SUPPLEMENT TO "REGULAR SYSTEM OF WEIGHTS AND ASSOCIATED SINGULARITIES"

(The sum formulae for powers of exponents)

Kyoji Saito

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

Sonderforschungsbereich 40
Theoretische Mathematik
Beringstraße 4
D-5300 Bonn 1

SFB/MPI 85-34

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§1. STATEMENT

(1.1) Let a system of four integers $(a,b,c;h) \in \mathbf{N}$ be regular in the sence [1, (1.2) Definition], which implies that $\mu := (h-a)(h-b)(h-c)/abc$ is a positive integer and there exist μ number of integers $m_1, m_2, \ldots, m_\mu \in \mathbf{Z}$, called the exponents for $(a,b,c;h)$, so that the following formula holds ([loc. cit.]):

$$(1.1.1) \qquad T^{-h} \frac{(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)} = T^{m_1}+T^{m_2}+\ldots+T^{m_\mu} \quad .$$

In fact this property is equivalent for $(a,b,c;h)$ to be regular.

(1.2) In this note we show that <u>for any positive integer $k \in \mathbf{N}$ there exists a unique homogeneous polynomial $P_k(a,b,c,h) \in \mathbf{Z}[a,b,c,h]$ of degree k such that for any regular system of weights</u> $(a,b,c;h)$, <u>we have</u>,

$$(1.2.1) \qquad \sum_{i=1}^{k} m_i^k = \frac{\mu}{(k+1)!} P_k(a,b,c,h) \quad .$$

(1.3) For $1 \leq k \leq 7$, explicite formulae are given as follows:

$$\sum_{i=1}^{\mu} m_i = \frac{\mu}{2} h$$

$$\sum_{i=1}^{\mu} m_i^2 = \frac{\mu}{6} h \, (2h-\varepsilon)$$

$$\sum_{i=1}^{\mu} m_i^3 = \frac{\mu}{4} h^2 \, (h-\varepsilon)$$

$$\sum_{i=1}^{\mu} m_i^4 = \frac{\mu}{30} h \, (6h^3-9\varepsilon h^2 + 4\varepsilon^2 h + \varepsilon^3 + 3\sigma)$$

$$\sum_{i=1}^{\mu} m_i^5 = \frac{\mu}{12} h^2 \, (2h^3-4\varepsilon h^2 + 4\varepsilon^2 h + \varepsilon^3 + 3\sigma)$$

$$\sum_{i=1}^{\mu} m_i^6 = \frac{\mu}{84} h \, \{12h^5-30\varepsilon h^4 + 44\varepsilon^2 h^3 - (9\varepsilon^3-32\sigma) h^2 + (-12\varepsilon^4 + 10\varepsilon^2\tau - 21\varepsilon\sigma) h - 2\varepsilon^5 - 10\varepsilon^2\sigma + 10\tau\sigma\}$$

$$\sum_{i=1}^{\mu} m_i^7 = \frac{\mu}{24} h^2 \, \{3h^5 - 9\varepsilon h^4 + 16\varepsilon^2 h^3 - (16\varepsilon^3 - 11\sigma) h^2 + (-12\varepsilon^4 + 10\varepsilon^2\tau - 21\varepsilon\sigma) h - 2\varepsilon^5 - 10\varepsilon^2\sigma + 10\tau\sigma\}$$

Here $\varepsilon := a + b + c - h$

$\tau := ab + bc + ca$

$\sigma := abc - (ab + bc + ca)(a + b + c - h)$

$= abc - \varepsilon\tau$

(1.4) Note

Let E_k and $\tilde{E}_k$ (k = 6,7,8) be regular system of weights for $\varepsilon = 1$ and $\varepsilon = 0$ ([1, Tables 1.,2.]). Due to the McKay correspondence, to each E_k certain finite subgroup of SU(2) , say Γ_k , is associated. (cf.[2],[3]. compare [1,(3.4) Note]). The exponents $m_i(\tilde{E}_k)$ for $\tilde{E}_k$ are the dimensions of the irreducible representations of Γ_k [loc. cit.], so that we have the formulae [ ],

$$\sum_{i=1}^{\mu} m_i(\tilde{E}_k) = h(E_k) \qquad k = 6,7,8 ,$$

$$\sum_{i=1}^{\mu} m_i(\tilde{E}_k)^2 = \#\Gamma_k \qquad k = 6,7,8 .$$

Comparing these formulae with the corresponding ones in (1.3), we obtain following relations:

(1.4.1) $$2\,h(E_k) = \mu(\tilde{E}_k)\; h(\tilde{E}_k) \qquad k = 6,7,8 ,$$

(1.4.2) $$\#\,\Gamma_k = \frac{2}{3}\, h(E_k)\; h(\tilde{E}_k) \qquad k = 6,7,8 .$$

The starting points of the present note is to proof the last formulae (1.4.1), (1.4.2).

