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The Kobayashi pseudometric on algebraic manifold and a canonical fibration

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Abstract.

Given a compact complex manifold X of dimension n, we define a bimeromorphic invariant $\kappa_+(X)$ as the maximum p for which there is a saturated line subsheave L of the sheaf of holomorphic p-forms whose Kodaira dimension $\kappa(L)$ equals p. We call X special if $\kappa_+(X) = 0$. We observe from earlier works that among the algebraic X with $\kappa_+(X) \neq 2$ the special ones are in fact characterized by vanishing Kobayashi pseudometric (and holomorphic connectedness) for $n = 2$ without any condition, for $n = 3$ when $\kappa(X) \neq 0, 3$, for birationally abelian and $K3$-fibrations over curves and for rationally connected fibrations over any of the above examples. We show here that rationally connected manifolds and algbraic manifolds of zero Kodaira dimension are special.

We use the well-known construction of F. Campana to give, for each projective X a unique fibration $f : X \to Y$ holomorphic outside a proper subvariety of Y and whose general fibers are special. We show that Y is a non-uni-special orbifold. We note that the Iitaka and rational connectioned fibrations of X are factors of f and we show that this fibration solves a general conjecture in Mori's classification program of algebraic varieties, namely, that an algebraic variety is either of general type, or (birationally) has a canonical fibration of positive fiber dimension that factors through the Iitaka and rationally connected fibrations of X. In fact, assuming the weak log-abundance conjecture that orbifolds of negative kodaira dimension admits a fibration with positive dimensional special type fibers, our fibration is then nothing but the iterations of the orbifold Iitaka and Mori's fibration that end in a general type orbifold base. This reduces the (weak) Abundance Conjecture in Mori's Program to the study of non-uniruled special manifold of negative Kodaira dimension.

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§1. Introduction and acknowledgements

The purpose of this paper is twofold: (1) to give a quick summary of our results on the Kobayashi pseudometric on algebraic manifolds, especially on our characterization of those with identically vanishing Kobayashi pseudometric that resulted in the introduction of a natural birational invariant and its associated canonical fibration, and (2) to show that this fibration resolves the conjectured alternative in Mori's Program that an algebraic variety is either of general type, or has a canonical fibration of positive fiber dimension that factors through the Iitaka and rationally connected fibrations in the birational category.

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§2. Preliminaries on fibrations and orbifolds

Since the key objects under study in this paper are all natural bimeromorphic invariants, we assume henceforth that all objects are compact smooth manifolds and that every (meromorphic) map, unless otherwise specified, is bimeromorphic to a flat morphism to the target space and that the map is smooth (of maximal rank) outside a reduced simple normal crossing divisor, the discriminant locus, on the target manifold. We can do this by modifying the target manifold via Hironaka's resolution of singularity theorem and his flattening theorem [7].

Recall that sections, if exist, of powers of a holomorphic line bundle L on a compact complex manifold X gives rise to a fibraion

$$I_L : X \longrightarrow B = I_L(X),$$

possibly after modifying X. The Kodaira dimension $\kappa(L)$ of L is defined to be $\dim I_L(X)$ in this case, and to be $-\infty$ otherwise. The general fibers F of I_L enjoys $\kappa(L|_F) = 0$. These definitions generalize directly to $\mathbf{Q}$-line bundles, which are line bundles tensored by $\mathbf{Q}$-divisors. We set

$I_X := I_{K_L}$ and call $\kappa(X) := \kappa(K_X)$ the Kodaira dimension of X. We note that the general fibers of I_X have zero Kodaira dimension.

A fibration $f : X \to Y$ naturally imposes an orbifold structure on Y as follows. Given a reduced divisor D on Y, we may write $f^*D = \sum_i m_i D_i$ for $m_i \in \mathbf{N}$ and reduced irreducible divisors D_i in X. Then we define the (minimum) multiplicity of f over D by

$$m^f(D) = \min\{\ m_i \mid f(D_i) = D\ \}.$$

We note that the classical multiplicity $m_f(D)$ is defined by replacing min above by gcd and is the more natural multiplicity to consider for questions concerning the fundamental group, espoused by its use in [4]. However, our multiplicity $m^f(D)$ is more natural for hyperbolicity questions as we will see. We define

$$D^f = \sum\{\ (1 - \frac{1}{m^f(D)})D \mid D \text{ a reduced irreducible divisor in} X\ \}$$

and call the pair (Y, D^f) an orbifold which we also write as (Y, f) or Y^∂ if no confusion arises. Note that D^f is really defined by a finite sum and that D^f_{red} is the discriminant locus of f. If C is a curve, not necessarily compact, we call $h : C \to Y$ a Z-orbifold map to Y^∂ if $g^*D \geq m^f(D)(g^*D)_{red}$ for all components D in D^f. This gives us the notion of the orbifold Kobayashi pseudometric d_{Y^∂}.

