A NOTE ON MODULI OF STABLE SHEAVES ON RULED SURFACES

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0. Introduction

Let $\pi : X \to C$ be a ruled surface defined over $\mathbb{C}$ and H an ample divisor on H. Let f be a fiber of π and σ a section of π. For a pair (L, c_2) of line bundle L on X and a non-negative integer c_2, $M_H(L, c_2)$ is the moduli space of stable sheaves E of rank 2 with $(\det(E), c_2(E)) = (L, n)$. We note that $M_{\sigma+nf}(L, c_2), n \gg 0$ does not depend on n. We denote this space by $M(L, c_2)$. We assume that $(K_X, H) < 0$. If $(c_1(L), f)$ is odd, then by using [G-S], Göttsche [G] and the author [Y2, appendix] computed the Hodge polynomial of $M_H(L, c_2)$. If $(c_1(L), f)$ is even, then the author [Y2] computed Poincaré polynomial of $M_H(L, c_2)$. In this note, we shall compute the Hodge polynomial of $M_H(L, c_2)$. By the same argument as in [G], [Y2, appendix], it is sufficient to compute the Hodge polynomial of $M(L, c_2)$. Our method is the same as that in [G]. That is, we shall use virtual Hodge polynomials. By virtue of the computations of Cheah [Ch], we can use almost all computations in [Y2] except the use of Siegel formula [H-N]. By using the idea of Drezet-Narasimhan [D-N], we can avoid to use Siegel formula.

For a vector bundle E of rank r on C, they considered filtrations $\mathcal{O}_C(-n) \subset E$ [1], where n is a sufficiently large integer. Then there is an exact sequence

$$0 \to \mathcal{O}_C(-n)(D) \to E \to F \to 0, \tag{0.1}$$

where D is an effective divisor on C and F is a vector bundle of rank $r-1$. If $E/\mathcal{O}_C(-n)$ is locally free, then $D = 0$. Since they are only interested in the computation of the Picard groups, it is sufficient to consider the case $D = 0$. But if we treat all D, we can get more precise informations. In particular, we can recover Siegel formula in [H-N]. So we shall follow Drezet-Narasimhan's method. Unfortunately, it seems that filtrations on non-locally free sheaves are not easy to treat. Hence we restrict ourselves to rank 2 case and consider extensions

$$0 \to \mathcal{O}_X(D - nf) \to E \to I_Z(L - nf - D) \to 0, \tag{0.2}$$

where E is locally free, D is an effective divisor coming from C and Z is a locally complete intersection subscheme of X. For such extensions, it is difficult to check the stability. Hence we must treat bigger bounded subset S of vector bundles. So we need to extend the definition of virtual Hodge polynomial to S. For a line bundle L coming from C, let $T(L, c_2)$ be the set of torsion free sheaves E of rank 2 such that $E_{|\pi^{-1}(P)} \cong \mathcal{O}_{\pi^{-1}(P)}^{\oplus 2}$ for general $P \in C$ and $(c_1(E), c_2(E)) = (L, c_2)$. Let $T(L, c_2)^{ss}$ be the subset of $T(L, c_2)$ consisting of μ-semi-stable sheaves with respect to $\sigma + nf$, $n \gg 0$. Then the following theorem holds.

Theorem 0.1. *We assume that* $\deg L := (\sigma, L)$ *is* 0 *or* 1. *Then,*

$$\begin{aligned}\sum_n e(T(L,n)^{ss})t^n =& \frac{(x^2y-1)^g(xy^2-1)^g}{((xy)^2-1)(xy-1)^2}\prod_{i=0,1}\prod_{b\geq 1}\prod_{p,q}\left(\frac{1}{1-x^py^q(xy)^{2(b-i)}t^b}\right)^{e^{p,q}(X)}\\ &-\frac{(x-1)^g(y-1)^g(xy)^{\deg L+g-1}}{((xy)^2-1)(xy-1)^2}\prod_{b\geq 1}\prod_{p,q}\left(\frac{1}{1-x^py^q(xy)^{2b-1}t^b}\right)^{2e^{p,q}(X)},\end{aligned}$$

where g *is the genus of* C *and* $e^{p,q}(X) = (-1)^{p+q}h^{p,q}(X)$.

For the definition of virtual Hodge polynomial, see Section 1.1. By this theorem, we can compute the Hodge polynomial of $M_H(L, c_2)$, if $M_H(L, c_2)$ is compact. For example, we get the following.

Corollary 0.2. *If* $\deg L = 1$, *then*

$$\begin{aligned}\sum_n e(M(L,n))t^n =& \frac{(x^2y-1)^g(xy^2-1)^g}{((xy)^2-1)(xy-1)}\prod_{i=0,1}\prod_{b\geq 1}\prod_{p,q}\left(\frac{1}{1-x^py^q(xy)^{2(b-i)}t^b}\right)^{e^{p,q}(X)}\\ &-\frac{(x-1)^g(y-1)^g(xy)^g}{((xy)^2-1)(xy-1)}\prod_{b\geq 1}\prod_{p,q}\left(\frac{1}{1-x^py^q(xy)^{2b-1}t^b}\right)^{2e^{p,q}(X)}.\end{aligned} \tag{0.3}$$

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[1] In fact, they used filtration $\mathcal{O}_C(-n)^{\oplus(r-1)} \subset E$, but the idea is similar.

1. Preliminaries

Notation.

Throughtout this note, we use the following notations. Let X be a smooth surface. For a scheme S, we denote the projection $S \times X \to S$ by p_S. For a coherent sheaf E on $S \times X$ and a morphism $f : S' \to S$, we denote the pull-back of E to $S' \times X$ by $f^{\#}E$.

