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ABSTRACT. Ochiai has proved that the Beilinson-Kato Euler systems for modular forms interpolate in nearly ordinary p-adic families (Howard has obtained a similar result for Heegner points). The principal goal of this article is to generalize Ochiai's work in the level of Kolyvagin systems so as to prove that Kolyvagin systems associated to Beilinson-Kato elements interpolate in the full deformation space (beyond the nearly ordinary locus). We exhibit applications of our *big* Kolyvagin system towards Greenberg's main conjectures, by defining (among other things) an ideal of the analytic generic fiber of the universal deformation space, that behaves like an algebraic p-adic L-function (in 3-variables).

CONTENTS

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1. INTRODUCTION

Fix forever a prime $p > 2$. Classical Iwasawa theory concerns the variation of arithmetic invariants of number fields in a $\mathbb{Z}_p$-tower. This has been extended by Mazur (resp., by Greenberg) to study abelian varieties (resp., motives) in a $\mathbb{Z}_p$-extension. All these may be seen as special cases of Mazur's general theory of Galois deformations. With this perspective, Greenberg in [Gre94, Conjecture 4.1] formulated a "main conjecture" for a p-adic deformation of a motive.

The results of this article are closely related to Greenberg's main conjecture: Given a 'big' Galois representation (attached to a p-adic deformation of a motive), we prove (in a wide variety of cases) that an associated 'big' Kolyvagin system exists as well (Theorem A below). As a standard application, we deduce that the relevant Selmer groups are controlled by the big Kolyvagin systems we prove to exist (Theorem C). Furthermore, when the motive in question is that attached to an elliptic modular form, we show that the Kolyvagin system we prove to exist interpolates (in an appropriate sense) the Beilinson-Kato Kolyvagin systems associated to the 'modular points' of the p-adic deformation space (Theorem B).

Before we explain our results in detail, we provide a quick overview of Mazur's theory of Galois deformations; see [Maz89, dSL97, Gou01] for details.

1.1. **Deformations of Galois representations.**

Let Φ be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}$ be the ring of integers of Φ. Let $\varpi \in \mathcal{O}$ be a uniformizer, and let $\mathtt{k} = \mathcal{O}/\varpi$ be its residue field. Consider the following category $\mathbf{C}$:

- An object of $\mathbf{C}$ is a commutative, complete, local, Noetherian $\mathcal{O}$-algebra A whose residue field $\mathtt{k}_A = A/\mathfrak{m}_A$ is isomorphic to $\mathtt{k}$, where $\mathfrak{m}_A$ denotes the maximal ideal of A.
- A morphism $f : A \to B$ in $\mathbf{C}$ is a local $\mathcal{O}$-algebra morphism.

Let Σ be a finite set of places of $\mathbb{Q}$ that contains p and ∞. Let $G_{\mathbb{Q},\Sigma}$ denote the Galois group of the maximal extension $\mathbb{Q}_\Sigma$ of $\mathbb{Q}$ unramified outside Σ. Fix an absolutely irreducible, continuous Galois representation

$$\bar{\rho} : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_n(\mathtt{k}),$$

and let $\bar{T}$ be the representation space (so that $\bar{T}$ is an n-dimensional $\mathtt{k}$-vector space on which $G_{\mathbb{Q},\Sigma}$ acts continuously).

Let $D_{\bar{\rho}} : \mathbf{C} \longrightarrow \mathtt{Sets}$ be the functor defined as follows. For every object A of $\mathbf{C}$, $D_{\bar{\rho}}(A)$ is the set of continuous homomorphisms

$$\rho_A : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_n(A)$$

that satisfy $\rho_A \otimes_A k \cong \bar{\rho}$, taken modulo conjugation by the elements of $\mathrm{GL}_n(A)$. For every morhism $f : A \to B$ in $\mathbf{C}$, $D_{\bar{\rho}}(f)(\rho_A)$ is the $\mathrm{GL}_n(B)$-conjugacy class of $\rho_A \otimes_A B$.

Theorem (Mazur)**.** *The functor $D_{\bar{\rho}}$ is representable.*

In other words, there is a ring $R(\bar{\rho}) \in \mathrm{Ob}(\mathbf{C})$ and a continuous representation

$$\boldsymbol{\rho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(R(\bar{\rho}))$$

such that for every $A \in \mathrm{Ob}(\mathbf{C})$ and any continuous representation $\rho_A : G_{\mathbb{Q},\Sigma} \to \mathrm{GL}_n(A)$, there is a unique morphism $f_A : R(\bar{\rho}) \to A$ for which we have

$$\boldsymbol{\rho} \otimes_{R(\bar{\rho})} A \cong \rho_A.$$

The ring $R(\bar{\rho})$ is called the universal deformation ring and $\boldsymbol{\rho}$ the universal deformation of $\bar{\rho}$.

Let $\mathrm{ad}\,\bar{\rho}$ be the adjoint representation. We say that the deformation problem for $\bar{\rho}$ is *unobstructed* if the following hypothesis holds true:

(**H.nOb**) $H^2(G_{\mathbb{Q},\Sigma}, \mathrm{ad}\,\bar{\rho}) = 0.$

When (**H.nOb**) holds true, Mazur proved that $R(\bar{\rho}) \cong \mathcal{O}[[X_1, \cdots, X_d]]$, where d is the dimension of the $\mathtt{k}$-vector space $H^1(G_{\mathbb{Q},\Sigma}, \mathrm{ad}\,\bar{\rho})$.

