Defining the residue current of a complete intersection

Mikael Passare *
August Tsikh **

*
Matematiska institutionen
Kungliga tekniska högskolan
100 44 Stockholm
Sweden

**
Otdelenie matematiki
Krasnoyarskiĭ gosudarstvennyĭ universitet
Prospekt Svobodnyĭ 79
660 062 Krasnoyarsk
Russia

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany

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Introduction

One way of expressing the classical residue of a one-variable meromorphic function φ/f at a point a is by the integral

$$\frac{1}{2\pi i}\int_{|z-a|=\varepsilon}\frac{\varphi(z)\,dz}{f(z)},$$

where ε is sufficiently small. The circle of integration $|z-a|=\varepsilon$ may of course be replaced by some other contour, for instance $\{z\in\mathcal{U}_a;\,|f(z)|=\varepsilon\}$, where $\mathcal{U}_a$ is a small neighborhood of the point a not containing any other zeros of the function f. With this choice of contour the notion of residue has been extended to the multidimensional case in two different directions.

On the one hand, given a mapping $f\colon\mathbb{C}^n\to\mathbb{C}^n$, holomorphic in a neighborhood of a point $a\in\mathbb{C}^n$ and having a as isolated zero ($f^{-1}(0)\cap\mathcal{U}_a=\{a\}$), one defines the Grothendieck residue

$$R_f(\varphi)=\frac{1}{(2\pi i)^n}\int_{T_a}\frac{\varphi(z)\,dz_1\wedge\ldots\wedge dz_n}{f_1(z)\cdots f_n(z)},\quad \varphi\in\mathcal{O}_a,$$

where $T_a=\{z\in\mathcal{U}_a;\,|f_1(z)|=\varepsilon_1,\ldots,|f_n(z)|=\varepsilon_n\}$, the radii ε_j being sufficiently small but otherwise arbitrary, see [7], [8], [12], [13].

On the other hand, if $f\colon X\to\mathbb{C}$ is a holomorphic function defined on a n-dimensional complex manifold X, then the limit

$$R_f(\varphi)=\lim_{\varepsilon\to0}\frac{1}{2\pi i}\int_{|f|=\varepsilon}\frac{\varphi}{f},\quad \varphi\in\mathcal{D}^{n,n-1}(X)$$

exists and defines a $(0,1)$-current, i.e. a continuous linear functional on the space of smooth compactly supported $(n,n-1)$-forms. The proof of the existence of this residue current

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was first given by Herrera & Lieberman in [9]. In order to generalize this result Coleff & Herrera considered in [4] integrals of the type

$$I_f^\varphi(\varepsilon) = \frac{1}{(2\pi i)^p} \int_{\substack{|f_1|=\varepsilon_1 \\ \vdots \\ |f_p|=\varepsilon_p}} \frac{\varphi}{f_1 \cdots f_p}, \quad \varphi \in \mathcal{D}^{n,n-p}(X), \tag{$*$}$$

associated to a holomorphic mapping $f = (f_1, \ldots, f_p): X \to \mathbb{C}^n$. They proved that if the vector $\varepsilon = (\varepsilon_1 \ldots, \varepsilon_p)$ tends to zero along an "admissible" path $\varepsilon = \varepsilon(\delta)$ such that

$$\lim_{\delta \to 0} \varepsilon_p(\delta) = 0, \quad \text{and} \quad \lim_{\delta \to 0} \frac{\varepsilon_j(\delta)}{(\varepsilon_{j+1}(\delta))^q} = 0, \quad j = 1, \ldots, p-1,$$

for any natural number q, then the limit of $I_f^\varphi(\varepsilon(\delta))$ as $\delta \to 0$ exists and defines a $(0,p)$-current independently of the particular choice of "admissible" path. We also wish to point out that for $\bar\partial$-closed forms φ the corresponding residues were first considered by Dolbeault in [5].

As shown by simple examples, the residue function $I_f^\varphi(\varepsilon)$ will not in general have a well defined limit at the origin $\varepsilon = 0$. Indeed, if we take $f_1 = z_1$, $f_2 = z_1 z_2$ and $\varphi = \tilde\varphi(z)\, dz_1 \wedge dz_2$ in $\mathbb{C}^2$, we obtain the residue function

$$I_f^\varphi(\varepsilon) = \frac{1}{(2\pi i)^2} \int_{\substack{|z_1|=\varepsilon_1 \\ |z_2|=\varepsilon_2/\varepsilon_1}} \frac{\tilde\varphi(z)\, dz_1 \wedge dz_2}{z_1^2\, z_2},$$

and we see that if one approaches the origin along a path with $\varepsilon_2/\varepsilon_1 \to \infty$ then the domain of integration will leave the compact support of φ and the limit of I_f^φ will be zero, while a path with $\varepsilon_2/\varepsilon_1 \to 0$ will yield the limit $\partial_{z_1}\tilde\varphi(0,0)$.

In the above example one spots right away the reason for the non-existence of a unique limit of the residue function ($*$) at the origin. Namely, the mapping f is not at complete intersection, i.e. its zero set $f^{-1}(0)$ has dimension bigger than $n - p = 0$. If we restrict our attention to complete intersection mappings $f: X \to \mathbb{C}^p$ with zero set of codimension p in X, it would seem reasonable to expect that the function ($*$) should be continuous at the origin, and hence that one would have an elegant and correct definition of the residue current by simply writing

$$R_f(\varphi) = \lim_{\varepsilon=(\varepsilon_1\ldots,\varepsilon_p)\to 0} I_f^\varphi(\varepsilon), \quad \varphi \in \mathcal{D}^{n,n-p}(X). \tag{$**$}$$

This possibility has been considered for instance in [4] and [2].

In this paper we disprove the continuity of a general complete intersection residue function. More precisely, we exhibit in Section 1 a polynomial mapping $f: \mathbb{C}^2 \to \mathbb{C}^2$ with $f^{-1}(0) = \{0\}$, and a smooth test form $\varphi = \tilde\varphi(z)\, dz_1 \wedge dz_2$, for which the limit ($**$) does not exist.