Let $\mathbb{Q}[[1/x, 1/y]]$ be the formal power series ring of two variable $1/x, 1/y$ and R the localization of $\mathbb{Q}[[1/x, 1/y]]$ by $1/(xy)$. For $f, g \in R$ and an ideal I of $\mathbb{Q}[[1/x, 1/y]]$, $f \equiv g \mod I$, if $f - g$ belongs to I.

1.1. **Virtual Hodge polynomial.** For a variety Y over $\mathbb{C}$, cohomology with compact support $H_c^*(Y, \mathbb{Q})$ has a natural mixed Hodge structure. Let $e^{p,q}(Y) := \sum_k (-1)^k h^{p,q}(H_c^k(Y))$ be the virtual Hodge number and $e(Y) := \sum_{p,q} e^{p,q}(Y) x^p y^q$ the virtual Hodge polynimial of Y. For more details, see [Ch, 0.1]. By the properties of virtual Hodge polynomials, we can extend the definition of $e(Y)$ to constructible sets.

Let S be a bounded set of coherent sheaves on X with Hilbert polynomial $h(x)$. Under some conditions, we shall define the virtual Hodge polynomial of S as an element of R. So it is not a polynomial in general. Let H be an ample divisor on X. Since S is bounded, there is an integer m such that for every element E,

1. $E(mH)$ is generated by global sections,
2. $H^i(X, E(mH)) = 0$ for $i > 0$.

We set $N := h(m)$. We shall consider the quot scheme $Q := \mathrm{Quot}^{h(x)}_{\mathcal{O}_X(-mH)^{\oplus N}/X/\mathbb{C}}$. Let $\mathcal{O}_{Q \times X}(-mH)^{\oplus N} \to \mathcal{Q}$ be the universal quotient. We assume that $Q_S := \{q \in Q | \mathcal{Q}_q \in S\}$ is a constructible set. We shall show that this condition does not depend on the choice of Q. Let T be a scheme and $\mathcal{E}$ a coherent sheaf on $T \times X$ such that $\mathcal{E}$ is flat over T. Let T^0 be the open subscheme of T consisting of point $t \in T_S$ such that

1. $\mathcal{E}_t(mH)$ is generated by global sections,
2. $H^i(X, \mathcal{E}_t(mH)) = 0$ for $i > 0$.

By base change theorem, $p_{T*}(\mathcal{E}(mH))$ is a locally free sheaf on T^0. Let $\tau : \mathcal{P} \to T$ be the associated principal $GL(N)$ bundle over T^0. Then there is a surjective homomorphism $\mathcal{O}_{\mathcal{P} \times X}(-mH)^{\oplus N} \to \tau^{\#}\mathcal{E}$. Hence we get a morphism $\eta : \mathcal{P} \to Q$. We set $T_S := \{t \in T | \mathcal{E}_t \in S\}$. Then we see that $T_S = \tau(\eta^{-1}(Q_S))$. Thus T_S is constructible, if Q_S is constructible. In particular, constructibility does not depend on the choice of Q.

Definition 1.1. 1. S is constructible, if Q_S is constructible.
2. For a constructible set S, we define the virtual Hodge polynomial by

$$e(S) := \frac{e(Q_S)}{e(GL(N))} \in R.$$

It is easy to see that this definition does not depend on the choice of Q.

1.2. **Quot scheme.** We shall prove a variant of [Y1, Thm. 0.4].

Lemma 1.1. *Let $\mathcal{E}$ be a vector bundle of rank r on $S \times X$. Then*

$$e(\mathrm{Quot}^n_{\mathcal{E}/S \times X/S}) = e(\mathrm{Quot}^n_{\mathcal{O}_X^{\oplus r}/X/\mathbb{C}}) e(S). \tag{1.1}$$

Proof. We shall prove that $e(\mathrm{Quot}^n_{\mathcal{E}/S \times X/S})$ does not depend on $\mathcal{E}$. Let Δ be the diagonal of $X \times X$ and I_X the ideal of Δ. We note that $\mathcal{E} \otimes \mathcal{O}_{X \times X}/I_\Delta^{n+1}$ is flat over $S \times X$. Hence $V := p_{S \times X*}(\mathcal{E} \otimes \mathcal{O}_{X \times X}/I_\Delta^{n+1})$ is a locally free sheaf on $S \times X$ and $p^*_{S \times X} V \to \mathcal{E} \otimes \mathcal{O}_{X \times X}/I_\Delta^{n+1}$ is surjective. Let $S \times X = \cup_i U_i$ be a stratification of $S \times X$ such that U_i are locally closed subset of $S \times X$, $V_{|U_i}$ are free and $\mathcal{O}_{U_i \times X \times X}/I_\Delta^{n+1} \cong \mathcal{O}_{U_i}[x, y]/(x, y)^{n+1}$. Indeed $I_\Delta/I_\Delta^2 \cong \Omega_X$ is a locally free sheaf on X, and hence we can choose such a pair (x, y). We set $Y := \mathrm{Spec}(\mathbb{C}[x, y]/(x, y)^{n+1})$ and $Q^k := \mathrm{Quot}^k_{\mathcal{O}_Y^{\oplus r}/Y/\mathbb{C}}$. Let $\mathcal{O}^{\oplus r}_{Q^k \times Y} \to \mathcal{Q}^k$ be the universal quotient on $Q^k \times Y$. Then we get a homomorphism $\mathcal{E} \to (\mathcal{O}_{U_i \times X \times X}/I_\Delta^{n+1})^{\oplus r} \to \mathcal{O}_{U_i} \otimes_{\mathbb{C}} \mathcal{Q}^k$ on $U_i \times Q^k \times Y$. Let $Q^k_{i,n}$ be the complement of the big diagonal of n-th fiber product $(U_i \times Q^k) \times_S (U_i \times Q^k) \times_S \cdots \times_S (U_i \times Q^k)$. For a partition $n = k_i^1 n_i^1 + \cdots + k_i^s n_i^s$, $1 \le k_i^1 < k_i^2 < \cdots < k_i^s$ of n, there is a morphism