1.1.1. *p-ordinary families.* One may also study a subclass of deformations of a given $\bar{\rho}$, rather than the full deformation space $R(\bar{\rho})$. The following paragraph illustrates a particular case which has been much studied by many authors. Suppose

$$\bar{\rho} : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_2(\mathtt{k})$$

is p-ordinary and a p-distinguished, in the sense that the restriction of $\bar{\rho}$ to a decomposition group at p is reducible and non-scalar. Assume further that $\bar{\rho}$ is odd, i.e., $\det(\rho)(c) = -1$ where c is any complex conjugation. Then Serre's conjecture [Ser87] (as proved in [KW09, Kis09a]) implies that $\bar{\rho}$ arises from an ordinary newform. Hida associates in [Hid86b, Hid86a] such f a family of ordinary modular forms and a Galois representation $\mathcal{T}$ attached to the family, with coefficients in the *universal ordinary Hecke algebra* $\mathfrak{H}$. Thanks to the "$R = T$" theorems proved in [Wil95, TW95] (and their refinements) it follows that $\mathfrak{H}$ is the universal ordinary deformation ring of $\bar{\rho}$, parametrizing all ordinary deformations of $\bar{\rho}$. Ochiai in [Och05] (resp., Howard in [How07]) has studied the Iwasawa theory of this family of Galois representations by interpolating Kato's Euler system (resp., Heegner points) for each member of the family to a 'big' Euler system for the whole p-ordinary family.

1.2. **Statements of the results.** The main goal of the current article is to generalize the works of Ochiai and Howard so as to treat the full deformation ring (e.g., not necessarily the p-ordinary locus) of a mod p Galois representation. Our approach is altogether different from theirs: Instead of interpolating Euler systems (as in [Och05, How07]), we instead deform Kolyvagin systems. An important point to note is that a Kolyvagin system has exactly the same use as an Euler system, when they are used to bound Selmer groups. We recall the definition of a Kolyvagin system in §3.2 below. See also [MR04, §3.2] for the relation between Euler systems and Kolyvagin systems over a DVR or over the cyclotomic Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\Gamma]]$, where $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ is the Galois group of the (unique) $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty/\mathbb{Q}$.

1.2.1. *Setup.* In this paper we will study the deformation problem for Kolyvagin systems to one of the following choices of rings:

(i) $\mathfrak{R} = \mathcal{R}[[\Gamma]]$, where $\mathcal{R}$ is a dimension-2 Gorenstein $\mathcal{O}$-algebra with a regular sequence $\{\varpi, X\}$ such that $\mathcal{R}/X$ is a finitely generated torsion-free $\mathcal{O}$-module.
(ii) $R = \mathcal{O}[[X_1, X_2, X_3]]$ (which we will think of as an *unobstructed* universal deformation ring of a two dimensional mod ϖ Galois representation $\overline{\rho}$ in examples.)

Let $\mathfrak{m}$ (resp., $\mathcal{M}$) be the maximal ideal of $\mathfrak{R}$ (resp., of R) and $\mathtt{k} = \mathfrak{R}/\mathfrak{m}$ (resp., $R/\mathcal{M}$ which we also denote by $\mathtt{k}$ as there will be no danger of confusion) be the residue field. When the coefficient ring we are interested in is the ring $\mathfrak{R}$ as in (i) above, we let $\mathcal{T}$ be a free $\mathcal{R}$-module of finite rank which is endowed with a continuous $G_{\mathbb{Q}}$-action, unramified outside a finite set of primes. Set $\mathfrak{T} = \mathcal{T} \otimes_{\mathbb{Z}_p} \Lambda$, where we allow $G_{\mathbb{Q}}$ act on both factors. When the coefficient ring we are interested in is R as in (ii), we let $\mathbb{T}$ be a free R-module of finite rank endowed with a continuous $G_{\mathbb{Q}}$-action unramified outside a finite number of primes. Let Σ be a finite consisting of primes at which $\mathbb{T}$ is ramified, p and ∞. Let $\mathbb{Q}_\Sigma$ denote the maximal extension of $\mathbb{Q}$ unramified outside Σ. We set $\bar{T} = \mathfrak{T}/\mathfrak{m}$ (resp., $\bar{T} = \mathbb{T}/\mathcal{M}$) and define $\chi(\mathfrak{T}) = \dim_{\Bbbk} \bar{T}^-$ (resp., $\chi(\mathbb{T})$), where $\bar{T}^-$ is the (-1)-eigensubspace of $\bar{T}$ under the action of a fixed complex conjugation.

Let $\boldsymbol{\mu}_{p^\infty}$ be the p-power roots of unity and for any $\mathcal{O}[[G_{\mathbb{Q}}]]$-module M, we let $M^* = \mathrm{Hom}\,(M, \boldsymbol{\mu}_{p^\infty})$ denote its Cartier dual.

