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ON THE UNIVERSAL ELLIPSITOMIC KZB CONNECTION

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ABSTRACT. We construct a twisted version of the genus one universal Knizhnik–Zamolodchikov–Bernard (KZB) connection introduced by Calaque–Enriquez–Etingof, that we call the *ellipsitomic* KZB connection. This is a flat connection on a principal bundle over the moduli space of Γ-structured elliptic curves with marked points, where $\Gamma = \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, and $M, N \geq 1$ are two integers. It restricts to a flat connection on Γ-twisted configuration spaces of points on elliptic curves, which can be used to construct a filtered-formality isomorphism for some interesting subgroups of the pure braid group on the torus. We show that the universal ellipsitomic KZB connection realizes as the usual KZB connection associated with elliptic dynamical r-matrices with spectral parameter, and finally, also produces representations of cyclotomic Cherednik algebras.

CONTENTS

INTRODUCTION

In this paper, which fits in a series of works about universal Knizhnik–Zamolodchikov–Bernard (KZB) connections by different authors [6, 13], we focus on a twisted version of the genus 1 situation. In his seminal work [9], Drinfeld considers the monodromy representation of the universal Knizhnik–Zamolodchikov (KZ) equation which leads to the formality of the pure braid group (see reminder below) and the so-called theory of associators that makes the link between rich algebraic structures (such as braided monoidal categories) and the Grothendieck–Teichmüller group GT.

B. Enriquez generalizes in [10] Drinfeld's work to the twisted (a-k-a trigonometric, or cyclotomic) situation and relates it to multiple polylogarithms at roots of unity. Namely, he uses the universal trigonometric KZ system to prove the formality of some subgroups of the pure braid group on $\mathbb{C}^\times$ and to emphazise relations between suitable algebraic structures (quasi-reflection algebras, or braided module categories) and analogues of the group GT.

The next step has been made by B. Enriquez, P. Etingof and the first author in [6], where a universal version of the elliptic KZB system (see [2]) is defined and used to:

- give a new proof (see [1] for the original one) of the filtered formality of the pure braid group on the torus,
- find a relation between the KZ associator and a generating series for iterated integrals of Eisenstein series (see also [12]),
- provide examples of elliptic structures on braided monoidal categories (see also [11]).

The main goal of the present paper is to introduce a twisted version of the universal elliptic KZB system, called the *ellipsitomic* KZB connection, and to derive from it the formality of some subgroups of the pure braid group on the torus. In a subsequent work [7], we use it to emphasize a relation between generating series for values of multiple polylogarithms at roots of unity and values of elliptic multiple polylogarithms at torsion points.

Throughout the paper and unless otherwise specified, $\mathbf{k}$ is a field of characteristic zero, M, N are fixed positive integers, and $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Genus zero situation (rational KZ). First recall from [26] that the holonomy Lie algebra of the configuration space

$$\mathrm{Conf}(\mathbb{C}, n) := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ if } i \neq j\}$$

of n points on the complex line is isomorphic to the graded Lie $\mathbb{C}$-algebra $\mathfrak{t}_n$ generated by t_{ij}, $1 \leq i \neq j \leq n$, with relations

(S) $t_{ij} = t_{ji}\,,$

(L) $[t_{ij}, t_{kl}] = 0 \quad \text{if } \#\{i,j,k,l\} = 4\,,$

(4T) $[t_{ij}, t_{ik} + t_{jk}] = 0 \quad \text{if } \#\{i,j,k\} = 3\,.$

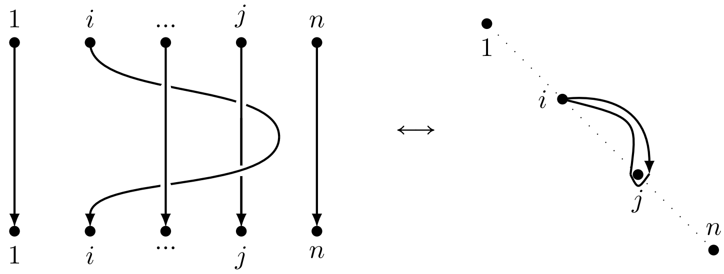
Then, on the one hand, denote by PB_n the fundamental group of $\mathrm{Conf}(\mathbb{C}, n)$, also known as the pure braid group with n strands, and by $\mathfrak{pb}_n$ its Malcev Lie algebra (which is filtered by its lower central series, and complete). One can easily check that PB_n is generated by elementary pure braids P_{ij}, $1 \leq i < j \leq n$, which satisfy (at least) the following relations:

(PB1) $(P_{ij}, P_{kl}) = 1 \quad \text{if } \{i,j\} \text{ and } \{k,l\} \text{ are non crossing}\,,$

(PB2) $(P_{kj}P_{ij}P_{kj}^{-1}, P_{kl}) = 1 \quad \text{if } i < k < j < l\,,$

(PB3) $(P_{ij}, P_{ik}P_{jk}) = (P_{jk}, P_{ij}P_{ik}) = (P_{ik}, P_{jk}P_{ij}) = 1 \quad \text{if } i < j < k\,.$

We can depict the generator P_{ij} in the following two equivalent ways:



Therefore one has a surjective morphism of graded Lie algebras $p_n : \mathfrak{t}_n \twoheadrightarrow \mathrm{gr}(\mathfrak{pb}_n)$ sending t_{ij} to $\sigma(\log(P_{ij}))$, $i < j$ and $\sigma : \mathfrak{pb}_n \to \mathrm{gr}(\mathfrak{pb}_n)$ being the symbol map.

On the other hand, denote by $\exp(\hat{\mathfrak{t}}_n)$ the exponential group associated with the degree completion $\hat{\mathfrak{t}}_n$ of $\mathfrak{t}_n$. The universal KZ connection on the trivial $\exp(\hat{\mathfrak{t}}_n)$-principal bundle over $\mathrm{Conf}(\mathbb{C}, n)$ is then given by the holomorphic 1-form

$$w_n^{\mathrm{KZ}} := \sum_{1 \leqslant i < j \leqslant n} \frac{dz_i - dz_j}{z_i - z_j} t_{ij} \in \Omega^1(\mathrm{Conf}(\mathbb{C}, n), \mathfrak{t}_n)\,,$$

which takes its values in $\mathfrak{t}_n$. It is a fact that the connection associated with this 1-form is flat, and descends to a flat connection on the moduli space $\mathcal{M}_{0,n+1} \simeq \mathrm{Conf}(\mathbb{C}, n)/\mathrm{Aff}(\mathbb{C})$ of rational curves with $n + 1$ marked points.

Firstly, the regularized holonomy of this connection along the real straight path from 0 to 1 in $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$ gives a formal power series Φ_{KZ} in two non-commuting variables, called the KZ associator, that is a generating series for values at 0 and 1 of multiple polylogarithms. Secondly, using the monodromy representation of the universal KZ connection, one obtains:

(1) A morphism of filtered Lie algebras $\mu_n : \mathfrak{pb}_n \to \hat{\mathfrak{t}}_n$ such that $\mathrm{gr}(\mu_n) \circ p_n = \mathrm{id}$. Hence one concludes that p_n and μ_n are bijective. This provides an isomorphic from $\mathfrak{pb}_n$ to the degree completion of its associated graded, which is actually $\hat{\mathfrak{t}}_n$. This recovers the known fact that the group PB_n is 1-*formal*, meaning that its Malcev Lie algebra is isomorphic to the degree completion of a quadratic Lie algebra.
(2) A system of relations (called Pentagon (P) and two Hexagons $(H_\pm)$) satisfied by the KZ associator. Then, if $\mathbf{k}$ is a field of characteristic 0, one can define a set of $\mathbf{k}$-associators $\mathrm{Ass}(\mathbf{k})$, for which the KZ associator will be a $\mathbb{C}$-point (showing at the same time that the set of such abstract $\mathbb{C}$-associators is indeed non-empty).

A twisted variant (trigonometric/cyclotomic KZ). Similarly, one can consider the configuration space

$$\mathrm{Conf}(\mathbb{C}^\times, n) := \{\mathbf{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n | z_i \neq z_j \text{ if } i \neq j\}$$

of n points on $\mathbb{C}^\times$. Then $\mathrm{Conf}(\mathbb{C}^\times, n) \simeq \mathrm{Conf}(\mathbb{C}, n+1)/\mathbb{C}$ and thus its fundamental group PB_n^1 is isomorphic to PB_{n+1}. More generally, for any $M \in \mathbb{Z} - \{0\}$ one can consider an M-twisted configuration space

$$\mathrm{Conf}(\mathbb{C}^\times, n, M) := \{\mathbf{z} = (z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n | z_i^M \neq z_j^M \text{ for some } i \neq j\}.$$

In [10] B. Enriquez exhibits, using the so-called universal trigonometric KZ connection, an isomorphism $\mathfrak{pb}_n^M \to \exp(\hat{\mathfrak{t}}_n^M)$, where $\mathfrak{pb}_n^M$ is the Malcev Lie algebra of the fundamental group $\mathrm{PB}_n^M \subset \mathrm{PB}_n^1$ of $\mathrm{Conf}(\mathbb{C}^\times, n, M)$, and $\mathfrak{t}_n^M$ is the holonomy Lie algebra of $\mathrm{Conf}(\mathbb{C}^\times, n, M)$. The monodromy of this connection along a suitable (non closed) path gives a universal pseudotwist $\Psi_{\mathrm{KZ}}^M \in \exp(\hat{\mathfrak{t}}_2^M)$ that is a generating series for values of multiple polylogarithms at Mth roots of unity, and satisfies relations with Φ_{KZ}.

Genus one situation (elliptic KZB). The genus one universal Knizhnik–Zamolodchikov–Bernard (KZB) connection $\nabla_{1,n}^{\mathrm{KZB}}$ was introduced in [6]. This is a flat connection over the moduli space of elliptic curves with n marked points $\mathcal{M}_{1,n}$, which was independently discovered by Levin–Racinet [27] in the specific cases $n = 1, 2$. It restricts to a flat connection over the configuration space

$$\mathrm{Conf}(\mathbb{T}, n) \coloneqq \Lambda_\tau^n \backslash \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n | z_i - z_j \notin \Lambda_\tau \text{ if } i \neq j\}$$

of n points on an (uniformized) elliptic curve $E_\tau \coloneqq \Lambda_\tau \backslash \mathbb{C}$, for $\tau \in \mathfrak{h}$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. More precisely, this connection is defined on a G-principal bundle over $\mathcal{M}_{1,n}$ where the Lie algebra associated with G has as components:

(1) a Lie algebra $\mathfrak{t}_{1,n}$ related to $\mathrm{Conf}(\mathbb{T}, n)$, somehow controlling the variations of the marked points: it has generators x_i, y_i, for $i = 1, ..., n$, corresponding to moving z_i along the topological cycles generating $H_1(E_\tau)$;
(2) a Lie algebra $\mathfrak{d}$ with as components the Lie algebra $\mathfrak{sl}_2$ with standard generators e, f, h and a Lie algebra $\mathfrak{d}_+ \coloneqq \mathrm{Lie}(\{\delta_{2m} | m \geq 1\})$ such that each δ_{2m} is a highest weight element for $\mathfrak{sl}_2$. The Lie algebra $\mathfrak{d}$ somehow controls the variation of the curve in $\mathcal{M}_{1,n}$ and is closely related to the one defined in [32].

Now, the connection $\nabla_{1,n}^{\mathrm{KZB}}$ can be locally expressed as $\nabla_{1,n}^{\mathrm{KZB}} \coloneqq d - \Delta(\mathbf{z}|\tau)d\tau - \sum_i K_i(\mathbf{z}|\tau)dz_i$ where

(1) the term $K_i(-|\tau) : \mathbb{C}^n \to \hat{\mathfrak{t}}_{1,n}$ is meromorphic on $\mathbb{C}^n$, having only simple poles on

$$\mathrm{Diag}_{n,\tau} \coloneqq \bigcup_{i \neq j} \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n | z_i - z_j \in \Lambda_\tau\}.$$

It is constructed out of a function

$$k(x, z|\tau) \coloneqq \frac{\theta(z + x|\tau)}{\theta(z|\tau)\theta(x|\tau)} - \frac{1}{x}.$$

This relates directly the connection $\nabla_{1,n}^{\mathrm{KZB}}$ with Zagier's work [34] on Jacobi forms (see Weil's book [33]) and to Brown and Levin's work [5].
(2) the term $\Delta(\mathbf{z}|\tau)$ is a meromorphic function $\mathbb{C}^n \times \mathfrak{h} \to \mathrm{Lie}(G)$, with only simple poles on $\mathrm{Diag}_n \coloneqq \{(\mathbf{z}, \tau) \in \mathbb{C}^n \times \mathfrak{h} | \mathbf{z} \in \mathrm{Diag}_{n,\tau}\}$. The coefficients of δ_{2m} in $\Delta(\mathbf{z}|\tau)$ are Eisenstein series.

We also refer to Hain's survey [23] and references therein for the Hodge theoretic and motivic aspects of the story.

Then, one can construct a holomorphic map sending each $\tau \in \mathfrak{h}$ to a couple $e(\tau) \coloneqq (A(\tau), B(\tau))$ where $A(\tau)$ (resp. $B(\tau)$) is the regularized holonomy of the universal elliptic

KZB connection along the straight path from 0 to 1 (resp. from 0 to τ) in the once punctured elliptic curve $\Lambda_\tau\backslash(\mathbb{C}-\Lambda_\tau)\simeq E_\tau\backslash\mathrm{Conf}(E_\tau,2)$. B. Enriquez developed in [11] the general theory of elliptic associators, whose scheme is denoted Ell and for which the couple $e(\tau)$ is an example of a $\mathbb{C}$-point. Some of the main features of the so-called elliptic KZB associators $e(\tau)$ are the following:

- They satisfy algebraic and modularity relations.
- They satisfy a differential equation in the variable τ expressed only in terms of iterated integrals of Eisenstein series, which will be called iterated Eisenstein integrals.
- When taking τ to $i\infty$ (which consists in computing the constant term of the q-expansion of the series $A(\tau)$ and $B(\tau)$, where $q=e^{2i\pi\tau}$), they can be expressed only in terms of the KZ associator Φ_{KZ}.
- They provide isomorphisms between the Malcev Lie algebra of the fundamental group $\mathrm{PB}_{1,n}$ of $\mathrm{Conf}(\mathbb{T},n)$ and the degree completion of its associated Lie algebra $\mathfrak{t}_{1,n}$.

Observe that, contrary to what happens in genus 0, $\mathrm{PB}_{1,n}$ (also known as the *pure elliptic braid group*) is not 1-formal (as $\mathfrak{t}_{1,n}$ is not quadratic), but only *filtered-formal* according to the terminology of [30].

Ellipsitomic KZB. As we wrote above, the purpose of the present work is to define a twisted version of the genus one KZB connection introduced in [6]. This is a flat connection on a principal bundle over the moduli space of elliptic curves with a Γ-structure and n marked points. It restricts to a flat connection on the so-called Γ-twisted configuration space of points on an elliptic curve, which can be used for constructing a filtered-formality isomorphism for some interesting subgroups of the pure braid group on the torus.

In a subsequent work [7], we will define ellipsitomic KZB associators as renormalized holonomies along certain paths on a once punctured elliptic curve with a Γ-structure, and exhibit a relation between ellipsitomic KZB associators, the KZ associator [9] and the cyclotomic KZ associator [10]. Moreover, ellipsitomic associators can be regarded as a generating series for iterated Eisenstein integrals whose coefficients are elliptic multiple zeta values at torsion points. In the case $M=N$ these coefficients are related to Goncharov's work [20], and also to the recent work [4] of Broedel–Matthes–Richter–Schlotterer.

We finally prove that the universal KZB connection realizes as the usual KZB connection associated with elliptic dynamical r-matrices with spectral parameter, that should be compared with [16, 18].

It is worth mentioning the recent work [31], where Toledano-Laredo and Yang define a similar KZB connection. More precisely, they construct a flat KZB connection on moduli spaces of elliptic curves associated with crystallographic root systems. The type A case coincides with the universal elliptic KZB connection defined in [6], and we suspect that the type B case coincides with the connection of the present paper for $M=N=2$. It is interesting to point out that a common generalization of their work and ours (for $M=N$) could be obtained by constructing a universal KZB connection associated with arbitrary complex reflection groups.

Plan of the paper. The paper is organized as follows:

- In Section 1, we introduce Γ-twisted configuration spaces on an elliptic curve and define the universal ellipsitomic KZB connection on them. It takes values in a the Lie algebra $\mathfrak{t}_{1,n}^{\Gamma}$ of infinitesimal ellipsitomic (pure) braids, that we also define.
- As in [6], the connection extends from the configuration space to the moduli space $\bar{\mathcal{M}}_{1,[n]}^{\Gamma}$ of elliptic curves with a Γ-level structure and unordered marked points. This is proven in Section 3 using some technical definitions introduced in Section 2, involving derivations of the Lie algebra $\mathfrak{t}_{1,n}^{\Gamma}$ related to the twisted configuration space in genus 1. As in the untwisted case, the results of this section also apply to the "unordered marked points" situation as well.
- In Section 4, we provide a notion of realizations for the Lie algebras previously introduced, and show that the universal ellipsitomic KZB connection realizes to a flat connection intimately related to elliptic dynamical r-matrices with spectral parameter.
- In Section 5, we derive from the monodromy representation the filtered-formality of the fundamental group of the twisted configuration space of the torus, which is a subgroup of $\mathrm{PB}_{1,n}$. As in the cyclotomic case, it extends to a relative filtered-formality result for the map $\mathrm{B}_{1,n} \to \Gamma^n \rtimes \mathfrak{S}_n$.
- Finally, in Section 6, we construct a homomorphism from the Lie algebra $\bar{\mathfrak{t}}_{1,n}^{\Gamma} \rtimes \mathfrak{d}^{\Gamma}$ to the twisted Cherednik algebra $H_n^{\Gamma}(k)$. This allows us to consider the twisted elliptic KZB connection with values in representations of the twisted Cherednik algebra. This study shall be closely related to the recent paper [3].
- We also include an appendix that summarizes our conventions for fundamental groups, covering maps, principal bundles, and monodromy maps.

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1. Bundles with flat connections on Γ-twisted configuration spaces

1.1. **The Lie algebra of infinitisemal ellipsitomic braids.** In this paragraph, Γ can be replaced by any finite abelian group (with the additive notation).

For any positive integer n we define $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ to be the bigraded $\mathbf{k}$-Lie algebra with generators x_i $(1 \leq i \leq n)$ in degree $(1,0)$, y_i $(1 \leq i \leq n)$ in degree $(0,1)$, and t_{ij}^{α} $(\alpha \in \Gamma,\ i \neq j)$ in degree $(1,1)$, and relations

(tS$_{e\ell\ell}$1) $t_{ij}^{\alpha} = t_{ji}^{-\alpha}$,

(tS$_{e\ell\ell}$2) $[x_i, y_j] = [x_j, y_i] = \sum\limits_{\alpha\in\Gamma} t_{ij}^{\alpha}$,

(tN$_{e\ell\ell}$) $[x_i, x_j] = [y_i, y_j] = 0$,

(tT$_{e\ell\ell}$) $[x_i, y_i] = -\sum\limits_{j:j\neq i}\sum\limits_{\alpha\in\Gamma} t_{ij}^{\alpha}$,

(tL$_{e\ell\ell}$1) $[t_{ij}^{\alpha}, t_{kl}^{\beta}] = 0$,

(tL$_{e\ell\ell}$2) $[x_i, t_{jk}^{\alpha}] = [y_i, t_{jk}^{\alpha}] = 0$,

(t4T$_{e\ell\ell}$1) $[t_{ij}^{\alpha}, t_{ik}^{\alpha+\beta} + t_{jk}^{\beta}] = 0$,

(t4T$_{e\ell\ell}$2) $[x_i + x_j, t_{ij}^{\alpha}] = [y_i + y_j, t_{ij}^{\alpha}] = 0$,

where $1 \leq i, j, k, l \leq n$ are pairwise distinct and $\alpha, \beta \in \Gamma$. We will call $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ the $\mathbf{k}$-Lie algebra of *infinitesimal ellipsitomic braids*. Observe that $\sum_i x_i$ and $\sum_i y_i$ are central in $\mathfrak{t}_{1,n}^{\Gamma}$. Then we denote by $\bar{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbf{k})$ the quotient of $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ by $\sum_i x_i$ and $\sum_i y_i$, and the quotient morphism $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k}) \to \bar{\mathfrak{t}}_{1,I}^{\Gamma}(\mathbf{k})$ by $u \mapsto \bar{u}$.

When $\mathbf{k} = \mathbb{C}$ we write $\mathfrak{t}_{1,n}^{\Gamma} := \mathfrak{t}_{1,n}^{\Gamma}(\mathbb{C})$, and $\bar{\mathfrak{t}}_{1,n}^{\Gamma} := \bar{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbb{C})$.

There is an alternative presentation of $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ and $\bar{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbf{k})$:

Lemma 1.1. *The Lie $\mathbf{k}$-algebra $\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k})$ (resp. $\bar{\mathfrak{t}}_{1,n}^{\Gamma}(\mathbf{k})$) can equivalently be presented with the same generators, and the following relations:* (tS$_{e\ell\ell}$1), (tS$_{e\ell\ell}$2), (tN$_{e\ell\ell}$), (tL$_{e\ell\ell}$1), (tL$_{e\ell\ell}$2), (t4T$_{e\ell\ell}$1)*, and, for every $1 \leq i \leq n$,*

$$[\sum_j x_j, y_i] = [\sum_j y_j, x_i] = 0$$

(resp. $\sum_j x_j = \sum_j y_j = 0$).

Proof. If x_i, y_i and t_{ij}^{α} satisfy the initial relations, then

$$[\sum_j x_j, y_i] = [x_i, y_i] + [\sum_{j\neq i} x_j, y_i] = -\sum_{j:j\neq i}\sum_{\alpha\in\Gamma} t_{ij}^{\alpha} + \sum_{j:j\neq i}\sum_{\alpha\in\Gamma} t_{ij}^{\alpha} = 0.$$

Now, if x_i, y_i and t_{ij}^{α} satisfy the above relations, then relations $[\sum\limits_j x_j, y_i] = 0$ and $[x_j, y_i] = \sum_{\alpha\in\Gamma} t_{ij}^{\alpha}$, for $i \neq j$, imply that $[x_i, y_i] = -\sum_{j:j\neq i}\sum_{\alpha\in\Gamma} t_{ij}^{\alpha}$. Now, relations $[\sum\limits_k x_k, y_j] = 0$ and $[\sum\limits_k x_k, x_i] = 0$ imply that $[\sum\limits_k x_k, \sum_{\alpha\in\Gamma} t_{ij}^{\alpha}] = 0$. Thus, as $[x_i, t_{jk}^{\alpha}] = 0$ if $\mathrm{card}\{i,j,k\} = 3$, we obtain relation $[x_i + x_j, t_{ij}^{\alpha}] = 0$, for $i \neq j$. In the same way we obtain $[y_i + y_j, t_{ij}^{\alpha}] = 0$, for $i \neq j$. □

There is an action $\Gamma^n \to \mathrm{Aut}(\mathfrak{t}_{1,n}^{\Gamma}(\mathbf{k}))$ defined as follows:

- it leaves x_i's and y_i's invariant.
- for every i and every $\alpha \in \Gamma$, α_i leaves t_{kl}^β's invariant if $k, l \neq i$, and sends t_{ij}^β to $t_{ij}^{\beta+\alpha}$. Here α_i denotes the element of Γ^n whose only nonzero component is the ith one and is α.

This action descends to an action on $\bar{\mathfrak{t}}_{1,n}^\Gamma(\mathbf{k})$.

Proposition 1.2. *For any group morphism $\rho : \Gamma_1 \to \Gamma_2$ we have a comparison morphism $\phi_\rho : \mathfrak{t}_{1,n}^{\Gamma_1}(\mathbf{k}) \to \mathfrak{t}_{1,n}^{\Gamma_2}(\mathbf{k})$ defined by $x_i \mapsto x_i$, $y_i \mapsto y_i$, and*

$$t_{ij}^\alpha \longmapsto \frac{1}{\#\ker(\rho)} \sum_{\beta \in \mathrm{coker}(\rho)} t_{ij}^{\rho(\alpha)+\beta}.$$

When ρ is not surjective this morphism depends on the choice of a (set theoretic) section $\mathrm{coker}(\rho) \to \Gamma_2$.

Proof. Let us prove that the relation $[x_i, y_j] = \sum_{\alpha\in\Gamma} t_{ij}^\alpha$, where $i \neq j$, is preserved by ϕ. On the one hand $[\phi(x_i), \phi(y_j)] = \sum_{\alpha\in\Gamma_2} t_{ij}^\alpha$. On the other hand

$$\phi([x_i, y_j]) \quad = \quad \sum_{\alpha\in\Gamma_1} \phi(t_{ij}^\alpha) = \sum_{\alpha\in\Gamma_1} \frac{1}{\#\ker(\rho)} \sum_{\beta\in\mathrm{coker}(\rho)} t_{ij}^{\rho(\alpha)+\beta} = \sum_{\alpha\in\Gamma_2} t_{ij}^\alpha.$$

The last equality holds because $\rho(\alpha)$ is in the image of ρ, and β is not. The fact that the remaining relations are preserved is immediate. □

Comparison morphisms are bigraded, and pass to the quotient by $\sum_i x_i$, $\sum_i y_i$. They also are compatible with the operadic module structure of $\mathfrak{t}_{1,\bullet}^\Gamma(\mathbf{k})$ from [7].

1.2. **Principal bundles over Γ-twisted configuration spaces.** Let E be an elliptic curve over $\mathbb{C}$ and consider the connected unramified Γ-covering $p : \tilde{E} \to E$ corresponding to the canonical surjective group morphism $\rho : \pi_1(E) \cong \mathbb{Z}^2 \to \Gamma$ where $\pi_1(E) \cong \mathbb{Z}^2$ is the natural choice of such an isomorphism. Let us then define the *twisted configuration space*

$$\mathrm{Conf}(\mathbb{T}, n, \Gamma) := \{\mathbf{z} = (z_1, \dots, z_n) \in \tilde{E}^n | p(z_i) \neq p(z_j) \text{ if } i \neq j\},$$

and $\mathrm{C}(\mathbb{T}, n, \Gamma) := \mathrm{Conf}(E, n, \Gamma)/\tilde{E}$ its reduced version. Notice that $\mathrm{C}(E, n, \Gamma)$ is just the inverse image of $\mathrm{C}(E, n)$ under the surjection $p^n : \tilde{E}^n \to E^n$.

Let us fix a uniformization $\tilde{E} \simeq E_\tau$, where $\tau \in \mathfrak{H}$: $E_\tau = \Lambda_\tau \backslash \mathbb{C}$, with $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. Then $E \simeq E_{\tau,\Gamma}$, where $E_{\tau,\Gamma} = \Lambda_{\tau,\Gamma} \backslash \mathbb{C}$ and $\Lambda_{\tau,\Gamma} := (1/M)\mathbb{Z} \times (\tau/N)\mathbb{Z}$. Therefore

$$\mathrm{Conf}(E, n, \Gamma) \simeq \Lambda_\tau^n \backslash (\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}),$$

where

$$\mathrm{Diag}_{\tau,n,\Gamma} := \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_{ij} := z_i - z_j \in \Lambda_{\tau,\Gamma} \text{ for some } i \neq j\}.$$

We now define a principal $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$-bundle $\mathcal{P}_{\tau,n,\Gamma}$ over $\mathrm{Conf}(E, n, \Gamma)$ as the quotient

$$\Lambda_\tau^n \backslash \big((\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}) \times \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)\big),$$

where the action is determined by the following non-abelian 1-cocycle:

$$\big(\mathbf{z}, (a + b\tau)_i\big) \longmapsto e^{-2\pi \mathrm{i} b x_i}.$$

Remark 1.3 (Notation)**.** Whenever we have en element g in a group G, and $1 \le i \le n$, we write g_i for the element of G^n given by g on the i-th component and the unit on the others.