§2 PROOFS OF THE STATEMENT

(2.1) First, we prepare a Lemma.

Lemma <u>For a positive integer</u> $k \in \mathbf{N}$, <u>there exists a polynomial</u> $Q_k(a,d) \in \mathbf{Z}[a,d]$ <u>of degree</u> k , <u>such that</u>

$$(2.1.1) \qquad \left[\left(T\frac{\partial}{\partial T}\right)^k \left(\frac{T^d-1}{T^a-1}\right)\right]_{T=1} = \frac{1}{(k+1)!}\,\frac{d}{a}\,Q_k(a,d)$$

<u>for any positive integers</u> a,d .

<u>PROOF</u> Consider a ring $R := \mathbf{Z}[a,d,T^a,T^d]$ of four indeterminates a,d,T^a,T^d , on which a differential operater $T\frac{\partial}{\partial T}$ is given by the relations: $T\frac{\partial}{\partial T}a = 0$, $T\frac{\partial}{\partial T}d = 0$, $T\frac{\partial}{\partial T}T^d = d\,T^d$, $T\frac{\partial}{\partial T}T^a = a\,T^a$.

By induction on k , one sees directly a formula,

$$(2.1.2) \qquad \left(T\frac{\partial}{\partial T}\right)^k \left(\frac{T^d-1}{T^a-1}\right) = \frac{\sum_{i=0}^{k} \varphi_{k,i}\, a^i (T^a)^i (T^a-1)^{k-i}}{(T^a-1)^{k+1}} \quad , k=0,1,2\ldots$$

where $\varphi_{k,i}$ is an element of the ideal $I := Rd + R(T^d-1)$ of R , given by

$$(2.1.3) \qquad \varphi_{k,i} = U_{k,i}\,dT^d + V_{k,i}\,a^{i-i}(T^d-1) \quad , 0 \le i \le k \quad .$$

Here $V_{k,i}$ are integers given by

$$(2.1.4) \qquad V_{k,i} := \begin{cases} \sum_{j=0}^{i} (-1)^j \frac{(i-j)^k}{j!(i-j)!} & , 0 \le i \le k \\ 0 & \text{otherwise} \end{cases}$$

and $U_{k,i}$ are homogeneous polynomials in a and d of degree $k-i-1$, inductively defined as follows.

$$(2.1.5) \qquad U_{k+1,i} = U_{k,i}(d+ia) + V_{k,i} + U_{k,i-1}$$

$$U_{k,i} = 0 \quad \text{for} \quad i \le 0 \quad \text{or} \quad i \ge k \quad .$$

If the positive integral values for a and d are given, there is a natural homomorphism $R = \mathbf{Z}[a,d,T^a,T^d] \to \mathbf{Z}[T]$ so that the formula (2.1.2) has a natural interpretation in the ring $\mathbf{Z}[T]$, whose verifications are omitted here. Particularly the function $\frac{T^d-1}{T^a-1}$ and hence also the functions $\left(T\frac{\partial}{\partial T}\right)\left(\frac{T^d-1}{T^a-1}\right)$ have removable singular point at $T \doteq 1$, so that they take finite values at $T = 1$, which can be evalueated in the following manner:

$$\left[\frac{\sum_{i=0}^{k}\varphi_{k,i}a^i(T^a)^i(T^a-1)^{k-i}}{(T^a-1)^{k+1}}\right]_{T=1} = \frac{\left[\left(T\frac{\partial}{\partial T}\right)^{k+1}\sum_{i=0}^{k}\varphi_{k,i}a^i(T^a)^i(T^a-1)^{k-i}\right]_{T^a=T^d=1}}{\left[\left(T\frac{\partial}{\partial T}\right)^{k+1}(T^a-1)^{k+1}\right]_{T^a=1}} ,$$

where the right hand side is defined and calculated in the ring R.

The denominater of the right hand side is $(k+1)!a^{k+1}$.

If we have shown that the numerater of the right hand is an integral polynomial in a and d of degree $\leq 2k+1$, which is divisible by $d\,a^k$, we have done the proof.