The following hypotheses will play a role in what follows:

(**H1**) $\bar{T}$ is an absolutely irreducible $G_{\mathbb{Q}}$-module.
(**H2**) There is a $\tau \in G_{\mathbb{Q}}$ such that τ acts trivially on $\boldsymbol{\mu}_{p^\infty}$ and the R-module $\mathbb{T}/(\tau-1)\mathbb{T}$ (resp., the $\mathfrak{R}$-module $\mathfrak{T}/(\tau-1)\mathfrak{T}$) is free of rank one.
(**H3**) $H^0(\mathbb{Q}, \bar{T}) = H^0(\mathbb{Q}, \bar{T}^*) = 0$.
(**H4**) Either

(i) $\mathrm{Hom}_{\mathbb{F}_p[[G_{\mathbb{Q}}]]}(\bar{T}, \bar{T}^*) = 0$, or
(ii) $p > 4$.

(**H.Tam**) For all bad primes ℓ,

(i) $H^0(\mathbb{Q}_\ell, \bar{T}) = 0$.
(ii) $H^0(I_\ell, A)$ is p-divisible.

(**H.nA**) $H^0(\mathbb{Q}_p, \bar{T}^*) = 0$.

Remark 1.1. The hypotheses (**H1**)-(**H4**) are also present in [MR04]. (**H.Tam**) will be used in §4.1 to check that the unramified local conditions are *cartesian*[1] in a sense to be made precise.

Remark 1.2. When $\mathcal{T}$ is the self-dual Galois representation attached to a (twisted) Hida family with coefficients in the universal ordinary Hecke algebra $\mathcal{R}$ (as studied in [How07]), the hypothesis (**H.Tam**)(ii) asks that there is a single member f of the twisted Hida family such that the local Tamagawa number (as defined in [FPR94, §I.4.2.2]) $c_\ell(f)$ is prime to p. As explained in [Büy13a, §3], this in turn implies that the Tamagawa number $c_\ell(g)$ is prime to p for every member g of the twisted family.

See §5.1 (particularly, Proposition 5.1 and Remark 5.2) for a discussion of the content of the hypotheses (**H.Tam**) and (**H.nA**) when $\bar{T}$ is the mod p Galois representation attached to an elliptic curve $E/\mathbb{Q}$. We only note here that these hypotheses simultaneously hold true for infinitely many primes p in that setting, so our results below are not vacuous.

1.2.2. *Results.* For a $G_{\mathbb{Q}}$-representation $\mathbb{T}$ (resp., $\mathfrak{T}$) as in §1.2.1, we let $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ (resp., $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$) denote the R-module (resp., the $\mathfrak{R}$-module) of *big* Kolyvagin systems for the *canonical Selmer structure* $\mathcal{F}_{\text{can}}$ on $\mathbb{T}$ (resp., for $\mathfrak{T}$). See §2 and §3.2 for precise definitions of these objects.

Theorem A (Theorem 3.12 below)**.** *Suppose* $\chi(\mathfrak{T}) = \chi(\mathbb{T}) = 1$. *Under the hypotheses* (**H1**)-(**H4**), (**H.Tam**) *and* (**H.nA**),

[1]Most importantly in the proof that the map β that appears in Lemma 4.6 is injective. Although one may possibly verify this fact under less restrictive hypothesis, the assumption (**H.Tam**) is simple to state, easy to check and thus allows one to produce many interesting examples where Theorem A applies; c.f., Proposition 5.1 below.

(i) *the R-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one, generated by a Kolyvagin system $\boldsymbol{\kappa}$ whose image $\bar{\boldsymbol{\kappa}} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is non-zero,*

(ii) *the $\mathfrak{R}$-module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one. When the ring $\mathcal{R}$ is regular, the module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is generated by $\boldsymbol{\kappa}$ whose image $\bar{\boldsymbol{\kappa}} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ is non-zero.*

In Theorem A, the modules $\mathbb{T}$ (resp., $\mathfrak{T}$) should be thought of as a family of Galois representations and the conclusion of Theorem A as an assertion that the Kolyvagin systems for each individual member of the family $\mathbb{T}$ (resp., $\mathfrak{T}$) interpolate to give rise to a 'big' Kolyvagin system. We note that a variant of Theorem A plays an important role in [Büy13b], see particularly Remark 3.8 in loc.cit.

Remark 1.3. The arguments used in the proof of Theorem A generalize without any effort to handle a general regular ring (not necessarily of relative dimension 3 over $\mathcal{O}$, as R above is). However, for arithmetic applications of our theorem we would need the big Kolyvagin system for $\mathbb{T}$ (resp., for $\mathfrak{T}$) of Theorem A interpolates Kolyvagin systems which are explicitly related to L-values. At the moment, this is only possible when the residual representation $\bar{T}$ is two dimensional and $\chi(\bar{T}) = 1$, in which case the unobstructed universal deformation ring is R. Note that when $\bar{T}$ is two dimensional and $\chi(\bar{T}) = 1$, we know that $\bar{T}$ is modular. Kisin's recent (unpublished) R = *big* T (only in this sentence T refers to (Katz) Hecke algebra acting on the space of p-adic modular forms) theorem shows that the big Kolyvagin system for the universal deformation $\mathbb{T}$ of $\bar{T}$ indeed interpolates Kato's Kolyvagin systems for elliptic modular forms whose associated Galois representations are congruent to $\bar{T}$.