In other words, it is the restriction on $\mathrm{Conf}(E,n,\Gamma)$ of the bundle over $\Lambda_\tau^n\backslash\mathbb{C}^n$ for which a section on $U \subset \Lambda_\tau^n\backslash\mathbb{C}^n$ is a regular map $f : \pi^{-1}(U) \to \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$ such that

- $f(\mathbf{z}+\delta_i) = f(\mathbf{z})$,
- $f(\mathbf{z}+\tau\delta_i) = e^{-2\pi \mathrm{i} x_i} f(\mathbf{z})$.

Here $\pi : \mathbb{C}^n \to \Lambda_\tau^n\backslash\mathbb{C}^n$ is the canonical projection and δ_i is the ith vector of the canonical basis of $\mathbb{C}^n$.

Since the $e^{-2\pi\mathrm{i}\bar{x}_i}$'s in $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma)$ pairwise commute and their product is 1, then the image of $\mathcal{P}_{\tau,n,\Gamma}$ under the natural morphism $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma)$ is the pull-back of a principal $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma)$-bundle $\bar{\mathcal{P}}_{\tau,n,\Gamma}$ over $\mathrm{C}(E,n,\Gamma)$.

1.3. **Variations.** The first variation we are interested in concerns *unordered configuration spaces*. The symmetric group $\mathfrak{S}_n$ acts on the left freely by automorphisms of $\mathrm{Conf}(E,n,\Gamma)$ by

$$\sigma * (z_1,\dots,z_n) \coloneqq (z_{\sigma^{-1}(1)},\dots,z_{\sigma^{-1}(n)})\,.$$

This descends to a free action of $\mathfrak{S}_n$ on $\mathrm{C}(E,n,\Gamma)$. We then defined the unordered twisted configuration spaces

$$\mathrm{Conf}(E,[n],\Gamma) \coloneqq \mathfrak{S}_n\backslash\mathrm{Conf}(E,n,\Gamma) \text{ and } \mathrm{C}(E,[n],\Gamma) \coloneqq \mathfrak{S}_n\backslash\mathrm{C}(E,n,\Gamma)\,.$$

The symmetric group $\mathfrak{S}_n$ also obviously acts on the Lie algebra $\mathfrak{t}_{1,n}^\Gamma$. One can then define, keeping the notation of the previous paragraph, a principal $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n$-bundle $\mathcal{P}_{\tau,[n],\Gamma}$ over $\mathrm{Conf}(E,[n],\Gamma)$: it is the restriction on $\mathrm{Conf}(E,[n],\Gamma)$ of the bundle over $(\Lambda_\tau^n \rtimes \mathfrak{S}_n)\backslash\mathbb{C}^n$ for which a section on $U \subset \Lambda_\tau^n\backslash\mathbb{C}^n \rtimes \mathfrak{S}_n$ is a regular map $f : \pi^{-1}(U) \to \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n$ such that

- $f(\mathbf{z}+\delta_i) = f(\mathbf{z})$,
- $f(\mathbf{z}+\tau\delta_i) = e^{-2\pi\,\mathrm{i}\,x_i} f(\mathbf{z})$,
- $f(\sigma * \mathbf{z}) = \sigma f(\mathbf{z})$.

In more compact form:

$$\mathcal{P}_{\tau,[n],\Gamma} = (\Lambda_\tau^n \rtimes \mathfrak{S}_n)\backslash\big((\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}) \times \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n\big)\,.$$

Remark 1.4. As before, $\mathcal{P}_{\tau,[n],\Gamma}$ descends to a principal $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^\Gamma) \rtimes \mathfrak{S}_n$-bundle $\bar{\mathcal{P}}_{\tau,[n],\Gamma}$ over the reduced unordered twisted configuration space $\mathrm{C}(E,[n],\Gamma)$.

The second variation concerns ordinary configuration spaces of the base $E = E_{\tau,\Gamma}$ of the covering map $E_\tau \to E_{\tau,\Gamma}$.

Recall from §1.1 that the group Γ^n acts on $\hat{\mathfrak{t}}_{1,n}^\Gamma$. Hence one has a principal $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \Gamma^n$-bundle

$$\bar{\mathcal{P}}_{(\tau,\Gamma),n} \coloneqq \Lambda_{\tau,\Gamma}^n\backslash\big((\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}) \times \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes \Gamma^n\big)$$

over $\mathrm{Conf}(E,n) \simeq \Lambda_{\tau,\Gamma}^n\backslash(\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma})$, where the action is determined by the non-abelian cocycle

$$\big(\mathbf{z}, (\frac{u}{M} + \frac{v}{N}\tau)_i\big) \longmapsto e^{-\frac{2\pi \mathrm{i} v}{N} x_i} (\bar{u},\bar{v})_i\,.$$

Remark 1.5. The map sending $\frac{u}{M}+\frac{v}{N}\tau$ to $(\bar{u},\bar{v})$ exhibits an isomorphism $\Lambda_{\tau,\Gamma}/\Lambda_\tau\simeq\Gamma$, that we will use on several occasions. Using this, if $\tilde{\alpha}=a+b\tau\in\Lambda_{\tau,\Gamma}$ is a lift of $\alpha\in\Gamma$, then the non-abelian cocycle is

$$(\mathbf{z},\alpha_i)\longmapsto e^{-2\pi\mathrm{i}bx_i}\alpha_i\,.$$

Remark 1.6. In a similar way as before, the above bundle obviously descends to a principal $\exp(\hat{\bar{\mathfrak{t}}}_{1,n}^{\Gamma})\rtimes(\Gamma^n/\Gamma)$-bundle $\bar{\mathcal{P}}_{(\tau,\Gamma),n}$ over the reduced ordinary configuration space $\mathrm{C}(E,n)$.

In concrete terms, a section over $U\subset\Lambda_{\tau,\Gamma}\backslash\mathbb{C}^n$ of $\mathcal{P}_{(\tau,\Gamma),n}$ is a regular map $f:\pi^{-1}(U)\to\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})\rtimes\Gamma^n$ such that

- $f(\mathbf{z}+\delta_i/M)=(\bar{1},\bar{0})_i f(\mathbf{z})$,
- $f(\mathbf{z}+\tau\delta_i/N)=(\bar{0},\bar{1})_i e^{-\frac{2\pi\mathrm{i}}{N}x_i}f(\mathbf{z})$.

Remark 1.7. We leave to the reader the task of combining the two variations.

1.4. **Flat connections on $\mathcal{P}_{\tau,n,\Gamma}$ and its variants.** A flat connection $\nabla_{\tau,n,\Gamma}$ on $\mathcal{P}_{\tau,n,\Gamma}$ is the same as an equivariant flat connection on the trivial $\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$-bundle over $\mathbb{C}^n-\mathrm{Diag}_{\tau,\mathrm{n},\Gamma}$, i.e., a connection of the form

$$\nabla_{\tau,n,\Gamma}:=d-\sum_{i=1}^{n}K_i(\mathbf{z}|\tau)dz_i\,,$$

where $K_i(-|\tau):\mathbb{C}^n\to\hat{\mathfrak{t}}_{1,n}^{\Gamma}$ are meromorphic with only poles at $\mathrm{Diag}_{\tau,n,\Gamma}$, and such that for any i,j:

(a) $K_i(\mathbf{z}+\delta_j|\tau)=K_i(\mathbf{z}|\tau)$,
(b) $K_i(\mathbf{z}+\tau\delta_j|\tau)=e^{-2\pi\mathrm{i}\,\mathrm{ad}(x_j)}K_i(\mathbf{z}|\tau)$,
(c) $[\partial_i-K_i(\mathbf{z}|\tau),\partial_j-K_j(\mathbf{z}|\tau)]=0$.

Moreover, the image of $\nabla_{\tau,n,\Gamma}$ under $\hat{\mathfrak{t}}_{1,n}^{\Gamma}\to\hat{\bar{\mathfrak{t}}}_{1,n}^{\Gamma}$ is the pull-back of a (necessarily flat) connection $\bar{\nabla}_{\tau,n,\Gamma}$ on $\bar{\mathcal{P}}_{\tau,n,\Gamma}$ if and only if:

(d) $\bar{K}_i(\mathbf{z}|\tau)=\bar{K}_i(\mathbf{z}+u\sum_i\delta_i|\tau)$ for any $u\in\mathbb{C}$ and $\sum_i\bar{K}_i(\mathbf{z}|\tau)=0$.

Similarly, the image of $\nabla_{\tau,n,\Gamma}$ under $\hat{\mathfrak{t}}_{1,n}^{\Gamma}\to\hat{\mathfrak{t}}_{1,n}^{\Gamma}\rtimes\Gamma^n$ is the pull-back of a (necessarily flat) connection $\nabla_{(\tau,\Gamma),n}$ on $\mathcal{P}_{(\tau,\Gamma),n}$ if and only if:

(e) $K_i(\mathbf{z}+\delta_i/M|\tau)=(\bar{1},\bar{0})_j\cdot K_i(\mathbf{z}|\tau)$,
(f) $K_i(\mathbf{z}+\tau\delta_i/N|\tau)=(\bar{0},\bar{1})_j\cdot e^{\frac{-2\pi\mathrm{i}}{N}\mathrm{ad}(x_j)}K_i(\mathbf{z}|\tau)$,

Remark 1.8. Observe that (e) implies (a), and that (f) implies (b).

Finally, the image of $\nabla_{\tau,n,\Gamma}$ under $\hat{\mathfrak{t}}_{1,n}^{\Gamma}\to\hat{\mathfrak{t}}_{1,n}^{\Gamma}\rtimes\mathfrak{S}_n$ is the pull-back of a (necessarily flat) connection $\nabla_{\tau,[n],\Gamma}$ on $\bar{\mathcal{P}}_{\tau,[n],\Gamma}$ if and only if:

(g) $K_i((ij)*\mathbf{z})=(ij)\cdot K_i(\mathbf{z})$.

1.5. **Constructing the connection.** We now construct a connection satisfying properties (d) to (g). Let us take the same conventions for theta functions as in [6]. This is the unique holomorphic function $\mathbb{C}\times\mathfrak{H}\to\mathbb{C}$, $(z,\tau)\mapsto\theta(z|\tau)$, such that

- $\{z|\theta(z|\tau)=0\}=\Lambda_\tau$,
- $\theta(z+1|\tau)=-\theta(z|\tau)=\theta(-z|\tau)$

- $\theta(z+\tau|\tau) = -e^{-\pi\mathrm{i}\tau}e^{-2\pi\mathrm{i}z}\theta(z|\tau)$
- $\partial_z\theta(0|\tau) = 1$.

In particular, $\theta(z|\tau+1) = \theta(z|\tau)$, while $\theta(-z/\tau|-1/\tau) = -(1/\tau)e^{(\pi\mathrm{i}/\tau)z^2}\theta(z|\tau)$. If $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ where $q = e^{2\pi\mathrm{i}\tau}$, and if we set $\vartheta(z|\tau) := \eta(\tau)^3\theta(z|\tau)$, then $\partial_\tau\vartheta = (1/4\pi\mathrm{i})\partial_z^2\vartheta$.

Observe that for any $\tilde{\alpha} = (a_0, a) \in \Lambda_{\tau,\Gamma}$ lifting $\alpha \in \Gamma$, the term $e^{-2\pi\mathrm{i}ax}(\theta(z-\tilde{\alpha}+x))/(\theta(z-\tilde{\alpha})\theta(x))$ only depends on the class $\alpha = (\bar{a}_0, \bar{a}) \in \Gamma$ of $\tilde{\alpha}$ mod Λ_τ. Then we set

$$k_\alpha(x,z|\tau) := e^{-2\pi\mathrm{i}ax}\frac{\theta(z-\tilde{\alpha}+x|\tau)}{\theta(z-\tilde{\alpha}|\tau)\theta(x|\tau)} - \frac{1}{x} = e^{-2\pi\mathrm{i}ax}k(x,z-\tilde{\alpha}|\tau) + \frac{e^{-2\pi\mathrm{i}ax}-1}{x},$$

where $k(x,z|\tau) := \frac{\theta(x+z)}{\theta(x)\theta(z)} - \frac{1}{x}$ (as in [6]), and

$$K_{ij}(z|\tau) := \sum_{\alpha\in\Gamma} k_\alpha(\mathrm{ad}x_i, z|\tau)(t_{ij}^\alpha), \quad K_i(\mathbf{z}|\tau) := -y_i + \sum_{j:j\neq i} K_{ij}(z_{ij}|\tau).$$

In the rest of the section we fix $\tau \in \mathfrak{H}$ and drop it from the notation. Recall from [6] that $k(x, z\pm 1) = k(x,z)$ and

$$k(x, z\pm\tau) = e^{\mp 2\pi\mathrm{i}x}k(x,z) + \frac{e^{\mp 2\pi\mathrm{i}x}-1}{x}.$$

We then define the universal ellipsitomic KZB connection on $\mathcal{P}_{\tau,n,\Gamma}$ by

$$\nabla^{\mathrm{KZB}}_{\tau,n,\Gamma} := d - \sum_{i=1}^n K_i(\mathbf{z}|\tau)dz_i.$$

Proposition 1.9. *The $K_{ij}(z)$'s have the following equivariance properties:*

$$(1) \qquad K_{ij}(z+\frac{1}{M}) = (\bar{1},\bar{0})_i\cdot(K_{ij}(z)),$$

$$(2) \qquad K_{ij}(z+\frac{\tau}{N}) = (\bar{0},-\bar{1})_i\cdot e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}x_j}\cdot(K_{ij}(z)) + (\bar{0},-\bar{1})_i\cdot(\sum_{\alpha\in\Gamma}\frac{e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}x_i}-1}{\mathrm{ad}x_i}(t_{ij}^\alpha)).$$

Proof. Let us choose representatives $0\leq u\leq M-1$ and $0\leq v\leq N-1$ so that $\tilde{\alpha} = \frac{u}{M}+\tau\frac{v}{N}$. The first equation comes from a straightforward verification. Let us show the second relation. On the one hand, we have

$$\begin{aligned}
K_{ij}\left(z+\frac{\tau}{N}\right) &= \sum_{\alpha\in\Gamma} k_\alpha\left(\mathrm{ad}\,x_i, z+\frac{\tau}{N}\right)(t_{ij}^\alpha) \\
&= \left(\sum_{\alpha\in\Gamma} e^{-\frac{2\pi\mathrm{i}}{N}v\,\mathrm{ad}(x_i)}k\left(\mathrm{ad}\,x_i, z+\frac{\tau}{N}-\tilde{\alpha}\right) + \frac{e^{-\frac{2\pi\mathrm{i}}{N}v\,\mathrm{ad}\,x_i}-1}{\mathrm{ad}\,x_i}\right)(t_{ij}^\alpha) \\
&= \left(\sum_{\alpha\in\Gamma} e^{-\frac{2\pi\mathrm{i}}{N}(v-1)\,\mathrm{ad}\,x_i}k(\mathrm{ad}\,x_i, z-\tilde{\alpha}) + \frac{e^{-\frac{2\pi\mathrm{i}}{N}(v-1)\,\mathrm{ad}\,x_i}-1}{\mathrm{ad}(x_i)}\right)(t_{ij}^{\alpha-(\bar{0},\bar{1})}) \\
&= (\bar{0},-\bar{1})_i\cdot\left(\sum_{\alpha\in\Gamma} e^{-\frac{2\pi\mathrm{i}}{N}(v-1)\,\mathrm{ad}\,x_i}k(\mathrm{ad}(x_i), z-\tilde{\alpha}) + \frac{e^{-\frac{2\pi\mathrm{i}}{N}(v-1)\,\mathrm{ad}\,x_i}-1}{\mathrm{ad}\,x_i}\right)(t_{ij}^\alpha).
\end{aligned}$$

On the other hand,

$$\begin{aligned} e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}K_{ij}(z) &= e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}\left(\sum_{\alpha\in\Gamma}k_\alpha(\operatorname{ad}x_i,z)\right)(t_{ij}^\alpha) \\ &= e^{\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_i}\left(\sum_{\alpha\in\Gamma}e^{-\frac{2\pi \mathrm{i}}{N}v\operatorname{ad}x_i}k(\operatorname{ad}x_i,z-\tilde\alpha)+\frac{e^{-\frac{2\pi \mathrm{i}}{N}v\operatorname{ad}x_i}-1}{\operatorname{ad}x_i}\right)(t_{ij}^\alpha) \\ &= \left(\sum_{\alpha\in\Gamma}e^{-\frac{2\pi \mathrm{i}}{N}(v-1)\operatorname{ad}x_i}k(\operatorname{ad}x_i,z-\tilde\alpha)+\frac{e^{-\frac{2\pi \mathrm{i}}{N}(v-1)\operatorname{ad}x_i}-e^{\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_i}}{\operatorname{ad}x_i}\right)(t_{ij}^\alpha), \end{aligned}$$

so

$$\begin{aligned} \sum_{\alpha\in\Gamma}e^{-\frac{2\pi \mathrm{i}}{N}(v-1)\operatorname{ad}x_i}k(\operatorname{ad}x_i,z-\tilde\alpha)(t_{ij}^\alpha) &= e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}K_{ij}(z) \\ &\quad -\sum_{\alpha\in\Gamma}\frac{e^{-\frac{2\pi \mathrm{i}}{N}(v-1)\operatorname{ad}x_i}-e^{\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_i}}{\operatorname{ad}x_i}(t_{ij}^\alpha). \end{aligned}$$

By putting these two equations together we finally get

$$\begin{aligned} K_{ij}\left(z+\frac{\tau}{N}\right) &= (\overline{0},\overline{-1})_i\cdot e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}K_{ij}(z) \\ &\quad +\sum_{\alpha\in\Gamma}\frac{-e^{-\frac{2\pi \mathrm{i}}{N}(v-1)\operatorname{ad}x_i}+e^{\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_i}+e^{-\frac{2\pi \mathrm{i}}{N}(v-1)\operatorname{ad}x_i}-1}{\operatorname{ad}x_i}(t_{ij}^\alpha) \\ &= (\overline{0},\overline{-1})_i\cdot e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}K_{ij}(z)+(\overline{0},\overline{-1})_i\cdot\left(\sum_{\alpha\in\Gamma}\frac{e^{\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_i}-1}{\operatorname{ad}x_i}(t_{ij}^\alpha)\right). \end{aligned}$$

□

Now recall that $\frac{e^{\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_i}-1}{\operatorname{ad}x_i}=\frac{1-e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}}{\operatorname{ad}x_j}$ and $\frac{1-e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}}{\operatorname{ad}x_j}(t_{ij})=\left(1-e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}\right)(y_i)$. We thus have

$$K_i\left(\mathbf{z}+\frac{\tau}{N}\delta_j\right) = -y_i+\sum_{j'\neq i,j}K_{ij'}(z_{ij'})+K_{ij}\left(z_{ij}+\frac{\tau}{N}\right)$$

and therefore we get the announced relation

$$K_i\left(\mathbf{z}+\frac{\tau}{N}\delta_j\right) = (\overline{0},\overline{1})_j\cdot e^{-\frac{2\pi \mathrm{i}}{N}\operatorname{ad}x_j}K_i(\boldsymbol{z}).$$

Consequently the $K_i(\mathbf{z})$'s satisfy conditions (e) and (f) above (and thus also (a) and (b)).

Moreover, the $K_i(\mathbf{z})$'s also satisfy conditions (d). Indeed, the first part of (d) is immediate and $k_\alpha(x,z)+k_{-\alpha}(-x,-z)=0$, therefore $K_{ij}(z)+K_{ji}(-z)=0$, and thus $\sum_i K_i(\mathbf{z})=-\sum_i y_i$.

Finally, from their very definition, the $K_i(\mathbf{z})$'s also satisfy condition (g).

In the next paragraph we show that the flatness condition (c) is satisfied.

1.6. **Flatness of the connection.**

Proposition 1.10. $[\partial_i-K_i(\mathbf{z}),\partial_j-K_j(\mathbf{z})]=0$, *i.e., condition (c) is satisfied.*

Proof. First we have

$$\partial_i(K_j(\mathbf{z}))-\partial_j(K_i(\mathbf{z}))=\partial_iK_{ji}(z_{ji})-\partial_jK_{ij}(z_{ij})=\partial_i(K_{ij}(z_{ij})+K_{ji}(z_{ji}))=0$$

since $K_{ij}(z) + K_{ji}(-z) = 0$. Therefore we have to prove that $[K_i(\mathbf{z}), K_j(\mathbf{z})] = 0$. As in [6] it follows from the universal classical dynamical Yang-Baxter equation:

$$\text{(CDYBE)} \qquad -[y_i, K_{jk}] + [K_{ji}, K_{ki}] + c.p.(i,j,k) = 0\,,$$

which we now prove (here $K_{ij} := K_{ij}(z_{ij})$). For any $f(x) \in \mathbb{C}[[x]]$ we have

$$[y_k, f(\mathrm{ad}x_i)(t^\alpha_{ij})] = \sum_{\beta\in\Gamma} \frac{f(\mathrm{ad}x_i) - f(-\mathrm{ad}x_j)}{\mathrm{ad}x_i + \mathrm{ad}x_j}[-t^\beta_{ki}, t^\alpha_{ij}],$$

$$[y_i, f(\mathrm{ad}x_j)(t^\alpha_{jk})] = \sum_{\beta\in\Gamma} \frac{f(\mathrm{ad}x_j) - f(\mathrm{ad}x_i + \mathrm{ad}x_j)}{-\mathrm{ad}x_i}[-t^\beta_{ij}, t^\alpha_{jk}],$$

$$[y_j, f(\mathrm{ad}x_k)(t^\alpha_{ki})] = \sum_{\beta\in\Gamma} \frac{f(-\mathrm{ad}x_i - \mathrm{ad}x_j) - f(-\mathrm{ad}x_i)}{-\mathrm{ad}x_j}[-t^\beta_{jk}, t^\alpha_{ki}].$$

It follows that the l.h.s. of (CDYBE) is now

$$\sum_{\alpha,\beta\in\Gamma} \Big(k_\alpha(-\mathrm{ad}x_j, z_{ij})k_\beta(-\mathrm{ad}x_k, z_{ik}) - k_\alpha(\mathrm{ad}x_i, z_{ij})k_{\beta-\alpha}(-\mathrm{ad}x_k, z_{jk})$$
$$+k_\beta(\mathrm{ad}x_i, z_{ik})k_{\beta-\alpha}(\mathrm{ad}x_j, z_{jk}) + \frac{k_{\beta-\alpha}(\mathrm{ad}x_j, z_{jk}) - k_{\beta-\alpha}(\mathrm{ad}x_i + \mathrm{ad}x_j, z_{jk})}{\mathrm{ad}x_i}$$
$$+\frac{k_\beta(\mathrm{ad}x_i, z_{ik}) - k_\beta(\mathrm{ad}x_i + \mathrm{ad}x_j, z_{ik})}{\mathrm{ad}x_j} - \frac{k_\alpha(\mathrm{ad}x_i, z_{ij}) - k_\alpha(-\mathrm{ad}x_j, z_{ij})}{\mathrm{ad}x_i + \mathrm{ad}x_j}\Big)[t^\alpha_{ij}, t^\beta_{ik}]\,,$$

and thus (CDYBE) follows from the identity

$$k_\alpha(-v,z)k_\beta(u+v,z') - k_\alpha(u,z)k_{\beta-\alpha}(u+v,z'-z) + k_\beta(u,z')k_{\beta-\alpha}(v,z'-z)$$
$$+\frac{k_{\beta-\alpha}(v,z'-z) - k_{\beta-\alpha}(u+v,z'-z)}{u} + \frac{k_\beta(u,z') - k_\beta(u+v,z')}{v}$$
$$-\frac{k_\alpha(u,z) - k_\alpha(-v,z)}{u+v} = 0\,.$$

This last identity can be written as

$$\Big(k_\alpha(-v,z) - \frac{1}{v}\Big)\Big(k_\beta(u+v,z') + \frac{1}{u+v}\Big) - \Big(k_\alpha(u,z) + \frac{1}{u}\Big)\Big(k_{\beta-\alpha}(u+v,z'-z) + \frac{1}{u+v}\Big)$$
$$+\Big(k_\beta(u,z') + \frac{1}{u}\Big)\Big(k_{\beta-\alpha}(v,z'-z) + \frac{1}{v}\Big) = 0\,, \tag{3}$$

which (taking into account that $k_\alpha(x,z) + (1/x) = e^{-2\pi\,\mathrm{i}\,ax}\,(k(x,z-\tilde\alpha) + (1/x))$) is a consequence of equation (3) of [6]. □

We have therefore proved:

Theorem 1.11. *$\nabla_{\tau,n,\Gamma}$ is a flat connection on $\mathcal{P}_{\tau,n,\Gamma}$, and its image under $\hat{\mathfrak{t}}^\Gamma_{1,n} \to \bar{\hat{\mathfrak{t}}}^\Gamma_{1,n}$ is the pull-back of a flat connection $\bar\nabla_{\tau,n,\Gamma}$ on $\bar{\mathcal{P}}_{\tau,n,\Gamma}$.* □

2. Lie algebras of derivations and associated groups

2.1. **The Lie algebras $\tilde{\mathfrak{d}}_0^\Gamma$ and $\tilde{\mathfrak{d}}^\Gamma$.** Let $\mathfrak{f}_\Gamma$ be the free Lie algebra with generators x, t^α ($\alpha \in \Gamma$). Let $p, q > 0$. We define $\tilde{\mathfrak{d}}_0^{p,q}$ to be the subspace of $\mathfrak{f}_\Gamma \oplus (\mathfrak{f}_\Gamma)^{\oplus|\Gamma|}$ consisting of elements

$$(D, C), \text{ where } C = (C_\alpha)_{\alpha\in\gamma},$$

such that $\deg_x(D) + \deg_t(D) = \deg_x(C_\alpha) + \deg_t(C_\alpha) = p$ and $\deg_t(D) - 1 = \deg_t(C_\alpha) = q$ for every $\alpha \in \Gamma$, and that satisfy the following of linear equations:

(i) $C_\alpha(x, t^\beta) = C_{-\alpha}(-x, t^{-\beta})$ in $\mathfrak{f}_\Gamma$,
(ii) $[x, D(x, t^\beta)] + \sum_\alpha [t^\alpha, C_\alpha(x, t^\beta)] = 0$ in $\mathfrak{f}_\Gamma$,
(iii) $[D(x_1, t_{13}^\beta), y_2] + c.p.(1,2,3) = 0$ in $\mathfrak{t}_{1,3}^\Gamma$,
(iv) $[D(x_1, t_{12}^\beta) + D(x_1, t_{13}^\beta) - [C_\alpha(x_2, t_{23}^\beta), y_1], t_{23}^\alpha] = 0$ in $\mathfrak{t}_{1,3}^\Gamma$,
(v) $[C_\alpha(x_1, t_{12}^\gamma), t_{13}^{\alpha+\beta} + t_{23}^\beta] + [t_{13}^{\alpha+\beta}, C_{\alpha+\beta}(x_1, t_{13}^\gamma)] + [t_{23}^\beta, C_\beta(x_2, t_{23}^\gamma)]$ commutes with t_{12}^α in $\mathfrak{t}_{1,3}^\Gamma$.

Remark that (i) and (ii) imply another relation

(vi) $D(x, t^\beta) = -D(-x, t^{-\beta})$,

which is very useful for computations. Then $\tilde{\mathfrak{d}}_0^\Gamma := \oplus_{p,q}(\tilde{\mathfrak{d}}_0^\Gamma)^{p,q}$.

We then define a Lie bracket $\langle , \rangle$ on $\mathfrak{f}_\Gamma \oplus (\mathfrak{f}_\Gamma)^{\oplus|\Gamma|}$ as follows:

$$\langle (D, C), (D', C') \rangle := (\delta_C(D') - \delta_{C'}(D), [C, C'] + \delta_C(C') - \delta_{C'}(C)) ,$$

where $\delta_C \in \mathrm{Der}(\mathfrak{f}_\Gamma)$ is the derivation

- $x \mapsto 0$, $t^\alpha \mapsto [t^\alpha, C_\alpha]$,
- δ_C acts on $(\mathfrak{f}_\Gamma)^{\oplus|\Gamma|}$ componentwise on a direct sum : $\delta_C(C')_\alpha = \delta_C(C'_\alpha)$,
- the bracket is understood componentwise as well: $[C, C']_\alpha = [C_\alpha, C'_\alpha]$.

We let the reader check that $\tilde{\mathfrak{d}}_0^\Gamma$ is stable under $\langle , \rangle$, and becomes a bigraded Lie algebra[1].

