Max-Planck-Institut für Mathematik
Bonn

The Freiheitssatz for Poisson algebras

by

Leonid Makar-Limanov
Ualbai Umirbaev

Max-Planck-Institut für Mathematik
Preprint Series 2010 (65)

The Freiheitssatz for Poisson algebras

Leonid Makar-Limanov
Ualbai Umirbaev

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Wayne State University
Detroit, MI 48202
USA

Department of Mathematics
Eurasian National University
Astana 010008
Kazakhstan

MPIM 10-65

THE FREIHEITSSATZ FOR POISSON ALGEBRAS

Leonid Makar-Limanov[1] and **Ualbai Umirbaev**[2]

Abstract. We prove the Freiheitssatz for Poisson algebras in characteristic zero. We also give a new proof of the tameness of automorphisms for two generated free Poisson algebras and show that an analogue of the commutator test theorem is equivalent to the two-dimensional classical Jacobian conjecture.

Mathematics Subject Classification (2010): Primary 17B63, 17B40; Secondary 17A36, 16W20.

Key words: Poisson algebras, Freiheitssatz, automorphisms.

1. Introduction

Many interesting and important results have been obtained about the structure of polynomial algebras, free associative algebras, and free Lie algebras. Although the free Poisson algebras are closely connected with these algebras, only a few results on their structure on their structure are known. Here is the surprisingly short list. (1) The centralizer of a nonconstant element of a free Poisson algebra in the case of characteristic zero is a polynomial algebra in a single variable (proved in [13], it is an analogue of the famous Bergman's Centralizer Theorem [1]) (2) Locally nilpotent derivations of two generated free Poisson algebras in the case of characteristic zero are triangulable and the automorphisms of these algebras are tame (proved in [14], these are analogues of the well-known Rentschler's Theorem [17] and Jung's Theorem [7] respectively). In this paper we continue the study of free Poisson algebras.

In 1930 W. Magnus proved one of the most important theorems of the combinatorial group theory (see [10]): *Let $G = \langle x_1, x_2, \ldots, x_n | r = 1\rangle$ be a group defined by a single cyclically reduced relator r. If x_n appears in r, then the subgroup of G generated by $x_1, \ldots, x_{n-1}$ is a free group, freely generated by $x_1, \ldots, x_{n-1}$.* He called it *the Freiheitssatz* and used it to give several applications, the decidability of the word problem for groups with a single defining relation among them.

Because of its importance and potential applications, the Freiheitssatz was studied in various settings. Thus, A. I. Shirshov [20] established it for Lie algebras (and deduced the decidability of the word problem for Lie algebras with a single defining relation); N. S. Romanovskii [18] researched it for solvable and nilpotent groups; L. Makar-Limanov

[1]Supported by an NSA grant H98230-09-1-0008, by an NSF grant DMS-0904713, and a Fulbright fellowship awarded by the United States–Israel Educational Foundation; Department of Mathematics & Computer Science, the Weizmann Institute of Science, Rehovot 76100, Israel and Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, e-mail: *lml@math.wayne.edu*

[2]Supported by an NSF grant DMS-0904713 and by a grant of Kazakhstan; Department of Mathematics, Eurasian National University, Astana, 010008, Kazakhstan and Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, e-mail: *umirbaev@math.wayne.edu*

[12] proved it for associative algebras over a field of characteristic zero (the question of decidability of the word problem for associative algebras and semigroups with a single defining relation and the Freiheitssatz for associative algebras in a positive characteristic remain open (see [2])); recently (see [8]) it was confirmed for right-symmetric algebras (and the decidability of the word problem for right-symmetric algebras with a single defining relation being obtained as a byproduct).

In this paper we prove the Freiheitssatz for Poisson algebras over fields of characteristic zero. If the characteristic is positive, it is not true. E. g. if $r = x + \{x, y\}^p$ where $p > 0$ is the characteristic, the ideal $(r) \ni x$ since $\{r, y\} = \{x, y\}$. There are two principal methods of proving the Freiheitssatz: one, employing the combinatorics of free algebras, applied in [10, 18, 20, 8], and the other, related to the study of algebraic and differential equations, applied in [12]. The latter is used here.