Until the end of the Introduction, we concentrate on a particular representation $\mathbb{T}$ and give an important application of Theorem A. See the main body of the article for the most general form of the results we record in the rest of this section. Let $E_{/\mathbb{Q}}$ be an elliptic curve, $\bar{T} = E[p]$ be the p-torsion subgroup of $E(\bar{\mathbb{Q}})$ and $\bar{\rho} = \bar{\rho}_E$ the mod p Galois representation on $\bar{T}$. Let Σ be the set of primes that consists of primes at which E has bad reduction, p and ∞. Define also $R = R(\bar{\rho})$ to be the universal deformation ring of $\bar{\rho}$. The universal deformation representation $\mathbb{T}$ is then a free R-module of rank two.

Theorem (Flach, [Fla92])**.** *Suppose*

(F1) *$\bar{\rho}_E$ is surjective,*
(F2) *$H^0(\mathbb{Q}_\ell, \bar{T} \otimes \bar{T}) = 0$ for all $\ell \in \Sigma$,*
(F3) *p does not divide $\Omega^{-1}L(\mathrm{Sym}^2(E), 2)$, where $\Omega = \Omega(\mathrm{Sym}^2(E), 2)$ is the transcendental period.*

Then E satisfies (**H.nOb**) *and $R \cong \mathbb{Z}_p[[X_1, X_2, X_3]]$.*

In addition to the assumptions (**F1**)$-$(**F3**)*, we suppose throughout the introduction that $\mathbb{T}$ satisfies the hypotheses* (**H1**)-(**H4**), (**H.Tam**)(i) *and* (**H.nA**). See §5.1 for the content of these assumptions in this particular setting when $\bar{\rho} = \bar{\rho}_E$.

Remark 1.4. Weston has proved the following result, which is an important generalization of Flach's theorem to modular forms of higher weight. Let f be an elliptic newform of level N, weight $k > 2$ and character ψ. Let K be the number field generated by the Fourier coefficients of f and $\mathcal{O}_K$ be its ring of integers. For a prime $\wp$ of K above p, let $\mathtt{k} = \mathcal{O}_K/\wp$ and $\mathcal{O} = W(\mathtt{k})$, the Witt vectors of $\mathtt{k}$. Let

$$\bar{\rho} = \bar{\rho}_{f,\wp} : G_{\mathbb{Q},\Sigma} \longrightarrow \mathrm{GL}_2(\mathtt{k})$$

be the Galois representation attached to f by Deligne. Weston proved in [Wes04] that the deformation problem for $\bar{\rho}$ is unobstructed and $R(\bar{\rho}) \cong \mathcal{O}[[X_1, X_2, X_3]]$, for almost all choices of a prime $\wp$ of K. Using Weston's theorem, all our results proved in weight two generalizes *verbatim* to higher weights.

We next state a consequence of Theorem A, which should be thought of an extension of a result of Ochiai (on the interpolation of Beilinson-Kato *Euler system* in p-ordinary familes), beyond the ordinary locus. Let us first set our notation. Let $f = \sum a_n q^n$ be a newform of weight $\omega \geq 2$ and let $\mathcal{O}_f$ be the finite flat normal extenion of $\mathbb{Z}_p$ which the a_n's generate. Let

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}_f)$$

be Deligne's Galois representation attached to f and T_f be the free $\mathcal{O}_f$-module of rank two on which $G_{\mathbb{Q}}$ acts via ρ_f. Suppose that $\bar{\rho}_f \cong \bar{\rho}$, so that by the universality of R there is a ring homomorphism $\varphi_f : R \to \mathcal{O}_f$ which induces an isomorphism $\mathbb{T} \otimes_{\varphi_f} \mathcal{O}_f \cong T_f$ and a map

$$\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) \xrightarrow{\varphi_f} \overline{\mathbf{KS}}(T_f, \mathcal{F}_{\text{can}}, \mathcal{P}).$$

For each f as above, Kato's Beilinson element Euler system also gives rise to a Kolyvagin system $\boldsymbol{\kappa}^{\text{Kato},(f)} \in \overline{\mathbf{KS}}(T_f, \mathcal{F}_{\text{can}}, \mathcal{P})$ (c.f., [MR04, §3.2]). Our result on the interpolation of these Kolyvagin systems reads as follows.

Theorem B (Theorem 5.3)**.** *Fix a generator $\boldsymbol{\kappa}$ of the R-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$. For every newform f as above, $\kappa^{\text{Kato},(f)} = \lambda_f \cdot \varphi_f(\boldsymbol{\kappa})$ for some $\lambda_f \in \mathcal{O}_f$.*

The next result we present (more specifically, its final part) is the standard application of the Kolyvagin system machinery, proved essentially by Ochiai in [Och05]. Let $\mathcal{F}_{\text{can}}^*$ be the dual Selmer structure on $\mathbb{T}^*$ (in the sense of Definition 2.4). For any abelian group N, let $N^\vee$ denote the Pontryagin dual. If M is a finitely generated torsion R-module, set

$$\text{char}(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{\text{length}(M_{\mathfrak{p}})}$$

where the product is over height one primes of R.

Given a Kolyvagin system $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$, let $\kappa_1 \in H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T})$ denote its *leading term*; see Definition 3.13 for a precise definition of this notion.