$$Q^{k_i^1}_{i,n_i^1} \times_S Q^{k_i^2}_{i,n_i^2} \times_S \cdots \times_S Q^{k_i^s}_{i,n_i^s} \to \mathrm{Quot}^n_{\mathcal{E}/S \times X/S}$$

which is $\mathfrak{S}_{(n_i^1, \dots, n_i^s)} := \mathfrak{S}_{n_i^1} \times \cdots \times \mathfrak{S}_{n_i^s}$-invariant. Hence we get that

$$e(\mathrm{Quot}^n_{\mathcal{E}/S \times X/S}) = \sum_i \sum_{\substack{n = k_i^1 n_i^1 + \cdots + k_i^s n_i^s \\ 1 \le k_i^1 < k_i^2 < \cdots < k_i^s}} e(Q^{k_i^1}_{i,n_i^1} \times_S Q^{k_i^2}_{i,n_i^2} \times_S \cdots \times_S Q^{k_i^s}_{i,n_i^s} / \mathfrak{S}_{(n_i^1, \dots, n_i^s)}).$$

Since we can choose the same stratification for $\mathcal{E}$ and $\mathcal{O}^{\oplus r}_{S \times X}$, we get that

$$e(\mathrm{Quot}^n_{\mathcal{E}/S \times X/S}) = e(\mathrm{Quot}^n_{\mathcal{O}^{\oplus r}_{S \times X}/S \times X/S}).$$

Since $\mathrm{Quot}^n_{\mathcal{O}_{S\times X}^{\oplus r}/S\times X/S} \cong \mathrm{Quot}^n_{\mathcal{O}_X^{\oplus r}/X/\mathbb{C}} \times S$, we get our lemma. $\square$

Lemma 1.2. *[N-Y] For a smooth surface X,*

$$\sum_n e(\mathrm{Quot}^n_{\mathcal{O}_X^{\oplus r}/X/\mathbb{C}})t^n = \prod_{i=1}^r \prod_{b\geq 1} \prod_{p,q} \left(\frac{1}{1-x^p y^q (xy)^{rb-i} t^b} \right)^{e^{p,q}(X)}. \tag{1.2}$$

By virtue of computations in Cheah [Ch], this lemma follows from the local case.

Lemma 1.3. *For $X = \mathrm{Spec}(\mathcal{O}_P)$, $P \in X$,*

$$\sum_n e(\mathrm{Quot}^n_{\mathcal{O}_P^{\oplus r}/X/\mathbb{C}})t^n = \prod_{i=1}^r \prod_{b\geq 1} \prod_{p,q} \left(\frac{1}{1-(xy)^{rb-i} t^b} \right). \tag{1.3}$$

Moreover $\mathrm{Quot}^n_{\mathcal{O}_P^{\oplus r}/X/\mathbb{C}}$ has a celluler decomposition. Hence $h^{p,q}(H_c^k(\mathrm{Quot}^n_{\mathcal{O}_P^{\oplus r}/X/\mathbb{C}})) = 0$ unless $k = p+q$.

The method of the proof can be seen in [N]. Also we can derive the first claim from a modification of the proof of [Y1, Thm. 0.4]. More precisely, it is not difficult to modify the proof of [Y1, Prop. 5.3] (cf. the proof of Proposition 2.1).

1.3. **Locally free extensions.** Let $\pi : X \to C$ be a ruled surface defined over C and f a fiber of π. We shall prove a variant of [Y1, Prop. 0.2]. For a constructible subset Y of C, we set

$$\begin{aligned} H^n &:= \{Z | Z \in \mathrm{Hilb}^n_X, Z \text{ is a locally complete intersection subscheme}\},\\ H_{n,a,Y} &:= \{Z \in H^n | \pi(Z) \in Y, \deg \pi_*(I_Z) = -a\}. \end{aligned}$$

For a point P of C, we also set

$$H_{n,a,P}(d) := \{Z \in H_{n,a,P} | I_{Z|\pi^{-1}(P)} \cong \mathcal{O}_{\pi^{-1}(P)}(-d) \oplus T, \# \operatorname{Supp} T < \infty\}.$$

Let $\mathcal{Z} \subset H^n \times X$ be the universal subscheme and D a divisor on X which comes from C. Then $W := p_{H^n *}(\wedge^2 (I_{\mathcal{Z}}/I_{\mathcal{Z}}^2)^\vee(D))$ is a locally free sheaf on H^n. We also denote the associated vector bundle by $q : \mathbb{V}(W^\vee) \to H^n$. Then there is a canonical homomorphism $\mathcal{O}_{\mathbb{V}(W^\vee)} \xrightarrow{s} W$. Let $\mathbb{V}(W^\vee)^0$ be the open subscheme of $\mathbb{V}(W^\vee)$ such that $p_{H^n}^* \mathcal{O}_{\mathbb{V}(W^\vee)} \xrightarrow{p_{H^n}^* s} p_{H^n}^* W \to \wedge^2(I_{\mathcal{Z}}/I_{\mathcal{Z}}^2)^\vee(D)$ is surjective. We set $N_{n,a,Y} := q^{-1}(H_{n,a,Y}) \cap \mathbb{V}(W^\vee)^0$ and $N_{n,a,P}(d) := q^{-1}(H_{n,a,P}(d)) \cap \mathbb{V}(W^\vee)^0$. We shall prove the following.

Lemma 1.4.

$$\sum_{n,a} e(N_{n,a,Y}) u^a t^n = \prod_{b\geq 1} \prod_{p,q} \left(\frac{1-x^p y^q (xy)^{2b-2} u t^b}{1-x^p y^q (xy)^{2b} u t^b} \right)^{e^{p,q}(Y)}. \tag{1.4}$$

By the same argument of Cheah [Ch][2], it is sufficient to consider the case that Y is a point P. In this case, similar computations in [Y1, sect. 1, 2] show our results. So we shall give an outline of our computations for $Y = \{P\}$.