The situation is thus not good enough to allow the definition of the residue current simply as the "value" for $\varepsilon = 0$ of the $(2n-p)$-dimensional integrals ($*$). Nevertheless, it turns out that it *is* possible to obtain the residue current by means of certain $2n$-dimensional integrals. Indeed, we prove in Section 3 that for a complete intersection mapping f the Mellin transform of the residue function $I_f^\varphi(\varepsilon)$ behaves almost as well as if it were the Mellin transform of a function which is continuous at the origin. More precisely, we find that for a complete intersection f, the transform

$$\Gamma_f^\varphi(\lambda) = \int_{\mathbb{R}_+^p} I_f^\varphi(\sqrt{\varepsilon})\,\varepsilon_1^{\lambda_1-1}\cdots\varepsilon_p^{\lambda_p-p}\,d\varepsilon$$

is such that the function $\lambda_1\cdots\lambda_p\,\Gamma_f^\varphi(\lambda)$ is holomorphic at the origin. This latter function is a p-dimensional integral of an integrand which itself is given by $(2n-p)$-dimensional integrals. It may therefore in a natural way be represented as a $2n$-dimensional integral, and a computation shows that

$$\lambda_1\cdots\lambda_p\,\Gamma_f^\varphi(\lambda) = \frac{1}{(4\pi i)^p}\int_X \frac{\bar\partial|f_1|^{\lambda_1}\wedge\ldots\wedge\bar\partial|f_p|^{\lambda_p}}{f_1\cdots f_p}\wedge\varphi,$$

and we obtain the relation

$$R_f(\varphi) = \lambda_1\cdots\lambda_p\,\Gamma_f^\varphi(\lambda)\Big|_{\lambda=0}. \qquad (***)$$

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1. A complete intersection whose residue function is discontinuous at the origin

Let us consider the mapping $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by the polynomials

$$\begin{aligned} f_1(z) &= z_1^4 \\ f_2(z) &= z_1^2 + z_2^2 + z_2^3 \end{aligned}.$$

Let further φ be a smooth compactly supported $(2,0)$-form which in a neighborhood of the origin is equal to

$$\varphi = \bar z_2\, f_2(z)\, dz_1\wedge dz_2.$$

With these choices of f and φ the residue function $(*)$ looks like

$$I_f^\varphi(\varepsilon) = \frac{1}{(2\pi i)^2}\int_{\substack{|f_1|=\varepsilon_1\\ |f_2|=\varepsilon_2}} \frac{\varphi}{f_1\,f_2} = \frac{1}{(2\pi i)^2}\int_{\substack{|z_1^4|=\varepsilon_1\\ |z_1^2+z_2^2+z_2^3|=\varepsilon_2}} \frac{\bar z_2}{z_1^4}\,dz_1\wedge dz_2.$$

After the birational coordinate change $z_1 = u$, $z_2 = uv$ we can write

$$I_f^{\varphi}(\varepsilon) = \frac{\sqrt{\varepsilon_1}}{(2\pi i)^2} \int_{\substack{|u^4|=\varepsilon_1 \\ |u^2(v^2+1+u)|=\varepsilon_2}} \frac{\bar{v}}{u^4}\, du \wedge dv. \tag{1.1}$$

Proposition 1. *For any fixed positive number $c \neq 1$ one has*

$$\lim_{\delta \to 0} I_f^{\varphi}(\delta^4, c\delta^2) = 0.$$

Proof. In view of (1.1) we are led to the following iterated integral:

$$I_f^{\varphi}(\delta^4, c\delta^2) = \frac{\delta^2}{2\pi i} \int_{|u|=\delta} \left(\frac{1}{2\pi i} \int_{|v^2+1+u|=c} \bar{v}\, dv \right) \frac{du}{u^4}. \tag{1.2}$$

We denote the inner integral by $J(u)$ and apply to it the following version of the trace formula:

$$\int_{|g(v)|=c} \psi(v)\, dv = \int_{|w|=c} \mathrm{Tr}\,[\frac{\psi}{g'}](w)\, dw,$$

where $g(v)$ is a holomorphic function whose level set $|g(v)| = c$ is a μ-fold branched covering over the circle $|w| = c$, and

$$\mathrm{Tr}\,[\psi/g'](w) = \sum_{j=1}^{\mu} [\psi/g'](v_j(w)),$$

with $v_j(w)$ denoting the different preimages $g^{-1}(w)$.

In our case we take as $g(v)$ the family $w = g_u(v) = v^2 + 1 + u$, depending on the parameter u, and we have $v_j(w) = \pm\sqrt{w - (1+u)}$. We get

$$J(u) = \frac{1}{2\pi i} \int_{|v^2+1+u|=c} \bar{v}\, dv = \frac{1}{2\pi i} \int_{|w|=c} \mathrm{Tr}\,[\frac{\bar{v}}{2v}](w)\, dw,$$

and hence

$$J(u) = \frac{1}{2\pi i} \int_{|w|=c} \frac{\overline{\sqrt{w-(1+u)}}}{\sqrt{w-(1+u)}}\, dw, \tag{1.3}$$

where the integrand should be understood as $|w - (1+u)|/(w - (1+u))$, and is hence independent of the choice of branch of the square root. From (1.3) it follows that, if $c \neq 1$, the function $J(u)$ is real-analytic in a neighborhood of the origin, and therefore the limit

$$\lim_{\delta \to 0} \int_{|u|=\delta} \frac{J(u)}{u^4}\, du$$

is a finite complex number. Now, in view of the factor δ^2 in formula (1.2) we reach the desired conclusion and Proposition 1 is proved. ∎

Proposition 2.

$$\lim_{\delta\to 0} I_f^{\varphi}(\delta^4,\delta^2) \neq 0.$$

Proof. In analogy to the equations (1.2.) and (1.3) we have

$$I_f^{\varphi}(\delta^4,\delta^2) = \frac{\delta^2}{2\pi i}\int_{|u|=\delta} \frac{J(u)}{u^4}\,du, \tag{1.2$'$}$$

where

$$J(u) = \frac{1}{2\pi i}\int_{|w|=1} \frac{\overline{\sqrt{w-(1+u)}}}{\sqrt{w-(1+u)}}\,dw. \tag{1.3$'$}$$

Here again the integrand is equal to $|w-(1+u)|/(w-(1+u))$ and from this it is straight forward to check that the integral (1.3$'$) is actually a function only of the modulus $|1+u|$. So if we denote $|1+u|$ by t we have reduced ourselves to the study of the integral

$$I(t) = \frac{1}{2\pi i}\int_{|w|=1} \frac{\overline{\sqrt{w-t}}}{\sqrt{w-t}}\,dw \tag{1.4}$$

for real parameters $t \geq 0$, and it is related to the previous integral $J(u)$ via the simple formula $I(|1+u|) = J(u)$.