Theorem C. *Let $\boldsymbol{\kappa}$ be a generator of the R-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$. Suppose that E has good reduction at p. Then,*

(i) *κ_1 is not R-torsion.*
(ii) *The R-module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)$ is cotorsion and $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T})$ is free of rank one.*
(iii) $\text{char}\left(H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)^\vee\right) = \text{char}\left(H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T})/R \cdot \kappa_1\right)$.

See Proposition 5.6 and Remark 4.23 for (i), Theorems 5.11 and 5.17 for (ii), Theorem 5.18 and Corollary 5.19 for (iii). The statement of (iii) is closely related to Greenberg's main conjecture [Gre94] on the Iwasawa theory of p-adic deformations of motives; compare our statement especially with Kato's formulation of the main conjecture (*without p-adic L-functions*) in [Kat93].

Remark 1.5. We will make use of Theorem C in a future work to study the p-adic variation of Iwasawa invariants of the members of the family $\mathbb{T}$, in the spirit of [EPW06] and [Och05].

Remark 1.6. Under the additional hypothesis that E has ordinary reduction at p, one may use the work of Skinner and Urban [SU13] along with (iii) to verify that the *interpolation factors* λ_f (which appeared in the statement of Theorem B) are all units (even when the newform f is not necessarily p-ordinary). This allows one to prove that the divisibility one obtains using the Beilinson-Kato elements in the statement of Kato's main conjecture (c.f., [Kat04, Theorem 12.5]) may be turned into an equality.

See Remark 5.4 below for further details.

1.3. **Greenberg's main conjecture, the Pottharst Selmer group and the Kolyvagin constructed p-adic L-function.** Greenberg's main conjecture implicitly predicts the existence of a 3-variable p-adic L-function attached to the universal deformation family $\mathbb{T}$. This p-adic L-function should in return control a 'big' Selmer group associated to $\mathbb{T}$. The existence of this p-adic L-function and the'big' Selmer group are both highly conjectural, although there has been (partial) progress in the recent years. See [Bel12a] for the construction of a 2-*variable* p-adic L-function on the eigencurve and [Pot12a, Bel12b] for the construction of a 'big' Selmer group attached to certain p-adic families. These important advances still fall short of the target. In order to link the assertions of Theorem C to Greenberg's conjecture, we propose a conjectural description of a "Pottharst Selmer group" attached to the family $\mathbb{T}$ that interpolates (in an appropriate sense) the Bloch-Kato Selmer structures on the members of this family.

Definition 1.7. Let $E/\mathbb{Q}_p$ be a finite extension and let $\mathcal{O}_E$ be its ring of integers. An E-valued *pseudo-geometric specialization* is a ring homomorphism $\psi : R \to \mathcal{O}_E$ such that the $G_{\mathbb{Q}_p}$-representation is $\mathbb{T} \otimes_\psi E$ is potentially semi-stable with distinct Hodge-Tate weights.

Let $X = \mathrm{Spec}\, R$ denote universal deformation space. Recall that we are working under the assumption (**H.nOb**) that the deformation problem is unobstructed. Let $\mathfrak{X}$ denote (Raynaud's) generic fiber of $\mathrm{Spf}R$ and let $R^\dagger := \Gamma(\mathfrak{X}, \mathcal{O}_{\mathrm{Spf}R})$. We set $\mathbb{T}^\dagger = \mathbb{T} \otimes_R R^\dagger$. Let E be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_E$ its ring of integers. Let $\Lambda_E = \mathcal{O}_E[[\Gamma]]$ be the cyclotomic Iwasawa algebra, $\mathfrak{I}_E$ be the generic fiber of $\mathrm{Spf}\,\Lambda$ and $\Lambda_E^\dagger = \Gamma(\mathfrak{I}_E, \mathcal{O}_{\mathrm{Spf}\,\Lambda_E})$. Note that our $\Lambda_E^\dagger$ is denoted by Λ_∞ in [Pot12b].

Remark 1.8. Let $\psi^\dagger : R^\dagger \to E$ be an E-valued point. By continuity, we have an induced $\mathcal{O}_E$-valued point $R \to \mathcal{O}_E$ that we denote by ψ. We say that $\psi^\dagger$ is pseudo-geometric if ψ is in the sense of Definition 1.7. We denote the $G_{\mathbb{Q},\Sigma}$-representation $\mathbb{T} \otimes_\psi E$ by $V_{\psi^\dagger}$.

Definition 1.9. A pseudo-geometric specialization $\psi^\dagger$ is said to be finite-slope if the φ-action on the potentially semi-stable Diuedonné module verifies $D_{\mathrm{pst}}(V_{\psi^\dagger})^{\varphi=\lambda} \neq 0$ for some λ.

Given $\psi^\dagger \in \mathfrak{X}(E)$ and $\psi \in X(\mathcal{O}_E)$ as above, the deformation $(\mathbb{T} \otimes_\psi \mathcal{O}_E) \otimes \Lambda_E$ of $\bar{\rho}$ to Λ_E induces a ring homomorphism $\psi_\Lambda : R \to \Lambda_E$, which in turn induces a map

$$\psi_\Lambda^\dagger : R^\dagger \longrightarrow \Lambda_E^\dagger$$

on the ring of sections of the generic fibers. Set $\widetilde{V}_{\psi^\dagger} := \mathbb{T}^\dagger \otimes_{\psi_\Lambda^\dagger} \Lambda_E^\dagger$, the *cyclotomic deformation* of $V_{\psi^\dagger}$.