We may assume that $C = \mathbb{P}^1$, $D = -2f, -f$ and $\pi^{-1}(P) = f$. By base change theorem, we get that

1. $\mathrm{Hom}_{p_{H^n}}(I_{\mathcal{Z}}, \mathcal{O}_{H^n \times X}(D)) = \mathrm{Ext}^2_{p_{H^n}}(I_{\mathcal{Z}}, \mathcal{O}_{H^n\times X}(D)) = 0$.
2. $\mathrm{Ext}^1_{p_{H^n}}(I_{\mathcal{Z}}, \mathcal{O}_{H^n\times X}(D))$ is a locally free sheaf.

Then $\mathbb{P}_D := \mathbb{P}(\mathrm{Ext}^1_{p_{H^n}}(I_{\mathcal{Z}}, \mathcal{O}_{H^n\times X}(D))^\vee)$ parametrizes all non-trivial extensions of I_Z, $Z \in H^n$ by $\mathcal{O}_X(D)$ up to multiplication by constants. Since $Rp^2_{H^n *}(\mathcal{H}om(I_{\mathcal{Z}}, \mathcal{O}_{H^n\times X}(D))) \cong Rp^2_{H^n *}(\mathcal{O}_{H^n\times X}(D)) = 0$, the restriction homomorphism

$$\phi : \mathrm{Ext}^1_{p_{H^n}}(I_{\mathcal{Z}}, \mathcal{O}_{H^n\times X}(D)) \to p_{H^n *}(\mathcal{E}xt^1(I_{\mathcal{Z}}, \mathcal{O}_{H^n\times X}(D))) \tag{1.5}$$

is surjective. Let $\mathbb{P}^0_D$ be the open subscheme parametrizing locally free extension sheaves. Since

$$\mathcal{E}xt^1(I_{\mathcal{Z}}, \mathcal{O}_{H^n\times X}(D)) \cong \wedge^2(I_{\mathcal{Z}}/I_{\mathcal{Z}}^2)^\vee(D), \tag{1.6}$$

we get that $\mathbb{P}^0_D = \phi^{-1}(\mathbb{V}(W^\vee)^0)/\mathbb{C}^\times$. Hence we see that

$$\begin{aligned} e(\mathbb{P}^0_D \times_{H^n} H_{n,a,P}(d)) &= (xy)^{-\chi(X,\mathcal{O}_X(D))} e(\mathbb{V}(W^\vee)^0 \times_{H^n} H_{n,a,P}(d))/(xy-1)\\ &= (xy)^{-\chi(X,\mathcal{O}_X(D))} e(N_{n,a,P}(d))/(xy-1). \end{aligned} \tag{1.7}$$

On the otherhand, by considering elementary transformations of locally free extension sheaves along $\mathcal{O}_f(k)$, $k \in \mathbb{Z}$, we obtain relations on $e(\mathbb{P}^0_{-2f} \times_{H^n} H_{n,a,P}(d))$ and $e(\mathbb{P}^0_{-f} \times_{H^{n'}} H_{n',a',P}(d'))$. By (1.7), we get the following relation.

$$e(N_{n,a,P}(d)) = \sum_{0\leq d' < d} (xy)^{2d-2}((xy)^2-1) e(N_{n-d,a-1,P}(d')) + (xy)^{2d} e(N_{n-d,a-1,P}(d)), \tag{1.8}$$

[2] see appendix

where we set $e(N_{0,0,P}(0)) = 1$. By using this relation successively, we get Lemma 1.4 for $Y = \{P\}$.

2. Computation of virtual Hodge polynomials

2.1. **Notations.** Let C be a smooth projective curve of genus g and $\pi : X \to C$ a ruled surface. Let σ be a section of π and f a fiber of π. For simplicity, we denote $\mathcal{O}_X(nf)$ by $\mathcal{O}_X(n)$. For a coherent sheaf E such that $\det E$ comes from C, we set $\deg E := (\sigma, c_1(E))$. We shall fix a line bundle L of $\deg L = 0, 1$ which comes from C.

Definition 2.1. 1. $T(L, c_2)$ is the set of torsion free sheaves E of rank 2 such that $E_{|\pi^{-1}(P)} \cong \mathcal{O}_{\pi^{-1}(P)}^{\oplus 2}$ for general points P of C and $(\det(E), c_2(E)) = (L, c_2)$, and $V(L, c_2)$ is the subset of $T(L, c_2)$ consisting of locally free sheaves.

2. For an integer $d \geq 1$, $T(L, c_2, d)$ is the subset of $T(L, c_2)$ consisting of E which is defined by an exact sequence
$$0 \to I_1 \to E \to I_2 \to 0, \tag{2.1}$$
where I_1 and I_2 are torsion free sheaves of rank 1 and $\deg(c_1(I_1)) = d$, and $V(L, c_2, d) := T(L, c_2, d) \cap V(L, c_2)$.

3. We set $T(L, c_2)^{ss} := T(L, c_2) \setminus \cup_{d \geq 1} T(L, c_2, d)$ and $V(L, c_2)^{ss} := T(L, c_2)^{ss} \cap V(L, c_2)$.

