_o = \#\{m_{p,i} \equiv 1 \pmod 2\},$$

$$\tilde{m}(p)_e = \#\{m_{p,i} \equiv 0 \pmod 4\}, \qquad \tilde{m}(p)_o = \#\{m_{p,i} \equiv 2 \pmod 4\}.$$

The following theorem follows from the properties of ψ.

Theorem 3.1 *Let M be an oriented, compact, connected, π_2-finite $2n$-dimensional manifold admitting a smooth S^1-action with isolated fixed points only. Then*

1. If for every $p \in F$

$$m(p)_e \leq m(p)_o, \qquad \textit{then} \qquad \Phi(M) = \mathrm{sign}(M).$$

If the inequality is strict, $\Phi(M) = \mathrm{sign}(M) = 0$.

2. If for every $p \in F$

$$\tilde{m}(p)_e \leq \tilde{m}(p)_o, \qquad \textit{then} \qquad \Phi(M) = \mathrm{sign}(M).$$

If the inequality is strict, $\Phi(M) = \mathrm{sign}(M) = 0$.

3. If $m_{p,i} \not\equiv 0 \pmod 3$ for all $p \in F$ and all i, then $\Phi(M) = 0$.

4. If $\dim(M) \equiv 4 \pmod 8$ and $\Phi(M) = \mathrm{sign}(M)$, then $\mathrm{sign}(M) = 0$.

Part 4 follows from modular properties of the elliptic genus. □

Corollary 3.1 *Let M be as in Theorem 3.1. Assume that the S^1-action is also semi-free. Then all the exponents of the action are ± 1 so that*

$$\Phi(M) = \mathrm{sign}(M) = 0.$$

□

Corollary 3.2 *Let M be as in Theorem 3.1. Assume that the S^1-action can be extended to a $SU(2)$ action with $M^{SU(2)} = M^{S^1}$. Then, all the exponents of the action are odd and*

$$\Phi(M) = \mathrm{sign}(M) = 0.$$

□

Finally, let us consider manifolds with finite even homotopy groups.

Theorem 3.2 *Let M be an oriented, compact, connected, $2n$-dimensional manifold with $|\pi_{2i}(M)| < \infty$ for all $1 \leq i \leq \dim(M)/2$. Then M admits no smooth S^1-action with isolated fixed points only. In particular, if M is the underlying manifold of a compact connected Lie group, then M does not admit smooth S^1-actions with isolated fixed points only.*

Proof. By [5], if $|\pi_{2i}(M)| < \infty$ for $1 \leq i \leq \dim(M)/2$, the fixed point set M^{S^1} is connected, which in the isolated fixed point case means there is only one fixed point. A smooth S^1-action on an oriented manifold, however, cannot have exactly one fixed point. □

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