Definition 1.10. Let $\mathcal{C}_{\text{pst-fs}}(\bar{\rho}) \subset \mathfrak{X} \times \mathbb{G}_m$ denote the set of points $(\psi^\dagger, \lambda)$ such that $\psi^\dagger$ is finite-slope with Hodge-Tate weights $0, k-1$ for $k \in \mathbb{Z}_{\geq 2}$. By a theorem of Kisin [Kis09b], it is known that the Zariski-analytic closure of this set is the full Coleman-Mazur $\bar{\rho}$-eigencurve $\mathcal{C}(\bar{\rho})$. Let $\mathcal{E}_{\text{pst-fs}}(\bar{\rho})$ be the image of $\mathcal{C}_{\text{pst-fs}}(\bar{\rho})$ under the projection onto the first component inside of Mazur's infinite fern $\mathcal{E}(\bar{\rho}) \subset \mathfrak{X}$.

The following conjecture predicts the existence of a *Pottharst local condition* on $\mathbb{T}^\dagger$ that interpolates the Bloch-Kato local conditions on the points of the (Zariski-dense) subset $\mathcal{E}_{\text{pst-cl}}(\bar{\rho})$.

Conjecture 1. *There is an $R^\dagger$-submodule* (the Pottharst submodule)

$$H^1_{\text{Pot}}(\mathbb{Q}_p, \mathbb{T}^\dagger) \subset H^1(\mathbb{Q}_p, \mathbb{T}^\dagger)$$

with the following interpolation property:

$$H^1_{\text{Pot}}(\mathbb{Q}_p, \mathbb{T}^\dagger) \otimes_{\psi^\dagger_\Lambda} \Lambda^\dagger_E \xrightarrow{\sim} H^1_{\text{Pot},\lambda(\psi^\dagger)}(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger})$$

for all finite-slope specializations $\psi^\dagger \in \mathcal{E}_{\text{pst-fs}}(\bar{\rho})$ and for some choice of $\lambda(\psi^\dagger)$ that verifies $D_{\text{pst}}(V_{\psi^\dagger})^{\varphi=\lambda(\psi^\dagger)} \neq 0$ (and varies p-adic analytically with $\psi^\dagger$ on $\mathfrak{X}$). Here $H^1_{\text{Pot},\lambda(\psi^\dagger)}(\mathbb{Q}_p, \tilde{V}_{\psi^\dagger})$ is the local Pottharst subgroup for the cyclotomic deformation attached to the ordinary filtration associated to $\lambda(\psi^\dagger)$.

This conjecture is admittedly overly optimistic; we still state it in this level of generality hoping that it will at least hold in many other similar forms. For example, note that a portion of this conjecture has been explicitly verified on the eigencurve building on the work of Pottharst alluded to above; c.f., Theorem 2 and §3.4 of [Bel12b].

Theorem (Bellaïche)**.** *Let $x_0 = (\psi^\dagger_0, \lambda(\psi^\dagger_0)) \in \mathcal{C}_{\text{pst-fs}}(\bar{\rho})$. There exists an irreducible affinoid neighborhood $D = \operatorname{Sp} S$ of x in an irreducible component of the eigencurve $\mathcal{C}(\bar{\rho})$ through x_0 (so that S is an affinoid domain of dimension 1) with the following property: There is a $\Lambda^\dagger_S := S \otimes_{\mathbb{Z}_p} \Lambda^\dagger$-submodule*

$$H^1_{\text{Pot},\lambda}(\mathbb{Q}_p, \mathbb{T}^\dagger_S) \subset H^1(\mathbb{Q}_p, \mathbb{T}^\dagger_S)$$

(where $\mathbb{T}^\dagger_S = \mathbb{T}^\dagger \otimes_{R^\dagger} \Lambda^\dagger_S$) such that for every E-valued (closed) point $x = (\psi^\dagger, \lambda(\psi^\dagger))$ contained in the (Zariski-dense) subset $D_{\text{pst-fs}} = D \cap \mathcal{C}_{\text{pst-fs}}(\bar{\rho})$ of D we have,

$$H^1_{\text{Pot},\lambda}(\mathbb{Q}_p, \mathbb{T}^\dagger_S) \otimes_{\Lambda^\dagger_S} \Lambda^\dagger_E = H^1_{\text{Pot},\lambda(\psi^\dagger_0)}(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger_0})\,.$$

Remark 1.11. The map $R^\dagger \to \Lambda^\dagger_S$ is the map obtained as follows: Sen's theory equips us with a polynomial $P(T) = T^2 + aT + b \in R^\dagger[T]$ such that for every $x \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$, the Sen polynomial of ρ_x is $P(x) = T^2 + a(x)T + b(x)$. It follows from a theorem of Faltings and Jordan that a point x associated to a modular form then $b(x) = 0$. It follows that $\mathcal{E}(\bar{\rho}) \subset \mathfrak{X}_0$, Sen's null locus, which is the closed subspace cut by $b = 0$. The subspace $\mathfrak{X}_0$ has codimension one everywhere and admits a morphism

$$\operatorname{sp} \Lambda^\dagger \times \mathfrak{X}_0 \longrightarrow \mathfrak{X}$$

(which, on the moduli interpretation, is given by Tate-twisitng). We therefore have a morphism

$$\operatorname{sp} \Lambda^\dagger \times \operatorname{sp} S \to \operatorname{sp} \Lambda^\dagger \times \mathfrak{X}_0 \to \mathfrak{X}$$

and the desired map is the map on the global sections.