We set
$$V_{c_2,a} := \{E \in V(L, c_2) | \deg(\pi_*(E)) = \deg(E) - a\}$$
For a positive integer k, we set
$$V_{c_2,a}(k) := \{E \in V_{c_2,a} | E \notin \cup_{j \geq k} V(L, c_2, j)\}.$$
For a fixed k, we shall choose a sufficiently large integer $n \geq k$ such that $H^1(C, \pi_*(E(n))) = 0$ for all $E \in V_{c_2,a}(k)$. We set
$$FV_{c_2,a} := \{\mathcal{O}_X(-n) \subset E | E \in V_{c_2,a}\}.$$
Then the filtration $\mathcal{O}_X(-n) \subset E$ defines an exact sequence
$$0 \to \mathcal{O}_X(D - nf) \to E \to I_Z(-D + L + nf) \to 0$$
where D is an effective divisor on X which comes from C. Let $FV_{c_2,a}^l$ be the set of equivalence classes of extensions
$$0 \to \mathcal{O}_X(D - nf) \to E \to I_Z(-D + L + nf) \to 0 \tag{2.2}$$
such that E is locally free, D is effectice, $\deg(D) < n + k$, $\deg(\pi_*(I_Z)) = -a$ and $\deg(Z) = c_2$, where equivalence relation is defined by the natural action of $\mathrm{Aut}(\mathcal{O}_X(D - nf)) \times \mathrm{Aut}(I_Z(-D + L + nf))$. Then there is a natural bijective correspondence $FV_{c_2,a} \to \cup_l FV_{c_2,a}^l$. By this identification, we can regard $FV_{c_2,a}^l$ as a subset of $FV_{c_2,a}$. There is an integer k' such that every filtration $\mathcal{O}_X(-n) \subset E$ satisfies that $E \in V_{c_2,a}(k')$.

2.2. **Virtual Hodge polynomial of $FV_{c_2,a}$.** Let H be an ample divisor on X. Since $FV_{c_2,a}$ is a bounded set, there is an integer m such that for every extension of (2.2) and the Harder-Narasimhan filtration (2.1) of $E \in T(L, c_2, d), d < k'$,

1. $\mathcal{O}_X(D - nf + mH)$ and $I_Z(-D + L + nf + mH)$ are generated by global sections,
2. $H^i(X, \mathcal{O}_X(D - nf + mH)) = H^i(X, I_Z(-D + L + nf + mH)) = 0$, $i > 0$,
3. $I_1(mH)$ and $I_2(mH)$ are generated by global sections,
4. $H^i(X, I_1(mH)) = H^i(X, I_2(mH)) = 0$, $i > 0$.

We set $N_1^l := \chi(X, \mathcal{O}_X(D - nf + mH))$, $N_2^l := \chi(X, I_Z(-D + L + nf + mH))$ and $N := N_1^l + N_2^l$, where $l = \deg D$. We note that N does not depend on the choice of l. We set $V_i = \mathcal{O}_X(-mH)^{\oplus N_i^l}$, $i = 1, 2$ and $V = \mathcal{O}_X(-mH)^{\oplus N}$. We shall consider an open subscheme Q of $\mathrm{Quot}_{V/X/\mathbb{C}}$ consisting of quotients $V \to E$, $E \in T(L, c_2) \setminus \cup_{j \geq k'} T(L, c_2, j)$.

Let $\mathcal{O}_Q \boxtimes V \to \mathcal{Q}$ be the universal quotient on $Q \times X$. We set
$$Q_{c_2,a} := \{q \in Q | \mathcal{Q}_q \in V_{c_2,a}\},$$
$$Q_{c_2,a}(k) := \{q \in Q | \mathcal{Q}_q \in V_{c_2,a}(k)\}.$$

We shall define virtual Hodge polynimial $e(FV_{c_2,a})$. For this purpose, we shall construct a parameter space of $FV_{c_2,a}$. Let $E_1 \to E_0 \to \mathcal{O}_X(-n)$ be a locally free resolution of $\mathcal{O}_X(-n)$ such that $H^j(X, E_0^\vee \otimes \mathcal{Q}_q) = H^j(X, E_1^\vee \otimes \mathcal{Q}_q) = 0$ for $q \in Q$ and $j > 0$. By base change theorem, $p_{Q*}(E_i^\vee \otimes \mathcal{Q})$, $i = 0, 1$ are locally free sheaves. Let I be the ideal of $\mathbb{P}(p_{Q*}(E_0^\vee \otimes \mathcal{Q})^\vee)$ generated by the image of the homomorphism $\xi : p_{Q*}(E_1^\vee \otimes \mathcal{Q})^\vee \to p_{Q*}(E_0^\vee \otimes \mathcal{Q})^\vee$. Let FQ be the subscheme of $\mathbb{P}(p_{Q*}(E_0^\vee \otimes \mathcal{Q})^\vee)$ defined by I. Then there is a universal family of filtrations
$$0 \subset \mathcal{O}_{FQ \times X}(-\lambda) \subset \mathcal{O}_{FQ} \otimes_Q \mathcal{Q} \tag{2.3}$$

on $FQ \times X$, where $\mathcal{O}_{FQ\times X}(\lambda)$ is the pull-back of the tautological line bundle on $\mathbb{P}(p_{Q*}(E_0^\vee \otimes \mathcal{Q})^\vee)$. We set $FQ_{c_2,a} := FQ \times_Q Q_{c_2,a}$ and $FQ_{c_2,a}(k) := FQ \times_Q Q_{c_2,a}(k)$. Let $FQ_{c_2,a}^l$ be the subset of $FQ_{c_2,a}$ parametrizing filtrations which belong to $FV_{c_2,a}^l$.

We shall define the virtual Hodge polynomials of $FV_{c_2,a}$, $FV_{c_2,a}(k)$ and $FV_{c_2,a}^l$ by

$$\begin{aligned} e(FV_{c_2,a}) &:= \frac{e(FQ_{c_2,a})}{e(GL(V))} \in R, \\ e(FV_{c_2,a}(k)) &:= \frac{e(FQ_{c_2,a}(k))}{e(GL(V))} \in R, \\ e(FV_{c_2,a}^l) &:= \frac{e(FQ_{c_2,a}^l)}{e(GL(V))} \in R. \end{aligned} \tag{2.4}$$

2.3. **Computation of** $e(FV_{c_2,a})$**.** We shall compute $e(FV_{c_2,a})$ by using the stratification $FV_{c_2,a} = \cup_l FV_{c_2,a}^l$. We get the following proposition.