Lemma 1. *The integral (1.4) is a piecewise real-analytic, continuous function for $t \geq 0$. It is explicitly given by*

$$I(t) = \begin{cases} F[-\frac{1}{2},\frac{1}{2};1;t^2], & 0 \leq t < 1, \\ \frac{1}{2t}F[\frac{1}{2},\frac{1}{2};2;\frac{1}{t^2}], & t \geq 1, \end{cases}$$

where F denotes the hypergeometric series

$$F[a,b;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!},$$

with $(a)_0 = 1$, $(a)_n = a\,(a+1)\cdots(a+n-1)$ and similary for b and c.

Proof of Lemma 1. Introducing the notation

$$\begin{aligned} I_-(t) &= I(t), \qquad 0 \leq t < 1, \\ I_+(t) &= I(t), \qquad t \geq 1, \end{aligned}$$

we first prove that the following power series expansions hold for the functions $I_{\mp}(t)$:

$$I_{-}(t) = 1 - \sum_{n=1}^{\infty} \frac{g_n^2}{2n-1} t^{2n}, \tag{1.5}$$

$$I_{+}(t) = \sum_{n=0}^{\infty} \frac{g_n^2}{2n+2} t^{-(2n+1)}, \tag{1.6}$$

where $g_n = \left(\frac{1}{2}\right)_n / n! = \binom{2n}{n}/2^{2n}$ are the Taylor coefficients at the origin of the function $x \mapsto 1/\sqrt{1-x}$ which takes the value $+1$ for $x = 0$.

To get started then, we notice that if $t > 1$ we may pick a single-valued branch of the function $\sqrt{w-t}$ on the disk $|w| \leq 1$. We then re-write it as $\sqrt{t} \cdot \sqrt{-1} \cdot \sqrt{1 - w/t}$, where the branch of $\sqrt{1 - w/t}$ is the one with value $+1$ for $w = 0$, the square root $\sqrt{t}$ is positive real, and $\sqrt{-1} = \pm i$ depending on the choice of branch we just made. A power series expansion of the integrand and term-by-term integration then yield

$$I_{+}(t) = \frac{1}{2\pi i} \int_{|w|=1} \frac{\overline{\sqrt{t}}}{\sqrt{t}} \cdot \frac{\overline{\sqrt{-1}}}{\sqrt{-1}} \cdot \frac{\overline{\sqrt{1 - w/t}}}{\sqrt{1 - w/t}}\, dw$$

$$= -\frac{1}{2\pi i} \int_{|w|=1} \left[1 + \sum_{n=1}^{\infty} g_n \left(\frac{w}{t}\right)^n\right] \times \left[1 - \sum_{m=1}^{\infty} \frac{g_{m-1}}{2m} \left(\frac{\bar{w}}{t}\right)^m\right] dw$$

$$= \sum_{n=0}^{\infty} \frac{g_n^2}{2n+2} t^{-(2n+1)},$$

and we have obtained formula (1.6).

If now instead $t < 1$ we can no longer choose a single-valued branch of the function $\sqrt{w-t}$ over the circle of integration $|w| = 1$, but we may still represent it as $\sqrt{w} \cdot \sqrt{1 - t/w}$, with a two-valued factor $\sqrt{w}$ and a single-valued factor $\sqrt{1 - t/w}$ which takes a positive real value for $w = 1$. A calculation similar to the one for the function $I_{+}(t)$ now gives

$$I_{-}(t) = \frac{1}{2\pi i} \int_{|w|=1} \frac{\overline{\sqrt{w}}}{\sqrt{w}} \cdot \frac{\overline{\sqrt{1 - t/w}}}{\sqrt{1 - t/w}}\, dw$$

$$= \frac{1}{2\pi i} \int_{|w|=1} \left[1 + \sum_{n=1}^{\infty} g_n \left(\frac{t}{w}\right)^n\right] \times \left[1 - \sum_{m=1}^{\infty} \frac{g_{m-1}}{2m} \left(\frac{t}{\bar{w}}\right)^m\right] \frac{dw}{w}$$

$$= 1 - \sum_{n=1}^{\infty} \frac{g_n \cdot g_{n-1}}{2n} t^{2n}.$$

To obtain formula (1.5) it then suffices to observe that

$$\frac{g_n \cdot g_{n-1}}{2n} = \frac{(2n)!}{(n!)^2 \cdot 2^{2n}} \cdot \frac{(2n-2)!}{[(n-1)!]^2 \cdot 2^{2n-2}} \cdot \frac{1}{2n} = \frac{g_n^2}{2n-1}.$$

Direct substitution shows that (1.5) and (1.6) do indeed coincide with the hypergeometric functions cited in the lemma, so the sole fact that remains to be proved is the continuity of the function $I(t)$. But this follows from the representation (1.4) in which $I(t)$ is written as the integral over $\{|w| = 1\} \times \{0 \le t < \infty\}$ of a bounded function which is continuous outside the single point $(1, 1)$. Lemma 1 is proved. ∎

Building on Lemma 1 we shall next describe the asymptotic behaviour of our function $I(t)$ at the point $t = 1$.