The paper is organized as follows. In Section 2 we prove that some type of differential equations admits a solution in formal power series over a field of characteristic zero. (A better result can be obtained over the field of complex numbers from a non-linear Cauchy-Kovalevsky theorem, see [16].) In Section 3 we confirm that symplectic Poisson algebras of infinite rank do not satisfy any nontrivial polynomial identity. In Section 4 we rewrite abstract Poisson algebraic equations as differential equations studied in Section 2 and prove the Freiheitssatz. In Section 5 we give a new proof of the tameness of automorphisms of free Poisson algebras of rank two and show that an analogue of the commutator test theorem found by W. Dicks for free associative algebras of rank two (see [4]) is equivalent to the two-dimensional classical Jacobian conjecture using the Freiheitssatz and Jung's Theorem.

2. Differential equations

Consider the set $\mathbb{Z}_+^n$, where $\mathbb{Z}_+$ is the set of all nonnegative integers. Denote by $\preceq$ the lexicographic order on $\mathbb{Z}_+^n$. Note that $\mathbb{Z}_+^n$ is well-ordered with respect to $\preceq$.

Let k be an arbitrary field of characteristic zero. Let $k[x_1, x_2, \ldots, x_n]$ be the polynomial algebra in the variables $x_1, x_2, \ldots, x_n$. For every $\alpha = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n$ we put

$$\partial^\alpha = (\frac{\partial}{\partial x_1})^{i_1} (\frac{\partial}{\partial x_2})^{i_2} \ldots (\frac{\partial}{\partial x_n})^{i_n}$$

and define a variable t^α.

Proposition 1. *Let $f(x_1, x_2, \ldots, x_n, t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_m}) \in k[x_1, x_2, \ldots, x_n, t^{\alpha_1}, t^{\alpha_2}, \ldots, t^{\alpha_m}]$ and $\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_m$. Suppose that there exists $(c_1, c_2, \ldots, c_n, c^{\alpha_1}, c^{\alpha_2}, \ldots, c^{\alpha_m}) \in k^{n+m}$ so that $f(c_1, c_2, \ldots, c_n, c^{\alpha_1}, c^{\alpha_2}, \ldots, c^{\alpha_m}) = 0$ and $\frac{\partial f}{\partial t^{\alpha_m}}(c_1, c_2, \ldots, c_n, c^{\alpha_1}, c^{\alpha_2}, \ldots, c^{\alpha_m}) \neq 0$. Then the differential equation*

$$f(x_1, x_2, \ldots, x_n, \partial^{\alpha_1}(T), \partial^{\alpha_2}(T), \ldots, \partial^{\alpha_m}(T)) = 0 \tag{1}$$

has a solution in the formal power series algebra $k[[x_1 - c_1, x_2 - c_2, \ldots, x_n - c_n]]$.

Proof. For convenience of notations we put $C = (c_1, c_2, \ldots, c_n)$, $X = (x_1, x_2, \ldots, x_n)$, $X - C = (x_1 - c_1, x_2 - c_2, \ldots, x_n - c_n)$, and $\overline{C} = (c_1, c_2, \ldots, c_n, c^{\alpha_1}, c^{\alpha_2}, \ldots, c^{\alpha_m})$. For every $\alpha = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n$ put also

$$\alpha! = i_1!i_2!\ldots i_n!, \quad (X-C)^\alpha = (x_1-c_1)^{i_1}(x_2-c_2)^{i_2}\ldots(x_n-c_n)^{i_n}.$$

We claim that the equation (1) has a unique formal solution in the form

$$T = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (X-C)^\alpha. \tag{2}$$

satisfying the initial conditions

$$\partial^{\alpha_1}(T)(C) = c^{\alpha_1}, \partial^{\alpha_2}(T)(C) = c^{\alpha_2}, \ldots, \partial^{\alpha_m}(T)(C) = c^{\alpha_m}$$

and

$$\partial^\beta(T)(C) = 0$$

for every $\beta \neq \alpha_1, \alpha_2, \ldots, \alpha_{m-1}$ and $\beta \prec \alpha_m$. Note that

$$\partial^\alpha(T)(C) = \alpha! a_\alpha \tag{3}$$

for every α in (2). So, by (3) we can define the values of a_α for every $\alpha \preceq \alpha_m$ since we already defined the values of $\partial^\alpha(T)(C)$.