Proposition 2.1.

$$e(FV_{c_2,a}^l) = \frac{e(S^lC)e(N_{c_2,a,C})}{(xy)^{2l-\deg L-2n+1-g}(xy-1)^2}. \tag{2.5}$$

Proof. The following argument is quite similar to that in [A-B], or [Y2, appendix]. Let $\mathcal{D} \subset S^lC \times C$ be the universal family of effective divisors. For simplicity, we denote the pull-back of $\mathcal{D}$ to $S^lC \times X$ by the same $\mathcal{D}$. We set

$$\begin{aligned} \mathcal{E}_1 &:= \mathcal{O}_{S^lC\times H^{c_2}\times X}(\mathcal{D} - nf + mH), \\ \mathcal{E}_2 &:= I_{\mathcal{Z}}(-\mathcal{D} + L + nf + mH). \end{aligned} \tag{2.6}$$

By the choice of m, $p_{S^lC\times H^{c_2}*}(\mathcal{E}_i)$, $i = 1,2$ are locally free sheaves of rank N_i^l on $S^lC \times H^{c_2}$. Let $\mathcal{P}_i \to S^lC \times H^{c_2}$ be the associated principal fiber bundles over $S^lC \times H^{c_2}$. We set $\mathcal{P} := (\mathcal{P}_1/\mathbb{C}^\times) \times_{S^lC\times H^{c_2}} (\mathcal{P}_2/\mathbb{C}^\times)$ and we denote the projection $\mathcal{P} \to S^lC \times H^{c_2}$ by α. $\alpha : \mathcal{P} \to S^lC \times H^{c_2}$ is a $PGL(N_1^l) \times PGL(N_2^l)$-bundle over $S^lC \times H^{c_2}$. Then there are line bundles P_1, P_2 on $\mathcal{P}$, and we get surjective homomorphisms:

$$\begin{aligned} \psi_1 &: \mathcal{O}_{\mathcal{P}} \boxtimes V_1 \to \alpha^\# \mathcal{E}_1 \otimes p_{\mathcal{P}}^* P_1 \to 0, \\ \psi_2 &: \mathcal{O}_{\mathcal{P}} \boxtimes V_2 \to \alpha^\# \mathcal{E}_2 \otimes p_{\mathcal{P}}^* P_2 \to 0. \end{aligned} \tag{2.7}$$

By base change theorem, $F := \operatorname{Hom}_{p_{\mathcal{P}}}(\ker\psi_2, \alpha^\#\mathcal{E}_1 \otimes p_{\mathcal{P}}^*P_1)$ is a locally free sheaf. Let $\beta : \mathbb{V}(F^\vee) \to \mathcal{P}$ be the projection. We set $\mathcal{E}_i' := \beta^\#(\alpha^\#\mathcal{E}_i \otimes p_{\mathcal{P}}^*P_i)$, $i = 1,2$. Then there is an exact and commutative diagram.

$$\begin{CD} 0 @>>> \mathcal{O}_{\mathbb{V}(F^\vee)} \boxtimes V_1 @>>> \mathcal{O}_{\mathbb{V}(F^\vee)} \boxtimes V @>>> \mathcal{O}_{\mathbb{V}(F^\vee)} \boxtimes V_2 @>>> 0 \\ @. @VVV @VVV @VVV \\ 0 @>>> \mathcal{E}_1' @>>> \mathcal{E} @>>> \mathcal{E}_2' @>>> 0 \\ @. @VVV @VVV @VVV \\ @. 0 @. 0 @. 0 \end{CD} \tag{2.8}$$

We note that

$$\varphi : F \to \operatorname{Ext}^1_{p_{\mathcal{P}}}(\mathcal{E}_2', \mathcal{E}_1') \to p_{\mathcal{P}*}(\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{P}\times X}}(\mathcal{E}_2', \mathcal{E}_1')) =: G$$

is surjective and

$$\dim\ker\varphi = \dim F - c_2. \tag{2.9}$$

We set $\widetilde{W} := p_{\mathcal{P}}(\wedge^2(I_{\mathcal{Z}}/I_{\mathcal{Z}}^2)^\vee(2\mathcal{D} - L - 2nf))$. In the same way as in section 1.3, we define $\mathbb{V}(\widetilde{W}^\vee)^0$. Then it is easy to see that

$$e(\mathbb{V}(\widetilde{W}^\vee)^0 \times_{H^{c_2}} H_{c_2,a,C}) = e(N_{c_2,a,C})e(S^lC)e(PGL(V_1))e(PGL(V_2)). \tag{2.10}$$

We set $Y := \varphi^{-1}(\mathbb{V}(\widetilde{W}^\vee)^0)$. Let P be the parabolic subgroup of $GL(V)$ fixing the subsheaf V_1. Then there is an injective morphism $Y \times_P GL(V) \to FQ$ whose image is the subset $FQ_{c_2,a}^l$. Hence we get that

$$\begin{aligned} e(FV_{c_2,a}^l) &= \frac{e(FQ_{c_2,a}^l)}{e(GL(V))} = \frac{e(Y)}{e(P)} \\ &= \frac{e(S^lC)e(N_{c_2,a,C})}{(xy)^{2l-\deg L-2n+1-g}(xy-1)^2}, \end{aligned} \tag{2.11}$$

where we used (2.9) and the equality

$$\begin{aligned}\dim F - \dim \operatorname{Hom}(V_2, V_1) &= -\chi((\mathcal{E}_2')_y, (\mathcal{E}_1')_y) \\ &= -(2l - \deg L - 2n - c_2 + 1 - g), y \in Y.\end{aligned} \tag{2.12}$$

□

Thus we get that

$$e(FV_{c_2,a}) = \sum_{0 \le l < n+k} \frac{e(S^l C)}{(xy)^{2l - \deg L - 2n + 1 - g}} \frac{e(N_{c_2,a,C})}{(xy-1)^2}. \tag{2.13}$$

2.4. **Proof of Theorem 0.1.** We shall first estimate $e(FV_{c_2,a}) - e(FV_{c_2,a}(k))$. We can easily show the following.