Lemma 2. *In a (real) neighborhood of the point $t = 1$ the functions $I_\mp(t)$ admit representations*

$$I_\mp(t) = A_\mp(t) \log|t^2 - 1| \; + \; B_\mp(t), \tag{1.7}$$

the functions $A_\mp$, $B_\mp$ being analytic with the properties

(i) $A_\mp(t) = (t^2 - 1)/2\pi + O\left[(t^2-1)^2\right], \quad$ *as* $t \to 1$,

(ii) $B_-(1) = B_+(1) = I(1)$.

Proof of Lemma 2. It is a classical fact that the hypergeometric function $F[a, b; c; z]$ satisfies the differential equation

$$z\,(1-z)\frac{d^2u}{dz^2} + \{c - (a+b+1)z\}\,\frac{du}{dz} - ab\,u = 0,$$

and all solutions to this equation are in a neighborhood of $z = 1$ of the form

$$A(z)\,\log(1-z) \; + \; B(z), \tag{1.8}$$

where A and B are locally convergent Puiseux series with respect to some rational powers of the variable $z - 1$, see [14,Sect.10.3-10.32]. It so happens that for those particular hypergeometric functions which occur in the representation of $I_\mp$, the coefficients A and B are actually holomorphic near the point $z = 1$, that is, the Puiseux series are in fact Taylor series. To see this we follow [14,Sect.10.3] and write the differential equation as

$$(z-1)^2\,\frac{d^2u}{dz^2} + (z-1)\,P(z)\,\frac{du}{dz} + Q(z)\,u = 0,$$

where the functions P and Q are holomorphic in a neighborhood of the point $z = 1$. Then we consider the roots ρ_1, ρ_2 of the quadratic equation

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0,$$

where $p_0 = P(1)$, $q_0 = Q(1)$. A sufficient condition for the functions A and B to be holomorphic is then that the roots ρ_1, ρ_2 be non-negative integers. Recalling that in our case the parameters are

$$a = -\tfrac{1}{2}, \quad b = \tfrac{1}{2}, \quad c = 1, \quad (\text{for } I_-(t)),$$

$$\text{and} \quad a = \tfrac{1}{2}, \quad b = \tfrac{1}{2}, \quad c = 2, \quad (\text{for } I_+(t)),$$

we find that both p_0 and q_0 are zero, and hence that the roots $\rho_1 = 0$, $\rho_2 = 1$ are indeed non-negative integers. It follows that for complex t the functions may be represented as

$$I_-(t) = A_-(t)\,\log(1-t^2) \; + \; B_-(t),$$

$$I_+(t) = A_+(t)\,\log(1-1/t^2) \; + \; \tilde{B}_+(t),$$

with the functions $A_\mp$, B_-, $\tilde{B}_+$ being analytic at $t = 1$. Finally putting $B_+ = \tilde{B}_+ - A_+ \, \log t^2$ we arrive at the representations (1.7).

Now let us prove property (i). To this end shall make use of series representations (1.5) and (1.6) of the functions $I_\mp$. Observe that, in view of Stirling's formula

$$n! = \sqrt{2\pi n} \cdot n^n \cdot e^{-n} \left(1 + \mathrm{O}\left(1/n\right)\right),$$

we have

$$g_n = \frac{(2n)!}{2^{2n}\cdot(n!)^2} = \frac{1}{\sqrt{\pi n}}\left(1 + \mathrm{O}\left(1/n\right)\right), \quad \text{as } n \to \infty.$$

Therefore the coefficients of the series (1.5) and (1.6) have the following asymptotic behaviour:

$$\frac{g_n^2}{2n-1} = \frac{1}{\pi n\,(2n-1)}\left[1 + \mathrm{O}\left(\frac{1}{n}\right)\right] = \frac{1}{2\pi n\,(n+1)} + \mathrm{O}\left(\frac{1}{n^3}\right),$$

$$\frac{g_n^2}{2n+2} = \frac{1}{\pi n 2\,(n+1)}\left[1 + \mathrm{O}\left(\frac{1}{n}\right)\right] = \frac{1}{2\pi n\,(n+1)} + \mathrm{O}\left(\frac{1}{n^3}\right).$$

Summing up the series

$$\sum_{n=1}^{\infty} \frac{1}{2\pi n\,(n+1)}\, t^{2n} = \frac{1}{2\pi t^2}\left((1-t^2)\,\log(1-t^2) \, + \, t^2\right),$$

$$\sum_{n=1}^{\infty} \frac{1}{2\pi n\,(n+1)}\, t^{-(2n+1)} = \frac{1}{2\pi t}\left((t^2-1)\,\log(t^2-1) + 1 \, - \, (t^2-1)\,\log t^2\right),$$

we deduce that

$$I_-(t) = \left(-\frac{1}{2\pi}(1-t^2) + \hat{A}_-(t)\right) \log(1-t^2) \ + \ \hat{B}_-(t),$$

$$I_+(t) = \left(\frac{1}{2\pi}(t^2-1) + \hat{A}_+(t)\right) \log(t^2-1) \ + \ \hat{B}_+(t), \tag{1.9}$$

with the functions $\hat{A}_\mp$, $\hat{B}_\mp$ being analytic at $t = 1$. It should now be observed that in forming the functions $\hat{A}_\mp$ only terms with order of vanishing ≥ 2 at $t = 1$ were involved, since they came from the expansions of $1/t^2$ and $1/t$ in powers of $t^2 - 1$, and from power series with $\mathrm{O}\,(1/n^3)$ coefficients. According to a well known result of Jüngen, see [3,p.86], if such a sum can be written $\hat{A}_\mp(t)\,\log(1-t^2) \ + \ \hat{B}_\mp(t)$, then the order of vanishing of $\hat{A}_\mp(t)$ at $t = 1$ is necessarily at least two. Consequently, from (1.9) we obtain property (i).