We now define $\tilde{\mathfrak{d}}^\Gamma$ as the quotient of the free product $\tilde{\mathfrak{d}}_0^\Gamma * \mathfrak{sl}_2$ by the relations $[\tilde{e}, (D, C)] = 0$, $[\tilde{h}, (D, C)] = (p - q)(D, C)$, and $(\mathrm{ad}^p \tilde{f})(D, C) = 0$ if $(D, C) \in \tilde{\mathfrak{d}}_0^\Gamma$ is homogeneous of bidegree (p, q). Here

$$\tilde{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tilde{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \tilde{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form the standard basis of $\mathfrak{sl}_2$. If we respectively give degree $(1, -1)$, $(0, 0)$ and $(-1, 1)$ to $\tilde{e}$, $\tilde{h}$ and $\tilde{f}$ then $\tilde{\mathfrak{d}}^\Gamma$ becomes $\mathbb{Z}^2$-graded.

We then define $\tilde{\mathfrak{d}}_+^\Gamma := \ker(\tilde{\mathfrak{d}}^\Gamma \to \mathfrak{sl}_2)$, which is $(\mathbb{Z}_{>0})^2$-graded. One observes that it is positively graded and finite dimensional in each degree. Thus, it is a direct sum of finite dimensional $\mathfrak{sl}_2$-modules.

[1] The proof is straightforward but quite long. We do not give it since we do use another simpler Lie algebra below.

2.2. **The Lie algebras $\mathfrak{d}_0^\Gamma$ and $\mathfrak{d}^\Gamma$.** We write $\mathfrak{d}_0^\Gamma$ for the free bigraded Lie algebra generated by $\delta_{s,\gamma}$'s ($s\geq 0$, $\gamma\in\Gamma$) in degree $(s+1,s)$ with relations

$$\delta_{s,\gamma}=(-1)^s\delta_{s,-\gamma},$$

for all $s\geq 0$ and $\gamma\in\Gamma$.

We then define $\mathfrak{d}^\Gamma$ as the quotient of the free product $\mathfrak{d}_0^\Gamma*\mathfrak{sl}_2$ by the relations $[\tilde e,\delta_{s,\gamma}]=0$, $[\tilde h,\delta_{s,\gamma}]=s\delta_{s,\gamma}$ and $\mathrm{ad}^{s+1}(\tilde f)(\delta_{s,\gamma})=0$; and $\mathfrak{d}_+^\Gamma$ as the kernel of $\mathfrak{d}^\Gamma\to\mathfrak{sl}_2$. As above, we have $\mathfrak{d}^\Gamma=\mathfrak{d}_+^\Gamma\rtimes\mathfrak{sl}_2$, and $\mathfrak{d}_+^\Gamma$ is positively graded (actually $(\mathbb{Z}_{>0})^2$-graded).

We now give examples of elements in $\tilde{\mathfrak{d}}_0^\Gamma$ that are of some use below. For any $s\in\mathbb{N}$ and $\gamma\in\Gamma$, we set

$$D_{s,\gamma}:=\sum_{p+q=s-1}\sum_{\beta\in\Gamma}[(\mathrm{ad}x)^pt^{\beta-\gamma},(-\mathrm{ad}x)^qt^\beta]$$

and

$$(C_{s,\gamma})_\alpha:=(\mathrm{ad}x)^st^{\alpha-\gamma}+(-\mathrm{ad}x)^st^{\alpha+\gamma}.$$

Observe that $(D_{s,\gamma},C_{s,\gamma})=(-1)^s(D_{s,-\gamma},C_{s,-\gamma})$.

The following result tells us that $\delta_{s,\gamma}\mapsto(D_{s,\gamma},C_{s,\gamma})$ defines a bigraded Lie algebra morphism $\mathfrak{d}_0^\Gamma\to\tilde{\mathfrak{d}}_0^\Gamma$, that obviously extends to $\mathfrak{d}^\Gamma\to\tilde{\mathfrak{d}}^\Gamma$.

Proposition 2.1. $(D_{s,\gamma},C_{s,\gamma})\in(\tilde{\mathfrak{d}}_0^\Gamma)^{s+1,1}$.

Proof. First observe that relations (i) and (vi) are obviously satisfied.

To prove (ii) it suffices to notice that in the free Lie algebra with three generators x,t_1,t_2 we have

$$[t_1,(\mathrm{ad}\,x)^st_2]+[t_2,(-\mathrm{ad}\,x)^st_1]=\sum_{p+q=s-1}[x,[(-\mathrm{ad}\,x)^qt_1,(\mathrm{ad}\,x)^pt_2]].$$

Let us prove (iii). In $\mathfrak{t}_{1,n}^\Gamma$ we compute for $\#\{i,j,k\}=3$,

$$[y_k,(\mathrm{ad}\,x_i)^pt_{ij}^\alpha]=-\sum_{k+l=p-1}\sum_\beta(\mathrm{ad}\,x_i)^k[t_{ik}^\beta,(\mathrm{ad}\,x_i)^lt_{ij}^\alpha]$$

$$=\sum_{k+l=p-1}\sum_\beta(\mathrm{ad}\,x_i)^k(-\mathrm{ad}\,x_j)^l[t_{ik}^\beta,t_{kj}^{\alpha-\beta}]=\sum_{k+l=p-1}\sum_\beta[(\mathrm{ad}\,x_i)^kt_{ik}^\beta,(-\mathrm{ad}\,x_j)^lt_{kj}^{\alpha-\beta}].$$

Therefore, in $\mathfrak{t}_{1,3}^\Gamma$, we have

$$[y_1,D(x_2,t_{23}^\beta)]=\sum_{k+l+m=s-2}\sum_{\alpha,\beta}[[(\mathrm{ad}\,x_2)^kt_{21}^\beta,(-\mathrm{ad}\,x_3)^lt_{13}^{\alpha-\beta-\gamma}],(-\mathrm{ad}\,x_2)^mt_{23}^\alpha]$$

$$+\sum_{k+l+m=s-2}\sum_{\alpha,\beta}(-1)^{l+m+1}[(\mathrm{ad}\,x_2)^kt_{23}^{\alpha-\gamma},[(\mathrm{ad}\,x_2)^lt_{21}^\beta,(-\mathrm{ad}\,x_3)^mt_{13}^{\alpha-\beta}]].$$

Then $[y_1,D(x_2,t_{23}^\beta)]+c.p.(1,2,3)=0$ follows from the Jacobi identity.

Let us prove (iv). On the one hand we have

$$[D(x_1,t_{12}^\beta)+D(x_1,t_{13}^\beta),t_{23}^\alpha]=$$

$$=\sum_{p+q=s-1}\sum_{\beta\in\Gamma}[[(\mathrm{ad}x_1)^pt_{12}^{\beta-\gamma},(-\mathrm{ad}x_1)^qt_{12}^\beta]+[(\mathrm{ad}x_1)^pt_{13}^{\beta-\gamma},(-\mathrm{ad}x_1)^qt_{13}^\beta],t_{23}^\alpha]$$

$$=-\sum_{p+q=s-1}\sum_{\beta\in\Gamma}\Big([[(\mathrm{ad}x_1)^p[t_{13}^{\alpha+\beta-\gamma},t_{23}^\alpha],(-\mathrm{ad}x_1)^qt_{12}^\beta]+[(\mathrm{ad}x_1)^pt_{12}^{\beta-\gamma},(-\mathrm{ad}x_1)^q[t_{13}^{\alpha+\beta},t_{23}^\alpha]]$$

$$+[(\mathrm{ad}x_1)^p[t_{12}^{\beta-\gamma},t_{23}^{\alpha}],(-\mathrm{ad}x_1)^q t_{13}^{\alpha+\beta}]+[(\mathrm{ad}x_1)^p t_{13}^{\alpha+\beta-\gamma},(-\mathrm{ad}x_1)^q[t_{12}^{\beta},t_{23}^{\alpha}]]\big)$$
$$=[t_{23}^{\alpha},\sum_{p+q=s-1}\sum_{\beta\in\Gamma}(\mathrm{ad}x_1)^p[t_{13}^{\alpha+\beta-\gamma},(-\mathrm{ad}x_1)^q t_{12}^{\beta}]+(\mathrm{ad}x_1)^p[t_{12}^{\beta},(-\mathrm{ad}x_1)^q t_{13}^{\alpha+\beta+\gamma}]]$$
$$=[t_{23}^{\alpha},\sum_{p+q=s-1}\sum_{\beta\in\Gamma}(\mathrm{ad}x_2)^p(-\mathrm{ad}x_3)^q[t_{13}^{\alpha+\beta-\gamma}+(-1)^s t_{13}^{\alpha+\beta+\gamma},t_{12}^{\beta}]].$$

On the other hand, we have

$$[C_\alpha(x_2,t_{23}^{\beta}),y_1]=[(\mathrm{ad}x_2)^s t_{23}^{\alpha-\gamma}+(-\mathrm{ad}x_2)^s t_{23}^{\alpha+\gamma},y_1]$$
$$=-\sum_{p+q=s-1}\sum_{\beta\in\Gamma}(\mathrm{ad}\,x_2)^p(-\mathrm{ad}\,x_3)^q[t_{12}^{\beta},t_{31}^{\alpha+\beta-\gamma}+(-1)^s t_{31}^{\alpha+\beta+\gamma}].$$

Therefore (iv) is satisfied.

Let us prove (v). We have

$$[C_\alpha(x_1,t_{12}^{\gamma}),t_{13}^{\alpha+\beta}+t_{23}^{\beta}]=[(\mathrm{ad}\,x_1)^s t_{12}^{\alpha-\gamma}+(-\mathrm{ad}\,x_1)^s t_{12}^{\alpha+\gamma},t_{13}^{\alpha+\beta}+t_{23}^{\beta}]$$
$$=(\mathrm{ad}\,x_2)^s[t_{12}^{\alpha+\gamma}+(-1)^s t_{12}^{\alpha-\gamma},t_{13}^{\alpha+\beta}]+(\mathrm{ad}\,x_1)^s[t_{12}^{\alpha-\gamma}+(-1)^s t_{12}^{\alpha+\gamma},t_{23}^{\beta}]$$
$$=(\mathrm{ad}\,x_2)^s[t_{13}^{\alpha+\beta},t_{23}^{\beta-\gamma}+(-1)^s t_{23}^{\beta+\gamma}]+(\mathrm{ad}\,x_1)^s[t_{23}^{\beta},t_{13}^{\alpha+\beta-\gamma}+(-1)^s t_{13}^{\alpha+\beta+\gamma}].$$

Therefore, by defining $A=t_{23}^{\beta-\gamma}+(-1)^s t_{23}^{\beta+\gamma}$ and $B=t_{13}^{\alpha+\beta-\gamma}+(-1)^s t_{13}^{\alpha+\beta+\gamma}$ we have

$$[t_{12}^{\alpha},[C_\alpha(x_1,t_{12}^{\gamma}),t_{13}^{\alpha+\beta}+t_{23}^{\beta}]]=[t_{12}^{\alpha},[t_{13}^{\alpha+\beta},(\mathrm{ad}\,x_2)^s A]+[t_{23}^{\beta},(\mathrm{ad}\,x_1)^s B]]$$
$$=[[t_{12}^{\alpha},t_{13}^{\alpha+\beta}],(-\mathrm{ad}\,x_3)^s A]+[t_{13}^{\alpha+\beta},(-\mathrm{ad}\,x_3)^s[t_{12}^{\alpha},A]]$$
$$+[[t_{12}^{\alpha},t_{23}^{\beta}],(-\mathrm{ad}\,x_3)^s B]+[t_{23}^{\beta},(-\mathrm{ad}\,x_3)^s[t_{12}^{\alpha},B]]$$
$$=[[t_{23}^{\beta},t_{12}^{\alpha}],(-\mathrm{ad}\,x_3)^s A]+[t_{13}^{\alpha+\beta},(-\mathrm{ad}\,x_3)^s[B,t_{12}^{\alpha}]]$$
$$+[[t_{13}^{\alpha+\beta},t_{12}^{\alpha}],(-\mathrm{ad}\,x_3)^s B]+[t_{23}^{\beta},(-\mathrm{ad}\,x_3)^s[A,t_{12}^{\alpha}]]$$
$$=[[t_{23}^{\beta},(\mathrm{ad}\,x_2)^s A]+[t_{13}^{\alpha+\beta},(\mathrm{ad}\,x_1)^s B],t_{12}^{\alpha}].$$

This finishes the proof. □

Remark 2.2. We do not know if $\mathfrak{d}_0^{\Gamma}\to\tilde{\mathfrak{d}}_0^{\Gamma}$ is injective or not.

2.3. Derivations of $\mathfrak{t}_{1,n}^{\Gamma}$ and $\bar{\mathfrak{t}}_{1,n}^{\Gamma}$.

Lemma 2.3. *We have a bigraded Lie algebra morphism $\tilde{\mathfrak{d}}_0^{\Gamma}\to\mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$, taking $(D,C)\in\tilde{\mathfrak{d}}_0^{\Gamma}$ to the derivation $\xi_{(D,C)}$:*

$$x_i\longmapsto 0,$$
$$y_i\longmapsto\sum_{j:j\neq i}D(x_i,t_{ij}^{\beta}),$$
$$t_{ij}^{\alpha}\longmapsto[t_{ij}^{\alpha},C_\alpha(x_i,t_{ij}^{\beta})].$$

This induces a bigraded Lie algebra morphism $\tilde{\mathfrak{d}}_0^{\Gamma}\to\mathrm{Der}(\bar{\mathfrak{t}}_{1,n}^{\Gamma})$.

Proof. We have to prove that defining relations of $\mathfrak{t}_{1,n}^{\Gamma}$ are preserved by $\xi := \xi_{(D,C)}$. First observe that relations $[x_i, x_j] = [x_i + x_j, t_{ij}^{\alpha}] = [x_i, t_{jk}^{\alpha}] = [t_{ij}^{\alpha}, t_{kl}^{\alpha}] = 0$ are obviously preserved. Then conditions (i) and (ii) respectively imply that $t_{ij}^{\alpha} = t_{ji}^{-\alpha}$ and $[x_i, y_j] = \sum_{\alpha} t_{ij}^{\alpha}$ are preserved. Condition (vi) implies that $[x_i, y_j] = [x_j, y_i]$ is preserved, and (vi) together with (iii) imply that $[y_i, y_j] = 0$ is preserved. Therefore it follows from the centrality of $\sum_i x_i$ and $\xi(\sum_i x_i) = 0$ that

$$\xi([x_i, y_i]) = \xi(-\sum_{j:j\neq i}[x_j, y_i]) = \xi(\sum_{j;j\neq i}\sum_{\alpha} t_{ij}^{\alpha}).$$

Condition (iv) ensures that $[y_i, t_{jk}^{\alpha}] = 0$ is preserved, and together with (vi) it implies that $[y_i + y_j, t_{ij}^{\alpha}] = 0$ is preserved. Finally condition (v) implies that the twisted infinitesimal braid relations are preserved, and the first part of the statement follows.

For the second part of the statement it remains to prove that the centrality of $\sum_i y_i$ is preserved. This follows directly from the identity $\xi(\sum_i y_i) = 0$ that we now prove. Relation (vi) implies that for any $i \neq j$ one has $D(x_i, t_{ij}^{\beta}) = -D(-x_i, t_{ij}^{-\beta}) = -D(x_j, t_{ji}^{\beta})$ in $\mathfrak{t}_{1,n}^{\Gamma}$ (the last equality happens since $\deg_t(D) = \deg_t(C_\alpha) + 1 > 0$), and hence

$$\xi(\sum_i y_i) = \sum_{i\neq j} D(x_i, t_{ij}^{\beta}) = \sum_{i<j} D(x_i, t_{ij}^{\beta}) - \sum_{j<i} D(x_j, t_{ji}^{\beta}) = 0\,.$$

We are done (the compatibility with bracket and grading are easy to check).

The last part of the statementis a consequence of the fact that $\xi(\sum_i y_i) = \xi(\sum_i x_i) = 0$, that we have already proved. □

We now prove that this morphism extends to a Lie algebra morphism $\tilde{\mathfrak{d}}^{\Gamma} \to \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$:

Proposition 2.4. *We have a bigraded Lie algebra morphism $\tilde{\mathfrak{d}}^{\Gamma} \to \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$ taking $(D,C) \in \tilde{\mathfrak{d}}_0^{\Gamma}$ to $\xi_{(D,C)}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sl}_2$ to the derivation*

$$\xi_g : t_{ij}^{\alpha} \mapsto 0, \begin{pmatrix} x_i & y_i \end{pmatrix} \mapsto \begin{pmatrix} x_i & y_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This induces a bigraded Lie algebra morphism $\tilde{\mathfrak{d}}^{\Gamma} \to \mathrm{Der}(\bar{\mathfrak{t}}_{1,n}^{\Gamma})$.

In what follows we write $\mathbf{d} := \tilde{h}$, $\mathbf{X} := \tilde{e}$ and $\Delta_0 := \tilde{f}$ and $\tilde{\mathbf{d}} := \xi_{\tilde{h}}$, $\tilde{\mathbf{X}} := \xi_{\tilde{e}}$ and $\tilde{\Delta}_0 := \xi_{\tilde{f}}$.

Proof. It is obvious that for any $g, g' \in \mathfrak{sl}_2$, ξ_g defines a derivation of the same degree of $\mathfrak{t}_{1,n}^{\Gamma}$, and that $\xi_{[g,g']} = [\xi_g, \xi_{g'}]$. Hence we have a bigraded Lie algebra morphism $\mathfrak{sl}_2 * \tilde{\mathfrak{d}}_0^{\Gamma} \to \mathrm{Der}(\mathfrak{t}_{1,n}^{\Gamma})$. Let us prove that it factorizes through the quotient $\tilde{\mathfrak{d}}^{\Gamma}$.

It is relatively clear that $[\tilde{\mathbf{X}}, \xi_{(D,C)}] = 0$ and $[\tilde{\mathbf{d}}, \xi_{(D,C)}] = (p-q)(D,C)$ if $(D,C) \in (\tilde{\mathfrak{d}}_0^{\Gamma})^{p,q}$. Thus it remains to prove that $(\mathrm{ad}\,\tilde{\Delta}_0)^p(\xi_{(D,C)}) = 0$ if $(D,C) \in (\tilde{\mathfrak{d}}_0^{\Gamma})^{p,q}$. We do this now. Let us write $\xi := \xi_{(D,C)}$ and $A := (\mathrm{ad}\,\tilde{\Delta}_0)^p(\xi)$. Then after an easy computation one obtains on generators:

$$\begin{aligned} A(x_i) &= -p\tilde{\Delta}_0^{p-1}\xi(y_i) = -p\tilde{\Delta}_0^{p-1}(\sum_{j:j\neq i} D(x_i, t_{ij}^\beta)), \\ A(y_i) &= \tilde{\Delta}_0^p\xi(y_i) = \tilde{\Delta}_0^p(\sum_{j:j\neq i} D(x_i, t_{ij}^\beta)), \\ A(t_{ij}^\alpha) &= \tilde{\Delta}_0^p\xi(t_{ij}^\alpha) = \tilde{\Delta}_0^p([t_{ij}^\alpha, C_\alpha(x_i, t_{ij}^\beta)]). \end{aligned}$$

Finally remark that we have an increasing filtration on $\mathfrak{t}_{1,n}^\Gamma$ defined by $\deg(x_i) = 1$ and $\deg(t_{ij}^\alpha) = \deg(y_i) = 0$. Δ_0 decreases the degree by 1 and vanishes on degree zero elements. The result then follows from the fact that $\deg_x(C_\alpha) = p - q < p$ and $\deg_x(D) = p - q - 1 < p - 1$. □

Now composing with $\mathfrak{d}_0^\Gamma \to \tilde{\mathfrak{d}}_0^\Gamma$ (resp. $\mathfrak{d}^\Gamma \to \tilde{\mathfrak{d}}^\Gamma$) one obtains a Lie algebra morphism $\mathfrak{d}_0^\Gamma \to \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)$ (resp. $\mathfrak{d}^\Gamma \to \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)$). We write $\xi_{s,\gamma} := \xi_{(D_{s,\gamma}, C_{s,\gamma})}$ for the image of $\delta_{s,\gamma}$. We then have $\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}^\Gamma = (\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma) \rtimes \mathfrak{sl}_2$, with $\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma$ positively graded (since both $\mathfrak{t}_{1,n}^\Gamma$ and $\mathfrak{d}_+^\Gamma$ are $(\mathbb{Z}_{\geq 0})^2$-graded) and a sum of finite dimensional $\mathfrak{sl}_2$-modules. Therefore we can construct the semi-direct product group

$$\mathbf{G}_n^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C}), \tag{4}$$

where $\exp(\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma)^\wedge$ is the exponential group associated to the degree completion of $\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma$.

Similarly, we define $\bar{\mathbf{G}}_n^\Gamma := \exp(\bar{\mathfrak{t}}_{1,n}^\Gamma \rtimes \mathfrak{d}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C})$.

Notice that one can also define semi-direct product groups $\tilde{\mathbf{G}}_n^\Gamma := \exp(\mathfrak{t}_{1,n}^\Gamma \rtimes \tilde{\mathfrak{d}}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C})$ and $\tilde{\bar{\mathbf{G}}}_n^\Gamma := \exp(\bar{\mathfrak{t}}_{1,n}^\Gamma \rtimes \tilde{\mathfrak{d}}_+^\Gamma)^\wedge \rtimes \mathrm{SL}_2(\mathbb{C})$. We therefore have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{G}_n^\Gamma & \longrightarrow & \tilde{\mathbf{G}}_n^\Gamma \\ \downarrow & & \downarrow \\ \bar{\mathbf{G}}_n^\Gamma & \longrightarrow & \tilde{\bar{\mathbf{G}}}_n^\Gamma. \end{array} \tag{5}$$

Lemma 2.5. *The kernel of* $\tilde{\mathfrak{d}}_0^\Gamma \to \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)$ *($n \geq 2$) is the space of elements* $(0, C)$ *for which* C_α *is proportional to* t^α*, and* $\ker(\mathfrak{d}_0^\Gamma \to \mathrm{Der}(\mathfrak{t}_{1,n}^\Gamma)) = \mathbb{C}\delta_{0,0}$.

Proof. Let us first prove it for $n = 2$. Recall that $\bar{\mathfrak{t}}_{1,2}^\Gamma = \mathfrak{t}_{1,2}^\Gamma/(x_1 + x_2, y_1 + y_2)$, so it is the Lie algebra generated by x (the class of x_1), y (the class of y_1) and t^α's (classes of t_{12}^α's) with the relation $[x, y] = \sum_{\alpha \in \Gamma} t^\alpha$. Then the derivation $\xi_{(D,C)}$ associated to $(D, C) \in \tilde{\mathfrak{d}}_0^\Gamma$ is given by

$$x \mapsto 0, y \mapsto D(x, t^\beta), t^\alpha \mapsto [t^\alpha, C_\alpha(x, t^\beta)].$$

This derivation vanishes if and only if $D = 0$ and C_α is proportional to t^α. Finally, the result for $n \geq 2$ follows from the fact that

$$\xi_{(D,C)}^{(2)} = (u \mapsto u^{1,2,\varnothing,\ldots,\varnothing}) \circ \xi_{(D,C)}^{(n)} \circ (u \mapsto u^{1,\ldots,n}),$$

where $\xi_{(D,C)}^{(n)}$ denotes the derivation of $\mathfrak{t}_{1,n}^\Gamma$ associated to (D, C). □

2.4. **Comparison morphisms.** Let $\rho : \Gamma_1 \to \Gamma_2$ a group morphism. We have a comparison morphism $\tilde{\mathfrak{d}}_0^{\Gamma_1} \to \tilde{\mathfrak{d}}_0^{\Gamma_2}$, $(D, C) \mapsto (D^\rho, C^\rho)$ defined by

$$D^\rho := D\left(x, \sum_{\gamma \in \operatorname{coker}(\rho)} \frac{t^{\rho(\beta)+\gamma}}{\#\ker(\rho)}\right), \; (C^\rho)_\alpha := C_\alpha\left(x, \sum_{\gamma \in \operatorname{coker}(\rho)} \frac{t^{\rho(\beta)+\gamma}}{\#\ker(\rho)}\right).$$

When ρ is not surjective it depends on the choice of a section $\operatorname{coker}(\rho) \to \Gamma_2$. It extends to $\tilde{\mathfrak{d}}^{\Gamma_1} \to \tilde{\mathfrak{d}}^{\Gamma_2}$ by sending the generators of $\mathfrak{sl}_2$ to themselves. These comparison morphisms are compatible with the morphisms $\tilde{\mathfrak{d}}^{\Gamma_i} \to \operatorname{Der}(\mathfrak{t}_{1,n}^{\Gamma_i})$, for $i = 1, 2$. Namely, there is a commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{d}}^{\Gamma_1} \ltimes \mathfrak{t}_{1,n}^{\Gamma_1} & \longrightarrow & \mathfrak{t}_{1,n}^{\Gamma_1} \\ \downarrow & & \downarrow \\ \tilde{\mathfrak{d}}^{\Gamma_2} \ltimes \mathfrak{t}_{1,n}^{\Gamma_2} & \longrightarrow & \mathfrak{t}_{1,n}^{\Gamma_2} \end{array}$$

Finally, we have comparison morphisms for the corresponding groups that fit into a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbf{G}}_n^{\Gamma_1} & \longrightarrow & \tilde{\mathbf{G}}_n^{\Gamma_2} \\ \downarrow & & \downarrow \\ \tilde{\bar{\mathbf{G}}}_n^{\Gamma_1} & \longrightarrow & \tilde{\bar{\mathbf{G}}}_n^{\Gamma_2}. \end{array} \tag{6}$$

Notice that the image of $(D_{s,\gamma}, C_{s,\gamma})$ under a comparison morphism is no longer of this form except if ρ is injective. In this case (and in this case only) we have a comparison morphism $\mathfrak{t}_{1,n}^{\Gamma_1} \rtimes \mathfrak{d}^{\Gamma_1} \to \mathfrak{t}_{1,n}^{\Gamma_2} \rtimes \mathfrak{d}^{\Gamma_2}$ taking x_i's, y_i's, $\mathbf{d}$, $\mathbf{X}$ and Δ_0 to themselves, and t_{ij}^α to $\sum_{\beta \in \operatorname{coker}(\rho)} t_{ij}^{\rho(\alpha)+\beta}$ and $\delta_{s,\gamma}$ to $\sum_{\beta \in \operatorname{coker}(\rho)} \delta_{s,\rho(\gamma)+\beta}$. In particular we have a canonical natural inclusion $\mathbf{G}_n^0 \to \mathbf{G}_n^\Gamma$ (which descends to an inclusion $\bar{\mathbf{G}}_n^0 \to \bar{\mathbf{G}}_n^\Gamma$).

3. Bundles with flat connections on moduli spaces

3.1. **On some subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and moduli spaces.** Let $M, N \geq 1$ two integers. Consider the group $\Gamma := \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and consider the following (finite index) subgroup of $\mathrm{SL}_2(\mathbb{Z})$:

$$\mathrm{SL}_2^\Gamma(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv 1 \bmod M, d \equiv 1 \bmod N, b \equiv 0 \bmod N \text{ and } c \equiv 0 \bmod M \right\}.$$

We write $Y(\Gamma)$ for the set of equivalences classes of pairs (E, ϕ) where E is an elliptic curve and $\phi : \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to E$ is an injective group morphism that is orientation preserving i.e. such that the basis $(\frac{d}{dt}_{|t=0}(t\phi(\bar{1}, \bar{0})), \frac{d}{dt}_{|t=0}(t\phi(\bar{0}, \bar{1}))$ of $T_0 E$ is direct. Then, one can see that $Y(\Gamma) = \mathrm{SL}_2^\Gamma(\mathbb{Z}) \backslash \mathfrak{H}$ and therefore inherits the structure of a complex orbifold.

Remark 3.1. The biggest congruence subgroup on which the connection we will construct in this section is well defined and flat is the subgroup $\tilde{\mathrm{SL}}_2^\Gamma(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $Mb \equiv 0 \bmod N$ and $Nc \equiv 0 \bmod M$. Nevertheless, in order to retrieve the twisted elliptic KZB connection defined at the level of configuration spaces, it suffices to consider the usual congruence subgroup $\mathrm{SL}_2^\Gamma(\mathbb{Z}) \subset \tilde{\mathrm{SL}}_2^\Gamma(\mathbb{Z})$.