Substituting (2) into the right hand side of the equation (1) we get

$$f(x_1, x_2, \ldots, x_n, \partial^{\alpha_1}(T), \partial^{\alpha_2}(T), \ldots, \partial^{\alpha_m}(T)) = \sum_{\beta \in \mathbb{Z}_+^n} b_\beta (X-C)^\beta. \tag{4}$$

We have to show that there exists a sequence $\{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ such that $b_\beta = 0$ for every $\beta \in \mathbb{Z}_+^n$. We prove this by transfinite induction using the relation $\preceq$ on $\mathbb{Z}_+^n$.

As above,

$$\beta! b_\beta = \partial^\beta(f(x_1, x_2, \ldots, x_n, \partial^{\alpha_1}(T), \partial^{\alpha_2}(T), \ldots, \partial^{\alpha_m}(T)))(C). \tag{5}$$

Then,

$$b_0 = f(x_1, x_2, \ldots, x_n, \partial^{\alpha_1}(T), \partial^{\alpha_2}(T), \ldots, \partial^{\alpha_m}(T))(C) = f(\overline{C}) = 0.$$

For the induction step take a nonzero element $\beta \in \mathbb{Z}_+^n$ such that a_α is defined for every $\alpha \prec \alpha_m + \beta$ and $b_\gamma = 0$ for every $\gamma \prec \beta$.

By (5),

$$\begin{aligned} b_\beta = \frac{1}{\beta!}\partial^\beta(f(x_1, x_2, \ldots, x_n, \partial^{\alpha_1}(T), \partial^{\alpha_2}(T), \ldots, \partial^{\alpha_m}(T)))(C) \\ = \frac{1}{\beta!}\left(\frac{\partial f}{\partial t^{\alpha_m}}(\overline{C})(\partial^{\beta+\alpha_m}(T))(C) + A\right), \end{aligned}$$

where A depends only on $\partial^\alpha(T)(C)$ with $\alpha \prec \alpha_m + \beta$. The values of $(\partial^\alpha(T))(C)$ for $\alpha \prec \alpha_m + \beta$ defined by (3). Since $\frac{\partial f}{\partial z^{\alpha_m}}(\overline{C}) \neq 0$ there exists a unique value of $(\partial^{\beta+\alpha_m}(T))(C)$ such that $b_\beta = 0$. Put $a_{\beta+\alpha_m} = \frac{1}{(\beta+\alpha_m)!}(\partial^{\beta+\alpha_m}(T))(C)$. $\square$

3. Identities of symplectic algebras

A vector space B over a field k endowed with two bilinear operations $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket) is called *a Poisson algebra* if B is a commutative associative algebra under $x \cdot y$, B is a Lie algebra under $\{x, y\}$, and B satisfies the following identity (the Leibniz identity):

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}.$$

There are two important classes of Poisson algebras.

1) Symplectic Poisson algebras PS_n. For each n algebra PS_n is a polynomial algebra $k[x_1, y_1, \ldots, x_n, y_n]$ endowed with the Poisson bracket defined by

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where δ_{ij} is the Kronecker symbol and $1 \leq i, j \leq n$. Note that PS_n is a subalgebra of PS_m if $n \leq m$. We consider also the symplectic Poisson algebra of infinite rank $PS_\infty = \bigcup_{n=1}^{\infty} PS_n$.

2) Symmetric Poisson algebras $PS(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra with a linear basis $e_1, e_2, \ldots, e_k, \ldots$. Then the usual polynomial algebra $k[e_1, e_2, \ldots, e_k, \ldots]$ endowed with the Poisson bracket defined by

$$\{e_i, e_j\} = [e_i, e_j]$$

for all i, j, where $[x, y]$ is the multiplication of the Lie algebra $\mathfrak{g}$, is a Poisson algebra and is called the symmetric Poisson algebra of $\mathfrak{g}$.

Note that the Poisson bracket of the algebra $PS(\mathfrak{g})$ depends on the Lie structure of $\mathfrak{g}$ but does not depend on a chosen basis.