Property (ii) follows from the fact that

$$A_\mp(t)\cdot\log|t^2-1|\bigg|_{t=1} = 0,$$

together with the continuity of the function $I(t)$. The proof of Lemma 2 is complete. ∎

Proof of Proposition 2, continued. We recall that we have to find the limit of the function (1.2′) as $\delta \to 0$, with the integrated function $J(u)$ being equal to the function $I(|1+u|)$, described in Lemma 2. According to the lemma we can represent the function $I(t)$ as a series

$$I(t) = b_0(t) + \sum_{n=1}^{\infty}\left\{a_n(t)\cdot(t^2-1)^n\cdot\log|t^2-1| \ + \ b_n(t)\cdot(t^2-1)^n\right\},$$

where the coefficients a_n, b_n are piecewise constant functions taking only two values:

$$a_n(t) = \begin{cases} a_n^-, & t<1, \\ a_n^+, & t\geq 1, \end{cases} \qquad b_n(t) = \begin{cases} b_n^-, & t<1, \\ b_n^+, & t\geq 1. \end{cases}$$

Moreover, the properties (i) and (ii) imply that the first two coefficients are truly constant:

$$b_0(t) \equiv b_0 = I(1), \quad a_1(t) \equiv 1/2\pi.$$

We can therefore write

$$J(u) = I(|1+u|) = b_0 + \frac{1}{2\pi}\left(|1+u|^2-1\right)\,\log\left||1+u|^2-1\right| \ + \ b_1(|1+u|)\left(|1+u|^2-1\right)$$

$$+\mathrm{O}\left(\left||1+u|^2-1\right|^{3/2}\right),$$

where

$$b_1(|1+u|) = \begin{cases} b_1^-, & |1+u| < 1, \\ b_1^+, & |1+u| \geq 1. \end{cases}$$

(Actually, the exponent 3/2 above can be replaced by any number < 2.)

We have thus written $J(u)$ as a sum of four terms. Let us first show that the first, third and fourth terms all give null contribution to the limit (1.2′). This is obvious for the first term, which is just the constant b_0. The fourth term is also easy to handle. Indeed, on the circle of integration $u = \delta e^{i\phi}$, $0 \leq \phi \leq 2\pi$, we have $|1+u|^2 - 1 = \delta(2\cos\phi + \delta)$, and hence

$$\delta^2 \int_{|u|=\delta} \mathrm{O}\left(\left||1+u|^2-1\right|^{3/2}\right) \frac{du}{u^4} = \delta^2 \int_0^{2\pi} \frac{\mathrm{O}(\delta^{3/2})\, d\phi}{\delta^3 e^{3i\phi}} \to 0, \quad \text{as } \delta \to 0.$$

Let us next consider the contribution of the third term:

$$\lim_{\delta\to 0} \delta^2 \int_{\substack{|u|=\delta \\ |1+u|<1}} \frac{b_1^-\,(|1+u|^2-1)}{u^4}\, du = \lim_{\delta\to 0} b_1^-\, \delta^2\, i \int_{\substack{\phi\in[0,2\pi] \\ 2\cos\phi+\delta<0}} \frac{\delta\,(2\cos\phi+\delta)}{\delta^3 e^{3i\phi}}\, d\phi$$

$$= 2b_1^-\, i \int_{\pi/2}^{3\pi/2} \cos\phi(\cos 3\phi - i\sin 3\phi)\, d\phi = 0.$$

Similarly,

$$\lim_{\delta\to 0} \delta^2 \int_{\substack{|u|=\delta \\ |1+u|>1}} \frac{b_1^+\,(|1+u|^2-1)}{u^4}\, du = 2b_1^+\, i \int_{-\pi/2}^{\pi/2} \cos\phi(\cos 3\phi - i\sin 3\phi)\, d\phi = 0.$$

This takes care of the third term.

What remains to be shown is that the second term, which contains the logarithm, gives a non-zero contribution to the limit (1.2′). We have

$$\lim_{\delta\to 0} I_f^{\varphi}(\delta^4, \delta^2) = \lim_{\delta\to 0} \frac{\delta^2}{2\pi i} \int_{|u|=\delta} \frac{(1/2\pi)(|1+u|^2-1)\,\log\left||1+u|^2-1\right|}{u^3}\, \frac{du}{u}$$

$$= \lim_{\delta\to 0} \frac{1}{4\pi^2} \int_0^{2\pi} \frac{(2\cos\phi+\delta)\,\log(\delta\,|2\cos\phi+\delta|)}{e^{3i\phi}}\, d\phi.$$

Observe now that the limit of the last integral does not change if we remove from it the factor δ inside the logarithm. This is because the integral of $(2\cos\phi+\delta)/e^{3i\phi}$ is equal to zero. After the removal of this factor δ the integrand will be a uniformly bounded family of continuous functions, and so by Lebesgue's theorem we may perform the limit procedure inside the integral and obtain:

$$\lim_{\delta\to 0} I_f^{\varphi}(\delta^4, \delta^2) = \frac{1}{2\pi^2} \int_0^{2\pi} \frac{\cos\phi\,\log|2\cos\phi|}{e^{3i\phi}}\, d\phi.$$

Expanding the function $\log|\cos\phi|$ as a Fourier series we get

$$\log|\cos\phi| = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cos 2n\phi,$$

containing even frequencies only, see [6]. Therefore the product $\cos\phi\,\log|2\cos\phi|$ is equal to the uniformly convergent series

$$\cos\phi\,\log|2\cos\phi| = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} \big(\cos(2n+1)\phi + \cos(2n-1)\phi\big),$$

in which $\cos 3\phi$ appears with the coefficient $1/4$. All the other odd harmonics $\cos(2n-1)\phi$ are orthogonal to $e^{-3i\phi}$, and we get

$$\lim_{\delta\to 0} I_f^{\varphi}(\delta^4,\delta^2) = \frac{1}{2\pi^2}\int_0^{2\pi} \frac{1}{4}\cos 3\phi\, e^{-3i\phi}\, d\phi = \frac{1}{8\pi^2}\int_0^{2\pi} \cos^2 3\phi\, d\phi = \frac{1}{8\pi}.$$

Proposition 2 follows. ∎

2. The Mellin transform of a residue function

Let X be a n-dimensional complex manifold. To a holomorphic mapping $f\colon X \to \mathbb{C}^n$ $(p \le n)$ and a test form $\varphi \in \mathcal{D}^{n,n-p}$ we associate a residue function $I_f^{\varphi}\colon \mathbb{R}_+^p \to \mathbb{C}$ defined by the integral

$$I_f^{\varphi}(\varepsilon) = \frac{1}{(2\pi i)^p}\int_{T_\varepsilon} \frac{\varphi}{f_1\cdots f_p}$$

over the tube $T_\varepsilon = \{z \in X; |f_1(z)|^2 = \varepsilon_1, \dots, |f_p(z)|^2 = \varepsilon_p\}$, which should be oriented in an alternating fashion with respect to the numbering of the components f_j of the mapping $f = (f_1,\dots,f_p)$. (Note that for convenience we here take $|f_j|^2 = \varepsilon_j$ in the definition of the tubes, while in the introduction there was no square.)