Recall the following standard group actions:

- The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the left on $\mathbb{C}^n \times \mathfrak{H}$:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * (\mathbf{z}|\tau) := \left(\frac{\mathbf{z}}{c\tau+d}\Big|\frac{a\tau+b}{c\tau+d}\right).$$
This obviously descends to a left action of $\mathrm{SL}_2(\mathbb{Z})$ on $(\mathbb{C}^n \times \mathfrak{H})/\mathbb{C}$, where $\mathbb{C}$ acts diagonally on $\mathbb{C}^n$: $u \cdot (\mathbf{z}|\tau) := (\mathbf{z} + u\sum_i \delta_i|\tau)$.
- The group $(\mathbb{Z}^n)^2$ acts on the left on $\mathbb{C}^n \times \mathfrak{H}$:
$$(\mathfrak{m},\mathfrak{n}) * (\mathbf{z}|\tau) := (\mathbf{z} + \mathfrak{m} + \tau\mathfrak{n}|\tau).$$
It obvioulsy descends to a left action of $(\mathbb{Z}^n)^2/\mathbb{Z}^2$ on $\mathbb{C}^n \times \mathfrak{H}/\mathbb{C}$, where $\mathbb{Z}^2$ is the diagonal subgroup in $(\mathbb{Z}^n)^2 = (\mathbb{Z}^2)^n$.
- Finally, there is a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $(m,n) \in \mathbb{Z}^2$ by automorphisms:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} n & m \end{pmatrix} \to \begin{pmatrix} n & m \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
We can thus form the semi-direct products $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ and $((\mathbb{Z}^n)^2/\mathbb{Z}^2) \rtimes \mathrm{SL}_2(\mathbb{Z})$.

A few observations are then in order:

- The above actions are compatible in the sense that we have a left action of $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}^n \times \mathfrak{H}$, which descends to an action of $\big((\mathbb{Z}^n)^2/\mathbb{Z}^2\big) \rtimes \mathrm{SL}_2(\mathbb{Z})$ on $(\mathbb{C}^n \times \mathfrak{H})/\mathbb{C}$, where $\mathbb{Z}^2$ is embedded in $(\mathbb{Z}^n)^2$ via the diagonal map. One can think of translation by $\mathbb{C}$ as a left or right action as it commutes with the $((\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}))$-action.
- The action of $(\mathbb{Z}^n)^2$ preserves the subset
$$\mathrm{Diag}_{n,\Gamma} := \{(\mathbf{z}|\tau) \in \mathbb{C}^n \times \mathfrak{H} | \mathbf{z} \in \mathrm{Diag}_{\tau,n,\Gamma}\}.$$
- The action of the subgroup $\mathrm{SL}_2^\Gamma(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{Z})$ also preserves $\mathrm{Diag}_{n,\Gamma}$.

We are thus ready to define several variants of $Y(\Gamma)$ "with marked points":

- We define the quotient
$$\bar{\mathcal{M}}_{1,n}^\Gamma := (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^\Gamma(\mathbb{Z}) \backslash \big((\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}\big)/\mathbb{C}$$
and call it the *moduli space of* Γ*-structured elliptic curves with* n *ordered marked points.*
- It has a non-reduced variant
$$\mathbf{p} : \mathcal{M}_{1,n}^\Gamma := (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^\Gamma(\mathbb{Z}) \backslash \big((\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}\big) \twoheadrightarrow \bar{\mathcal{M}}_{1,n}^\Gamma.$$
- One can also define the *moduli space of* Γ*-structured elliptic curves with* n *unordered marked points*
$$\bar{\mathcal{M}}_{1,[n]}^\Gamma := \mathfrak{S}_n \backslash \bar{\mathcal{M}}_{1,n}^\Gamma$$
and its non-reduced variant
$$\mathcal{M}_{1,[n]}^\Gamma := \mathfrak{S}_n \backslash \mathcal{M}_{1,n}^\Gamma.$$

Remark 3.2. We have $\bar{\mathcal{M}}^{\Gamma}_{1,1} = \bar{\mathcal{M}}^{\Gamma}_{1,[1]} = Y(\Gamma)$, and $\mathcal{M}^{\Gamma}_{1,1} = \mathcal{M}^{\Gamma}_{1,[1]}$ is the universal curve over it. The fiber of $\mathcal{M}^{\Gamma}_{1,n} \to Y(\Gamma)$ (resp. $\bar{\mathcal{M}}^{\Gamma}_{1,n} \to Y(\Gamma)$) at (the class of) τ is precisely the twisted (resp. reduced twisted) configuration space $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ (resp. $\mathrm{C}(E_{\tau,\Gamma}, n, \Gamma)$). Moreover, the map

$$h : \bar{\mathcal{M}}^{\Gamma}_{1,2} \longrightarrow \bar{\mathcal{M}}^{\Gamma}_{1,1}$$

factors through (and is open in) $\mathcal{M}^{\Gamma}_{1,1}$. We can interpret $\bar{\mathcal{M}}^{\Gamma}_{1,2}$ as the Γ-punctured universal curve over $Y(\Gamma)$.

3.2. **Principal bundles over $\mathcal{M}^{\Gamma}_{1,n}$ and $\bar{\mathcal{M}}^{\Gamma}_{1,n}$.** In this paragraph, $\mathbf{G}^{\Gamma}_{n}$ is defined as in (4) and we define a principal $\mathbf{G}^{\Gamma}_{n}$-bundle $\mathcal{P}_{n,\Gamma}$ over $\mathcal{M}^{\Gamma}_{1,n}$ whose image under the natural morphism $\mathbf{G}^{\Gamma}_{n} \to \bar{\mathbf{G}}^{\Gamma}_{n}$ is the pull-back of a principal $\bar{\mathbf{G}}^{\Gamma}_{n}$-bundle $\bar{\mathcal{P}}_{n,\Gamma}$ over $\bar{\mathcal{M}}^{\Gamma}_{1,n}$. Let us fix the notation first: for $u \in \mathbb{C}^{\times}$ and $v, w_i \in \mathbb{C}$ $(i = 1, \dots, n)$,

$$u^{\mathbf{d}} := \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, e^{v\mathbf{X}} := \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

Since $[\mathbf{X}, x_i] = 0$ then it makes sense to define $e^{v\mathbf{X} + \sum_i w_i x_i} := e^{v\mathbf{X}} e^{\sum_i w_i x_i}$. In particular, we have $\mathrm{Ad}(u^{\mathbf{d}})(x_i) = u x_i$ and $\mathrm{Ad}(u^{\mathbf{d}})(y_i) = y_i/u$ $(\forall i)$, $\mathrm{Ad}(u^{\mathbf{d}})(\mathbf{X}) = u^2 \mathbf{X}$ and $\mathrm{Ad}(u^{\mathbf{d}})(\Delta_0) = \Delta_0/u^2$. Let $\pi : \mathbb{C}^n \times \mathfrak{H} \to \mathcal{M}_{1,n}$ be the canonical projection.

Proposition 3.3. *There exists a unique principal $\mathbf{G}^{\Gamma}_{n}$-bundle $\mathcal{P}_{n,\Gamma}$ over $\mathcal{M}^{\Gamma}_{1,n}$ for which a section on $U \subset \mathcal{M}^{\Gamma}_{1,n}$ is a function $f : \pi^{-1}(U) \to \mathbf{G}^{\Gamma}_{n}$ such that*

$$\begin{aligned} f(\mathbf{z} + \delta_i|\tau) &= f(\mathbf{z}|\tau), \\ f(\mathbf{z} + \tau\delta_i|\tau) &= e^{-2\pi \mathrm{i} x_i} f(\mathbf{z}|\tau), \\ f(\mathbf{z}, \tau + 1) &= f(\mathbf{z}|\tau), \\ f(\frac{\mathbf{z}}{\tau}| - \frac{1}{\tau}) &= \tau^{\mathbf{d}} e^{\frac{2\pi \mathrm{i}}{\tau}(\mathbf{X} + \sum_i z_i x_i)} f(\mathbf{z}|\tau). \end{aligned}$$

Moreover, the image of $\mathcal{P}_{n,\Gamma}$ under $\mathbf{G}^{\Gamma}_{n} \to \bar{\mathbf{G}}^{\Gamma}_{n}$ is the pull-back of a unique principal $\bar{\mathbf{G}}^{\Gamma}_{n}$-bundle $\bar{\mathcal{P}}_{n,\Gamma}$ over $\bar{\mathcal{M}}^{\Gamma}_{1,n}$ for which a section on $U \subset \bar{\mathcal{M}}^{\Gamma}_{1,n}$ is a function $f : (\mathbf{p} \circ \pi)^{-1}(U) \to \bar{\mathcal{M}}^{\Gamma}_{1,n}$ satisfying the above conditions (with x_i's replaced by $\bar{x}_i$'s) and such that $f(\mathbf{z} + v \sum_i \delta_i|\tau) = f(\mathbf{z}|\tau)$ for any $v \in \mathbb{C}$.

Proof. First recall that for $\Gamma = 0$ this is precisely [6, Proposition 3.4]. Then observe that we have an obvious map $\iota : \mathcal{M}^{\Gamma}_{1,n} \to \mathcal{M}^{0}_{1,n}$. Therefore we define $\mathcal{P}_{n,\Gamma}$ (resp. $\bar{\mathcal{P}}_{n,\Gamma}$) to be the image under the natural inclusion $\mathbf{G}^{0}_{n} \to \mathbf{G}^{\Gamma}_{n}$ (resp. $\bar{\mathbf{G}}^{0}_{n} \to \bar{\mathbf{G}}^{\Gamma}_{n}$) of $\iota^* \mathcal{P}_{n,0}$ (resp. $\iota^* \bar{\mathcal{P}}_{n,0}$).

We thus proved existence. Unicity is obvious. □

In other words, there exists a unique non-abelian 1-cocycle $(c_g)_{g \in (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})}$ on $\mathbb{C}^n \times \mathfrak{H}$ with values in $\mathbf{G}^{\Gamma}_{n}$ such that $c_{(\delta_i, 0)} = 1$, $c_{(0,\delta_i)} = e^{-2\pi \mathrm{i}\, x_i}$, $c_S = 1$ and

$$c_T(\mathbf{z}|\tau) = \tau^{\mathbf{d}} e^{(2\pi \mathrm{i}/\tau)(\mathbf{X} + \sum_j z_j x_j)} = e^{2\pi \mathrm{i}(\tau\mathbf{X} + \sum_j z_j x_j)} \tau^{\mathbf{d}},$$

where $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are the generators of $\mathrm{SL}_2(\mathbb{Z})$. Here *cocycle* means (as in [6]) that c_g's are holomorphic functions $\mathbb{C}^n \times \mathfrak{H} \to \mathbf{G}_n^\Gamma$ satisfying the cocycle condition $c_{gg'}(\mathbf{z}|\tau) = c_g(g' * (\mathbf{z}, \tau))c_{g'}(\mathbf{z}|\tau)$.

Remark 3.4. Notice that we do have a $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$-cocycle (since our bundle is define as the pull-back of a bundle on $\mathcal{M}_{1,1}^0$) but the cocycle defining $\mathcal{P}_{n,\Gamma}$ is its restriction to $(\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^\Gamma(\mathbb{Z})$.

3.3. **Connections on $\mathcal{P}_{n,\Gamma}$ and $\bar{\mathcal{P}}_{n,\Gamma}$.** A connection on $\mathcal{P}_{n,\Gamma}$ is the same as an equivariant connection on the trivial $\mathbf{G}_n^\Gamma$-bundle over $\mathbb{C}^n \times \mathfrak{H} - \mathrm{Diag}_{n,\Gamma}$. Namely, it is of the form $\nabla_{n,\Gamma} := d - \eta(\mathbf{z}|\tau)$, where η is a $\mathfrak{t}_{1,n}^\Gamma \rtimes \mathfrak{d}^\Gamma$-valued meromorphic one-form on $\mathbb{C}^n \times \mathfrak{H}$ with only poles on $\mathrm{Diag}_{n,\Gamma}$, and the equivariance condition reads: for any $g \in (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2^\Gamma(\mathbb{Z})$,

$$g^*\eta = (dc_g(\mathbf{z}|\tau))c_g(\mathbf{z}|\tau)^{-1} + \mathrm{Ad}(c_g(\mathbf{z}|\tau))(\eta(\mathbf{z}|\tau)) \,. \tag{7}$$

We now construct such a connection. For any $\gamma \in \Gamma$, we define $g_\gamma(x, z|\tau) := \partial_x k_\gamma(x, z|\tau)$ and

$$\tilde{\varphi}_\gamma(x|\tau) = \sum_{s\geq 0} A_{s,\gamma}(\tau)x^s := g_{-\gamma}(x, 0|\tau) \,.$$

Then we set

$$\Delta(\mathbf{z}|\tau) := -\frac{1}{2\pi\,\mathrm{i}}\left(\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}(\tau)\delta_{s,\gamma} - \sum_{i<j} g_{ij}(z_{ij}|\tau)\right),$$

where $g_{ij}(z|\tau) := \sum_{\alpha\in\Gamma} g_\alpha(\mathrm{ad}x_i, z|\tau)(t_{ij}^\alpha)$. And finally, with $K_i(\mathbf{z}|\tau)$'s as in §1.4, we define

$$\eta(\mathbf{z}|\tau) := \Delta(\mathbf{z}|\tau)d\tau + \sum_i K_i(\mathbf{z}|\tau)dz_i.$$

Remark 3.5. One can see that $\tilde{\varphi}_{\mathbf{0}}(x) = (\theta'/\theta)'(x) + 1/x^2$ and that for any $\gamma \in \Gamma - \{0\}$

$$\tilde{\varphi}_\gamma(x) = \partial_x\left(e^{2\pi\,\mathrm{i}\,cx}\frac{\theta(\tilde{\gamma}+x)}{\theta(\tilde{\gamma})\theta(x)} - \frac{1}{x}\right),$$

where $\tilde{\gamma} = (c_0, c) \in \Lambda_{\tau,\Gamma} - \Lambda_\tau$ is any lift of γ.

Proposition 3.6. *The equivariance identity* (7) *is satisfied for any* $g \in (\mathbb{Z}^n)^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.

Before proving this statement, let us notice that the $\mathrm{SL}_2(\mathbb{Z})$-equivariance is stronger than what we need (the $\mathrm{SL}_2^\Gamma(\mathbb{Z})$-equivariance), but easier to prove. The action of $\mathrm{SL}_2(\mathbb{Z})$ moves the poles while $\mathrm{SL}_2^\Gamma(\mathbb{Z})$ fixes them. In both cases, it makes sense to prove this proposition for meromorphic forms on $\mathbb{C}^n \times \mathfrak{h}$.

Proof. For $g = (\delta_j, 0)$, the identity translates into $K_i(\mathbf{z} + \delta_j|\tau) = K_i(\mathbf{z}|\tau)$ $(i = 1, \ldots, n)$ and $\Delta(\mathbf{z} + \delta_j|\tau) = \Delta(\mathbf{z}|\tau)$, which are immediate.

For $g = (0, \delta_j)$, the identity translates into $K_i(\mathbf{z} + \tau\delta_j|\tau) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}(x_j)}K_i(\mathbf{z}|\tau)$ $(\forall i)$ and

$$\Delta(\mathbf{z} + \tau\delta_j|\tau) + K_j(\mathbf{z} + \tau\delta_j|\tau) = e^{-2\pi\,\mathrm{i}\,\mathrm{ad}(x_j)}\Delta(\mathbf{z}|\tau). \tag{8}$$

The first equality is proved in §1.4, and we prove the second one now. First remember that for any $\tau \in \mathfrak{H}$, $z \in \mathbb{C} - (\frac{1}{M}\mathbb{Z} + \frac{\tau}{N}\mathbb{Z}))$ and $\alpha \in \Gamma$, we have the following identity in $\mathbb{C}[[x]]$:

$$e^{-2\pi\,\mathrm{i}\,x}(g_\alpha(x, z) - 1/x^2) + 1/x^2 - 2\pi\,\mathrm{i}(k_\alpha(x, z + \tau) + 1/x) = g_\alpha(x, z + \tau) \,. \tag{9}$$

Then, we can compute $2\pi\mathrm{i}\big(K_j(\mathbf{z}+\tau\delta_j|\tau)-e^{-2\pi\mathrm{i}\,\mathrm{ad}(x_j)}\Delta(\mathbf{z}|\tau)\big)$: it is equal to

$$2\pi\mathrm{i}\left(\sum_{k:k\neq j}k_\alpha(\mathrm{ad}x_j,z_{jk}+\tau)-y_j\right)+\Delta_0+\frac{1-e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j}}{\mathrm{ad}x_j}(y_j)+\frac{1}{2}\sum_{\substack{s\geq0,\\ \gamma\in\Gamma}}A_{s,\gamma}\delta_{s,\gamma}-e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j}\sum_{k<l}g_{kl}(z_{kl}),$$

and, therefore, using

$$\frac{1-e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j}}{\mathrm{ad}x_j}(y_j)-2\pi\mathrm{i}\,y_j=\left(\frac{e^{-2\pi\mathrm{i}\,\mathrm{ad}x_j}-1}{(\mathrm{ad}x_j)^2}+\frac{2\pi\mathrm{i}}{\mathrm{ad}x_j}\right)\left(\sum_{\alpha\in\Gamma}\sum_{k:k\neq j}t^\alpha_{jk}\right),$$

together with (9), we obtain

$$\Delta_0+\frac{1}{2}\sum_{s\geq0,\gamma\in\Gamma}A_{s,\gamma}\delta_{s,\gamma}-\sum_{\substack{k<l\\ k,l\neq j}}g_{kl}(z_{kl})-\sum_{\substack{k:k\neq j\\ \alpha\in\Gamma}}g_\alpha(\mathrm{ad}x_j,z_{jk}+\tau)(t^\alpha_{jk}),$$

which is precisely equal to $-2\pi\mathrm{i}\,\Delta(\mathbf{z}+\tau\delta_j)$.

For $g=S$, the identity translates into $K_i(\mathbf{z}|\tau+1)=K_i(\mathbf{z})$ $(\forall i)$ and $\Delta(\mathbf{z}|\tau+1)=\Delta(\mathbf{z})$. Both equalities obviously follow from $\theta(z|\tau+1)=\theta(z|\tau)$.

For $g=T$, the identity translates into

$$\frac{1}{\tau}K_i\Big(\frac{\mathbf{z}}{\tau}\Big|-\frac{1}{\tau}\Big)=\mathrm{Ad}\,(c_T(\mathbf{z}|\tau))\,(K_i(\mathbf{z}|\tau))+2\pi\mathrm{i}\,x_i, \tag{10}$$

for all $i\in\{1,\dots,n\}$ and

$$\frac{1}{\tau^2}\left(\Delta\Big(\frac{\mathbf{z}}{\tau}\Big|-\frac{1}{\tau}\Big)-\sum_i z_iK_i\Big(\frac{\mathbf{z}}{\tau}\Big|-\frac{1}{\tau}\Big)\right)=\mathrm{Ad}\,(c_T(\mathbf{z}|\tau))\,(\Delta(\mathbf{z}|\tau))+\frac{\mathbf{d}}{\tau}-2\pi\mathrm{i}\,\mathbf{X}. \tag{11}$$

Let us check (10) first. $\mathrm{Ad}(e^{2\pi\mathrm{i}(\sum_j z_jx_j+\tau\mathbf{X})}\tau^{\mathbf{d}})(-y_i)+2\pi\mathrm{i}\,x_i$ equals

$$\begin{aligned}-\mathrm{Ad}(e^{2\pi\mathrm{i}\sum_j z_jx_j})(y_i/\tau)&=-\frac{y_i}{\tau}-\frac{e^{2\pi\mathrm{i}\,\mathrm{ad}(\sum_j z_jx_j)}-1}{\mathrm{ad}(\sum_j z_jx_j)}\Big(\Big[\sum_j z_jx_j,\frac{y_i}{\tau}\Big]\Big)\\ &=-\frac{y_i}{\tau}-\frac{e^{2\pi\mathrm{i}\sum_j z_j\mathrm{ad}x_j}-1}{\sum_j z_j\mathrm{ad}x_j}\Big(\sum_{\substack{j:j\neq i\\ \alpha\in\Gamma}}\frac{z_{ji}}{\tau}t^\alpha_{ij}\Big)=-\frac{y_i}{\tau}-\sum_{j:j\neq i}\frac{e^{2\pi\mathrm{i}\,z_{ij}\mathrm{ad}x_i}}{z_{ij}\mathrm{ad}x_i}\Big(\sum_{\alpha\in\Gamma}\frac{z_{ji}}{\tau}t^\alpha_{ij}\Big).\end{aligned}$$

Therefore we have

$$-\frac{y_i}{\tau}=\mathrm{Ad}(c_T(\mathbf{z}|\tau))(-y_i)+2\pi\mathrm{i}\,x_i-\sum_{j:j\neq i}\frac{e^{2\pi\mathrm{i}\,z_{ij}\mathrm{ad}x_i}}{\mathrm{ad}x_i}\Big(\sum_{\alpha\in\Gamma}\frac{t^\alpha_{ij}}{\tau}\Big). \tag{12}$$

Now, since

$$\theta\left(-\frac{z}{\tau}\middle|-\frac{1}{\tau}\right)=-\frac{1}{\tau}e^{\frac{\pi\mathrm{i}}{\tau}z^2}\theta(z|\tau),$$

we obtain

$$\begin{aligned}k_\alpha\left(x,\frac{z}{\tau}\middle|-\frac{1}{\tau}\right)&=e^{-2\pi\mathrm{i}ax}\frac{\theta\left(\frac{z}{\tau}-\left(a_0-\frac{a}{\tau}\right)+x\middle|\tau\right)}{\theta\left(\frac{z}{\tau}-\left(a_0-\frac{a}{\tau}\right)\middle|\tau\right)\theta(x|\tau)}-\frac{1}{x}\\ &=-\tau e^{2\pi\mathrm{i}zx-2\pi\mathrm{i}a_0\tau x}\frac{\theta(\tau x+z+a-\tau a_0|\tau)}{\theta(z+a-\tau a_0|\tau)\theta(\tau x|\tau)}-\frac{1}{x}\\ &=\tau e^{2\pi\mathrm{i}zx}k_{T\alpha}(\tau x,z|\tau)+\frac{e^{2\pi\mathrm{i}\,zx}-1}{\tau x},\end{aligned}$$

where $T(\bar{a}_0,\bar{a}) = (-\bar{a},\bar{a}_0)$. Now substituting $(x,z) = (\mathrm{ad}x_j, z_j)$ in

$$\frac{1}{\tau}(k_\alpha(x,\frac{z}{\tau}|-\frac{1}{\tau}) = e^{2\pi\,\mathrm{i}\,zx}k_{T\alpha}(\tau x,z|\tau) + \frac{e^{2\pi\,\mathrm{i}\,zx}-1}{\tau x}\,, \tag{13}$$

then applying to t_{ij}^α, summing over $j\neq i$ and $\alpha\in\Gamma$ and adding up (12), we obtain (10) by using that

$$e^{2\pi\,\mathrm{i}\,z_{ij}\mathrm{ad}x_i}k_\alpha(\tau\mathrm{ad}x_i,z_{ij}|\tau)(t_{ij}^\alpha) = \mathrm{Ad}(e^{2\pi\,\mathrm{i}(\tau\mathbf{X}+\sum_j z_jx_j)}\tau^{\mathbf{d}})(k_\alpha(\mathrm{ad}x_i,z_{ij}|\tau)(t_{ij}^\alpha)).$$

We now check (11). Differentiating (13) w.r.t. x and dividing by τ, we get

$$\frac{1}{\tau^2}g_\alpha(x,\frac{z}{\tau}|-\frac{1}{\tau}) = e^{2\pi\,\mathrm{i}\,zx}g_{T\alpha}(\tau x,z|\tau) + \frac{2\pi\,\mathrm{i}\,z}{\tau^2}k_\alpha(x,\frac{z}{\tau}|-\frac{1}{\tau}) + \frac{1+2\pi\,\mathrm{i}\,zx-e^{2\pi\,\mathrm{i}\,zx}}{\tau^2x^2}\,.$$

Now substituting $(x,z) = (\mathrm{ad}x_i, z_{ij})$, applying to t_{ij}^α, and summing over $\alpha\in\Gamma$ we obtain

$$\begin{aligned}\frac{1}{\tau^2}g_{ij}(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) &= \mathrm{Ad}(c_T(\mathbf{z}|\tau))\,(g_{ij}(\mathbf{z}|\tau)) + \frac{2\pi\,\mathrm{i}\,z_{ij}}{\tau^2}K_{ij}(\frac{z_{ij}}{\tau}|-\frac{1}{\tau})\\ &\quad+\left(\frac{1+2\pi\,\mathrm{i}\,z_{ij}\mathrm{ad}x_i-e^{2\pi\,\mathrm{i}\,z_{ij}\mathrm{ad}x_i}}{\tau^2(\mathrm{ad}x_i)^2}\right)(\sum_{\alpha\in\Gamma}t_{ij}^\alpha)\,.\end{aligned}$$

Then taking the sum over $i<j$ one gets

$$\frac{1}{\tau^2}\sum_{i<j}g_{ij}(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) = \mathrm{Ad}(c_T(\mathbf{z}|\tau))\left(\sum_{i<j}g_{ij}(\mathbf{z}|\tau)\right) + \frac{2\pi\,\mathrm{i}}{\tau^2}\sum_i z_iK_i(\frac{\mathbf{z}}{\tau}|-\frac{1}{\tau}) + B(\mathbf{z})\,, \tag{14}$$

where

$$B(\mathbf{z}) := \sum_i\frac{2\pi\,\mathrm{i}\,z_iy_i}{\tau^2} + \sum_{i<j}\left(\frac{1+2\pi\,\mathrm{i}\,z_{ij}\mathrm{ad}x_i-e^{2\pi\,\mathrm{i}\,z_{ij}\mathrm{ad}x_i}}{\tau^2(\mathrm{ad}x_i)^2}\right)(\sum_\alpha t_{ij}^\alpha).$$

Lemma 3.7. $\mathrm{Ad}\,(c_T(\mathbf{z}|\tau))\,(\Delta_0) = \frac{\Delta_0}{\tau^2} + \frac{2\pi\,\mathrm{i}\,\mathbf{d}}{\tau} - (2\pi\,\mathrm{i})^2(\frac{1}{\tau}\sum_i z_ix_i + \mathbf{X}) + B(\mathbf{z})$.

Proof of the lemma. We first compute

$$\begin{aligned}\mathrm{Ad}\,(c_T(\mathbf{z}|\tau))\,(\Delta_0) &= \mathrm{Ad}(e^{2\pi\,\mathrm{i}(\tau\mathbf{X}+\sum_i z_ix_i)})(\frac{\Delta_0}{\tau^2}) = \mathrm{Ad}(e^{2\pi\,\mathrm{i}\sum_i z_ix_i})(\frac{\Delta_0}{\tau^2} + \frac{2\pi\,\mathrm{i}\,\mathbf{d}}{\tau} - (2\pi\,\mathrm{i})^2\mathbf{X})\\ &= \mathrm{Ad}(e^{2\pi\,\mathrm{i}\sum_i z_ix_i})(\frac{\Delta_0}{\tau^2}) + \frac{2\pi\,\mathrm{i}\,\mathbf{d}}{\tau} - (2\pi\,\mathrm{i})^2(\frac{1}{\tau}\sum_i z_ix_i + \mathbf{X})\,.\end{aligned}$$

It remains to show that $\mathrm{Ad}(e^{2\pi\,\mathrm{i}\sum_i z_ix_i})(\frac{\Delta_0}{\tau^2}) = \frac{\Delta_0}{\tau^2} + B(\mathbf{z})$. The proof of this fact goes along the same lines of computation as in [6, pp.16-17]. □

Using the above lemma and equation (14), one sees that equation (11) follows from

$$\mathrm{Ad}(c_T(\mathbf{z}|\tau)(\sum_{s,\gamma}A_{s,\gamma}(\tau)\delta_{s,\gamma}) = \sum_{s,\gamma}A_{s,\gamma}(-\frac{1}{\tau})\delta_{s,\gamma}\,.$$

This last equality is proved using $[x_i,\delta_{s,\gamma}] = 0 = [\mathbf{X},\delta_{s,\gamma}]$, $[\mathbf{d},\delta_{s,\gamma}] = s\delta_{s,\gamma}$, and, since

$$\tilde{\varphi}_\gamma(x|-\frac{1}{\tau}) = \tau^2\tilde{\varphi}_{T\gamma}(\tau x|\tau),$$

we get $A_{s,\gamma}(-\frac{1}{\tau}) = \tau^{s+2}A_{s,T\gamma}(\tau)$. □

We therefore have:

Theorem 3.8. $\nabla_{n,\Gamma}$ *defines a connection on* $\mathcal{P}_{n,\Gamma}$. *Moreover, its image under* $\mathbf{G}_n^\Gamma \to \bar{\mathbf{G}}_n^\Gamma$ *is the pull-back of a connection* $\bar{\nabla}_{n,\Gamma}$ *on* $\bar{\mathcal{P}}_{n,\Gamma}$.