Lemma 2.2. *For $E \in V_{c_2,a} \cap V(L, c_2, d)$, $h^0(X, E(n)) \le \max\{n + d + 1, 2n + \deg L - a + 2\}$.*

By using deformation theory of Drezet-Le Potier [D-L] or the argument of Atiyah-Bott [A-B], we see that

$$\operatorname{codim}_Q\{q \in Q | \mathcal{Q}_q \in T(L, c_2, d)\} = 2d + c_2 - \deg L + g - 1 \tag{2.14}$$

for $d < k'$. By Lemma 2.2, $\chi(X, \pi_* E(n)) - h^0(X, E(n)) \ge \min\{n - d + \deg L + 1 - 2g, -2g\}$. Since $n \ge k$, we see that $\operatorname{codim}(FQ_{c_2,a} \setminus FQ_{c_2,a}(k)) \ge 2k + A_1$, where A_1 is a constant which does not depend on k and n. Since $\dim FQ_{c_2,a} - \dim GL(V) - 2n$ does not depend on k and n, we get that

$$e(FV_{c_2,a}) \equiv e(FV_{c_2,a}(k)) \mod \left(\frac{(xy)^{2n}}{(xy)^{2k + A_2}}\right), \tag{2.15}$$

where A_2 is a constant which does not depend on k and n. Since $FQ_{c_2,a}(k) \to Q_{c_2,a}(k)$ is a projective bundle with fiber $\mathbb{P}(H^0(X, E(n))^\vee)$, $E \in V_{c_2,a}(k)$, we get that

$$e(FV_{c_2,a}(k)) = \frac{(xy)^{\deg L + 2n - a + 2(1-g)} - 1}{xy - 1} e(V_{c_2,a}(k)). \tag{2.16}$$

By (2.15) and (2.16),

$$e(V_{c_2,a}(k)) \equiv \sum_{0 \le l < n+k} \frac{e(S^l C)}{(xy)^{2l}} (xy)^{3g-3} (xy)^a \frac{e(N_{c_2,a,C})}{(xy-1)} \mod \left(\frac{1}{(xy)^{2k + A_3}}\right), \tag{2.17}$$

where A_3 is a constant which does not depend on k and n. By using the argument of Atiyah-Bott [A-B] (see [Y2, appendix]), we get that

$$\begin{aligned} e(V_{c_2,a}(1)) &= e(V_{c_2,a}(k)) - \sum_{j=1}^{k-1} e(V(L, c_2, j) \cap V_{c_2,a}) \\ &= e(V_{c_2,a}(k)) - \sum_{j=1}^{k-1} \frac{e(\operatorname{Pic}^0(C))}{(xy)^{2j - \deg L + (1-g)}} \frac{e(N_{c_2,a,C})}{(xy-1)^2}. \end{aligned}$$

Lemma 2.3.

$$\sum_{0 \le l < n+k} \frac{e(S^l C)}{(xy)^{2l}} \equiv \frac{(1 - x/(xy)^2)^g (1 - y/(xy)^2)^g}{(1 - 1/(xy)^2)(1 - 1/(xy))} \mod \left(\frac{1}{(xy)^{n+k}}\right). \tag{2.18}$$

Proof. By Macdonald's formula [Ch], we get that

$$\sum_{l \ge 0} e(S^l C) t^l = \frac{(1 - xt)^g (1 - yt)^g}{(1-t)(1-xyt)}. \tag{2.19}$$

Since $\deg_x(e(S^l C)) \le l$ and $\deg_y(e(S^l C)) \le l$, $\sum_{l \ge 0} e(S^l C)/(xy)^{2l}$ converges to

$$\frac{(1 - x/(xy)^2)^g (1 - y/(xy)^2)^g}{(1 - 1/(xy)^2)(1 - 1/(xy))}$$

and

$$\sum_{0 \le l < n+k} \frac{e(S^l C)}{(xy)^{2l}} \equiv \sum_{l \ge 0} \frac{e(S^l C)}{(xy)^{2l}} \mod \left(\frac{1}{(xy)^{n+k}}\right).$$

□

By Lemma 2.3, we obtain that

$$\begin{aligned} e(V_{c_2,a}(1)) \equiv & \frac{(1-x/(xy)^2)^g(1-y/(xy)^2)^g}{(1-1/(xy)^2)(1-1/(xy))}(xy)^{3g-3}(xy)^a\frac{e(N_{c_2,a,C})}{xy-1} \\ & -\frac{e(\operatorname{Pic}^0(C))}{(1-1/(xy)^2)(xy-1)^2}(xy)^{\deg L+g-3}e(N_{c_2,a,C}) \mod \left(\frac{1}{(xy)^{2k+A_4}}\right), \end{aligned} \tag{2.20}$$

where A_4 is a constant which does not depend on k and n. Since we can choose any sufficiently large integer k, we get that

$$\begin{aligned} e(V_{c_2,a}(1)) = & \frac{(1-x/(xy)^2)^g(1-y/(xy)^2)^g}{(1-1/(xy)^2)(1-1/(xy))}(xy)^{3g-3}(xy)^a\frac{e(N_{c_2,a,C})}{xy-1} \\ & -\frac{e(\operatorname{Pic}^0(C))}{(1-1/(xy)^2)(xy-1)^2}(xy)^{\deg L+g-3}e(N_{c_2,a,C}). \end{aligned} \tag{2.21}$$