The Mellin transform of the function $I_f^{\varphi}(\varepsilon)$ is given by the integral

$$\Gamma_f^{\varphi}(\lambda) = \int_{\mathbb{R}_+^p} I_f^{\varphi}(\varepsilon)\, \varepsilon^{\lambda - I}\, d\varepsilon,$$

where $\lambda = (\lambda_1,\dots,\lambda_p) \in \mathbb{C}^p$ is a complex vector and

$$\varepsilon^{\lambda-I}\, d\varepsilon = \varepsilon_1^{\lambda_1-1}\cdots\varepsilon_p^{\lambda_p-1}\, d\varepsilon_1 \wedge \dots \wedge d\varepsilon_p.$$

Proposition 3. *The Mellin transform of the residue function associated to a holomorphic mapping $f\colon X \to \mathbb{C}^p$ may be expressed as an integral over X as follows:*

$$\Gamma_f^\varphi(\lambda) = \frac{1}{(4\pi i)^p} \int_X |f|^{2(\lambda - I)}\, d\bar{f} \wedge \varphi, \tag{2.1}$$

with the vector notation

$$|f|^{2(\lambda-I)} = |f_1|^{2(\lambda_1 - 1)} \cdots |f_p|^{2(\lambda_p - 1)}, \qquad d\bar{f} = d\bar{f}_1 \wedge \ldots \wedge d\bar{f}_p.$$

Proof. Since the form φ is of bidegree $(n, n-p)$ we have

$$\left(\bigwedge_{j=1}^{p} \frac{d\bar{f}_j}{\bar{f}_j}\right) \wedge \varphi = \left(\bigwedge_{j=1}^{p} \frac{d|f_j|^2}{|f_j|^2}\right) \wedge \varphi.$$

Hence, fibering X over $\mathbb{R}_+^p$ by the tubes T_ε, $\varepsilon \in \mathbb{R}_+^p$, we can represent the integral in the right hand side of (2.1) as an integral in ε of integrals along the fibers in the following way:

$$\frac{1}{(4\pi i)^p} \int_X |f_1|^{2\lambda_1} \cdots |f_p|^{2\lambda_p} \cdot \frac{\varphi}{f_1 \cdots f_p} \wedge \left(\bigwedge_{j=1}^{p} \frac{d|f_j|^2}{|f_j|^2}\right)$$

$$= \frac{1}{(4\pi i)^p} \int_{\mathbb{R}_+^p} \varepsilon_1^{\lambda_1} \cdots \varepsilon_p^{\lambda_p} \left(\int_{T_\varepsilon} \frac{\varphi}{f_1 \cdots f_p}\right) \frac{d\varepsilon_1^2}{\varepsilon_1^2} \wedge \ldots \wedge \frac{d\varepsilon_p^2}{\varepsilon_p^2}.$$

By the definition of the residue integral $I_f^\varphi(\varepsilon)$ this last integral is indeed equal to $\Gamma_f^\varphi(\lambda)$ and the proposition follows. ∎

The following result was proved in [11].

Theorem 1. *The Mellin transform Γ_f^φ defined by (2.1) is holomorphic for $Re\,\lambda$ in $\mathbb{R}_+^p$ and it has a meromorphic continuation to all of $\mathbb{C}^p$. There is a finite collection of non-zero vectors a^k in $\mathbb{N}^p$, depending only on f and on the support of φ, such that the poles of Γ_f^φ, which are all simple, are contained in the hyperplanes $(a^k, \lambda) = -m$, $m \in \mathbb{N}$ (here (a^k, λ) denotes the usual scalar product). In particular, near the origin one has*

$$\Gamma_f^\varphi(\lambda) = \sum_{|K|=p} \frac{c_K}{(a^{k_1}, \lambda) \cdots (a^{k_p}, \lambda)} + Q(\lambda),$$

where the c_K are constants and Q is a finite sum of functions with simple poles along fewer than p hyperplanes.

In the next section we shall show that when the mapping f is a complete intersection one can say a lot more about the structure of the polar set of the function $\Gamma_f^\varphi(\lambda)$.

3. The Mellin transform associated to a complete intersection

Theorem 2. *If $f: X \to \mathbb{C}^p$ is a complete intersection, i.e. $\dim f^{-1}(0) = n - p$, then in a neighborhood of the origin $\lambda = 0$ the function $\Gamma_f^{\varphi}(\lambda)$ can have (simple) poles only along the coordinate hyperplanes $\lambda_j = 0$. In other words, for a complete intersection the function $\lambda_1 \cdots \lambda_p \Gamma_f^{\varphi}(\lambda)$ is holomorphic near the origin.*

Remark. For $p = 2$ the result was proved in [1].