Proof. The first part follows from Proposition 3.6 above. For the second part, we need to prove the three following identities:

- $\sum_i \bar{K}_i(\mathbf{z}|\tau) = 0$;
- $\bar{K}_i(\mathbf{z} + u\sum_j \delta_j|\tau) = \bar{K}_i(\mathbf{z}|\tau)$, for all i;
- $\bar{\Delta}(\mathbf{z} + u\sum_j \delta_j|\tau) = \bar{\Delta}(\mathbf{z}|\tau)$.

The first two equalities have already been proven, and the last one is obvious. □

3.4. **Flatness.** In this paragraph we prove the flatness of $\nabla_{n,\Gamma}$ (and thus of $\bar{\nabla}_{n,\Gamma}$).

Proposition 3.9. *For any* $i \in \{1, \dots, n\}$ *we have* $[\partial_\tau - \Delta(\mathbf{z}|\tau), \partial_i - K_i(\mathbf{z}|\tau)] = 0$.

In what follows, we often drop τ from the notation when it does not lead to any confusion.

Proof. Let us first prove that $\partial_\tau K_i(\mathbf{z}) = \partial_i \Delta(\mathbf{z})$. This follows from the identity $\partial_z g_\alpha(x,z) = 2\pi\mathrm{i}\,\partial_\tau k_\alpha(x,z)$, which is proved as follows (here $\tilde{\alpha} = (a_0, a)$ is any lift of α):

$$\begin{aligned}\partial_z g_\alpha(x,z) &= \partial_z\partial_x k_\alpha(x,z) = \partial_z\partial_x\left(e^{-2\pi\mathrm{i}\,ax}k(x,z-\tilde{\alpha}) + \frac{e^{-2\pi\mathrm{i}\,ax}-1}{x}\right)\\ &= e^{-2\pi\mathrm{i}\,ax}\partial_z\partial_x k(x,z-\tilde{\alpha}) - 2\pi\mathrm{i}\,ae^{-2\pi\mathrm{i}\,ax}\partial_z k(x,z-\tilde{\alpha})\\ &= 2\pi\mathrm{i}\,e^{-2\pi\mathrm{i}\,ax}\partial_\tau k(x,z-\tilde{\alpha}) - 2\pi\mathrm{i}\,ae^{-2\pi\mathrm{i}\,ax}\partial_z k(x,z-\tilde{\alpha})\\ &= 2\pi\mathrm{i}\,\partial_\tau\left(e^{-2\pi\mathrm{i}\,ax}k(x,z-\tilde{\alpha})\right) = 2\pi\mathrm{i}\,\partial_\tau k_\alpha(x,z).\end{aligned}$$

It remains to prove that $[\Delta(\mathbf{z}), K_i(\mathbf{z})] = 0$.

Let us first prove it in the case $n = 2$. Namely, we will prove that

$$[\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}\delta_{s,\gamma} - \sum_{\alpha\in\Gamma} g_\alpha(\mathrm{ad}\,x_1, z)(t_{12}^\alpha)\,,\, y_2 + \sum_{\beta\in\Gamma} k_\beta(\mathrm{ad}\,x_1, z)(t_{12}^\beta)] = 0. \tag{15}$$

One the one hand,

$$\begin{aligned}&[\Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}\delta_{s,\gamma} - \sum_{\alpha\in\Gamma} g_\alpha(\mathrm{ad}\,x_1, z)(t_{12}^\alpha)\,,\, y_2]\\ &= [y_1, \sum_{\alpha\in\Gamma} g_\alpha(\mathrm{ad}\,x_1, z)(t_{12}^\alpha)] - \frac{1}{2}\sum_{\alpha,\gamma\in\Gamma}\sum_{p,q} a_{p,q}^\gamma[\mathrm{ad}^p\,x_1(t_{12}^{\alpha-\gamma}), \mathrm{ad}^q\,x_1(t_{12}^\alpha)]\,,\end{aligned}$$

where

$$\frac{\tilde{\varphi}_\gamma(u) - \tilde{\varphi}_{-\gamma}(v)}{u+v} = \sum_{p,q} a_{p,q}^\gamma u^p v^q\,.$$

On the other hand, we have

$$\begin{aligned}[\Delta_0, \sum_\beta k_\beta(\mathrm{ad}\,x_1, z)(t_{12}^\beta)] =& [y_1, \sum_\beta g_\beta(\mathrm{ad}\,x_1, z)(t_{12}^\beta)]\\ &+ \sum_{p,q}\sum_{\alpha,\beta\in\Gamma} b_{p,q}^{\alpha,\beta}(z)[\mathrm{ad}^p\,x_1(t_{12}^\alpha), \mathrm{ad}^q\,x_1(t_{12}^\beta)]\,,\end{aligned}$$

where the series $\sum_{p,q} b_{p,q}^{\alpha,\beta}(z)u^pv^q$ is given by

$$\frac{1}{2}\left(\frac{1}{v^2}\left(k_\beta(u+v,z)-k_\beta(u,z)-v\partial_u k_\beta(u,z)\right)-\frac{1}{u^2}\left(k_\alpha(u+v,z)-k_\alpha(v,z)-u\partial_v k_\alpha(v,z)\right)\right).$$

Therefore the l.h.s. of (15) equals

$$\frac{1}{2}\left(\sum_{p,q}\sum_{\alpha,\beta\in\Gamma} c_{p,q}^{\alpha,\beta}(z)[\mathrm{ad}^p\, x_1(t_{12}^\alpha),\mathrm{ad}^q\, x_1(t_{12}^\beta)]\right),$$

where $\sum_{p,q} c_{p,q}^{\alpha,\beta}u^pv^q(z)$ is given by

$$\frac{1}{v^2}\left(k_\beta(u+v,z)-k_\beta(u,z)-vg_\beta(u,z)\right)-\frac{1}{u^2}\left(k_\alpha(u+v,z)-k_\alpha(v,z)-ug_\alpha(v,z)\right)$$
$$+\frac{\tilde\varphi_{\beta-\alpha}(u)-\tilde\varphi_{\alpha-\beta}(v)}{u+v}+k_\alpha(u+v,z)\tilde\varphi_{\alpha-\beta}(v)-k_\beta(u+v,z)\tilde\varphi_{\beta-\alpha}(u)$$
$$+k_\beta(u,z)g_\alpha(v,z)-g_\beta(u,z)k_\alpha(v,z),$$

which can be rewritten as

$$\left(g_{\beta-\alpha}(u,z-z')-\frac{1}{u^2}\right)\left(k_\alpha(u+v,z')+\frac{1}{u+v}\right)-\left(g_{\alpha-\beta}(v,z'-z)-\frac{1}{v^2}\right)\left(k_\beta(u+v,z)+\frac{1}{u+v}\right)$$
$$+\left(g_\alpha(v,z')-\frac{1}{v^2}\right)\left(k_\beta(u,z)+\frac{1}{u}\right)-\left(g_\beta(u,z)-\frac{1}{u^2}\right)\left(k_\alpha(v,z')+\frac{1}{v}\right), \tag{16}$$

with $z=z'$. Thus, to end the proof of equation (15), the following lemma is sufficient:

Lemma 3.10. *Expression (16) equals zero.*

Proof of the lemma. The case $\alpha=\beta=0$ follows from an explicit computation. Then we choose lifts $\tilde\alpha=(a_0,a)$ and $\tilde\beta=(b_0,b)$ of α and β, respectively. One has

$$k_\alpha(x,z)+1/x=e^{-2\mathrm{i}\pi ax}\left(k(x,z-\tilde\alpha)+1/x\right)\quad\text{and}$$
$$g_\alpha(x,z)-1/x^2=e^{-2\mathrm{i}\pi ax}\left(g(x,z-\tilde\alpha)-1/x^2\right)-2\mathrm{i}\pi b\left(k_\alpha(x,z)+1/x\right).$$

Therefore (16) equals

$$-2\mathrm{i}\pi(a-b)\Big(\left(k_\alpha(v,z')+\frac{1}{v}\right)\left(k_\beta(u,z)+\frac{1}{u}\right)+\left(k_{\beta-\alpha}(u,z-z')+\frac{1}{u}\right)\left(k_\alpha(u+v,z')+\frac{1}{u+v}\right)$$
$$+\left(k_{\alpha-\beta}(v,z'-z)-\frac{1}{v}\right)\left(k_\beta(u+v,z)+\frac{1}{u+v}\right)\Big),$$

which vanishes because of (3). □

Let us now assume that $n>2$.

Let $\mathfrak{t}_{n,+}^\Gamma\subset\mathfrak{t}_{1,n}^\Gamma$ be the subalgebra generated by x_i,t_{jk}^α $(i,j,k=1,\ldots,n,\ j\neq k,\ \alpha\in\Gamma)$.

We have functions $E_{ij}(\mathbf{z})$ with values in $\mathfrak{t}_{n,+}^\Gamma$ defined by $E_{ij}(\mathbf{z})=[\Delta_0,k_{ij}]-[y_i,g_{ij}]$, which decomposes as $e_{ij}(\mathbf{z})+\sum_{k\neq i,j}e_{ijk}(\mathbf{z})$, where $e_{ij}(\mathbf{z})$ takes its values in

$$\mathrm{Span}_{p,q,\alpha,\beta}[(\mathrm{ad}x_i)^p(t_{ij}^\alpha),(\mathrm{ad}x_j)^q(t_{ij}^\beta)]$$

and $e_{ijk}(\mathbf{z})$ takes its values in $\mathrm{Span}_{\alpha,\beta}\,\mathbb{C}[\mathrm{ad}x_i,\mathrm{ad}x_j][t_{ij}^\alpha,t_{jk}^\beta]$. Explicitely,

$$e_{ij}(\mathbf{z})=\sum_{\alpha,\beta}\sum_{p,q}b_{p,q}^{\alpha,\beta}(z_{ij})[\mathrm{ad}^px_i(t_{ij}^\alpha),\mathrm{ad}^qx_i(t_{ij}^\beta)],$$

where $b_{p,q}^{\alpha,\beta}(z)$ is as before, and

$$e_{ijk}(\mathbf{z}) = \sum_{\alpha,\beta}\left(\frac{k_\alpha(\mathrm{ad}x_i, z_{ij}) - k_\alpha(-\mathrm{ad}x_j, z_{ij})}{(\mathrm{ad}x_i + \mathrm{ad}x_j)^2} - \frac{g_\alpha(-\mathrm{ad}x_j, z_{ij})}{\mathrm{ad}x_i + \mathrm{ad}x_j}\right)[t_{ij}^\alpha, t_{ik}^\beta].$$

On the other hand, we have $Y_{ijk}(\mathbf{z}) \in \mathfrak{t}_{n,+}^\Gamma$ defined by $Y_{ijk}(\mathbf{z}) = [y_i, g_{jk}]$. It takes its values in $\mathrm{Span}_{\alpha,\beta}\,\mathbb{C}[\mathrm{ad}x_i, \mathrm{ad}x_j][t_{ij}^\alpha, t_{jk}^\beta]$. Explicitly,

$$Y_{ijk}(\mathbf{z}) = -\sum_{\alpha,\beta}\frac{g_\beta(\mathrm{ad}x_j, z_{jk}) - g_{-\beta}(\mathrm{ad}x_k, -z_{jk})}{\mathrm{ad}x_j + \mathrm{ad}x_k}[t_{ij}^\alpha, t_{jk}^\beta]$$

(remember that $g_\alpha(u,z) = g_{-\alpha}(-u,-z)$). We have

$$\begin{aligned}
[\Delta(\mathbf{z}), K_1(\mathbf{z})] &= \sum_{i>1}\left([\Delta_0, k_{1i}] - [y_1, g_{1i}] + [\frac{1}{2}\sum_\alpha \delta_{\tilde\varphi_\alpha}, k_{1i}] - [g_{1i}, k_{1i}]\right) - [\frac{1}{2}\sum_\alpha \delta_{\tilde\varphi_\alpha}, y_1] \\
&\quad - \sum_{1<i<j}\left([g_{1i}, k_{1j}] + [g_{1j}, k_{1i}] + [g_{ij}, k_{1i} + k_{1j}]\right) \\
&= \sum_{i>1}\left(e_{12} + [\frac{1}{2}\sum_\alpha \delta_{\tilde\varphi_\alpha}, k_{12}] - [g_{12}, k_{12}] - [\frac{1}{2}\sum_\alpha \delta_{\tilde\varphi_\alpha}, y_1]\right)_{1i} \qquad (17) \\
&\quad + \sum_{1<i<j}\left(e_{1ij} + e_{1ji} - Y_{1ij} - [g_{ij}, k_{1i} + k_{1j}] - [g_{1i}, k_{1j}] - [g_{1j}, k_{1i}]\right),
\end{aligned}$$

where $\{-\}_{1i}$ is the natural morphism $\mathfrak{t}_{1,2}^\Gamma \to \mathfrak{t}_{1,n}^\Gamma$, $u_1 \mapsto u_1$, $u_2 \mapsto u_i$ ($u = x, y$), $t_{12}^\alpha \mapsto t_{1i}^\alpha$. It is easy to see that the line (17) equals $\sum_{i>1}\left([\Delta(z_{1i}), K_1(z_{1i})]\right)_{1i}$ which is zero as we have seen before (case $n = 2$).

Therefore $[\Delta(\mathbf{z}), K_1(\mathbf{z})]$ equals

$$\begin{aligned}
&\sum_{1<i<j}\sum_{\alpha,\beta}\Big(\frac{k_\alpha(\mathrm{ad}x_1, z_{1i}) - k_\alpha(-\mathrm{ad}x_i, z_{1i}) - g_\alpha(-\mathrm{ad}x_i, z_{1i})(\mathrm{ad}x_1 + \mathrm{ad}x_i)}{(\mathrm{ad}x_1 + \mathrm{ad}x_i)^2}[t_{1i}^\alpha, t_{1j}^\beta] \\
&- \frac{k_\beta(\mathrm{ad}x_1, z_{1j}) - k_\beta(-\mathrm{ad}x_j, z_{1j}) - g_\beta(-\mathrm{ad}x_j, z_{1j})(\mathrm{ad}x_1 + \mathrm{ad}x_j)}{(\mathrm{ad}x_1 + \mathrm{ad}x_j)^2}[t_{1i}^\alpha, t_{1j}^\beta] \\
&- \frac{g_{\beta-\alpha}(\mathrm{ad}x_i, z_{ij}) - g_{\alpha-\beta}(\mathrm{ad}x_j, -z_{ij})}{\mathrm{ad}x_i + \mathrm{ad}x_j}[t_{1i}^\alpha, t_{1j}^\beta] \\
&- \left(k_\alpha(\mathrm{ad}x_1, z_{1i})g_{\beta-\alpha}(-\mathrm{ad}x_j, z_{ij}) - k_\beta(\mathrm{ad}x_1, z_{1j})g_{\beta-\alpha}(\mathrm{ad}x_i, z_{ij})\right)[t_{1i}^\alpha, t_{1j}^\beta] \\
&- \left(k_\beta(-\mathrm{ad}x_j, z_{1j})g_\alpha(-\mathrm{ad}x_i, z_{1i}) - k_\alpha(-\mathrm{ad}x_i, z_{1i})g_\beta(-\mathrm{ad}x_j, z_{1j})\right)[t_{1i}^\alpha, t_{1j}^\beta]\Big),
\end{aligned}$$

which is zero because of Lemma 3.10. □

We have therefore proved (Proposition 1.10 and Proposition 3.9 above):

Theorem 3.11. *The connection $\nabla_{n,\Gamma}$ is flat, and thus so is $\bar\nabla_{n,\Gamma}$.* □

Let us now show how the universal KZB connexion over moduli spaces coincides with the one defined over configuration spaces.

Remark 3.12. The connection $\nabla_{n,\Gamma}$ defined above is an extension to the twisted moduli space $\mathcal{M}_{1,n}^\Gamma$ of the connection $\nabla_{n,\tau,\Gamma}$ defined over the twisted configuration space $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ from Subsection 1.4.

Indeed, the pull-back of the principal $\mathbf{G}_n^\Gamma$-bundle with flat connection $(\mathcal{P}_{n,\Gamma},\nabla_{n,\Gamma})$ along the inclusion

$$\mathrm{Conf}(E_{\tau,\Gamma},n,\Gamma)\hookrightarrow\mathcal{M}_{1,n}^\Gamma$$

of the fiber at (the class of) τ in $Y(\Gamma)$ admits a reduction of structure group to

$$\exp(\mathfrak{t}_{1,n}^\Gamma)\subset\mathbf{G}_n^\Gamma\,,$$

and one easily sees from our explicit formulæthat it coincides with $(P_{\tau,n,\Gamma},\nabla_{\tau,n,\Gamma})$ constructed in Subsection 1.4.

Similarly, the connection $\bar\nabla_{n,\Gamma}$ is an extension to the twisted moduli space $\bar{\mathcal{M}}_{1,n}^\Gamma$ of the connection $\bar\nabla_{n,\tau,\Gamma}$ defined over the reduced twisted configuration space $\mathrm{C}(E_{\tau,\Gamma},n,\Gamma)$.

3.5. **Variations.** Let us first consider the unordered variants

$$\mathcal{M}_{1,[n]}^\Gamma=\mathfrak{S}_n\backslash\mathcal{M}_{1,n}^\Gamma\quad\text{and}\quad\bar{\mathcal{M}}_{1,[n]}^\Gamma=\mathfrak{S}_n\backslash\bar{\mathcal{M}}_{1,n}^\Gamma\,,$$

where, as before, the action of $\mathfrak{S}_n$ is again by permutation on $\mathbb{C}^n$.

Proposition 3.13. *1. There exists a unique principal $\mathbf{G}_n^\Gamma\rtimes\mathfrak{S}_n$-bundle $\mathcal{P}_{[n],\Gamma}$ over $\mathcal{M}_{1,[n]}^\Gamma$, such that a section over $U\subset\mathcal{M}_{1,[n]}^\Gamma$ is a function*

$$f:\tilde\pi^{-1}(U)\to\mathbf{G}_n^\Gamma\rtimes\mathfrak{S}_n$$

satisfying the conditions of Proposition 3.3 as well as $f(\sigma\mathbf{z}|\tau)=\sigma^{-1}f(\mathbf{z}|\tau)$ for $\sigma\in\mathfrak{S}_n$ (here $\tilde\pi:(\mathbb{C}^n\times\mathfrak{H})-\mathrm{Diag}_{n,\Gamma}\to\mathcal{M}_{1,[n]}^\Gamma$ is the canonical projection).

2. There exists a unique flat connection $\nabla_{[n],\Gamma}$ on $\mathcal{P}_{[n],\Gamma}$, whose pull-back to $(\mathbb{C}^n\times\mathfrak{H})-\mathrm{Diag}_{n,\Gamma}$ is the connection

$$\mathrm{d}-\Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau-\sum_i K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$$

on the trivial $\mathbf{G}_n^\Gamma\rtimes\mathfrak{S}_n$-bundle.

3. The image of $(\mathcal{P}_{[n],\Gamma},\nabla_{[n],\Gamma})$ under $\mathbf{G}_n^\Gamma\rtimes\mathfrak{S}_n\to\bar{\mathbf{G}}_n^\Gamma\rtimes\mathfrak{S}_n$ is the pull-back of a flat principal $\bar{\mathbf{G}}_n^\Gamma\rtimes\mathfrak{S}_n$-bundle $(\bar{\mathcal{P}}_{[n],\Gamma},\bar\nabla_{[n],\Gamma})$ on $\bar{\mathcal{M}}_{1,[n]}^\Gamma$.

Proof. For the proof of the first point, one easily checks that $\sigma c_{\tilde g}(\mathbf{z}|\tau)\sigma^{-1}=c_{\sigma\tilde g\sigma^{-1}}(\sigma^{-1}\mathbf{z})$, where $\tilde g\in(\mathbb{Z}^n)^2\rtimes\mathrm{SL}_2^\Gamma(\mathbb{Z})$, $\sigma\in\mathfrak{S}_n$. It follows that there is a unique cocycle $c_{(\tilde g,\sigma)}:\mathbb{C}^n\times\mathfrak{H}\to\bar{\mathbf{G}}_n^\Gamma\rtimes\mathfrak{S}_n$ such that $c_{(\tilde g,1)}=c_{\tilde g}$ and $c_{(1,\sigma)}(\mathbf{z}|\tau)=\sigma$.

For the proof of the second point, taking into account Theorem 3.11, one only has to show that this connection is $\mathfrak{S}_n$-equivariant. We have already mentioned that $\sum_i\bar K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$ is equivariant, and $\bar\Delta(\mathbf{z}|\tau)$ is also checked to be so.

The third point is obvious. □

For every (class of) τ in $Y(\Gamma)$, one has an action of Γ^n on the fiber $\mathrm{Conf}(E_{\tau,\Gamma},n,\Gamma)$ at τ of $\mathcal{M}_{1,n}^\Gamma\twoheadrightarrow Y(\Gamma)$, resp. an action of Γ^n/Γ on the fiber $\mathrm{C}(E_{\tau,\Gamma},n,\Gamma)$ at τ of $\bar{\mathcal{M}}_{1,n}^\Gamma\twoheadrightarrow Y(\Gamma)$. Recall that

$$\Gamma^n\backslash\mathrm{Conf}(E_{\tau,\Gamma},n,\Gamma)=\mathrm{Conf}(E_{\tau,\Gamma},n)\quad\text{and}\quad(\Gamma^n/\Gamma)\backslash\mathrm{C}(E_{\tau,\Gamma},n,\Gamma)=\mathrm{C}(E_{\tau,\Gamma},n)\,.$$

This action depends holomorphically of τ, so that we have an action of Γ^n on $\mathcal{M}_{1,n}^\Gamma$, resp. an action of Γ^n/Γ on $\bar{\mathcal{M}}_{1,n}^\Gamma$.

Proposition 3.14. *1. There exists a unique principal $\mathbf{G}_n^\Gamma \rtimes \Gamma^n$-bundle $\mathcal{P}_{(\Gamma),n}$ over $\Gamma^n \backslash \mathcal{M}_{1,n}^\Gamma$, such that a section over $U \subset \Gamma^n \backslash \mathcal{M}_{1,n}^\Gamma$ is a function*

$$f : \tilde{\pi}^{-1}(U) \to \mathbf{G}_n^\Gamma \rtimes \Gamma^n$$

satisfying the following conditions:

$$\begin{aligned} f(\mathbf{z} + \frac{\delta_i}{M}|\tau) &= (\bar{1},\bar{0})_i f(\mathbf{z}|\tau), \\ f(\mathbf{z} + \tau\frac{\delta_i}{N}|\tau) &= e^{-\frac{2\pi\mathrm{i}}{N}x_i}(\bar{0},\bar{1})_i f(\mathbf{z}|\tau), \\ f(\mathbf{z}, \tau+1) &= f(\mathbf{z}|\tau), \\ f(\frac{\mathbf{z}}{\tau}| -\frac{1}{\tau}) &= \tau^{\mathbf{d}} e^{\frac{2\pi\mathrm{i}}{\tau}(\mathbf{X}+\sum_i z_i x_i)} f(\mathbf{z}|\tau). \end{aligned}$$

Here, $\tilde{\pi} : (\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma} \to \Gamma^n \backslash \mathcal{M}_{1,n}^\Gamma$ is the canonical projection.

2. There exists a unique flat connection on this bundle whose pull-back to $(\mathbb{C}^n \times \mathfrak{H}) - \mathrm{Diag}_{n,\Gamma}$ is the connection

$$\mathrm{d} - \Delta(\mathbf{z}|\tau)\,\mathrm{d}\,\tau - \sum_i K_i(\mathbf{z}|\tau)\,\mathrm{d}\,z_i$$

on the trivial $\mathbf{G}_n^\Gamma \rtimes \Gamma^n$-bundle.

3. The image of the above flat bundle under $\mathbf{G}_n^\Gamma \rtimes \Gamma^n \to \bar{\mathbf{G}}_n^\Gamma \rtimes (\Gamma^n/\Gamma)$ is the pull-back of a flat principal $\bar{\mathbf{G}}_n^\Gamma \rtimes (\Gamma^n/\Gamma)$-bundle on $(\Gamma^n/\Gamma)\backslash \bar{\mathcal{M}}_{1,n}^\Gamma$.

Proof. The first assertion is left to the reader. Assertion 3 is evident. Let us prove assertion 2. By Proposition 1.9, we know that the K_i satisfy

(e) $K_i(\mathbf{z} + \frac{\delta_j}{M}|\tau) = (\bar{1},\bar{0})_j \cdot K_i(\mathbf{z}|\tau)$,
(f) $K_i(\mathbf{z} + \frac{\tau\delta_j}{N}|\tau) = (\bar{0},\bar{1})_j \cdot e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}(x_j)} K_i(\mathbf{z}|\tau)$.

The fact that $\Delta(\mathbf{z} + \frac{\delta_j}{M}|\tau) = (\bar{1},\bar{0})_j \cdot \Delta(\mathbf{z}|\tau)$ is immediate. Thus, it remains to show that $\Delta(\mathbf{z} + \frac{\tau\delta_j}{N}|\tau) = e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}(x_j)}(\bar{0},\bar{1})_j \cdot (\Delta(\mathbf{z}|\tau) - K_j(\mathbf{z}|\tau))$ which is proved in Lemma 3.15 below. □

Lemma 3.15. *We have*

$$\Delta(\mathbf{z} + \frac{\tau\delta_j}{N}|\tau) = e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}(x_j)}(\bar{0},\bar{1})_j \cdot (\Delta(\mathbf{z}|\tau) - K_j(\mathbf{z}|\tau)). \tag{18}$$

Proof. On the one hand, we have

$$-2\pi\,\mathrm{i}\,\Delta(\mathbf{z} + \frac{\tau\delta_j}{N}) = \Delta_0 + \frac{1}{2}\sum_{s\geq 0,\gamma\in\Gamma} A_{s,\gamma}\delta_{s,\gamma} - \sum_{\substack{k<l \\ k,l\neq j}} g_{kl}(z_{kl}) - \sum_{\substack{k:k\neq j \\ \alpha\in\Gamma}} g_\alpha(\mathrm{ad}x_j, z_{jk} + \frac{\tau}{N})(t_{jk}^\alpha).$$

On the other hand, as

$$\begin{aligned} e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}(x_j)}(\Delta_0) &= (1 - (1 - e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}(x_j)})(\Delta_0) = (\Delta_0) + \frac{1 - e^{-\frac{2\pi\mathrm{i}}{N}adx_j}}{\mathrm{ad}x_j}(y_j) \\ &= \frac{e^{-\frac{2\pi\mathrm{i}}{N}\mathrm{ad}x_j} - 1}{(\mathrm{ad}x_j)^2}\left(\sum_{\alpha\in\Gamma}\sum_{k:k\neq j} t_{jk}^\alpha\right) \end{aligned}$$

and the $\delta_{s,\gamma}$ commute with the x_j, we compute

$$2\pi\,\mathrm{i}\left(K_j(\mathbf{z}+\frac{\tau}{N}\delta_j|\tau) - e^{-\frac{2\pi\,\mathrm{i}}{N}\mathrm{ad}(x_j)}(\bar{0},\bar{1})_j\cdot\Delta(\mathbf{z}|\tau)\right)$$

$$= 2\pi\,\mathrm{i}\left((\bar{0},\overline{-1})_j\cdot K_j(\mathbf{z}+\frac{\tau}{N}\delta_j|\tau) - e^{-\frac{2\pi\,\mathrm{i}}{N}\mathrm{ad}(x_j)}\Delta(\mathbf{z}|\tau)\right)$$

$$= \quad 2\pi\,\mathrm{i}(\bar{0},\overline{-1})_j\cdot\left(\sum_{k:k\neq j}k_\alpha(\mathrm{ad}x_j, z_{jk}+\frac{\tau}{N}) - y_j\right) + \Delta_0 + \frac{1-e^{-\frac{2\pi\,\mathrm{i}}{N}\mathrm{ad}x_j}}{\mathrm{ad}x_j}(y_j)$$
$$+\frac{1}{2}\sum_{\substack{s\geq 0,\\ \gamma\in\Gamma}}A_{s,\gamma}\delta_{s,\gamma} - e^{-\frac{2\pi\,\mathrm{i}}{N}\mathrm{ad}x_j}\sum_{k<l}g_{kl}(z_{kl}).$$

Next, by combining

$$K_{ij}(z+\frac{\tau}{N}) = (\bar{0},\bar{-1})_i\cdot e^{-\frac{2\pi\,\mathrm{i}}{N}\mathrm{ad}x_j}\cdot(K_{ij}(z)) + (\bar{0},\bar{-1})_i\cdot(\sum_{\alpha\in\Gamma}\frac{e^{-\frac{2\pi\,\mathrm{i}}{N}\mathrm{ad}x_i}-1}{\mathrm{ad}x_i}(t_{ij}^\alpha))\,,$$

with equation

$$g_\alpha(x,z) - 1/x^2 = e^{-2\mathrm{i}\pi ax}\left(g(x,z-\tilde{\alpha}) - 1/x^2\right) - 2\mathrm{i}\pi a_0\left(k_\alpha(x,z)+1/x\right)\,,$$

we can follow the same lines as in the proof of relation (8) to obtain the wanted equation. □

We also leave to the reader the task of combining several variants.