LEMMA 2.1 ([12]). *With the above setup, assume that $d_F \equiv 0$ for a topologically dense set of fibers F of f. Then $d_X = f^*d_{Y^\partial}$ in the sense that $d_X(p,q) = d_{Y^\partial}(f(p), f(q))$ for all $p, q \in X$.*

We define the orbifold canonical bundle of Y^∂ by the $\mathbf{Q}$-line bundle $K_{Y^\partial} = K_Y(D^f)$, understood as a line bundle on Y after taking a tensor power. We define the orbifold Kodaira dimension $\kappa(Y^\partial)$ of Y^∂ by $\kappa(K_{Y^\partial})$. If $\kappa(Y^\partial) = \dim Y$, then Y^∂ and f are said to be of general type. The following is easily derived from the usual Castelnuovo-De Franchis lemma as was observed by Bogomolov in [2], see also [14].

PROPOSITION 2.2 (Castelnuovo-De Franchis, Bogomolov). *Let L be a saturated line subsheaf of Ω^p_X with X projective. Then*

(I) $\kappa(L) \leq p$

(II) *If $\kappa(L) = p$, then I_L defines a fibration from X to a projective base B of dimension p and $I_L^*K_B$ saturates to L in Ω_X.*

(II') *More specifically, outside the singularity of the discriminant locus in B, we have $L = I_L^*K_B(dI_L)$ where dI_L is a divisor given by $\sum I_L^*(D_i) - I_L^*(D_i)_{red}$ and $dI_L \geq I_L^*D^{I_L}$ as $\mathbf{Q}$-divisors with equality achieved on every fiber. The latter implies that $\kappa(B^\partial) = \kappa(L)$ so that B^∂ is an orbifold of general type.*

Parts (I) and (II) are found in [2]. Note that a line sheaf given by (II) and (II') gives rise to a fibration of general type and vise-versa.

DEFINITION 2.3. *We define $\kappa'_+(X)$ to be the maximum $p > 0$ for which a line subsheaf L of Ω^p_X (the sheaf of holomorphic p-forms) exists with $\kappa(L) = p$. X is said to be special if $\kappa'_+(X) = 0$.*

This invariant was introduced in [12] as a natural refinement of an invariant κ^+ given in [3]. Both were really forced upon us by our respective problems. The author brought it to the attention of Bogomolov during this conference in honor of K. Oka, who was surprisingly unaware of its relevance to the problem, in particular of the new notion of orbifold. But he did suggest the relevance of the orbifold Iitaka conjecture to the problem, which this paper partially resolves. He would like to thank him for the suggestion. The following is quoted from [12].

THEOREM 2.4. *Let X be an n-dimensional projective manifold not of general type. We impose the condition for $n = 3$ that $\kappa(X) \neq 0$ and that $\kappa'_+(X) \neq 2$ while for $n > 3$, we let X be a rationally connected fibration over a birationally abelian fibration (possibly with trivial fibres) over any of the above examples. Then $d_X \equiv 0$ if and only if $\kappa'_+(X) = 0$.*

§3. An orbifold Iitaka conjecture on Kodaira dimension

Fibrations whose general fibers are special are called special fibrations. Their importance is already clear from tha last section. The key question that arises is whether there is a unique line sheaf associated to $\kappa'_+(X)$ and whether the fibration(s) associated with $\kappa'_+(X)$ have special fibers. The answer can be checked directly for a large class, including those in the theorem above, but we refer the reader to [12] for details. These questions are equivalent and are resolved in the affirmative via:

QUESTION 3.1. *Let $f : X \to Y$, $g : Y \to Z$ be fibrations such that $h = g \circ f$ is a fibration of general type. If the restrictions of f to the general fibers of h are also of general type, is f then of general type?*

We address this question in [13], where we prove the orbifold versions of the theorems proved here. In the case f is the identity map and Z is of general type, theorems of Vieweg (see [6]), Kawamata ([8]) and Kollar ([9]) independently answer this question in the affirmative. These theorems are in turn special cases of the $\mathrm{C}_{n,m}$ conjecture of Iitaka whose orbifold generalization answers the above question in the affirmative.

We prove here the following case of this orbifold $\mathrm{C}_{n,m}$ conjecture.

THEOREM 3.2. *Let $f : X \to Y$ and $g : Y \to Z$ be projective fibrations with g of general type. Then $\kappa(Y^{\partial}) \geq dim(Z) + \kappa(Y_z^{\partial})$, where Y_z^{∂} is the orbifold general fiber of g.*

This is essentially a direct corrolary of the following theorem, which generalizes slightly the results of the people mentioned above. The details of this implication can be found in [4] (or [13]) as given in [8, 6].