Proof. Let us denote the differential form in the integrand of (2.1) by ω. Following [1] we observe that

$$d\Big\{\frac{|f_1|^{2\lambda_1}}{\lambda_1 f_f} \cdot |f_2|^{2(\lambda_2-1)} \cdots |f_p|^{2(\lambda_p-1)}\, d\bar{f}_2 \wedge \ldots \wedge d\bar{f}_p \wedge \varphi\Big\} = \bar{\partial}\Big\{\ldots\Big\}$$

$$= \omega + \frac{(-1)^{p-1}}{\lambda_1} \cdot \Big(\prod_{j=1}^{p} \frac{|f_j|^{2\lambda_j}}{f_j}\Big) \cdot \Big(\bigwedge_{j=2}^{p} \frac{d\bar{f}_j}{\bar{f}_j}\Big) \wedge \bar{\partial}\varphi.$$

Since φ has compact support we therefore obtain

$$(4\pi i)^p\, \Gamma_f^{\varphi}(\lambda) = \int_X \omega = \frac{(-1)^p}{\lambda_1} \int_X \Big(\prod_{j=1}^{p} \frac{|f_j|^{2\lambda_j}}{f_j}\Big) \cdot \Big(\bigwedge_{j=2}^{p} \frac{d\bar{f}_j}{\bar{f}_j}\Big) \wedge \bar{\partial}\varphi, \tag{3.1}$$

by Stokes' theorem.

Our objective is now to show that the polar set of this last integral at the origin $\lambda = 0$ can only contain hyperplanes whose equations do not contain λ_1. By an identical argument we can then deduce the analogous property for $\lambda_2, \ldots, \lambda_p$ as well, and this will imply the statement of the theorem.

After a partition of unity we can assume that the support of φ is small enough to allow a resolution of the singularities of $\operatorname{supp}\varphi \cap \{f_1 \cdots f_p = 0\}$. Letting X denote a small neihgborhood of $\operatorname{supp}\varphi$ this means that we can find a new complex manifold $\tilde{X}$ and a proper holomorphic mapping $\pi: \tilde{X} \to X$ such that π^{-1} is biholomorphic outside the analytic set $\{f_1 \cdots f_p = 0\}$, and such that the preimage $\pi^{-1}(\{f_1 \cdots f_p = 0\})$ has only normal crossings. The last fact means that near any point z_0 on $\tilde{X}$ there are local coordinates $\zeta_1, \ldots \zeta_n$ centered at z_0 such that

$$\pi^* f_j = \zeta^{\alpha_j} u_j(\zeta) = \zeta_1^{\alpha_j^1} \cdots \zeta_n^{\alpha_j^n} u_j(\zeta), \quad j = 1, \ldots, p,$$

where the holomorphic functions $u_j(\zeta)$ are nowhere vanishing. It follows that after a partition of unity on the manifold $\tilde{X}$ we have decomposed the integral in (3.1) into a finite sum of terms such as

$$\int_{\mathbb{C}^n} \prod_{j=1}^{p} \Big(\frac{|\zeta|^{2\alpha_j\lambda_j}}{\zeta^{\alpha_j}} \cdot \frac{|u_j|^{2\lambda_j}}{u_j}\Big) \bigwedge_{j=2}^{p} \Big(\frac{d\overline{(u_j\zeta^{\alpha_j})}}{\overline{u_j\,\zeta^{\alpha_j}}}\Big) \wedge \pi^*(\bar{\partial}\varphi). \tag{3.2}$$

Each of the expressions $d(\overline{u_j\,\zeta^{\alpha_j}})/\overline{u_j\,\zeta^{\alpha_j}}$ can only have simple poles along the coordinate hyperplanes $\zeta_k = 0$, since they can be written

$$\frac{d\bar{u}_j}{\bar{u}_j} + \frac{d\bar{\zeta}^{\alpha_j}}{\bar{\zeta}^{\alpha_j}} = \sum_{k=1}^{n}\left[\frac{\overline{(\partial u_j/\partial\zeta_k)}}{\bar{u}_j} + \frac{\alpha_j^k}{\bar{\zeta}_k}\right] d\bar{\zeta}_k.$$

We may therefore re-write the integral (3.2) in the following way:

$$\int_{\mathbb{C}^n} \prod_{k=1}^{p}\left(\frac{|\zeta_k|^{2L_k(\lambda)}}{\bar{\zeta}_k^{|\alpha^k|}}\right) \bigwedge_{j=2}^{p}\left[\sum_{i=1}^{n}\left(\xi_{ji}(\zeta) + \frac{\alpha_j^i}{\bar{\zeta}_i}\right) d\bar{\zeta}_i\right] \wedge \xi(\zeta,\lambda)\,\pi^*(\bar{\partial}\varphi), \tag{3.3}$$

with linear functions $L_k(\lambda) = \langle\alpha^k,\lambda\rangle = \alpha_1^k\lambda_1 + \ldots + \alpha_p^k\lambda_p$ and norms $|\alpha^k| = \alpha_1^k + \ldots + \alpha_p^k$. Notice also that inside the square brackets we have the expression $\pi^*(d\bar{f}_j/\bar{f}_j)$, and that the function $\xi(\zeta,\lambda)$ is smooth in ζ and holomorphic in λ. It is obvious that the integral (3.3) can be written as sum of integrals such as

$$\int_{\mathbb{C}^n} \prod_{k=1}^{n} \frac{|\zeta_k|^{2L_k(\lambda)}}{\bar{\zeta}_k^{|\alpha^k|}} \cdot \frac{A_J(\zeta,\lambda)}{\bar{\zeta}_J}\, d\zeta \wedge d\bar{\zeta} = \int_{\mathbb{C}^n} |\zeta|^{2(L(\lambda)-I_J)} \cdot \frac{A_J(\zeta,\lambda)}{\bar{\zeta}^{|\alpha|-I_J}}\, d\zeta \wedge d\bar{\zeta}, \tag{3.4}$$

where the multi-indices $J = \{j_1 \ldots, j_q\}$ run through the set $\{1,\ldots,n\}$, and have lenght $|J| = q \le p-1$; further, $\bar{\zeta}_J = \bar{\zeta}_{j_1}\cdots\bar{\zeta}_{j_q}$, and the functions $A_J(\zeta,\lambda)$ are smooth in ζ, holomorphic in λ; finally, by I_J we mean the n-vector with 1 in the places $j_1,\ldots,j_q$ and with zeros elsewhere.