4. Realizations

4.1. **Realizations of $\bar{\mathfrak{t}}_{1,n}^\Gamma$ and $\bar{\mathfrak{t}}_{n,+}^\Gamma$.** Let $\mathfrak{g}$ be a Lie algebra and $t_\mathfrak{g}\in S^2(\mathfrak{g})^\mathfrak{g}$ be nongenerate. Assume that we have a group morphism $\Gamma\to\mathrm{Aut}(\mathfrak{g},t_\mathfrak{g})$ and set $\mathfrak{l}:=\mathfrak{g}^\Gamma$ and $\mathfrak{u}:=\oplus_{\chi\in\widehat{\Gamma}-\{0\}}\mathfrak{g}_\chi$, where $\mathfrak{g}_\chi$ is the eigenspace of $\mathfrak{g}$ corresponding to the character $\chi:\Gamma\longrightarrow\mathbb{C}^\star$. Then we have $\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{u}$ with $[\mathfrak{l},\mathfrak{u}]\subset\mathfrak{u}$, and $\mathfrak{t}=\mathfrak{t}_\mathfrak{l}+\mathfrak{t}_\mathfrak{u}$ with $\mathfrak{t}_\mathfrak{l}\in S^2(\mathfrak{l})^\mathfrak{l}$ and $\mathfrak{t}_\mathfrak{u}\in S^2(\mathfrak{u})^\mathfrak{l}$. We denote by $(a,b)\mapsto\langle a,b\rangle$ the invariant pairing on $\mathfrak{l}$ corresponding to $t_\mathfrak{l}$ and write $t_\mathfrak{l}=\sum_\nu e_\nu\otimes e_\nu$.

Let $\mathrm{Diff}(\mathfrak{l}^*)$ be the algebra of algebraic differential operators on $\mathfrak{l}^*$. It has generators x_l, ∂_l ($l\in\mathfrak{l}$) and relations $\mathrm{x}_{tl+l'}=t\,\mathrm{x}_l+\mathrm{x}_{l'}$, $\partial_{tl+l'}=t\partial_l+\partial_{l'}$, $[\mathrm{x}_l,\mathrm{x}_{l'}]=0=[\partial_l,\partial_{l'}]$ and $[\partial_l,\mathrm{x}_{l'}]=\langle l,l'\rangle$. Moreover, one has a Lie algebra morphism $\mathfrak{l}\to\mathrm{Diff}(\mathfrak{l}^*); l\mapsto X_l:=\sum_\nu\mathrm{x}_{[l,e_\nu]}\partial_{e_\nu}$. We denote by $\mathfrak{l}^{\mathrm{diag}}$ the image of the induced morphism

$$\mathfrak{l}\ni l\mapsto Y_l:=X_l\otimes 1+1\otimes\sum_{i=1}^n l^{(i)}\in\mathrm{Diff}(\mathfrak{l}^*)\otimes U(\mathfrak{g})^{\otimes n}\,,$$

and define $H_n(\mathfrak{g},\mathfrak{l}^*)$ as the Hecke algebra of $A_n:=\mathrm{Diff}(\mathfrak{l}^*)\otimes U(\mathfrak{g})^{\otimes n}$ with respect to $\mathfrak{l}^{\mathrm{diag}}$. Namely, $H_n(\mathfrak{g},\mathfrak{l}^*):=(A_n)^\mathfrak{l}/(A_n\mathfrak{l}^{\mathrm{diag}})^\mathfrak{l}$. It acts in an obvious way on $(\mathcal{O}_{\mathfrak{l}^*}\otimes(\otimes_{i=1}^n V_i))^\mathfrak{l}$ if $(V_i)_{1\leq i\leq n}$ is a collection of $\mathfrak{g}$-modules.

Let us set $\mathrm{x}_\nu:=\mathrm{x}_{e_\nu}$ and $\partial_\nu:=\partial_{e_\nu}$, and write $\alpha^{(i)}\cdot$ for the action of $\alpha\in\Gamma$ on the i-th component in $U(\mathfrak{g})^{\otimes n}$.

Proposition 4.1. *There is a unique Lie algebra morphism* $\rho_{\mathfrak{g}} : \bar{\mathfrak{t}}^{\Gamma}_{1,n} \to H_n(\mathfrak{g}, \mathfrak{l}^*)$ *defined by*

$$\begin{aligned} \bar{x}_i &\longmapsto M \sum_{\nu} \mathrm{x}_{\nu} \otimes e^{(i)}_{\nu}, \\ \bar{y}_i &\longmapsto -N \sum_{\nu} \partial_{\nu} \otimes e^{(i)}_{\nu}, \\ \bar{t}^{\alpha}_{ij} &\longmapsto 1 \otimes (\alpha^{(1)} \cdot t_{\mathfrak{g}})^{(ij)}. \end{aligned}$$

Proof. Let us use the presentation of $\bar{\mathfrak{t}}^{\Gamma}_{1,n}$ coming from Lemma 1.1. The only non trivial check is that the relation $\sum_j \bar{x}_j = 0$ is preserved. We have

$$\begin{aligned} \rho_{\mathfrak{g}}\left(\sum_{i=1}^{n} x_i\right) &= M \sum_{\nu} \mathrm{x}_{\nu} \otimes \sum_{i=1}^{n} e^{(i)}_{\nu} = M \sum_{\nu} (\mathrm{x}_{\nu} \otimes 1)\left(1 \otimes \sum_{i=1}^{n} e^{(i)}_{\nu}\right) \\ &\equiv M \sum_{\nu} (\mathrm{x}_{\nu} \otimes 1)(Y_{\nu} - X_{\nu} \otimes 1) \\ &\equiv M - \sum_{\nu} \mathrm{x}_{\nu} X_{\nu} \otimes 1 = M \sum_{\nu_1,\nu_2} \mathrm{x}_{e_{\nu_1}} \mathrm{x}_{[e_{\nu_1}, e_{\nu_2}]} \partial_{\nu_2} \otimes 1 = 0 \end{aligned}$$

as $\mathrm{x}_{e_{\nu_1}}$ commutes with $\mathrm{x}_{[e_{\nu_1}, e_{\nu_2}]}$ and $t_{\mathfrak{l}}$ is invariant. Here the sign $\equiv$ means that both terms define the same equivalence class in $H_n(\mathfrak{g}, \mathfrak{l})$.

The proof that $\sum_j \bar{y}_j = 0$ is preserved is a consequence of the fact that $\rho_{\mathfrak{g}}\left(\sum_j \bar{y}_j\right) = 0$, which was proven in [6, Proposition 6.1]. ☐

Let $\bar{\mathfrak{t}}^{\Gamma}_{n,+} \subset \bar{\mathfrak{t}}^{\Gamma}_{1,n}$ be the Lie subalgebra generated by $\bar{x}_i$'s and $\bar{t}^{\alpha}_{jk}$'s. Then the restriction of $\rho_{\mathfrak{g}}$ to $\bar{\mathfrak{t}}^{\Gamma}_{n,+}$ lifts to a Lie algebra morphism $\bar{\mathfrak{t}}^{\Gamma}_{n,+} \to (\mathcal{O}_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}}$. Moreover, $(\mathcal{O}_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}}$ is a subalgebra of $H_n(\mathfrak{g}, \mathfrak{l}^*)$ that is a Lie ideal for the commutator, and one has a commutative diagram

$$\begin{array}{ccc} \bar{\mathfrak{t}}^{\Gamma}_{1,n} \times \bar{\mathfrak{t}}^{\Gamma}_{n,+} & \xrightarrow{(u,v)\mapsto[u,v]} & \bar{\mathfrak{t}}^{\Gamma}_{n,+} \\ \downarrow & & \downarrow \\ H_n(\mathfrak{g}, \mathfrak{l}^*) \times (\mathcal{O}_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}} & \longrightarrow & (\mathcal{O}_{\mathfrak{l}^*} \otimes U(\mathfrak{g})^{\otimes n})^{\mathfrak{l}}. \end{array}$$

4.2. **Realizations of $\bar{\mathfrak{t}}^{\Gamma}_{1,n} \rtimes \mathfrak{d}^{\Gamma}$.** Let us write $t_{\mathfrak{g}} = \sum_u a_u \otimes a_u$.

Proposition 4.2. *The Lie algebra morphism* $\rho_{\mathfrak{g}}$ *of Proposition 4.1 extends to a Lie algebra morphism* $\bar{\mathfrak{t}}^{\Gamma}_{1,n} \rtimes \mathfrak{d}^{\Gamma} \to H_n(\mathfrak{g}, \mathfrak{l}^*)$ *defined by*

$$\begin{aligned} \mathbf{d} &\longmapsto -\frac{1}{2}\left(\sum_{\nu} \mathrm{x}_{\nu}\, \partial_{\nu} + \partial_{\nu}\, \mathrm{x}_{\nu}\right) \otimes 1, \\ \mathbf{X} &\longmapsto \frac{1}{2}\left(\sum_{\nu} \mathrm{x}^2_{\nu}\right) \otimes 1, \\ \Delta_0 &\longmapsto -\frac{1}{2}\left(\sum_{\nu} \partial^2_{\nu}\right) \otimes 1, \\ \xi_{s,\gamma} &\longmapsto \frac{1}{|\Gamma|} \sum_{\nu_1,\cdots,\nu_s,u} \mathrm{x}_{\nu_1} \cdots \mathrm{x}_{\nu_s} \otimes \sum_{i=1}^{n} \left(\mathrm{ad}(e_{\nu_1}) \cdots \mathrm{ad}(e_{\nu_s})(a_u) \odot (\gamma \cdot a_u)\right)^{(i)}. \end{aligned}$$

Here $\odot$ denotes the symmetric product: $A \odot B := AB + BA$.

Proof. Since $t_\mathfrak{g}$ is invariant under the commuting actions of Γ and $\mathfrak{l}$ then the relation $\xi_{s,\gamma} = (-1)^s\xi_{s,-\gamma}$ is also preserved. This invariance argument also implies that $[\rho_\mathfrak{g}(\xi_{s,\gamma}),\rho_\mathfrak{g}(\bar{x}_i)]$ equals

$$\frac{1}{|\Gamma|}\sum_{\nu_1,\cdots,\nu_s,\nu,u}\mathrm{x}_{\nu_1}\cdots\mathrm{x}_{\nu_s}\,\mathrm{x}_\nu\otimes\sum_{t=1}^{s}\big(\mathrm{ad}(e_{\nu_1})\cdots\mathrm{ad}([e_\nu,e_{\nu_t}])\cdots\mathrm{ad}(e_{\nu_s})(a_u)\odot(\gamma\cdot a_u)\big)^{(i)}\ ,$$

which is zero since the first and second factors are respectively symmetric and antisymmetric in (ν,ν_t). Let us now prove that the relation $[\xi_{s,\gamma},\bar{t}^\alpha_{ij}] = [\bar{t}^\alpha_{ij},(\mathrm{ad}\bar{x}_i)^s(\bar{t}^{\alpha-\gamma}_{ij}) + (\mathrm{ad}\bar{x}_j)^s(\bar{t}^{\alpha+\gamma}_{ij})]$ is preserved. It is sufficient to do it for $n=2$:

$$\rho_\mathfrak{g}(\xi_{s,\gamma} + (\mathrm{ad}x_1)^s(t^{\alpha-\gamma}_{12}) + (\mathrm{ad}x_2)^s(t^{\alpha+\gamma}_{12})) = \sum_{\nu_1,\cdots,\nu_s}\mathrm{x}_{\nu_1}\cdots\mathrm{x}_{\nu_s}\otimes(\alpha^{(1)}\cdot\Delta(B_{\nu_1,\cdots,\nu_s}))\,,$$

where Δ is the standard coproduct of $U\mathfrak{g}$ and $B_{\nu_1,\cdots,\nu_s} := \sum_u \mathrm{ad}(e_{\nu_1})\cdots\mathrm{ad}(e_{\nu_s})(a_u)\odot(\gamma\cdot a_u)$; therefore $\rho_\mathfrak{g}(\xi_{s,\gamma} + (\mathrm{ad}x_1)^s(t^{\alpha-\gamma}_{12}) + (\mathrm{ad}x_2)^s(t^{\alpha+\gamma}_{12}))$ commutes with $\rho_\mathfrak{g}(t^\alpha_{12})$. Hence it remains to prove that the relation $[\xi_{s,\gamma},\frac{y_i}{N}] = \sum_{j:j\neq i} D_{s,\gamma}(\frac{x_i}{M},\frac{t^\beta_{ij}}{|\Gamma|})$ is preserved. For this we compute $[\rho_\mathfrak{g}(\xi_{s,\gamma}),\rho_\mathfrak{g}(\frac{y_i}{N})]$: it equals

$$\begin{aligned}
&\frac{1}{|\Gamma|}\sum_{\substack{\nu_1,\cdots,\nu_s\\ \nu,u}}\Big(\sum_{j=1}^{n}[\partial_\nu,\mathrm{x}_{\nu_1}\cdots\mathrm{x}_{\nu_s}]\otimes e^{(i)}_\nu\big(\mathrm{ad}(e_{\nu_1})\cdots\mathrm{ad}(e_{\nu_s})(a_u)\odot(\gamma\cdot a_u)\big)^{(j)}\\
&+\mathrm{x}_{\nu_1}\cdots\mathrm{x}_{\nu_s}\,\partial_\nu\otimes[e_\nu,\mathrm{ad}(e_{\nu_1})\cdots\mathrm{ad}(e_{\nu_s})(a_u)\odot(\gamma\cdot a_u)]^{(i)}\Big)\\
&=\frac{1}{|\Gamma|}\sum_{l=1}^{s}\sum_{\nu_1,\ldots,\nu_s,\nu}\mathrm{x}_{\nu_1}\cdots\check{\mathrm{x}}_{\nu_l}\cdots\mathrm{x}_{\nu_s}\otimes\sum_{j=1}^{n}\Big(e^{(i)}_\nu\big(\mathrm{ad}(e_{\nu_1})\cdots\mathrm{ad}(e_{\nu_s})(a_u)\odot(\gamma\cdot a_u)\big)^{(j)} - (i\leftrightarrow j)\Big)\,.
\end{aligned}$$

The term corresponding to $j=i$ is the linear map $S^{s-1}(\mathfrak{l})\to U(\mathfrak{g})^{\otimes n}$ such that for $x\in\mathfrak{l}$

$$x^{s-1}\longmapsto\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}[e_\nu,\mathrm{ad}(x)^p\mathrm{ad}(e_\nu)\mathrm{ad}(x)^q(a_u)\odot(\gamma\cdot a_u)]^{(i)}\,.$$

Using $\mathfrak{l}$-invariance of $\sum_u a_u\odot(\gamma\cdot a_u)$ one obtains that this last expression equals

$$\begin{aligned}
&=\frac{1}{|\Gamma|}\sum_{\substack{p+q+r=s-1\\ \nu,u}}\big(\mathrm{ad}(x)^p\mathrm{ad}([e_\nu,x])\mathrm{ad}(x)^q\mathrm{ad}(e_\nu)(\mathrm{ad}x)^r(a_u)\odot(\gamma\cdot a_u)\\
&\qquad+\mathrm{ad}(x)^p\mathrm{ad}(e_\nu)\mathrm{ad}(x)^q\mathrm{ad}([e_\nu,x])\mathrm{ad}(x)^r(a_u)\odot(\gamma\cdot a_u)\big)^{(i)}\,,
\end{aligned}$$

which is zero from the $\mathfrak{l}$-invariance of $t_\mathfrak{l} = \sum_\nu e_\nu\otimes e_\nu$. The term corresponding to $j\neq i$ is the linear map $S^{s-1}(\mathfrak{l})\to U(\mathfrak{g})^{\otimes n}$ such that for $x\in\mathfrak{l}$

$$\begin{aligned}
x^{s-1}&\longmapsto\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}(\mathrm{ad}(x)^p\mathrm{ad}(e_\nu)\mathrm{ad}(x)^q(a_u)\odot(\gamma\cdot a_u))^{(j)}\,e^{(i)}_\nu - (i\leftrightarrow j)\\
&=\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}(\mathrm{ad}(x)^p([e_\nu,a_u])\odot(-\mathrm{ad}(x))^q(\gamma\cdot a_u))^{(j)}\,e^{(i)}_\nu - (i\leftrightarrow j)\\
&=\frac{1}{|\Gamma|}\sum_{\substack{p+q=s-1\\ \nu,u}}(-1)^q\,(\mathrm{ad}(x)^p([e_\nu,a_u])\odot(\mathrm{ad}(x))^q(\gamma\cdot a_u))^{(j)}\,e^{(i)}_\nu - (i\leftrightarrow j)
\end{aligned}$$

$$= \frac{1}{|\Gamma|} \sum_{\substack{p+q=s-1 \\ \nu,u}} (-1)^q \left(\mathrm{ad}(x)^p([e_\nu, a_u]) \odot (\mathrm{ad}(x))^q(\gamma \cdot a_u)\right)^{(j)} e_\nu^{(i)} - (i \leftrightarrow j)$$

$$= \frac{1}{|\Gamma|^2} \sum_{\beta\in\Gamma} \sum_{\substack{p+q=s-1 \\ v,u}} (-1)^q \left(\mathrm{ad}(x)^p([a_v, a_u]) \odot (\mathrm{ad}(x))^q(\gamma \cdot a_u)\right)^{(j)} (\beta \cdot a_v)^{(i)} - (i \leftrightarrow j)$$

$$= \frac{1}{|\Gamma|^2} \sum_{\beta\in\Gamma} \sum_{p+q=s-1} (-1)^q \sum_{\nu,u} \left(\mathrm{ad}(x)^p(a_v) \odot \mathrm{ad}(x)^q(\gamma \cdot a_u)\right)^{(i)} (\beta \cdot [a_u, a_v])^{(j)} - (i \leftrightarrow j)$$

$$= \frac{1}{|\Gamma|^2} \sum_{\beta\in\Gamma} \sum_{p+q=s-1} (-1)^q \sum_{\nu,u} \left(\mathrm{ad}(x)^p(\beta \cdot a_v) \odot \mathrm{ad}(x)^q((\beta+\gamma) \cdot a_u)\right)^{(i)} [a_u, a_v]^{(j)} - (i \leftrightarrow j)$$

$$= \frac{1}{|\Gamma|^2} \sum_{\beta\in\Gamma} \sum_{p+q=s-1} (-1)^q \sum_{\nu,u} \left(\mathrm{ad}(x)^p((\beta-\gamma) \cdot a_v) \odot \mathrm{ad}(x)^q((\beta) \cdot a_u)\right)^{(i)} [a_u, a_v]^{(j)} - (i \leftrightarrow j)$$

which coincides with the image of

$$D_{s,\gamma}\left(\frac{x_i}{M}, \frac{t_{ij}^\beta}{|\Gamma|}\right) = \sum_{p+q=s-1} \sum_{\beta\in\Gamma} \left[\left(\mathrm{ad}\frac{x_i}{M}\right)^p \left(\frac{t_{ij}^\beta}{|\Gamma|}\right), \left(-\mathrm{ad}\frac{x_i}{M}\right)^q \left(\frac{t_{ij}^\beta}{|\Gamma|}\right)\right]$$

under $\rho_{\mathfrak{g}}$. In conclusion we get the relation

$$\rho_{\mathfrak{g}}\left(\left[\xi_{s,\gamma}, \frac{y_i}{N}\right]\right) = \left[\rho_{\mathfrak{g}}(\xi_{s,\gamma}), \rho_{\mathfrak{g}}\left(\frac{y_i}{N}\right)\right].$$

A direct computation shows that the commutation relations of $[\mathbf{X}, \xi_{s,\gamma}] = 0$, $[\mathbf{d}, \xi_{s,\gamma}] = s\xi_{s,\gamma}$ and $\mathrm{ad}^{s+1}(\Delta_0)(\xi_{s,\gamma}) = 0$ are preserved, which finishes the proof. □

4.3. **Reductions.** Assume that $\mathfrak{l}$ is finite dimensional and we have a reductive decomposition $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}$, i.e. $\mathfrak{h} \subset \mathfrak{l}$ is a subalgebra and $\mathfrak{m} \subset \mathfrak{l}$ is a vector subspace such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. We also assume that $t_{\mathfrak{l}} = t_{\mathfrak{h}} + t_{\mathfrak{m}}$ with $t_{\mathfrak{h}} = \sum_{\bar\nu} e_{\bar\nu} \otimes e_{\bar\nu} \in S^2(\mathfrak{h})^{\mathfrak{h}}$ and $t_{\mathfrak{m}} \in S^2(\mathfrak{m})^{\mathfrak{h}}$, and that for a generic $h \in \mathfrak{h}$, $\mathrm{ad}(h)_{|\mathfrak{m}} \in \mathrm{End}(\mathfrak{m})$ is invertible. This last condition means that

$$P(\lambda) \coloneqq \det(\mathrm{ad}(\lambda^\vee))_{|\mathfrak{m}}) \in S^{\dim(\mathfrak{m})}(\mathfrak{h})$$

is nonzero, where $\lambda^\vee \coloneqq (\lambda \otimes \mathrm{id})(t_{\mathfrak{h}})$ for any $\lambda \in \mathfrak{h}^*$.

We now define $H_n(\mathfrak{g}, \mathfrak{h}^*_{reg})$. As in the previous paragraph, $\mathrm{Diff}(\mathfrak{h}^*)$ has generators $\bar{\mathrm{x}}_h$, $\bar\partial_h$ ($h \in \mathfrak{h}$) and relations

$$\begin{aligned}
\bar{\mathrm{x}}_{th+h'} &= t\bar{\mathrm{x}}_h + \bar{\mathrm{x}}_{h'}, \\
\bar\partial_{th+h'} &= t\bar\partial_h + \bar\partial_{h'}, \\
[\bar{\mathrm{x}}_h, \bar{\mathrm{x}}_{h'}] = 0 &= [\bar\partial_h, \bar\partial_{h'}], \\
[\bar\partial_h, \bar{\mathrm{x}}_{h'}] &= \langle h, h'\rangle,
\end{aligned}$$

and $\mathrm{Diff}(\mathfrak{h}^*_{reg}) = \mathrm{Diff}(\mathfrak{h}^*)[\frac{1}{P}]$ with $[\bar\partial_l, \frac{1}{P}] = -\frac{[\bar\partial_l, P]}{P^2}$. One has a Lie algebra morphism

$$\mathfrak{h} \to \mathrm{Diff}(\mathfrak{h}^*); h \longmapsto \bar{X}_h \coloneqq \sum_{\bar\nu} \mathrm{x}_{[h, e_{\bar\nu}]} \partial_{e_{\bar\nu}}.$$

We denote by $\mathfrak{h}^{\mathrm{diag}}$ the image of the map

$$\mathfrak{h} \ni h \longmapsto \bar{Y}_h \coloneqq \bar{X}_h + \sum_{i=1}^n l^{(i)} \in \mathrm{Diff}(\mathfrak{h}^*_{reg}) \otimes U(\mathfrak{g})^{\otimes n} =: B_n,$$

and define $H_n(\mathfrak{g},\mathfrak{h}^*_{reg})$ as the Hecke algebra of B_n with respect to $\mathfrak{h}^{\mathrm{diag}}$:

$$H_n(\mathfrak{g},\mathfrak{h}^*_{reg}) := (B_n)^{\mathfrak{h}}/(B_n\mathfrak{h}^{\mathrm{diag}})^{\mathfrak{h}}.$$

It acts in an obvious way on $(\mathcal{O}_{\mathfrak{h}^*_{reg}}\otimes(\otimes_{i=1}^n V_i))^{\mathfrak{h}}$ if $(V_i)_{1\le i\le n}$ is a collection of $\mathfrak{g}$-modules. Finally, let us set, for $\lambda\in\mathfrak{h}^*$,

$$r(\lambda) := (\mathrm{id}\otimes(\mathrm{ad}\,\lambda^\vee)^{-1}_{|\mathfrak{m}})(t_{\mathfrak{m}}).$$

Then, following [14], $r:\mathfrak{h}^*_{\mathrm{reg}}\to\wedge^2(\mathfrak{m})$ is an $\mathfrak{h}$-equivariant map satisfying the classical dynamical Yang-Baxter equation (CDYBE)

$$\sum_{\bar\nu} e_{\bar\nu}^{(1)}\partial_{\bar\nu}r^{(23)} + [r^{(12)},r^{(13)}] + c.p.(1,2,3) = 0\,,$$

and we write $r=\sum_\delta a_\delta\otimes b_\delta\otimes\ell_\delta\in(\mathfrak{m}^{\otimes 2}\otimes S(\mathfrak{h})[1/P])^{\mathfrak{h}}$.

Proposition 4.3. *There is a unique Lie algebra morphism $\rho_{\mathfrak{g},\mathfrak{h}}:\bar{\mathfrak{t}}^{\Gamma}_{1,n}\to H_n(\mathfrak{g},\mathfrak{h}^*_{reg})$ given by*

$$\begin{aligned}
\bar x_i &\longmapsto M\sum_{\bar\nu}\bar{\mathrm{x}}_{\bar\nu}\otimes h_{\bar\nu}^{(i)},\\
\bar y_i &\longmapsto -N\sum_{\bar\nu}\bar\partial_{\bar\nu}\otimes h_{\bar\nu}^{(i)} + \sum_j\sum_\delta \ell_\delta\otimes a_\delta^{(i)}b_\delta^{(j)},\\
\bar t_{ij}^{\alpha} &\longmapsto 1\otimes(\alpha^{(1)}\cdot t_{\mathfrak{g}})^{(ij)}.
\end{aligned}$$

Proof. First of all, the images of the above elements are all $\mathfrak{h}$-invariant. As in [6], we will imply summation over repeated indices, and adopt the following conventions: $\bar\partial_{e_{\bar\nu}}=\bar\partial_{\bar\nu}$, $\bar{\mathrm{x}}_{e_{\bar\nu}}=\bar{\mathrm{x}}_{\bar\nu}$, and $1\otimes -$'s and $-\otimes 1$'s may be dropped from the notation.

In particular, $\rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i)=h_{\bar\nu}^{(i)}\bar{\mathrm{x}}_{\bar\nu}$, $\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_i)=-h_\nu^{(i)}\bar\partial_\nu+\sum_{j=1}^n r(\lambda)^{(ij)}$ (here, for $x\otimes y\in\mathfrak{g}^{\otimes 2}$, $(x\otimes y)^{(ii)}:=x^{(i)}y^{(i)}$).