THEOREM 3.3. *Assume that $f : X \to Y$ is a fibration where X, Y are projective. Set $K_{X/Y^{\partial}} = K_X \otimes f^*Y^{\partial}$ as Q-line bundles. Let m be a positive integer such that D^f is a Cartier divisor in Y. Then $f_*(K_{X/Y^{\partial}}^{\otimes m})$ is weakly positive in the sense of* [15], *defined also in* [6].

Without the orbifold ∂, this is the Kawamata-Vieweg's theorem. Weak positivity is a generalization of nefness of a vector sheaf to the case of torsion free coherent sheaf and so enjoys the following properties:
(P1) A torsion free coherent sheaf is weakly stable if it contains a weakly stable subsheaf of the same rank.
(P2) Let Υ be a torsion free coherent sheaf on Y. Let $v : Y' \to Y$ be a flat base change. If $v^*\Upsilon$ is weakly positive, then so is Υ.

Proof of Theorem 3.3: Recall that $D = D^f_{red} = \sum_i D_i$ is a simple normal crossing divisor in Y. Write $f^*D_i = \sum_j m_{ij}D_{ij}$ and let M_i be the least common multiple of $\{ m_{ij} \mid f(D_{ij}) = D_i \}$. The Kawamata branched covering trick ([8]) guarantees the existence of a finite Galois cover $v : Y' \to Y$ with Y' smooth such that $v^*D_i = M_i D'_i$ for some effective divisor. We let $u : X' \to X$ be the composition $u = u_1 \circ d$, where $u_1 : X_1 \to X$ is the base change of X by v composed with the normalisation map, and $d : X' \to X_1$ is a desingularisation which is an isomorphism above the smooth locus of X_1. We also denote by $f_1 : X_1 \to Y'$ and $f' = f_1 \circ d : X' \to Y'$ the resulting pullbacks of f by v. Note that if $f(D_{ij}) = D_i$, then $u^*(m_{ij}D_{ij}) = M_i D'_{ij}$ for some effective divisor D'_{ij} by construction. Hence, applying u^* to the $\mathbf{Q}$-divisor f^*D^f renders it into an effective (integral) divisor on X'.

Now, a local computation shows the following natural inclusion of torsion free coherent sheaves of the same rank over Y':

$$u_{1*} : (f')_*\big(K^{\otimes m}_{X'/Y'}\big) \to v^* f_*\big(K_{X/Y}(-f^*D^f)^{\otimes m}\big) = v^* f_*\big(K^{\otimes m}_{X/Y^{\partial}}\big). \qquad (*)$$

A round about way to see this is as follows: This fact without the $(-f^*D^f)$ is well-known (eg., [8], [6, Lemma 13]). Hence, we only need to check $(*)$ above an analytic neighborhood V in Y of a generic point $p \in D'_i = v^{-1}D_i$. Let D'_{ij} be a component of $f_1^{-1}D'_i$ dominating D_{ij} and let U be a small neighborhood in X_1 of a generic point q on D'_{ij} above

p. These neighborhoods as j varies cover $f_1^{-1}(V)$ outside a codimension two subset if we include also the open subset $f_1^{-1}(V \setminus D'_i)$, above which $K_{X_1/Y'}$ and $u_1^* K_{X/Y}$ are naturally identical. Since $d_* K_{X'/Y'} = K_{X_1/Y}$ on X_1, see the proof of Lemma 13 in [6], and since X_1 is normal, it is sufficient to show that $u_1^* K_{X/Y}(-u_1^*(df))$ is naturally identified with $K_{X_1/Y'}$ on U, where $(df) = \sum f^*(D_i) - f^*(D_i)_{red} \geq f^* D^f$ is as given in part (II') of proposition 2.2. This identification is easily seen from the fact that, on a neighborhood of q, we can write $u = u_1 = u_{ij} \circ u'_{ij}$ where $u_{ij} : X_{ij} \to X$ is obtained from the base change by an m_{ij}-cyclic cover Y_{ij} of Y branched precisely on D_i and u'_{ij} is obtained from the remainder (M_i/m_{ij})-cyclic cover of Y' over Y_{ij}. Let $f_{ij} : X_{ij} \to Y_{ij}$ be the map corresponding to f. Then by construction, $K_{X_{ij}} = u_{ij}^* K_X$ and f_{ij} is smooth on a neighborhood W of $u'_{ij}(q)$. The first implies $K_{X_{ij}/Y_{ij}} = u_{ij}^* K_{X/Y}(-(df))$ on W while the second implies that $K_{X_1/Y'} = (u'_{ij})^* K_{X_{ij}/Y_{ij}}$ on $U \cap u'^{-1}_{ij}(W)$ as required for $(*)$. Combining the theorem of Kawamata-Vieweg with $(*)$ via (P1) and (P2) now gives us theorem 3.3 and hence theorem 3.2. Q.E.D.