Corollary 1. *Let $e_1, e_2, \ldots, e_m, \ldots$ be linearly independent elements of $\mathfrak{g}$. Then the elements*

$$u = e_{i_1} e_{i_2} \ldots e_{i_k}, \quad i_1 \leq i_2 \leq \ldots \leq i_k$$

are linearly independent in $PS(\mathfrak{g})$.

Let $\mathfrak{g}$ be a free Lie algebra with free (Lie) generators $x_1, x_2, \ldots, x_n, \ldots$. It is well known (see, for example, [19]) that $PS(\mathfrak{g})$ is a free Poisson algebra on the same set of generators. We denote this algebra by $P = k\{x_1, x_2, \ldots, x_n, \ldots\}$.

By deg we denote the standard homogeneous degree function on P, i.e. $\deg(x_i) = 1$, where $1 \leq i \leq n$. By $\deg_{x_i}$ we denote the degree function on P with respect to x_i. The homogeneous elements of P with respect to $\deg_{x_i}$ can be defined in the usual way. If f is homogeneous with respect to each $\deg_{x_i}$, then f is called multihomogeneous. A multihomogeneous element $f \in P$ is called multilinear if $\deg_{x_i} = 0, 1$ for every i.

Denote by L_n the subspace of P of all multilinear elements of degree n in the variables $x_1, x_2, \ldots, x_n$. Denote by Q_{2n} the subspace of L_{2n} spanned by the elements

$$\{x_{i_1}, x_{i_2}\}\{x_{i_3}, x_{i_4}\} \ldots \{x_{i_{2n-1}}, x_{i_{2n}}\}. \qquad (6)$$

The elements of Q_{2n} are called *customary* polynomials (see [5]). By Corollary 1, the elements of the form (6) with $i_1 < i_2, i_3 < i_4, \ldots, i_{2n-1} < i_{2n}, i_1 < i_3 < \ldots < i_{2n-1}$, compose a linear basis of Q_{2n}.

Denote by T_{2n} the set of all permutations τ from S_{2n} such that

$$\tau(1) < \tau(2), \tau(3) < \tau(4), \ldots, \tau(2n-1) < \tau(2n), \tau(1) < \tau(3) < \ldots < \tau(2n-1).$$

Then every customary polynomial $f \in Q_{2n}$ can be uniquely written in the form

$$f = \sum_{\tau \in T_{2n}} \alpha_\tau \{x_{\tau(1)}, x_{\tau(2)}\}\{x_{\tau(3)}, x_{\tau(4)}\} \ldots \{x_{\tau(2n-1)}, x_{\tau(2n)}\}.$$

Recall that a Poisson algebra is called a PI algebra if it satisfies a nontrivial identity, i.e., there is a nonzero element $f \in P$ which is an identity of this algebra. Identities of Poisson algebras are studied in [5, 6, 15].

Theorem 1. [5] *Every Poisson PI algebra over a field of characteristic zero satisfies a nontrivial customary identity.*

Note that the symplectic Poisson algebra PS_1 satisfies the standard customary identity

$$St_4 = \{x_1, x_2\}\{x_3, x_4\} - \{x_1, x_3\}\{x_2, x_4\} + \{x_1, x_4\}\{x_2, x_3\}$$

and that PS_n also satisfies a standard customary identity (see [15]).

Lemma 1. *The symplectic Poisson algebra PS_∞ over a field of characteristic zero does not satisfy any nontrivial identity.*

Proof. Suppose that PS_∞ satisfies a nontrivial identity. Then PS_∞ satisfies a nontrivial customary identity by Theorem 1. Every nontrivial customary identity can be written in the form

$$\{z_1, z_2\}\{z_3, z_4\} \ldots \{z_{2n-1}, z_{2n}\} = \sum_{1 \neq \tau \in T_{2n}} \alpha_\tau \{z_{\tau(1)}, z_{\tau(2)}\}\{z_{\tau(3)}, z_{\tau(4)}\} \ldots \{z_{\tau(2n-1)}, z_{\tau(2n)}\}.$$

Substitution $z_{2k-1} = x_k, z_{2k} = y_k$, where $1 \leq k \leq n$, gives $1 = 0$, i.e., a contradiction. □

Corollary 2. *For every nonzero f from P there is a natural $n = n(f)$ such that f is not an identity of PS_n.*

4. Homomorphisms into symplectic Poisson algebras

In this section we consider both Poisson symplectic algebras and free Poisson variables. To distinguish variables, we consider the free Poisson algebra $k\{z_1, z_2, \ldots, z_m\}$ in the variables $z_1, z_2, \ldots, z_m$.