We will use the same presentation of $\bar{\mathfrak{t}}^{\Gamma}_{1,n}$ as in Lemma 1.1. The relations $[\bar x_i,\bar x_j]=0$ and $\bar t_{ij}^{\alpha}=\bar t_{ji}^{-\alpha}$ are obviously preserved.

Let us check that $[\bar x_i,\bar y_j]=\sum\bar t_{ij}^{\alpha}$ is preserved. We have for $i\neq j$,

$$\begin{aligned}
\frac{1}{MN}[\rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_j)] &= -\sum_{\bar\nu_1,\bar\nu_2}[\bar{\mathrm{x}}_{\bar\nu_1},\partial_{\bar\nu_2}]h_{\bar\nu_1}^{(i)}h_{\bar\nu_2}^{(j)} + \sum\bar\nu,\delta,k\bar{\mathrm{x}}_{\bar\nu}[h_{\bar\nu}^{(i)},\ell_\delta\otimes a_\delta^{(j)}b_\delta^{(k)}]\\
&= t_{\mathfrak{h}}^{(ij)}+t_{\mathfrak{m}}^{(ij)}=t_{\mathfrak{l}}^{(ij)}=\frac{1}{MN}\sum_{\alpha\in\Gamma}\alpha^{(i)}\cdot t_{\mathfrak{g}}^{(ij)}
\end{aligned}$$

by the same argument as in Proposition 4.1.

Let us check that $\sum_i\bar x_i=\sum_i\bar y_i=0$ are preserved. We have $\sum_i\rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i)=0$ and $\sum_i\rho_{\mathfrak{g},\mathfrak{h}}(\bar y_i)=\sum_{\bar\nu,i}h_{\bar\nu}^{(i)}\partial_{\bar\nu}$ (by the antisymmetry of r), which equals zero as in Proposition 4.1.

The fact that the relation $[\bar y_i,\bar y_j]=0$ is satisfied for $i\neq j$ is a consequence of the dynamical Yang-Baxter equation (this follows from the exact same argument as in the proof of [6, Proposition 63]).

Next, $[\bar x_i,\bar t_{jk}^{\alpha}]=0$ is preserved (i,j,k distinct). Indeed, we have

$$[\rho_{\mathfrak{g},\mathfrak{h}}(\bar x_i),\rho_{\mathfrak{g},\mathfrak{h}}(\bar t_{jk}^{\alpha})]=\sum_{\bar\nu}\bar{\mathrm{x}}_{\bar\nu}[h_{\bar\nu}^{(i)},\alpha^{(i)}\cdot t_{\mathfrak{g}}^{(jk)}]=0\,.$$

Finally $[\bar{y}_i, \bar{t}^{\alpha}_{jk}] = 0$ is preserved (i,j,k distinct): we have

$$\begin{aligned}[\rho_{\mathfrak{g},\mathfrak{h}}(\bar{y}_i), \rho_{\mathfrak{g},\mathfrak{h}}(\bar{t}^{\alpha}_{jk})] =& [-\sum_{\bar{\nu}} h^{(i)}_{\bar{\nu}}\bar{\partial}_{\bar{\nu}} + \sum_{l} r^{(il)}, \alpha^{(j)}\cdot t^{(jk)}_{\mathfrak{g}})] \\ =& [r(\lambda)^{(ij)} + r(\lambda)^{(ik)}, \alpha^{(j)}\cdot t^{(jk)}_{\mathfrak{g}})] = 0\,,\end{aligned}$$

where the last equality follows the the $\mathfrak{g}$-invariance of $t_{\mathfrak{g}}$. □

Remark 4.4. We expect that there exists a Lie algebra morphism

$$\mathrm{red}_{\mathfrak{l},\mathfrak{h}} : H_n(\mathfrak{g},\mathfrak{l}^*) \to H_n(\mathfrak{g},\mathfrak{h}^*_{reg})$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{t}^{\Gamma}_{1,n} & \xrightarrow{\rho_{\mathfrak{g}}} & H_n(\mathfrak{g},\mathfrak{l}^*) \\ & \searrow^{\rho_{\mathfrak{g},\mathfrak{h}}} & \downarrow \mathrm{red}_{\mathfrak{l},\mathfrak{h}} \\ & & H_n(\mathfrak{g},\mathfrak{h}^*_{reg}) \end{array}$$

4.4. **Elliptic dynamical r-matrix systems as realizations of the universal Γ-KZB system on twisted configuration spaces.** Let $K(z)$ be a meromorphic function on $\mathbb{C}$ with values in the subalgebra $\hat{\mathfrak{t}}^{\Gamma}_{2,+} \subset \hat{\mathfrak{t}}^{\Gamma}_{1,2}$ generated by x_1, x_2, t^{α}_{12} ($\alpha\in\Gamma$), such that $K(-z) = -K(z)^{2,1}$ and satisfying the universal CDYBE with a spectral parameter

$$-[y_1, K(z_{23})^{2,3}] + [K(z_{12})^{1,2}, K(z_{13})^{1,3}] + c.p.(1,2,3) = 0\,.$$

On the one hand, it follows from §4.1 that the image $r(\mathrm{x},z) \coloneqq \rho_{\mathfrak{g}}(K(z))$ of $K(z)$ under $\rho_{\mathfrak{g}} : \hat{\mathfrak{t}}^{\Gamma}_{2,+} \to (\hat{\mathcal{O}}_{\mathfrak{l}^*}\otimes\mathfrak{g}^{\otimes 2})^{\mathfrak{l}}$ is a dynamical r-matrix[2] with spectral parameter, i.e. a solution of the CDYBE with a spectral parameter for the pair $(\mathfrak{l},\mathfrak{g})$

$$\sum_{\nu} e^{(1)}_{\nu}\partial_{\nu} r(\mathrm{x},z_{23})^{(23)} + [r(\mathrm{x},z_{12})^{(12)}, r(\mathrm{x},z_{13})^{(13)}] + c.p.(1,2,3) = 0\,,$$

which satisfies $r(\mathrm{x},-z) = -r(\mathrm{x},z)^{(21)}$. On the other hand, the image of $K(z)$ under $\rho_{\mathfrak{g},\mathfrak{h}} : \hat{\mathfrak{t}}^{\Gamma}_{2,+} \to (\hat{\mathcal{O}}_{\mathfrak{h}^*_{reg}}\otimes\mathfrak{g}^{\otimes 2})^{\mathfrak{h}}$ is precisely equal to the restriction $\rho_{\mathfrak{g}}(K(z))|_{\mathfrak{h}^*} \in (\hat{\mathcal{O}}_{\mathfrak{h}^*_{reg}}\otimes\mathfrak{g}^{\otimes 2})^{\mathfrak{h}}$ of $\rho_{\mathfrak{g}}(K(z))$ to $\mathfrak{h}^*$. Then applying [14, Proposition 0.1], we conclude that

$$\tilde{r}(\bar{\mathrm{x}},z) \coloneqq \rho_{\mathfrak{g},\mathfrak{h}}(K(z)) + r(\lambda)$$

is a solution of the CDYBE with spectral parameter for $(\mathfrak{h},\mathfrak{g})$:

$$\sum_{\bar{\nu}} e^{(1)}_{\bar{\nu}}\partial_{\bar{\nu}}\tilde{r}(\bar{\mathrm{x}},z_{23})^{(23)} + [\tilde{r}(\bar{\mathrm{x}},z_{12})^{(12)}, \tilde{r}(\bar{\mathrm{x}},z_{13})^{(13)}] + c.p.(1,2,3) = 0\,.$$

Then for any n-tuple $\underline{V} = (V_1,\ldots,V_n)$ of $\mathfrak{g}$-modules one has a flat connection $\nabla^{(\underline{V})}_{\tau,n,\Gamma}$ on the trivial vector bundle over $\mathbb{C}^n - \mathrm{Diag}_{\tau,n\Gamma}$ with fiber $(\mathcal{O}_{\mathfrak{h}^*_{reg}}\otimes(\otimes_i V_i))^{\mathfrak{h}}$, defined by the following compatible system of first order differential equations:

$$\partial_{z_i} F(\bar{\mathrm{x}},\mathbf{z}) = \sum_{\bar{\nu}} e^{(i)}_{\bar{\nu}}\cdot\bar{\partial}_{\bar{\nu}} F(\bar{\mathrm{x}},\mathbf{z}) + \sum_{j:j\neq i}\tilde{r}^{(ij)}(\bar{\mathrm{x}},z_{ij})\cdot F(\bar{\mathrm{x}},\mathbf{z})\,. \tag{19}$$

---

[2]Remember that $\mathcal{O}_{\mathfrak{l}^*} \coloneqq S(\mathfrak{l})$ and $\hat{\mathcal{O}}_{\mathfrak{l}^*} \coloneqq \hat{S}(\mathfrak{l})$.

Here $\mathbf{z} \mapsto F(\bar{\mathrm{x}}, \mathbf{z})$ is a function with values in $(\mathcal{O}_{\mathfrak{h}^*_{reg}} \otimes (\otimes_i V_i))^{\mathfrak{h}}$.

Starting from $K(z) = K_{12}(z)$ as in §1.4, it would be interesting to know if one can recover (up to gauge equivalence), using the above realization morphisms, the generalization of Felder's elliptic dynamical r-matrices [18] constructed in [16, 17].

5. Formality of subgroups of the pure braid group on the torus

5.1. **Relative formality.** Let G and S be two groups, with S finite, and let $\varphi : G \to S$ be a surjective group morphism with finitely generated kernel $\operatorname{Ker}\varphi$. We then consider the category of commuting triangles

$$\begin{array}{ccc} G & \longrightarrow & G' \\ & \overset{\varphi}{\searrow} & \downarrow{\scriptstyle \varphi'} \\ & & S \end{array}$$

where G' is pro-algebraic, and φ' is surjective with $\mathbf{k}$-prounipotent kernel. This category has an initial object, denoted $\varphi(\mathbf{k}) : G(\varphi, \mathbf{k}) \to S$, which we call the *relative ($\mathbf{k}$-prounipotent) completion* of G with respect to φ.

Observe that, if we regard the finite group S as an affine algebraic group, then this is a particular case of the relative completion defined in [22]. It also coincides with the partial completion defined in [10, §1.1] (which seems to force S to be finite).

Right exactness of relative completion (see e.g. [24, Proposition 2.4]), together with standard characterization of obstructions to left exactness, provides us with an exact sequence[3]

$$H_2(S, \mathbf{k}) \longrightarrow \big(\operatorname{Ker}\varphi\big)(\mathbf{k}) \longrightarrow G(\varphi, \mathbf{k}) \longrightarrow S \longrightarrow 1.$$

Since S is finite, $H_2(S, \mathbf{k}) = 0$, and thus we get that the kernel $\operatorname{Ker}\big(\varphi(\mathbf{k})\big)$ of $\varphi(\mathbf{k})$ is the usual $\mathbf{k}$-prounipotent completion $\big(\operatorname{Ker}\varphi\big)(\mathbf{k})$ of the kernel of φ, which we can therefore unambiguously denote $\operatorname{Ker}\varphi(\mathbf{k})$.

Lemma 5.1. *Every extension*

$$1 \longrightarrow U \longrightarrow H \longrightarrow S \longrightarrow 1$$

of a finite group by a $\mathbf{k}$-prounipotent one splits.

Proof. We consider the filtration $(F_i)_i$ given by the lower central series of U, and prove by induction that

$$1 \longrightarrow U/F_i \longrightarrow H/F_i \longrightarrow S \longrightarrow 1$$

splits.
Initial step ($i = 2$): Recall that $F_1 = U$, and that F_1/F_2 is abelian and finitely generated, so that

$$1 \longrightarrow U/F_2 \longrightarrow H/F_2 \longrightarrow S \longrightarrow 1$$

[3] This can also be seen as the end of the long exact sequence from [29, Theorem 1.17].

splits, as every extension of a finite group by a finite dimensional representation splits (this is because the cohomology of a finite group with coefficients in a divisible module vanishes).
Induction step: We have a (surjective) morphism of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U/F_{i+1} & \longrightarrow & H/F_{i+1} & \longrightarrow & S & \longrightarrow & 1. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U/F_i & \longrightarrow & H/F_i & \longrightarrow & S & \longrightarrow & 1 \end{array}$$

Assuming (by induction) that the bottom extension splits, we have that the corresponding obstruction class in the first non-abelian cohomology $H^1\big(S, U/F_i\big)$ is trivial. Hence, by exactness of

$$H^1\big(S, F_i/F_{i+1}\big) \longrightarrow H^1\big(S, U/F_{i+1}\big) \longrightarrow H^1\big(S, U/F_i\big),$$

we get that the obstruction class for the splitting of the top extension lies in the image of

$$H^1\big(S, F_i/F_{i+1}\big) \longrightarrow H^1\big(S, U/F_{i+1}\big)\,.$$

We conclude by using the vanishing of group cohomology of a finite group in a finite dimensional representation. □

The above Lemma tells us in particular that $G(\varphi,\mathbf{k}) \simeq \mathrm{Ker}(\varphi)(\mathbf{k}) \rtimes S$, and justifies the following definition from [10, §1.2][4].

Definition 5.2. If S is finite, we say that the surjective group morphism $\varphi : G \to S$ with finitely generated kernel is *(relatively) filtered-formal* if there exists a group isomorphism

$$G(\mathbf{k},\varphi) \xrightarrow{\sim} \exp\big(\hat{\mathrm{gr}}\,\mathrm{Lie}\,\mathrm{Ker}\,\varphi(\mathbf{k})\big) \rtimes S$$

over S. This is equivalent to having an S-equivariant filtered-formality isomorphism

$$\mathrm{Ker}\,\varphi(\mathbf{k}) \xrightarrow{\sim} \hat{\mathrm{gr}}\,\mathrm{Lie}\,\mathrm{Ker}\,\varphi(\mathbf{k})\,.$$

Example 5.3. The surjective morphism $\mathrm{B}_n \twoheadrightarrow \mathfrak{S}_n$, where B_n is the standard n strands braid group is filtered-formal. This morphism, or rather the exact sequence

$$1 \longrightarrow \mathrm{PB}_n \longrightarrow \mathrm{B}_n \longrightarrow \mathfrak{S}_n \longrightarrow 1\,,$$

can be deduced from the covering map $\mathrm{Conf}(\mathbb{C},n) \to \mathrm{Conf}(\mathbb{C},n)/\mathfrak{S}_n$. Note that filtered-formality of smooth complex algebraic varieties is proven in [28] in a functorial way, which implies in particular the wanted relative filtered-formality. An explicit filtered-formality isomorphism was first given in [25] when $\mathbf{k} = \mathbb{C}$ (in terms of the monodromy of the KZ connection) and then in [9] for $\mathbf{k} = \mathbb{Q}$ (using an associator). We also refer to [22, Example 1.5] for interesting considerations about this example. More precisely, one has an $\mathfrak{S}_n$-equivariant isomorphism $\mathrm{PB}_n(\mathbf{k}) \tilde{\to} \exp(\hat{\mathfrak{t}}_n)$.

[4]In [10], Enriquez speaks about *relative formality*. We prefer to speak about *relative filtered-formality* in order to remain consistent with our conventions in the absolute case $S = 1$ (recall that we were following the convention from [30] in the absolute case).

Example 5.4. Let $M \in \mathbb{N}$ be a positive integer. From the covering map $\mathrm{Conf}(\mathbb{C}^\times, n, M) \to \mathrm{Conf}(\mathbb{C}^\times, n)/\mathfrak{S}_n$ one also gets an exact sequence

$$1 \longrightarrow \mathrm{PB}_n^M \longrightarrow \mathrm{B}_n^1 \longrightarrow S \longrightarrow 1\,,$$

where $S := (\mathbb{Z}/M\mathbb{Z})^n \rtimes \mathfrak{S}_n$. It follows from [10, §1.3–1.6] that the surjective morphism $\mathrm{B}_n^1 \twoheadrightarrow S$ is filtered-formal. More precisely, Enriquez exhibits an S-equivariant isomorphism $\mathrm{PB}_n^M(\mathbf{k}) \tilde{\to} \exp(\hat{\mathfrak{t}}_n^M)$.

5.2. **Subgroups of** $\mathrm{B}_{1,n}$**.** For $\tau \in \mathfrak{H}$, let $U_{\tau,n,\Gamma} \subset \mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}$ be the open subset of all $\mathbf{z} = (z_1, \dots, z_n)$ of the form $z_i = a_i + \tau b_i$, where $0 < a_1 < \cdots < a_n < 1/M$ and $0 < b_n < \cdots < b_1 < 1/N$. If $\mathbf{z}_0 \in U_{\tau,n,\Gamma}$, then it both defines a point in the Γ-twisted configuration space $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ and in the (non-twisted) unordered configuration space $\mathrm{Conf}(E_{\tau,\Gamma}, [n])$.

Recall that the map

$$\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma) \twoheadrightarrow \mathrm{Conf}(E_{\tau,\Gamma}, [n])$$

is a covering map with structure group $\Gamma^n \rtimes \mathfrak{S}_n$. Hence we get a short exact sequence

$$1 \longrightarrow \mathrm{PB}_{1,n}^\Gamma \longrightarrow \mathrm{B}_{1,n} \xrightarrow{\varphi_n} \Gamma^n \rtimes \mathfrak{S}_n \longrightarrow 1\,,$$

where $\mathrm{PB}_{1,n}^\Gamma := \pi_1(\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma), \mathbf{z}_0)$ and $\mathrm{B}_{1,n} := \pi_1\big(\mathrm{Conf}(E_{\tau,\Gamma}, [n]), \mathbf{z}_0\big)$.

We will also consider $\mathrm{PB}_{1,n} = \pi_1\big(\mathrm{Conf}(E_{\tau,\Gamma}, n), \mathbf{z}_0\big)$, and the short exact sequence

$$1 \longrightarrow \mathrm{PB}_{1,n}^\Gamma \longrightarrow \mathrm{PB}_{1,n} \longrightarrow \Gamma^n \longrightarrow 1$$

associated with the Γ^n-covering map

$$\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma) \twoheadrightarrow \mathrm{Conf}(E_{\tau,\Gamma}, n)\,.$$

Our main aim in this Section is to construct a relative filtered-formality isomorphism for

$$\mathrm{B}_{1,n} \twoheadrightarrow \Gamma^n \rtimes \mathfrak{S}_n\,.$$

Moreover, we will have an explicit description of the relative completion in terms of the Lie algebra $\mathfrak{t}_{1,n}^\Gamma$.

5.3. **The monodromy morphism** $\mathrm{B}_{1,n} \to \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$**.** The monodromy of the flat $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)$-bundle $(\mathcal{P}_{(\tau,\Gamma),[n]}, \nabla_{(\tau,\Gamma),[n]})$ on $\mathrm{Conf}(E_{\tau,\Gamma}, [n])$ provides us with a group morphism

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),[n]} : \mathrm{B}_{1,n} \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n)\,.$$

This actually fits into a morphism of short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{PB}_{1,n}^\Gamma & \longrightarrow & \mathrm{B}_{1,n} & \longrightarrow & \Gamma^n \rtimes \mathfrak{S}_n & \longrightarrow & 1\,, \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) & \longrightarrow & \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma) \rtimes (\Gamma^n \rtimes \mathfrak{S}_n) & \longrightarrow & \Gamma^n \rtimes \mathfrak{S}_n & \longrightarrow & 1
\end{array}$$

where the first vertical morphism is the monodromy morphism

$$\mu_{\mathbf{z}_0,\tau,n,\Gamma} : \mathrm{PB}_{1,n}^\Gamma \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$$

of associated with the flat $\exp(\hat{\mathfrak{t}}_{1,n}^\Gamma)$-bundle $(\mathcal{P}_{\tau,n,\Gamma}, \nabla_{\tau,n,\Gamma})$ on $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$.

Indeed, this comes from the fact that $\nabla_{(\tau,\Gamma),[n]}$ is obtained by descent, from $\nabla_{\tau,n,\Gamma}$ and using its equivariance properties (see §1.3). More precisely, the monodromy of $\nabla_{(\tau,\Gamma),[n]}$ along a loop γ based at $\mathbf{z}_0$ in $\mathrm{Conf}(E_{\tau,\Gamma},[n])$ can be computed along the following steps:

- First consider the unique lift $\tilde{\gamma}$ of γ departing from $\mathbf{z}_0 \in \mathrm{Conf}(E_{\tau,\Gamma},n,\Gamma)$. Note that it ends at $g\cdot\mathbf{z}_0$, $g\in\Gamma^n\rtimes\mathfrak{S}_n$. If $g=(g_1,\dots,g_n)\in\Gamma^n$ and $\mathbf{z}_0=(z_1,\dots,z_n)$ we will simply write $g\cdot\mathbf{z}_0 := (z_1^{g_1},\dots z_n^{g_n})$.
- Then compute the holonomy of $\nabla_{\tau,n,\Gamma}$ along $\tilde{\gamma}$: this is an element in $\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$, as $\nabla_{\tau,n,\Gamma}$ is defined on a principal $\exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$-bundle obtained as a quotient of the trivial one on $\mathbb{C}^n - \mathrm{Diag}_{\tau,n,\Gamma}$ (see §1.2), that we abusively denote $\mu_{\mathbf{z}_0,\tau,n,\Gamma}(\tilde{\gamma})$.
- Finally, $\mu_{\mathbf{z}_0,(\tau,\Gamma),[n]}(\gamma) = g\mu_{\mathbf{z}_0,\tau,n,\Gamma}(\tilde{\gamma})$.

Having such a morphism of exact sequences guarantees that it factors through a morphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \hat{\mathrm{PB}}_{1,n}^{\Gamma}(\mathbb{C}) & \longrightarrow & \hat{\mathrm{B}}_{1,n}(\varphi_n,\mathbb{C}) & \longrightarrow & \Gamma^n\rtimes\mathfrak{S}_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma}) & \longrightarrow & \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})\rtimes(\Gamma^n\rtimes\mathfrak{S}_n) & \longrightarrow & \Gamma^n\rtimes\mathfrak{S}_n & \longrightarrow & 1 \end{array},$$

where $\hat{\mathrm{B}}_{1,n}(\varphi_n,\mathbb{C})$ is the relative prounipotent completion of the morphism $\mathrm{B}_{1,n}\to\Gamma^n\rtimes\mathfrak{S}_n$, and $\hat{\mathrm{PB}}_{1,n}^{\Gamma}(\mathbb{C})$ is the prounipotent completion of $\mathrm{PB}_{1,n}^{\Gamma}$.

We will call the vertical maps the completed monodromy morphisms.

In the remainder of this Section we will prove that these completed monodromy morphisms are isomorphisms, exhibiting in particular a relative filtered-formality isomorphism for $\mathrm{B}_{1,n}\to\Gamma^n\rtimes\mathfrak{S}_n$.

Theorem 5.5. *The completed monodromy morphism*

$$\hat{\mathrm{B}}_{1,n}(\varphi_n,\mathbb{C}) \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})\rtimes(\Gamma^n\rtimes\mathfrak{S}_n)$$

is an isomorphism. Equivalently, the completed monodromy morphism

$$\hat{\mu}_{\mathbf{z}_0,\tau,n,\Gamma}(\mathbb{C}) : \hat{\mathrm{PB}}_{1,n}^{\Gamma}(\mathbb{C}) \longrightarrow \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$$

is an isomorphism.

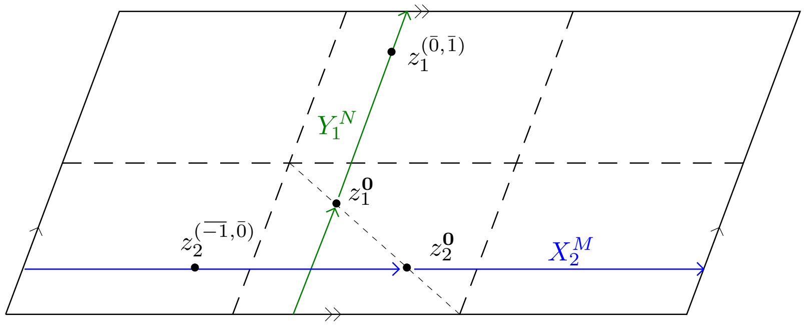
Our aim now is to prove Theorem 5.5. For this we will prove, as usual, that the induced morphism on Malcev Lie algebras

$$\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau,n,\Gamma}) : \mathfrak{pb}_{1,n}^{\Gamma} \to \hat{\mathfrak{t}}_{1,n}^{\Gamma}$$

is an isomorphism of filtered Lie algebras.

5.4. **A morphism $\mathfrak{t}_{1,n}^{\Gamma}\to\mathrm{gr}(\mathfrak{pb}_{1,n}^{\Gamma})$.** Let us start with a few algebraic facts about $\mathrm{PB}_{1,n}$ and $\mathrm{PB}_{1,n}^{\Gamma}$. The group $\mathrm{PB}_{1,n}$ is generated by the X_i's and Y_i's ($i=1,\dots,n$), where X_i (resp. Y_i) is the class of the path given by $[0,1]\ni t\mapsto \mathbf{z}_0+t\delta_i/M$ (resp. $[0,1]\ni t\mapsto\mathbf{z}_0+t\tau\delta_i/N$).

One sees easily that X_i^M (resp. Y_i^N) is the class of the path given by $[0,1] \ni t \mapsto \mathbf{z}_0 + t\delta_i$ (resp. $[0,1] \ni t \mapsto \mathbf{z}_0 + t\tau\delta_i$), so that X_i^M and Y_i^N are elements of $\mathrm{PB}_{1,n}^{\Gamma}$.



One has an obvious inclusion $\mathrm{PB}_n \hookrightarrow \mathrm{PB}_{1,n}^{\Gamma}$ coming from the identification of $\mathbb{C}$ with the fundamental domain

$$\{z = a + b\tau \in \mathbb{C} | 0 < a < \frac{1}{M}, 0 < b < \frac{1}{N}\}$$

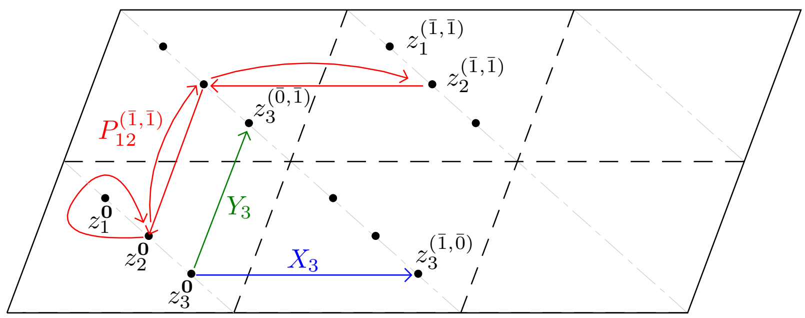
of $E_{\tau,\Gamma}$.

Recall that we write the composition of paths from left to right. Then one can check (by simply drawing) that the following relations are satisfied in $\mathrm{PB}_{1,n}$:

(T1) $(X_i, X_j) = 1 = (Y_i, Y_j)$ $(i < j)$,
(T2) $(X_i, Y_j) = P_{ij}$, and is conjugated to (X_j^{-1}, Y_i^{-1}) $(i < j)$,
(T3) $(X_1, Y_1^{-1}) = P_{1n} \cdots P_{13}P_{12}$,
(T4) $(X_i, P_{jk}) = 1 = (Y_i, P_{jk})$ $(\forall i,\ j < k)$,
(T5) $(X_iX_j, P_{ij}) = 1 = (Y_iY_j, P_{ij})$ $(i < j)$.

One also observes that $X_1 \cdots X_n$ and $Y_1 \cdots Y_n$ are central in $\mathrm{PB}_{1,n}$.

Now it follows from the geometric description of $\mathrm{PB}_{1,n}^{\Gamma}$ that it is generated by X_i^M, Y_i^N $(i = 1, \ldots, n)$, and $P_{ij}^{\alpha} := X_j^{-p} Y_j^{-q} P_{ij} Y_j^q X_j^p$ $(i < j,\ 1 \le p \le M,\ 1 \le q \le N$ and $\alpha = (\bar{p}, \bar{q}))$. One can for instance represent lifts of X_3, Y_3 and $P_{12}^{(\bar{1},\bar{1})}$ in $\mathrm{Conf}(E_{\tau,\Gamma}, n, \Gamma)$ as follows



Observe that the standard descending filtration on $\hat{\mathfrak{t}}_{1,n}^{\Gamma}$ coincides with the descending filtration coming from the grading of $\mathfrak{t}_{1,n}^{\Gamma}$ defined in §1.1.