To the smooth functions $A_J(\zeta,\lambda)$ we apply the Taylor type formula of [4] and [11] according to which, for any $\beta \in \mathbb{N}^n$, we have

$$A_J(\zeta,\lambda) = \sum_{j=1}^{n}\sum_{k+\ell<\beta_j} \varphi_{k\ell}^j(\zeta,\lambda)\,\zeta_j^k\bar{\zeta}_k^\ell \;+ \sum_{K+L=\beta} \varphi_{KL}(\zeta,\lambda)\,\zeta^K\bar{\zeta}^L,$$

where the functions $\varphi_{k\ell}^j$ are independent of ζ_j, and all coefficients are smooth in ζ, holomorphic in λ. In this formula we now let β be the vector $|\alpha| - I_J$.

In order to evaluate the integral (3.4) we introduce polar coordinates

$$\zeta_j = r_j\,e^{i\theta_j}, \quad r \in \mathbb{R}_+^n, \theta \in [0,2\pi]^n, \quad j = 1,\ldots,n.$$

Integration with respect to θ_j then shows that the terms $\varphi_{k\ell}^j(\zeta,\lambda)\,\zeta_j^k\bar{\zeta}_j^\ell$ give no contribution to the integral. This is because they will contain the factor

$$\int_0^{2\pi} e^{i(k-\ell-\beta_j)\theta_j}\, d\theta_j,$$

which is equal to zero, since $k - \ell \le k + \ell < \beta_j$.

Integration of the remaining terms $\sum_{K+L=\beta} \varphi_{KL}(\zeta,\lambda)\,\zeta^K\bar{\zeta}^L$ with respect to $\theta \in [0,2\pi]^n$ gives

$$\Phi(r) = (2i)^n r_1 \cdots r_n \sum_{K+L=|\alpha|-I_J} \int_{[0,2\pi]^n} e^{i(-|\alpha|-I_J+K-L)\theta} \varphi_{KL}(re^{i\theta})\,d\theta,$$

where the sum now is a smooth function in r. (The factor $(2i)^n r_1 \cdots r_n$ comes from expressing the form $d\zeta \wedge d\bar{\zeta}$ in polar coordinates r, θ.) As a consequence the integral (3.4) is reduced to

$$\int_{\mathbb{R}_+^n} r^{2L(\lambda)-I_J} \cdot \frac{\Phi(r)}{r^{I_J}}\,dr,$$

with a smooth function $\Phi(r)/r^{I_J}$. From this it follows that the only poles that will occur near $\lambda = 0$ are the ones given by $L_j(\lambda) = 0$, $j \in J$.

We claim that none of those of the hyperplanes $L_k(\lambda) = 0$ above for which L_k depends on λ_1 actually gives rise to any pole for the integral (3.3). Indeed, the fact that L_k depends on λ_1 means that $\alpha_1^k \neq 0$. Since in the integral (3.4) the denominators $\bar{\zeta}_J$ are obtained from the denominators $\bar{\zeta}_i$ in (3.3), with the coefficients α_j^i as numerators, the plane $L_k = 0$ will be polar for (3.3) only if at least one of the numbers α_j^k, $j = 2, \ldots, p$, is different from zero, i.e. when L_k depends on at least one of the remaining variables $\lambda_2, \ldots, \lambda_p$. Assume that the corresponding non-zero coefficients are $\alpha_1^k, \alpha_2^k, \ldots, \alpha_s^k$, $s \le p$, while the remaining ones $\alpha_{s+1}^k, \ldots, \alpha_p^k$ are all zero. (This can always be accomplished by re-ordering the components f_j.) This means that the coordinate ζ_k appears as a factor in each of the pull-backs $\pi^* f_1, \ldots, \pi^* f_s$, but not in any of $\pi^* f_{s+1}, \ldots, \pi^* f_p$. Hence the hyperplane $\zeta_k = 0$ is mapped by π into the set $V_s = \{f_1 = \ldots = f_s = 0\}$, which has dimension $n - s$. We conclude from this that for any multi-index I of length $|I| = n - p + 1$ the restriction

$$\left[\pi^*(d\bar{f}_{s+1} \wedge \ldots \wedge d\bar{f}_p \wedge d\bar{z}_I)\right]\Big|_{\zeta_k=0} = \pi^*\left[d\bar{f}_{s+1} \wedge \ldots \wedge d\bar{f}_p \wedge d\bar{z}_I\Big|_{V_s}\right]$$

is equal to zero. Hence, if in (3.3) we write the form $\bar{\partial}\varphi$ as a sum of $(n, n-p+1)$-forms $\sum_{|I|=n-p+1} \varphi_I \wedge \omega_I$, where the φ_I are smooth $(n,0)$-forms and $\omega_I = d\bar{z}_I$, and if we decompose the product

$$\bigwedge_{j=2}^{p} [\ldots] = \bigwedge_{j=2}^{p} \left[\frac{d\bar{f}_j}{\bar{f}_j}\right]$$

as a product $\left(\wedge_{j=1}^{s}[\ldots]\right) \wedge \left(\wedge_{j=s+1}^{p}[\ldots]\right)$, we find that the function $A_J(\zeta,\lambda)$ in (3.4) is divisible by $\bar{\zeta}_k$. It follows that the hyperplane $L_k = 0$ is not polar for the integrals (3.4) and (3.3), and and this finishes the proof of the theorem. ∎

4. Resume

Since for a smooth function $I(\varepsilon)$ the Mellin transform $\Gamma(\lambda)$ has the property that $\lambda_1 \cdots \lambda_p \Gamma(\lambda)\big|_{\lambda=0}$ is equal to the value $I(0)$, we see from Section 3 that we can define the residue current of a complete intersection by the simple formula $(***)$. This fits in well with the results obtained in [10] and [11], where the residue current was defined by means of a mean value operation. More precisely, the residue current $R_f(\varphi)$ may be obtained by considering the limits of the residue function $I_f^{\varphi}(\varepsilon)$ along one-parameter curves of the form $\varepsilon = (\delta^{s_1}, \ldots, \delta^{s_p})$, and forming the mean value of these limits over the simplex $s_1 + \ldots + s_p = 1$, $s_1 \geq 0, \ldots, s_p \geq 0$.

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