Assuming the truth of Conjecture 1 above and using the big Kolyvagin system constructed in Theorem C, we have the following (*3-variable*) generalization of Perrin-Riou's [PR95] notion of a module of algebraic p-adic L-function.

Theorem D (Theorem 5.24 below)**.** *Attached to a generator $\boldsymbol{\kappa}$ of $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$, there is an ideal $\mathcal{L}(\boldsymbol{\kappa}) \subset R^\dagger$ that satisfies the following interpolation property:*

$$\psi^\dagger_\Lambda\left(\mathcal{L}(\boldsymbol{\kappa})\right) = \delta^{-1}_{\psi^\dagger} \cdot \Lambda^\dagger_E L_p(f_{\psi^\dagger}, \lambda)$$

for every $\psi^\dagger \in \mathcal{E}_{\text{cl-fs}}(\bar\rho)$ *and for some* $\delta_{\psi^\dagger} \in \Lambda_E^\dagger$. *Here* $L_p(f_{\psi^\dagger}, \lambda)$ *is a* p*-adic* L*-function attached to the modular form* $f_{\psi^\dagger}$ [2].

We call the ideal $\mathcal{L}(\boldsymbol{\kappa})$ the *Kolyvagin constructed* p*-adic* L*-function*.

Remark 1.12. Passing to a suitable affinoid subdomain as above in Bellaïche's theorem, one can obtain less general but unconditional versions of Theorem D. It would be interesting to know the precise relation between Bellïche's two variable p-adic L-function (constructed in [Bel12a]) and the restriction of $\mathcal{L}(\boldsymbol{\kappa})$ to the appropriate affinoid subdomain on the eigencurve. This problem is also considered by D. Hansen in his forthcoming work.

1.3.1. *Outline of the paper.* The paper is organized as follows. After setting our notation and providing the definitions of the basic objects of interest in Section 2, we state our main technical result (Theorem 3.12) in Section 3. We prove this theorem in Section 3 modulo the existence of *core vertices*. The existence of core vertices then is proved in Section 4.

Finally in Section 5, we discuss several applications of our result. Bounds on Selmer groups in terms of the big Kolyvagin system we prove to exist are obtained in Section 5.3. Concrete arithmetic applications concerning the p-adic interpolation of the Beilinson-Kato Kolyvagin systems for a family of elliptic modular forms is discussed in Sections 5.1 and 5.2.

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1.4. **Notations.** For any field K, fix a separable closure $\bar K$ of K and set $G_K = \mathrm{Gal}(\bar K/K)$. Let F be a number field and λ be a non-archimedean place of F. Fix a decomposition subgroup $G_\lambda < G_F$ and let $I_\lambda < G_\lambda$ denote the inertia subgroup. Often we will identify G_λ by G_{F_λ}. For a finite set Σ of places of K, define K_Σ to be the maximal extension of K unramified outside Σ.

Let p be an odd prime and let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension. Let $\boldsymbol{\mu}_{p^n}$ denote the p^n-th roots of unity and set $\boldsymbol{\mu}_{p^\infty} = \varinjlim \boldsymbol{\mu}_{p^n}$. Set $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and fix a topological generator γ of Γ. Let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ be the cyclotomic Iwasawa algebra.

For a ring S, an S-module M and an ideal I of S, let $M[I]$ denote the submodule of M consisting of elements that are killed by I.

For the ring $R = \mathcal{O}[[X_1, X_2, X_3]]$, we set $R_{u,v,w} := R/(X_1^u, X_2^v, X_3^w)$ and $R_{r,u,v,w} := R/(\varpi^r, X_1^u, X_2^v, X_3^w)$. We define the quotient modules $\mathbb{T}_{u,v,w} := \mathbb{T} \otimes_R R_{u,v,w}$ and $\mathbb{T}_{r,u,v,w} := \mathbb{T} \otimes_R R_{r,u,v,w}$.

Similarly for the ring $\mathfrak{R} = \mathcal{R}[[\Gamma]]$ as above, define the rings $\mathfrak{R}_{u,v} = \mathfrak{R}/(X^u, (\gamma-1)^v)$ and $\mathfrak{R}_{r,u,v} = \mathfrak{R}/(\varpi^r, X^u, (\gamma-1)^v)$. Define also the quotient modules $\mathfrak{T}_{u,v} := \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{u,v}$ and $\mathfrak{T}_{r,u,v} := \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{r,u,v}$.

[2] The p-adic L-function $L_p(f_{\psi^\dagger}, \lambda)$ depends on the choice of an eigenvalue $\lambda(\psi^\dagger)$ of the Atkin operator U_p and λ that appears in the notation here is to remind us that fact.

Finally, we define the p-divisible goups $\mathcal{A}_{u,v,w} := \mathbb{T}_{u,v,w} \otimes \Phi/\mathcal{O}$ and $\mathfrak{A}_{u,v} := \mathfrak{T}_{u,v} \otimes \Phi/\mathcal{O}$.