By taking $X = Y$ and $f = id$, we get the following corollaries.

THEOREM 3.4. *Let $g : Y \to Z$ be a projective fibration of general type. Then $\kappa(Y) \geq dim(Z) + \kappa(Y_z)$, where Y_z is the general fiber. In particular, if the general fibers of g are of general type, then Y is also.*

THEOREM 3.5. *A projective manifold X is special if $\kappa(X) = 0$.*

§4. A canonical special fibration

Due to page limitations, this section will be brief but with complete references. We restrict to the projective category here.

We first observe that the general fiber of any fibration has the same κ'_+ as one can deduce from semi-continuity and other facts in the subject ([13]). Let X be a smooth projective variety and T the union of all irreducible components of the Chow scheme $Chow(X)$ whose general points t correspond to irreducible reduced special subvarieties V_t of X, including the trivial ones. T gives an equivalence relation on X where $x, x' \in X$ are T-equivalent if and only if they are contained in a T-chain (i.e., a connected union of finitely many of the V_t's). We say that X is T-connected if all pairs of points in X are T-equivalent. If X is T-connected, then X is special: For if $f : X \to B$ is a fibration of general type, then T-connectedness gives us a family $Y \to S \to T$ such that the general Y_s is a special subvariety in X and that the composition $Y \to X \to B$ is generically finite, showing that Y is of general type and hence so are the general fibers Y_s, a contradiction. Hence we can use the construction of [4, §14], see also [11, 10, 5], to obtain:

THEOREM 4.1. *There is a unique fibration $C_X^\partial : X \to C^\partial(X)$, holomorphic outside a normal crossing divisor in $C^\partial(X)$, such that*

(1) *The general fibre of C_X^∂ is special.*

(2) *If $Z \subset X$ is a special subvariety of X going through a general point $a \in X$,then Z is contained in the fiber of C_X^∂ through a.*

(3) *If X is not of general type, then C_X^∂ has positive fiber dimension.*

(4) *The general point of the orbifold base $C^\partial(X)^\partial$ is not contained in a nontrivial special orbifold subvariety (definition given in the proof).*

Proof: The first parts can be found in the references above, so we only need to show parts (3) and (4) here. By the assumption of (3), $\kappa_+(X) < \dim X$ so that we have a fibration $f : X \to Z$ of general type with positive fiber dimension. Now the general fibers X_z of X are not of general type, for otherwise X would be of general type by theorem 3.4. Hence we can repeat this process with X_z until we end with a positive dimensional special subvariety through the general point of X. For (4), we define κ'_+ for an orbifold X^∂ by the same definition as for $\kappa'_+(X)$ except we replace subsheaves of Ω_X with subsheaves of a canonical locally free subsheaf Ω_{X^∂} of $\Omega_{X'}$ (X' being a Galois cover that kills the orbifold structure of X) invariant under the Galois group. We call X special if $\kappa'_+(X^\partial) = 0$. Then (4) follows from the following easily deduced lemma. See [13] for more details on these.

LEMMA 4.2. *X is special if it has a special fibration $X \to Y$ such that Y^∂, understood as (Y, D^f), is a special orbifold.* *Q.E.D.*

Now, a projective variety X that is rationally connected (having a rational curve through every pair of points of X) or have zero Kodaira dimension is special by above and theorem 3.5, respectively. Hence (1) and (2) of theorem 4.1 shows that the Iitaka and rationally connected fibrations of X are factors of C_X^∂. This coupled with (3) solves the Mori's alternative stated at the introduction.

Also, assuming the weak log-abundance conjecture that orbifolds of negative Kodaira dimension admits a canonical fibration (called Mori fibration) with positive dimensional fibers, then lemma 4.2 makes it clear that C_X^∂ is nothing but the iteration of the orbifold Iitaka and Mori fibrations ending at an orbifold base of general type, see [13].

We end with a few remarks comparing our definition of a special manifold with that given by Campana in [4], which uses the usual gcd definition for the multiplicity of a fiber rather than the definition via the minimum multiplicity of the components of each fiber used here. The obvious advantage of Campana's definition is its strong relation to the fundamental group. The equivalence of our two different definitions for special fibrations and manifolds is related to the Shafarevich conjecture concerning the universal cover and the jury is still out on this although we are able to prove this equivalence for a large class of manifolds. Lemma 4.2 remains a conjecture in the Campana's category. However, except for this, Mori's alternative and part (4) of theorem 4.1, essentially all the other results here but in the Campana category can be found in [4], which follow from our results but not in general conversely.

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