Theorem 2. (Freiheitssatz) *Let $k\{z_1, z_2, \ldots, z_m\}$ be the free Poisson algebra over a field k of characteristic 0 in the variables $z_1, z_2, \ldots, z_m$. If $f \in k\{z_1, z_2, \ldots, z_m\}$ and $f \notin k\{z_1, z_2, \ldots, z_{m-1}\}$, then $(f) \cap k\{z_1, z_2, \ldots, z_{m-1}\} = 0$.*

Proof. Without loss of generality we may assume that k is algebraically closed and that $f(z_1, z_2, \ldots, z_{m-1}, 0) \neq 0$. The Theorem will be proved if for f and any nonzero $g \in k\{z_1, z_2, \ldots, z_{m-1}\}$ there exist a Poisson algebra A and a homomorphism $\theta : k\{z_1, z_2, \ldots, z_m\} \to A$ of Poisson algebras such that $\theta(g) \neq 0, \theta(f) = 0$.

Let $\hat{f}$ be the highest homogeneous part of f with respect to z_m. By Corollary 2, there exists a natural n and a homomorphism $\phi : k\{z_1, z_2, \ldots, z_{m-1}, z_m\} \to PS_n$, where

$PS_n = k[x_1, y_1, \ldots, x_n, y_n]$ is the Poisson symmetric algebra, for which $\phi(gf\hat{f}) \neq 0$. Denote by $Z_1, Z_2, \ldots, Z_{m-1}$ the images of $z_1, z_2, \ldots, z_{m-1}$ under ϕ, by Z a general element of PS_n, and consider the equation

$$f(Z_1, Z_2, \ldots, Z_{m-1}, Z) = 0. \tag{7}$$

Note that

$$\{a, b\} = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_i}\frac{\partial b}{\partial y_i} - \frac{\partial a}{\partial y_i}\frac{\partial b}{\partial x_i}\right) \tag{8}$$

for $a, b \in PS_n$. For every $\alpha = (i_1, j_1, \ldots, i_n, j_n) \in \mathbb{Z}_+^{2n}$ we put

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{i_1}\left(\frac{\partial}{\partial y_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n}\left(\frac{\partial}{\partial y_n}\right)^{j_n}$$

and define the variable z^α. Denote by $\preceq$ the lexicographic order on $\mathbb{Z}_+^{2n}$. Using (8) rewrite (7) as

$$h(x_1, y_1, \ldots, x_n, y_n, \partial^{\alpha_1}(Z), \partial^{\alpha_2}(Z), \ldots, \partial^{\alpha_r}(Z)) = 0 \tag{9}$$

where $h = h(x_1, y_1, \ldots, x_n, y_n, z^{\alpha_1}, z^{\alpha_2}, \ldots, z^{\alpha_r})$ is a polynomial in variables $x_1, y_1, \ldots, x_n, y_n, z^{\alpha_1}, z^{\alpha_2}, \ldots, z^{\alpha_r}$. Since $f \notin k\{z_1, z_2, \ldots, z_{m-1}\}$ the polynomial h depends on $z^{\alpha_1}, z^{\alpha_2}, \ldots, z^{\alpha_r}$, i. e. $r > 0$ in (9).

Assume that $\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_r$ and that h is irreducible. (If h is not irreducible we can replace it with its irreducible factor which contains z^{α_r}.) We assert that there exists $L = (a_1, b_1, \ldots, a_n, b_n, c^{\alpha_1}, \ldots, c^{\alpha_r}) \in k^{2n+r}$ such that $h(L) = 0$ and $\frac{\partial h}{\partial z^{\alpha_r}}(L) \neq 0$. If this is not true then by Hilbert's Nulstellenssatz h divides $(\frac{\partial h}{\partial z^{\alpha_r}})^s$ for some $s > 0$. But then, since h is irreducible, h divides $(\frac{\partial h}{\partial z^{\alpha_r}})$, which is clearly impossible.