Proposition 5.6. *There is a surjective graded Lie algebra morphism $p_n : \mathfrak{t}_{1,n}^{\Gamma} \to \mathrm{gr}(\mathfrak{pb}_{1,n}^{\Gamma})$, sending*

- $x_i \longmapsto \sigma\big(\log(X_i^M)\big)$ *for* $i = 1, \dots, n$,
- $y_i \longmapsto \sigma\big(\log(Y_i^N)\big)$ *for* $i = 1, \dots, n$,
- $t_{ij}^{\alpha} \longmapsto \sigma\big(\log(P_{ij}^{\alpha})\big)$ *for* $i < j$,

where σ denotes the symbol map $\mathfrak{pb}_{1,n}^{\Gamma} \to \mathrm{gr}(\mathfrak{pb}_{1,n}^{\Gamma})$.

Proof. It is sufficient to check that the defining relations of $\mathfrak{t}_{1,n}^{\Gamma}$ are preserved by the above assignment. The relation $[x_i, x_j] = 0 = [y_i, y_j]$ is obviously preserved, thanks to (T1).

Now using (T2) and the identity

$$(X^M, Y^N) = \prod_{i=0}^{M-1} X^{M-i+1}\Big(\prod_{j=0}^{N-1} Y^j (X,Y) Y^{-j}\Big) X^{i-M-1}$$

(which is true in the free group F_2, and thus in any group) with $X = X_i$ and $Y = Y_j$ $(i < j)$, one obtains that $[x_j, y_i] = \sum_\alpha t_{ij}^{\alpha}$ is preserved. The same reasoning with $X = X_i$ and $Y = Y_j^{-1}$ $(i \neq j)$ shows that $[x_i, y_j] = \sum_\alpha t_{ij}^{\alpha}$ is preserved as well.

Using (T3) and the above identity with $X = X_1$ and $Y = Y_1^{-1}$, one also obtains that $[x_1, y_1] = -\sum_\alpha \sum_{j:1\neq j} t_{1j}^{\alpha}$ is preserved. Now it is obvious that the centrality of $\sum_i x_i$ and $\sum_i y_i$ is preserved, and thus it follows that $[x_i, y_i] = -\sum_\alpha \sum_{j:j\neq i} t_{ij}^{\alpha}$ is also preserved for any $i \in \{1, \dots, n\}$.

For any $\alpha = (\bar{p}, \bar{q})$ we compute

$$\begin{aligned}
(X_i^M, P_{jk}^{\alpha}) &= X_i^M X_k^{-p} Y_k^{-q} P_{jk} Y_k^q X_k^p X_i^{-M} X_k^{-p} Y_k^{-q} P_{jk}^{-1} Y_k^q X_k^p \\
&= X_k^{-p} (X_i^M, Y_k^{-q}) Y_k^{-q} X_i^M P_{jk} X_i^{-M} Y_k^q (X_i^M, Y_k^{-q})^{-1} Y_k^{-q} P_{jk}^{-1} Y_k^q X_k^p \\
&= X_k^{-p} (X_i^M, Y_k^{-q}) Y_k^{-q} P_{jk} Y_k^q (X_i^M, Y_k^{-q})^{-1} Y_k^{-q} P_{jk}^{-1} Y_k^q X_k^p .
\end{aligned}$$

On the one hand, $\sigma\big(\log(X_i^M, P_{jk}^{\alpha})\big) = [\sigma(\log(X_i^M)), \sigma(\log(P_{jk}^{\alpha}))]$, and one the other hand, the leading term of the log of the r.h.s. lies in higher degree. Hence one obtains that $[x_i, t_{jk}^{\alpha}] = 0$ is preserved. The proof that $[y_i, t_{jk}^{\alpha}] = 0$ is preserved is identical, and the proof that $[x_i + x_j, t_{ij}^{\alpha}] = 0 = [y_i + y_j, t_{ij}^{\alpha}]$, $[t_{ij}^{\alpha}, t_{kl}^{\beta}] = 0$ and $[t_{ij}^{\alpha}, t_{ik}^{\alpha+\beta} + t_{jk}^{\beta}] = 0$ are preserved is similar. ☐

5.5. **The filtered-formality of $\mathrm{PB}_{1,n}^{\Gamma}$ (end of the proof of Theorem 5.5).** To prove that $\mathrm{Lie}(\mu_{\mathbf{z}_0,\tau,n,\Gamma})$ is an isomorphism, it is sufficient to prove that it is an isomorphism on associated graded. According to Proposition 5.6, we simply have to prove that $\phi := \mathrm{grLie}(\mu_{\mathbf{z}_0,\tau,n,\Gamma}) \circ p_n$ is an isomorphism of graded Lie algebras.

We will actually be more specific and prove the following:

Lemma 5.7. *We have $\phi(x_i) = y_i$, $\phi(y_i) = -2\pi\mathrm{i}\, x_i + \tau y_i$ and $\phi(t_{ij}^{\alpha}) = 2\pi\mathrm{i}\, t_{ij}^{\alpha}$. In particular, ϕ is an automorphism.*

Proof. Recall (see the appendix for more details) that $\mu_{\mathbf{z}_0,\tau,n,\Gamma}$ can be computed as follows. Let $F_{\mathbf{z}_0} : U_\tau \to \exp(\hat{\mathfrak{t}}_{1,n}^{\Gamma})$ be such that

$$\begin{cases} (\partial/\partial z_i) F_{\mathbf{z}_0}(\mathbf{z}) = K_i^{\Gamma}(\mathbf{z}|\tau) F_{\mathbf{z}_0}(\mathbf{z}) , \\ F_{\mathbf{z}_0}(\mathbf{z}_0) = 1 . \end{cases}$$

Then consider

$$H^{\Gamma}_{\tau,n} := \left\{ \mathbf{z} = (z_1, ..., z_n) | z_i = a_i + \tau b_i, 0 < b_n < ... < b_1 < \frac{1}{N} \right\}$$

and

$$V^{\Gamma}_{\tau,n} := \left\{ \mathbf{z} = (z_1, ..., z_n) | z_i = a_i + \tau b_i, 0 < a_1 < ... < a_n < \frac{1}{M} \right\}.$$

Let $F^H_{\mathbf{z}_0}$ (resp. $F^V_{\mathbf{z}_0}$) be the analytic prolongations of $F_{\mathbf{z}_0}$ to $H^{\Gamma}_{\tau,n}$ (resp. $V^{\Gamma}_{\tau,n}$). Then

$$\mu_{\mathbf{z}_0,\tau,n,\Gamma}(X_i^M) = F^H_{\mathbf{z}_0}(\mathbf{z}) F^H_{\mathbf{z}_0}(\mathbf{z} + \delta_i)^{-1} \quad \text{and} \quad \mu_{\mathbf{z}_0,\tau,n,\Gamma}(Y_i^N) e^{2\pi \mathrm{i}\, x_i} = F^V_{\mathbf{z}_0}(\mathbf{z}) F^V_{\mathbf{z}_0}(\mathbf{z} + \tau\delta_i)^{-1}.$$

Knowing that $\log F_{\mathbf{z}_0}(\mathbf{z}) = -\sum_i (z_i - z_i^0) y_i$ + terms of degree ≥ 2, we get

$$\log \mu_{\mathbf{z}_0,\tau,n,\Gamma}(X_i^M) = y_i + \text{terms of degree } \geq 2$$

and

$$\log \mu_{\mathbf{z}_0,\tau,n,\Gamma}(Y_i^N) = -2\pi \mathrm{i}\, x_i + \tau y_i + \text{terms of degree } \geq 2.$$

This gives us that $\phi(x_i) = y_i$ and $\phi(y_i) = -2\pi \mathrm{i}\, x_i + \tau y_i$.

In order to compute $\log \mu_{\mathbf{z}_0,\tau,n,\Gamma}(P^{\alpha}_{ij})$, which is also equal to $\log \mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P^{\alpha}_{ij})$, we will need to compute $\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i)$, $\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_i)$ and $\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P_{ij})$:

- As usual, and with our conventions, we have
$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P_{ij}) = \exp(2\pi \mathrm{i}\, t^{\mathbf{0}}_{ij} + \text{terms of degree } \geq 3),$$
where $\mathbf{0} = (\bar{0}, \bar{0})$.
- We also have
$$F^{H^{\Gamma}}_{\mathbf{z}_0}(\mathbf{z}) = \mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i)(-\bar{1}, \bar{0})_i F^{H^{\Gamma}}_{\mathbf{z}_0}(\mathbf{z} + \frac{\delta_i}{M}),$$
which implies that
$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i) \in (-\bar{1}, \bar{0})_i \exp(\mathfrak{t}^{\Gamma}_{1,n}).$$
- We finally have
$$F^V_{\mathbf{z}_0}(\mathbf{z}) = \mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_i)(\bar{0}, -\bar{1})_i e^{\frac{2\pi \mathrm{i}}{N} \mathrm{x}_i} F^V_{\mathbf{z}_0}(\mathbf{z} + \frac{\tau\delta_i}{N}),$$
which implies that
$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_i) \in (\bar{0}, -\bar{1})_i \exp(\mathfrak{t}^{\Gamma}_{1,n}).$$

Hence, if $\alpha = (\bar{p}, \bar{q}) \in \Gamma$, then

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(X_i^{-p} Y_j^{-q}) = g^{-1}(\bar{p}, \bar{0})_i (\bar{0}, \bar{q})_j,$$

with $g \in \exp(\mathfrak{t}^{\Gamma}_{1,n})$, and

$$\mu_{\mathbf{z}_0,(\tau,\Gamma),n}(Y_j^q X_i^p) = (\bar{0}, -\bar{q})_j (-\bar{p}, \bar{0})_i g.$$

Therefore

$$\begin{aligned} \mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P^{\alpha}_{ij}) &= g^{-1}(\bar{p}, \bar{0})_i (\bar{0}, \bar{q})_j \exp(2\pi \mathrm{i}\, t^{\mathbf{0}}_{ij} + \text{terms of degree } \geq 3)(\bar{0}, -\bar{q})_j (-\bar{p}, \bar{0})_i g \\ &= g^{-1} \exp(2\pi \mathrm{i}\, t^{\alpha}_{ij} + \text{terms of degree } \geq 3) g. \end{aligned}$$

This shows that $\log \mu_{\mathbf{z}_0,(\tau,\Gamma),n}(P^{\alpha}_{ij}) = 2\pi \mathrm{i}\, t^{\alpha}_{ij}$ + terms of degree ≥ 3, so that $\phi(t^{\alpha}_{ij}) = 2\pi \mathrm{i}\, t^{\alpha}_{ij}$. This ends the proof of the Lemma. □

Finally, if we denote $\hat{\overline{\mathrm{PB}}}_{1,n}^{\Gamma}(\mathbb{C}) := \hat{\pi}_1(\mathrm{C}(E_{\tau,\Gamma},n,\Gamma),\bar{\mathbf{z}}_0)(\mathbb{C})$, where $\bar{\mathbf{z}}_0$ is the image of $\mathbf{z}_0$ by the projection $\mathrm{Conf}(E_{\tau,\Gamma},n) \to \mathrm{C}(E_{\tau,\Gamma},n)$, then the isomorphism $\hat{\mu}_{\mathbf{z}_0,\tau,n,\Gamma}(\mathbb{C})$ descends to an isomorphism $\bar{\hat{\mu}}_{\bar{\mathbf{z}}_0,\tau,n,\Gamma}(\mathbb{C}) : \hat{\overline{\mathrm{PB}}}_{1,n}^{\Gamma}(\mathbb{C}) \to \exp(\hat{\bar{\mathfrak{t}}}_{1,n}^{\Gamma})$.

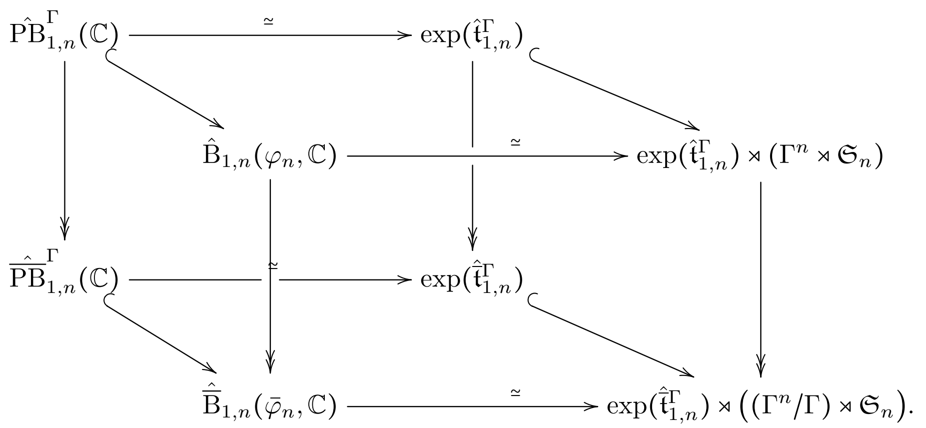
Now let $\overline{\mathrm{B}}_{1,n}$ be the fundamental group $\pi_1(\mathrm{C}(E_{\tau,\Gamma},[n]),[\bar{\mathbf{z}}_0])$. By considering the short exact sequence

$$1 \longrightarrow \overline{\mathrm{PB}}_{1,n}^{\Gamma} \longrightarrow \overline{\mathrm{B}}_{1,n} \xrightarrow{\bar{\varphi}_n} (\Gamma^n/\Gamma) \rtimes \mathfrak{S}_n \longrightarrow 1,$$

we deduce that the map

$$\hat{\overline{\mathrm{B}}}_{1,n}(\bar{\varphi}_n,\mathbb{C}) \longrightarrow \exp(\hat{\bar{\mathfrak{t}}}_{1,n}^{\Gamma}) \rtimes \big((\Gamma^n/\Gamma) \rtimes \mathfrak{S}_n\big)$$

is also relatively filtered-formal. In conclusion, we obtain the summarizing commutative cube



6. Representations of Cherednik algebras

6.1. **The Cherednik algebra of a wreath product.** In this paragraph Γ is any finite group such that $\Gamma \subset \mathrm{Aut}(\mathbb{C})$, $\underline{k} = (k_\alpha)_\alpha \in \mathbb{C}^\Gamma$ is such that $k_\alpha = k_{-\alpha}$ and $G := \Gamma \wr \mathfrak{S}_n$. We define the Cherednik algebra $H_n^\Gamma(\underline{k})$ as the quotient of the algebra $\mathbb{C}\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle \rtimes \mathbb{C}[G]$ by the relations

- $\sum_i \mathrm{x}_i = \sum_i \mathrm{y}_i = 0$
- $[\mathrm{x}_i,\mathrm{x}_j] = 0 = [\mathrm{y}_i,\mathrm{y}_j]$,
- $[\mathrm{x}_i,\mathrm{y}_j] = \frac{1}{n} - \sum_{\alpha\in\Gamma} k_\alpha s_{ij}^\alpha \ (i \neq j)$,

where $s_{ij}^\alpha = (\alpha_i - \alpha_j)s_{ij}$, and s_{ij} is the permutation of i and j.

Remark 6.1. Since $\Gamma \subset \mathrm{Aut}(\mathbb{C})$, $H_n^\Gamma(\underline{k})$ admits a geometric construction. Define $X := \{\mathbf{z} \in \mathbb{C}^n | \sum_i z_i = 0\}$ and consider the following action of G on it: $\mathfrak{S}_n$ acts in an obvious way and

$$\alpha_i(\mathbf{z}) = (\alpha^{(i)} - \frac{1}{n}\sum_j \alpha^{(j)})(\mathbf{z}),$$

where $\alpha^{(k)}$ is the action of $\alpha \in \Gamma$ on the k-th factor of $\mathbb{C}^n$. Following [15] one can construct a Cherednik algebra $H_{1,\underline{k},0}(X,G)$ on X/G. It can be defined as the subalgebra of $\mathrm{Diff}(X) \rtimes \mathbb{C}[G]$

generated by the function algebra $\mathcal{O}_X$, the group G and the Dunkl-Opdam operators $D_i - D_j$, where

$$D_i = \partial_{z_i} + \sum_{\substack{j:j\neq i\\ \alpha\in\Gamma}} k_\alpha \frac{1-s_{ij}^\alpha}{(-\alpha)(z_i)-\alpha(z_j)}\,.$$

One can then prove that there is a unique isomorphism of algebras $H_n^\Gamma(\underline{k}) \to H_{1,\underline{k},0}(X,G)$ defined by

$$\begin{aligned} \mathrm{x}_i &\longmapsto z_i, \\ \mathrm{y}_i &\longmapsto D_i - \frac{1}{n}\sum_j D_j, \\ G \ni g &\longmapsto g. \end{aligned}$$

6.2. **Morphisms from $\bar{\mathfrak{t}}_{1,n}^\Gamma$ to the Cherednik algebra.**

Proposition 6.2. *For any $a,b\in\mathbb{C}$ there is a morphism of Lie algebras $\phi_{a,b}:\bar{\mathfrak{t}}_{1,n}^\Gamma \to H_n^\Gamma(\underline{k})$ defined by*

$$\begin{aligned} \bar{x}_i &\longmapsto a\,\mathrm{x}_i \\ \bar{y}_i &\longmapsto b\,\mathrm{y}_i\,, \\ \bar{t}_{ij}^\alpha &\longmapsto ab\left(\frac{1}{n} - k_\alpha s_{ij}^\alpha\right)\,. \end{aligned}$$

Proof. Straightforward from the alternative presentation of $\bar{\mathfrak{t}}_{1,n}^\Gamma$ in Lemma 1.1. □

Hence any representation V of $H_n^\Gamma(\underline{k})$ yields a family of flat connections $\nabla_{a,b}^{(V)}$ over the configuration space $\mathrm{C}(E,[n],\Gamma)$.

6.3. **Monodromy representations of Hecke algebras.** Let E be an elliptic curve and $\tilde{E}\to E$ the Γ-covering as in §1.2. Define $X=\tilde{E}^n/\tilde{E}$ and $G=(\Gamma\wr\mathfrak{S}_n)/\Gamma^{\mathrm{diag}}$. Then the set $X'\subset X$ of points with trivial stabilizer is such that $X'/G=\mathrm{C}(E,[n],\Gamma)$.

Let us recall from [15] the construction of the Hecke algebra $\mathcal{H}_n^\Gamma(q,\underline{t})$ of X/G. It is the quotient of the group algebra of the orbifold fundamental group $\bar{B}_{1,n}^\Gamma$ of $\mathrm{C}(E,[n],\Gamma)$ by the additional relations $(T_\alpha - q^{-1}t_\alpha)(T_\alpha + q^{-1}t_\alpha^{-1})=0$, where T_α is an element of $\bar{B}_{1,n}^\Gamma$ homotopic as a free loop to a small loop around the divisor $Y_\alpha := \cup_{i\neq j}\{z_i=\alpha\cdot z_j\}$ in X/G, in the counterclockwise direction.[5]

Let us consider the flat connection $\nabla_{a,b}^{(V)}$ and set

$$q=e^{-2\pi\,\mathrm{i}\,ab/n}\,,\ t_\alpha = e^{-2\pi\,\mathrm{i}\,k_\alpha ab}\,.$$

Then the monodromy representation $\bar{B}_{1,n}^\Gamma\to GL(V)$ of $\nabla_{a,b}^{(V)}$ obviously gives a representation of $\mathcal{H}_n^\Gamma(q,\underline{t})$ either if V is finite dimensional or if a,b are formal parameters. In particular, taking $a=b$ a formal parameter and $V=H_n^\Gamma(\underline{k})$, one obtains an algebra morphism

$$\mathcal{H}_n^\Gamma(q,\underline{t}) \longrightarrow H_n^\Gamma(\underline{k})[[a]]\,.$$

We do not know if this morphism is an isomorphism upon inverting a.

[5]Here the sugroup of G acting trivially on Y_α is the order 2 cyclic subgroup generated by s_{ij}^α.

6.4. **The modular extension of $\phi_{a,b}$.** Now assume that $a,b\neq 0$.

Proposition 6.3. *The Lie algebra morphism $\phi_{a,b}$ can be extended to the algebra $U(\bar{\mathfrak{t}}^{\Gamma}_{1,n}\rtimes\mathfrak{d}^{\Gamma})\rtimes G$ by the following formulæ:*

$$\phi_{a,b}(s^{\alpha}_{ij}) = s^{\alpha}_{ij}\,,$$

$$\phi_{a,b}(d) = \frac{1}{2}\sum_i(\mathrm{x}_i\mathrm{y}_i+\mathrm{y}_i\mathrm{x}_i)\,,\quad \phi_{a,b}(X) = -\frac{1}{2}ab^{-1}\sum_i \mathrm{x}_i^2\,,$$

$$\phi_{a,b}(\Delta_0) = \frac{1}{2}ba^{-1}\sum_i \mathrm{y}_i^2\,,\quad \phi_{a,b}(\xi_{s,\gamma}) = -a^{s-1}b^{-1}\sum_{i<j}(\gamma\cdot(\mathrm{x}_i-\mathrm{x}_j))^s\,.$$

Thus, the flat connections $\nabla^{\Gamma}_{a,b}$ extend to flat connections on $\mathcal{M}^{\Gamma}_{1,[n]}$.

Proof. The proof is a straightforward calculation. □

Appendix A. Conventions

In this appendix we spell out our conventions regarding, fundamental groups, covering spaces, principal bundles, and monodromy morphisms.

A.1. **Fundamental groups.** Our convention is that we read the concatenation of paths from left to right. For instance, if X is a space, p is a path from x to y in X, and q is a path from y to z in X, then we write pq for the concatenated path, going from x to z.

A.2. **Covering spaces and group actions.** Our convention is that the group of deck transformations acts from the left. Apart from the case of principal bundles (see next §), group actions will always be from the left. We will often use $\cdot$ for such a left action.

The situation we are interested in is the one of a discrete group H acting properly discontinuously from the left on a space Y, with quotient space $X = H\backslash Y$, so that the quotient map $Y\to X$ is a covering map.

We thus have a short exact sequence

$$1\to\pi_1(Y,y)\to\pi_1(X,x)\to H\to 1$$

of groups, where $y\in Y$ and $x = H\cdot y\in X$ is its projection. Note that the surjective map $\pi_1(X,x)\to H$ sends (the class of) a loop γ based at x to h_γ, which is defined as follows: $\tilde\gamma(1) = h_\gamma\cdot\tilde\gamma(0)$, where $\tilde\gamma$ is a path lifting (uniquely) γ to Y and such that $\tilde\gamma(0) = y$. For the sake of completeness, let us check that this is indeed a group homomorphism.

Proof. We have

$$h_{\gamma_1\gamma_2}\cdot y = \widetilde{\gamma_1\gamma_2}(1) = \tilde{\tilde{\gamma}}_2(1)\,,$$

where $\tilde{\tilde{\gamma}}_2 = h_{\gamma_1}\cdot\tilde\gamma_2$ is the (unique) lift of γ_2 such that $\tilde{\tilde{\gamma}}_2(0) = \tilde\gamma_1(1) = h_{\gamma_1}\cdot y$. Therefore, $h_{\gamma_1\gamma_2} = h_{\gamma_1}h_{\gamma_2}$. □

A.3. **Principal bundles and descent.** Let G be a group. All principal G-bundles (apart from covering spaces, see above) are right principal G-bundles. Let $\mathcal{P}$ be a principal G-bundle over X, so that $\mathcal{P}/G = X$.

Let us assume that $X = H\backslash Y$, where H is a discrete group acting on Y. We now describe a way of constructing a G-bundle on the quotient space X from the trivial G-bundle $\tilde{\mathcal{P}} := Y \times G$ on Y, by means of non-abelian 1-cocycles.

A left H-action on $\tilde{\mathcal{P}}$, compatible with the one on Y, is given as follows:

$$h \cdot (y, g) = \big(h \cdot y, c_h(y) g\big), \quad c_h(y) \in G$$

The property of being a left action is equivalent to the non-abelian 1-cocycle identity

$$c_{h_1 h_2}(y) = c_{h_1}(h_2 \cdot y) c_{h_2}(y)\,.$$

A.4. **Monodromy and group actions.** Let us start with the monodromy in the case of a trivial principal G-bundle $\tilde{P} = Y \times G$ on a manifold Y equipped with a flat connection $\nabla = d - \omega$. Here ω is a one-form on Y with values in $\mathfrak{g} = \mathrm{Lie}(G)$, and G is assumed to be prounipotent.

Let $\gamma : [0,1] \to Y$ be a differentiable path, and consider its (unique) horizontal lift $\tilde{\gamma} = (\gamma, g) : [0,1] \to \tilde{\mathcal{P}}$ such that $g(0) = 1_G$. We define the monodromy $\mu(\gamma) := g(1)^{-1}$.

Remark A.1. Observe that if $(\gamma, \tilde{g})$ is another lift so that $\tilde{g} = g_0 \in G$, then $\tilde{g}(t) = g(t) g_0$ (by unicity of horizontal lifts), and thus $\mu(\gamma) = \tilde{g}(0)\tilde{g}(1)^{-1}$.

Again, for the sake of completeness, we check that μ is a morphism, in the sense that it sends the concatenation of paths to the product in G.

Proof. Let γ_1, γ_2 be composable paths in Y, and let g_1, g_2 determine composable horizontal lifts. Then

$$\begin{aligned} \mu(\gamma_1\gamma_2) &= (g_1 g_2)(0)(g_1 g_2)(1)^{-1} = g_1(0) g_2(1)^{-1} \\ &= g_1(0) g_1(1)^{-1} g_2(0) g_2(1)^{-1} = \mu(\gamma_1)\mu(\gamma_2)\,. \end{aligned}$$

□

Let us now assume that Y is acted on properly discontinuously from the left by a discrete group H, that also acts in a compatible way on $\tilde{P}$ thanks to a non-abelian 1-cocycle $c : H \times Y \to G$ (see previous § above). We borrow the notation from §A.2, and assume that $\tilde{P}$ is equipped with an H-equivariant flat connection, that therefore descends to a flat connection on $\mathcal{P}$ We define a monodromy morphism

$$\begin{aligned} \mu_x : \pi_1(X, x) &\longrightarrow G \\ \gamma &\longmapsto \mu(\tilde{\gamma}) c_{h_\gamma}(y)\,, \end{aligned}$$

where $\tilde{\gamma}$ is the lift of γ along the quotient map $Y \to X$ such that $\tilde{\gamma}(0) = y$. Let us again check, for the sake of completeness, that μ_x is indeed a group morphism.

Proof. Recall that for every loop γ based at x, $\tilde{\gamma}(1) = h_\gamma \cdot y$. Hence, if γ_1, γ_2 are loops based at x, then $\widetilde{\gamma_1\gamma_2} = \tilde{\gamma}_1\tilde{\tilde{\gamma}}_2$, with $\tilde{\tilde{\gamma}}_2 = h_{\gamma_1} \cdot \tilde{\gamma}_2$. Therefore

$$\begin{aligned}
\mu_x(\gamma_1\gamma_2) &= \mu(\widetilde{\gamma_1\gamma_2})c_{h_{\gamma_1\gamma_2}}(y) \\
&= \mu(\widetilde{\gamma_1\gamma_2})c_{h_{\gamma_1}h_{\gamma_2}}(y) \\
&= \mu(\tilde{\gamma}_1)\mu(h_{\gamma_1} \cdot \tilde{\gamma}_2)c_{h_{\gamma_1}}(h_{\gamma_2} \cdot y)c_{h_{\gamma_2}}(y) \\
&= \mu(\tilde{\gamma}_1)c_{h_{\gamma_1}}(y)\mu(\tilde{\gamma}_2)c_{h_{\gamma_1}}(h_{\gamma_2} \cdot y)^{-1}c_{h_{\gamma_1}}(h_{\gamma_2} \cdot y)c_{h_{\gamma_2}}(y) \\
&= \mu_x(\gamma_1)\mu_x(\gamma_2)
\end{aligned}$$

Here we made used of the (easy) fact that, if the flat connection is equivariant, then so is the monodromy map μ: $\mu(h \cdot \gamma) = c_h(\gamma(0))\mu(\gamma)c_h(\gamma(1))^{-1}$. □

List of notation

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