By using Lemma 1.4, we see that

$$\begin{aligned} \sum_n e(V(L,n)^{ss})t^n = & \frac{(x^2y-1)^g(xy^2-1)^g}{((xy)^2-1)(xy-1)^2}\left(\sum_{n,a} e(N_{c_2,a,C})(xy)^a t^n\right) \\ & -\frac{(x-1)^g(y-1)^g}{((xy)^2-1)(xy-1)^2}(xy)^{\deg L+g-1}\left(\sum_{n,a} e(N_{c_2,a,C})t^n\right) \\ = & \frac{(x^2y-1)^g(xy^2-1)^g}{((xy)^2-1)(xy-1)^2}\prod_{b\geq 1}\left(\frac{1-x^py^q(xy)^{2b-1}t^b}{1-x^py^q(xy)^{2b+1}t^b}\right)^{e^{p,q}(C)} \\ & -\frac{(x-1)^g(y-1)^g}{((xy)^2-1)(xy-1)^2}(xy)^{\deg L+g-1}\prod_{b\geq 1}\left(\frac{1-x^py^q(xy)^{2b-2}t^b}{1-x^py^q(xy)^{2b}t^b}\right)^{e^{p,q}(C)}. \end{aligned} \tag{2.22}$$

By using Lemma 1.1, 1.2, we see that

$$\sum_n e(T(L,n)^{ss})t^n = \left(\sum_n e(V(L,n)^{ss})t^n\right)\prod_{i=1}^{2}\prod_{b\geq 1}\prod_{p,q}\left(\frac{1}{1-x^py^q(xy)^{2b-i}t^b}\right)^{e^{p,q}(X)}. \tag{2.23}$$

Since $e(X)=e(C)e(\mathbb{P}^1)$, (2.22) and (2.23) implies Theorem 0.1.

For the proof of Corollary 0.2, we note that there is a universal family on $M(L,n)\times X$. In the notation of this section, we set $Q(L,n)^{ss} := \{q \in Q \mid \mathcal{Q}_q \in T(L,n)^{ss}\}$. Then the quotient map $Q(L,n)^{ss} \to M(L,n)$ is a (Zariski locally trivial) $PGL(V)$-bundle over $M(L,n)$. Hence

$$e(Q(L,n)^{ss}) = e(M(L,n))e(PGL(V)).$$

By the definition of $e(T(L,n)^{ss})$ and Theorem 0.1, we get Corollary 0.2.

Remark 2.1. Kirwan [K, sect. 14] defined a natural Hodge structure of the equivariant cohomology group $H^*_{GL(V)}(Q(L,n)^{ss},\mathbb{C})$. By [K, sect. 14], $h^{p,q}(H^k_{GL(V)}(Q(L,n)^{ss},\mathbb{C}))=0$ unless $k=p+q$. Moreover [K, sect. 14] implies that

$$\sum_{p,q}(-1)^{p+q}h^{p,q}(H^{p+q}_{GL(V)}(Q(L,n)^{ss},\mathbb{C}))x^{-p}y^{-q} = \frac{e(T(L,n)^{ss})}{(xy)^{\dim T(L,n)^{ss}}}, \tag{2.24}$$

where $\dim T(L,n)^{ss} = \dim M(L,n)-1$ (cf. [Y2, Prop. 4.3]).

3. Appendix

In this appendix, we shall explain what will be useful to modify Cheah's argument [Ch].

Definition 3.1.
1. A constructible set Y has a grading if there is a stratification $Y=\cup_{n\geq 0}Y_n$ and Y_n are constructible.
2. For constructible sets, Y, Z with gradings $Y=\cup_{n\geq 0}Y_n$, $Z=\cup_{n\geq 0}Z_n$, we define the grading of $Y\times Z$ by $(Y\times Z)_n := \cup_{i+j=n}Y_i\times Z_j$.

Let $Y=\cup_{n\geq 0}Y_n$ be a graded constructible set. Then m-th product Y^m has a natural grading and the natural action of $\mathfrak{S}_m$ preserves the grading. Hence m-th symmetric product $S^m(Y)$ also has a natural grading.

For a graded constructible set $Y = \cup_{n\geq 0} Y_n$, we define the virtual Hodge polynomial by $e(Y) := \sum_{p,q,r} e^{p,q,r}(Y) x^p y^q z^r$, where $e^{p,q,r}(Y) := e^{p,q}(Y_r)$. Then [Ch, Prop. 1.2] is replaced by

$$\sum_m e(S^m(Y))t^m = \prod_{p,q,r} \left(\frac{1}{1 - x^p y^q z^r t} \right)^{e^{p,q,r}(Y)}. \tag{3.1}$$

Proof. By the definition of the grading, we get

$$(S^m Y)_n = \bigcup_{\substack{\sum_r r m_r = n \\ \sum_r m_r = m}} \prod_r S^{m_r} Y_r. \tag{3.2}$$

By using [Ch, Prop. 1.2], we see that

$$\begin{aligned}
\sum_{m,n} e((S^m Y)_n) z^n t^m &= \sum_{m,n} \left(\sum_{\substack{\sum_r r m_r = n \\ \sum_r m_r = m}} \prod_r e(S^{m_r} Y_r) \right) z^n t^m \\
&= \prod_r \left(\sum_m e(S^m Y_r)(z^r t)^m \right) \\
&= \prod_r \prod_{p,q} \left(\frac{1}{1 - x^p y^q z^r t} \right)^{e^{p,q}(Y_r)}.
\end{aligned} \tag{3.3}$$

Thus (3.1) holds. □

In the notation of section 1.3, $N_{n,Y} := \cup_{a\geq 0} N_{n,a,Y}$ becomes a graded constructible set. Replacing [Ch, Prop. 1.2] by (3.1), we can argue the same way as in [Ch].

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