Therefore we can use Proposition 1 and find a solution Z_m of the differential equation (9) in the formal power series algebra $A = k[[x_1 - a_1, b_1, \ldots, x_n - a_n, y_n - b_n]]$. Note that $PS_n = k[x_1, y_1, \ldots, x_n, y_n] \subseteq A$ and that the Poisson structure of PS_n can be naturally extended to A. Take a homomorphism of Poisson algebras $\theta : k\{z_1, z_2, \ldots, z_m\} \to A$ defined by

$$\theta(z_1) = Z_1, \theta(z_2) = Z_2, \ldots, \theta(z_{m-1}) = Z_{m-1}, \theta(z_m) = Z_m.$$

Then $\theta_{|k\{z_1, z_2, \ldots, z_{m-1}\}} = \phi_{|k\{z_1, z_2, \ldots, z_{m-1}\}}$ and $\theta(f) = 0$. $\square$

Here is a more traditional formulation of the Freiheitssatz.

Corollary 3. (Freiheitssatz) *Let $k\{z_1, z_2, \ldots, z_m\}$ be the free Poisson algebra over a field k of characteristic 0 in the variables $z_1, z_2, \ldots, z_m$. Suppose that $f \in k\{z_1, z_2, \ldots, z_m\}$ and $f \notin k\{z_1, z_2, \ldots, z_{m-1}\}$. Then the subalgebra of the quotient algebra $k\{z_1, z_2, \ldots, z_m\}/(f)$ generated by $z_1 + (f), z_2 + (f), \ldots, z_{m-1} + (f)$ is a free Poisson algebra with free generators $z_1 + (f), z_2 + (f), \ldots, z_{m-1} + (f)$.*

As is well known (see [3, 7, 9, 11]), the automorphisms of polynomial algebras and free associative algebras in two variables are tame. The automorphisms of free Poisson algebras in two variables over a field of characteristic zero are also tame [14]. A proof of the tameness theorem for Poisson algebras and associative algebras can be obtained from the Freiheitssatz and Jung's Theorem [7].

Theorem 3. [14] *Automorphisms of two generated free Poisson algebras over a field of characteristic* 0 *are tame.*

Proof. Let φ be an automorphism of the free Poisson algebra $k\{x, y\}$ in the variables x, y over k. Consider the polynomial algebra $k[x, y]$ as a Poisson algebra with trivial Poisson bracket and the homomorphism $k\{x, y\} \to k[x, y]$ of Poisson algebras such that $x \mapsto x, y \mapsto y$. Every automorphism φ of $k\{x, y\}$ induces the automorphism $\overline{\varphi}$ of the polynomial algebra $k[x, y]$. By Jung's Theorem [7], we may assume that $\overline{\varphi} = id$. Then,

$$\varphi(x) = x + f, \varphi(y) = y + g, \qquad f, g \in I,$$

where I is the ideal of $k\{x, y\}$ generated by $\{x, y\}$.

We want to show that $f = g = 0$. Suppose that $f \neq 0$. Then $x + f \in (x) + I = (x)$ where (x) is the ideal of $k\{x, y\}$ generated by x but $x + f \notin k\{x\}$ since $f \notin k\{x\}$. By the Freiheitssatz, $(x+f) \bigcap k\{x\} = 0$. Note that an algebra $k\{x, y\}/(x+f)$ is generated by the image of $y+g$. Consequently, the Poisson bracket of $k\{x, y\}/(x+f)$ is trivial. This means that $I \subseteq (x + f)$ and $g \in I \subseteq (x + f)$. Hence $k\{x, y\}/(x + f)$ is the polynomial algebra in a single variable y and there exists a polynomial $h(y)$ such that $x-h(y) \in (x+f)$. Substituting $x = 0$, we get $h(y) = 0$. Therefore $x \in (x+f)$ which contradicts the Freiheitssatz. □

This approach can be used in the case of associative algebras to prove that automorphisms of two generated free associative algebras in characteristic zero are tame. (They are also tame in positive characteristic [3, 11].)