Put

$$\left(T\frac{\partial}{\partial T}\right)^{k+1}\sum_{i=0}^{k}\varphi_{k,i}a^i(T^a)^i(T^a-1)^{k-i}$$

$$=\sum_{i=0}^{k}\sum_{j=0}^{k+1}\left(\left(T\frac{\partial}{\partial T}\right)^j\varphi_{k,i}\right)a^i\left(T\frac{\partial}{\partial T}\right)^{k+1-j}(T^a)^i(T^a-1)^{k-i} .$$

The facts $\varphi_{k,i}\in I$, $T\frac{\partial}{\partial T}I\subset I$ and $I\big|_{T^a=T^d=1}\subset d\,\mathbf{Z}[a,d]$, imply that the numerater is divisible by d. The facts that $\left[\left(T\frac{\partial}{\partial T}\right)^{k+1-j}(T^a)^i(T^a-1)^{k-i}\right]_{T^a=1}$ is 0 for $k+1-j<k-i$ and is divisible by a^{k+1-j} for $k+1-j\geq k-i$, imply that the numerater is divisible by a^k.

Q.E.D.

<u>(2.2) A PROOF OF (1.2.1)</u>

Apply $\left[\left(T\frac{\partial}{\partial T}\right)^k\right]_{T=1}$ on the both sides of (1.1.1), where the right hand gives $\sum_{i=1}^{\mu}m_i^k$. Applying the (2.1) Lemma the left hand side is calculated as follows.

$$\left[\left(T\frac{\partial}{\partial T}\right)^{k}\left(T^{\varepsilon}\left(\frac{T^{h-a}-1}{T^{a}-1}\right)\left(\frac{T^{h-b}-1}{T^{b}-1}\right)\left(\frac{T^{h-c}-1}{T^{c}-1}\right)\right)\right]_{T=1} =$$

$$= \sum_{\substack{0 \leq k_0,k_1,k_2,k_3 \leq k \\ k_0+k_1+k_2+k_3=k}} C^{k}_{k_0k_1k_2k_3} \left[\left(T\frac{\partial}{\partial T}\right)^{k_0} T^{\varepsilon}\right]_{T=1}$$

$$\times \left[\left(T\frac{\partial}{\partial T}\right)^{k_1}\left(\frac{T^{h-a}-1}{T^{a}-1}\right)\right]_{T=1}\left[\left(T\frac{\partial}{\partial T}\right)^{k_2}\left(\frac{T^{h-b}-1}{T^{b}-1}\right)\right]_{T=1}\left[\left(T\frac{\partial}{\partial T}\right)^{k_3}\left(\frac{T^{h-c}-1}{T^{c}-1}\right)\right]_{T=1} =$$

$$= \sum_{\substack{0 \leq k_0,k_1,k_2,k_3 \leq k \\ k_0+k_1+k_2+k_3=k}} \frac{C_{k_0k_1k_2k_3}}{(k_1+1)!(k_2+1)!(k_3+1)!} \; \frac{\varepsilon^{k_0}(h-a)(h-b)(h-c)}{a\,b\,c} \times$$

$$\times \; \frac{(h-a)(h-b)(h-c)}{a\,b\,c} \; Q_{k_1}(a,h-a)\; Q_{k_1}(a,h-a)\; Q_{k_2}(b,h-b)\; Q_{k_3}(c,h-c) \quad .$$

Recalling $\mu := (h-a)(h-b)(h-c)/abc$, we obtain the formula (1.2.1).

For the unicity of the polynomial $P_k(a,b,c,h)$, it is enough to see the set $\{(a,b,c,h) \in \mathbf{Z}^4 : (a,b,c;h)$ is a regular system of weights$\}$ is Zariski dense in $\mathbf{Z}^4$. But for any positive $a,b,c,t \in \mathbf{N}$, $(a,b,c;t\,a\,b\,c)$ is regular and the set $\{(a,b,c,t\,a\,b\,c) : a,b,c,t \in \mathbf{N}\}$ is already Zariski dense in $\mathbf{Z}^4$.

The homocety transformation $(a,b,c;h) \longmapsto (ta,tb,tc;th)$ for some $t \in \mathbf{N}$ leaves μ invariant but the exponents are transformed to $tm_1,\ldots,tm_\mu$. Therefore

$$P_k(ta,tb,tc,th) = t^k P_k(a,b,c,h)$$

Again using the fact of Zariski denseness above, we see that P_k is a homogeneous polynomial of degree k .

(2.3) A further explicite calculations shows the formulae of (1.3), where the following duality property is strongly used.

$$\sum_{i=1}^{k} m_i^k = \sum_{i=1}^{k} (h-m_i)^k = \sum_{j=0}^{k} C_j^k (-1)^{k-j} h^j \sum_{i=1}^{k} m_i^{k-j}$$

REFERENCES

[1] K. SAITO: Regular System of Weights and Associated Singularities, to appear in the Proc. of Japan - U.S. Seminar on Singularities '84.

[2] P. SLODOWY: Simple Singularities and Simple Algebraic Groups, Springer Lecture Note #815, Appendix III.

[3] R. STEINBERG: Subgroups of SU_2 and Dynkin Diagrams, Preprint.