The well-known commutator test theorem says that an endomorphism φ of a free associative algebra $k\langle x, y\rangle$ in two variables is an automorphism if and only if $\varphi([x, y]) = \alpha[x, y]$, where $\alpha \in k^*$. In the case of a free Poisson algebra $k\{x, y\}$ in two variables it is easy to check that $\sigma(\{x, y\}) = \alpha\{x, y\}$ where $\alpha \in k^*$ for a linear or a triangular automorphism. Then by Theorem 3 it is true for every automorphism in characteristic 0.

Theorem 4. *Let k be a field of characteristic* 0*. Then the following statements are equivalent:*

(i) Every endomorphism φ of the free Poisson algebra $k\{x, y\}$ in the variables x, y with $\varphi(\{x, y\}) = \alpha\{x, y\}$, where $\alpha \in k^*$, is an automorphism;

(ii) Every endomorphism φ of the polynomial algebra $k[x, y]$ in the variables x, y with $J(\varphi) \in k^*$, where $J(\varphi)$ is the Jacobian of φ, is an automorphism.

Proof. Let φ be an endomorphism of the polynomial algebra $k[x, y]$ such that $J(\varphi) = \alpha \in k^*$. Then φ can be uniquely extended to an endomorphism of $k\{x, y\}$ since $k[x, y] \subset k\{x, y\}$. Note that $\varphi(\{x, y\}) = \alpha\{x, y\}$. If (i) is true then φ is an automorphism of $k\{x, y\}$. Then obviously φ is an automorphism of $k[x, y]$, i. e., (i) implies (ii).

The opposite direction is a bit more involved. Let us choose a homogeneous linear basis

$$e_1, e_2, \ldots, e_m, \ldots$$

of the free Lie algebra $g = Lie < x, y >$ such that $e_1 = x, e_2 = y$, and $e_3 = \{x, y\}$. Then $\deg\, e_i \geq 3$ for all $i \geq 4$. The elements

$$e_{i_1} e_{i_2} \ldots e_{i_k}, \quad i_1 \leq i_2 \leq \ldots \leq i_k \tag{10}$$

form a linear basis of $k\{x, y\}$. As in Theorem 3, denote by I the ideal of $k\{x, y\}$ generated by $\{x, y\}$. Every element of I is a linear combination of words of the form (10) which contain at least one e_i with $i \geq 3$.

Let φ be an endomorphism of $k\{x, y\}$ such that $\varphi(\{x, y\}) = \alpha\{x, y\}$, where $\alpha \in k^*$. Put $f = \varphi(x)$ and $g = \varphi(y)$. Then $f = f_1 + f_2, g = g_1 + g_2$ where $f_1, g_1 \in k[x, y]$ and $f_2, g_2 \in I$. Note that

$$\varphi(\{x, y\}) = \{f, g\} = \{f_1, g_1\} + h, \quad h = \{f_1, g_2\} + \{f_2, g_1\} + \{f_2, g_2\},$$

and h is a linear combination of words of the form (10) containing at least two e_i with $i \geq 3$ or one e_i with $i \geq 4$. Note also that $\{f_1, g_1\} = t\{x, y\}$ where $t \in k[x, y]$. Therefore the equality

$$\varphi(\{x, y\}) = \{f, g\} = \{f_1, g_1\} + h = \alpha\{x, y\}$$

is possible if and only if $\{f_1, g_1\} = \alpha\{x, y\}$ and $h = 0$.

Denote by ψ the endomorphism of $k[x, y]$ with $\psi(x) = f_1$ and $\psi(y) = g_1$. Since

$$\psi(\{x, y\}) = \{f_1, g_1\} = J(\psi)\{x, y\} = \alpha\{x, y\},$$

$J(\psi) = \alpha \in k^*$. If (ii) is true then ψ is an automorphism of $k[x, y]$ which can be extended to an automorphism of $k\{x, y\}$.

Consider the endomorphism $\theta = \psi^{-1}\varphi$ of $k\{x, y\}$. Then $\theta(\{x, y\}) = \{x, y\}$ and

$$\theta(x) = x + s, \ \theta(y) = y + t; \qquad s, t \in I.$$

We want to show that $s = t = 0$. Suppose that $s \neq 0$. Then $s \notin k\{x\}$ and $x + s \notin k\{x\}$. By the Freiheitssatz, $(x + s) \bigcap k\{x\} = 0$. In our case $\{x + s, y + t\}) = \{x, y\}$. Hence $\{x, y\} \in (x + s)$ and $I \subseteq (x + s)$. Therefore $x = x + s - s \in (x + s) + I = (x + s)$ which contradicts the Freiheitssatz. $\square$

Acknowledgments

The authors are grateful to Max-Planck Institute für Mathematik for hospitality and excellent working conditions, where part of this work has been done. We are also thankful to Yakar Kannai for an interesting discussion of nonlinear Cauchy-Kovalevsky theorem.

References

[1] Bergman, G.M., Centralizers in free associative algebras. Trans. Amer. Math. Soc., 137 (1969), 327–344.

[2] Bokut', L.A., Kukin, G.P., Algorithmic and combinatorial algebra. Mathematics and its Applications, 255. Kluwer Academic Publishers Group, Dordrecht, 1994.

[3] Czerniakiewicz, A.J., Automorphisms of a free associative algebra of rank 2. I, II, Trans. Amer. Math. Soc., 160 (1971), 393–401; 171 (1972), 309–315.

[4] Dicks, W., A commutator test for two elements to generate the free algebra of rank two. Bull. London Math. Soc. 14 (1982), no. 1, 48–51.

[5] Farkas, D.R., Poisson polynomial identities. Comm. Algebra 26 (1998), no. 2, 401–416.

[6] Farkas, D.R., Poisson polynomial identities. II. Arch. Math. (Basel) 72 (1999), no. 4, 252–260.

[7] Jung, H.W.E., Über ganze birationale Transformationen der Ebene, J. reine angew. Math., 184 (1942), 161–174.

[8] Kozybaev, D., Makar-Limanov, L., Umirbaev, U., The Freiheitssatz and the automorphisms of free right-symmetric algebras. Asian-European Journal of Mathematics, Vol. 1, No. 2 (2008), 243–254.

[9] Van der Kulk, W., On polynomial rings in two variables. Nieuw Archief voor Wiskunde, (3)1 (1953), 33–41.

[10] Magnus, M., Über discontinuierliche Gruppen mit einer definierenden Relation (Der Freiheitssatz). J. Reine Angew. Math., 163 (1930), 141-165

[11] Makar-Limanov, L., The automorphisms of the free algebra with two generators. Funksional. Anal. i Prilozhen. 4(1970), no.3, 107-108; English translation: in Functional Anal. Appl. 4 (1970), 262–263.

[12] Makar-Limanov, L., Algebraically closed skew fields. J. Algebra, 93 (1985), no. 1, 117–135.

[13] Makar-Limanov, L., Umirbaev, U., Centralizers in free Poisson algebras. Proc. Amer. Math. Soc. 135 (2007), no. 7, 1969–1975.

[14] Makar-Limanov, L., Turusbekova, U., Umirbaev, U., Automorphisms and derivations of free Poisson algebras in two variables. J. Algebra 322 (2009), 3318-3330.

[15] Mishchenko, S.P., Petrogradsky, V.M., Regev, A., Poisson PI algebras. Trans. Amer. Math. Soc. 359 (2007), no. 10, 4669–4694.

[16] Petrovsky, I. G. Lectures on partial differential equations. Translated by A. Shenitzer Interscience Publishers, New York-London, 1954. x+245 pp.

[17] Rentschler, R., Operations du groupe additif sur le plan. C.R. Acad. Sci. Paris, 267 (1968), 384–387.

[18] Romanovskii, N.S., A theorem on freeness for groups with one defining relation in varieties of solvable and nilpotent groups of given degrees. (Russian) Mat. Sb. (N.S.) 89(131) (1972), 93–99, 166.

[19] Shestakov, I.P., Quantization of Poisson superalgebras and speciality of Jordan Poisson superalgebras. Algebra and Logic, 32(1993), no. 5, 309–317.

[20] Shirshov, A.I., Some algorithm problems for Lie algebras. Sibirsk. Mat. Z., 3 (1962), 292–296.