Let $\mathcal{R}_0 = \mathcal{R}/(\varpi, X)$ be the dimension-zero Gorenstein artinian ring, where $\mathcal{R}$ is as above. As explained in [Til97, Proposition 1.4],

$$\mathcal{R}_0[\mathfrak{m}_{\mathcal{R}}] \text{ is a one-dimensional } \Bbbk\text{-vector space} \tag{1.1}$$

where $\mathfrak{m}_{\mathcal{R}}$ denotes the maximal ideal of $\mathcal{R}$ and $\Bbbk = \mathcal{R}/\mathfrak{m}_{\mathcal{R}}$. Define also $\mathcal{R}_1 = \mathcal{R}/X$. Using the fact $\{\varpi, X\}$ is a regular sequence in $\mathcal{R}$, we see that $\mathcal{R}_1$ is a dimension-1 Gorenstein domain. Set $\tilde{\Phi} = \mathrm{Frac}(\mathcal{R}_1)$. As $\mathcal{R}_1$ is finitely generated and free as an $\mathcal{O}$-module, it follows that $\tilde{\Phi}$ is a finite extension of Φ. Let $\mathfrak{O}$ be the integral closure of $\mathcal{R}_1$ in $\tilde{\Phi}$. Then $\mathfrak{O}$ is a discrete valuation ring and $\mathfrak{O}/\mathcal{R}_1$ has finite cardinality. Let $\mathfrak{m}_{\mathfrak{O}}$ be the maximal ideal of $\mathfrak{O}$ and $\pi_{\mathfrak{O}}$ be a uniformizer of $\mathfrak{O}$. Define $T_{\mathfrak{O}} := \mathfrak{T}_{1,1} \otimes_{\mathcal{R}_1} \mathfrak{O}$ (*deformation of* $\bar{T}$ *to* $\mathfrak{O}$) and $A = T_{\mathfrak{O}} \otimes \mathbb{Q}_p/\mathbb{Z}_p$. As $\mathfrak{O}/\mathcal{R}_1$ is of finite order, it follows that $A \cong \mathfrak{A}_{1,1}$.

2. Local Conditions and Selmer groups

We recall a definition from [MR04, §2]. Let M be any $\mathcal{O}[[G_{\mathbb{Q}}]]$-module.

Definition 2.1. A *Selmer structure* $\mathcal{F}$ on M is a collection of the following data:

- A finite set $\Sigma(\mathcal{F})$ of places of $\mathbb{Q}$, including ∞, p, and all primes where M is ramified.
- For every $\ell \in \Sigma(\mathcal{F})$, a local condition on M (which we now view as a $\mathcal{O}[[G_\ell]]$-module), i.e., a choice of an $\mathcal{O}$-submodule $H^1_{\mathcal{F}}(\mathbb{Q}_\ell, M) \subset H^1(\mathbb{Q}_\ell, M)$.

Definition 2.2. For a Selmer structure $\mathcal{F}$ on M, define the Selmer group $H^1_{\mathcal{F}}(\mathbb{Q}, M)$ to be

$$H^1_{\mathcal{F}}(\mathbb{Q}, M) = \ker\left(H^1(\mathbb{Q}_{\Sigma(\mathcal{F})}/\mathbb{Q}, M) \longrightarrow \prod_{\ell \in \Sigma(\mathcal{F})} \frac{H^1(\mathbb{Q}_\ell, M)}{H^1_{\mathcal{F}}(\mathbb{Q}_\ell, M)}\right).$$

Definition 2.3. A *Selmer triple* is a triple $(M, \mathcal{F}, \mathcal{P})$ where $\mathcal{F}$ is a Selmer structure on M and $\mathcal{P}$ is a set of rational primes, disjoint from $\Sigma(\mathcal{F})$.

Definition 2.4. Let $\mathcal{F}$ be a Selmer structure on M. For each prime $\ell \in \Sigma(\mathcal{F})$, define $H^1_{\mathcal{F}^*}(\mathbb{Q}_\ell, M^*) := H^1_{\mathcal{F}}(\mathbb{Q}_\ell, M)^\perp$ as the orthogonal complement of $H^1_{\mathcal{F}}(\mathbb{Q}_\ell, M)$ under the local Tate pairing. The Selmer structure $\mathcal{F}^*$ on M^* defined in this manner is called the *dual Selmer structure*.

Define the Selmer structure $\mathcal{F}_{\text{can}}$ (the *canonical Selmer structure*) on $\mathbb{T}_{u,v,w}$ as follows:

- $\Sigma(\mathcal{F}_{\text{can}}) = \Sigma := \{\ell : \mathbb{T} \text{ is ramified at } \ell\} \cup \{p, \infty\}$.
- $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) := \begin{cases} H^1(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) & , \text{ if } \ell = p, \\ H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) & , \text{ if } \ell \in \Sigma(\mathcal{F}_{\text{can}}) - \{p, \infty\}. \end{cases}$

Here

$$H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) := \ker\left\{H^1(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathbb{T}_{u,v,w} \otimes_{\mathcal{O}} \Phi)\right\}.$$

We denote the Selmer structure on the quotients $\mathbb{T}_{r,u,v,w}$ obtained by *propagating* $\mathcal{F}_{\text{can}}$ on $\mathbb{T}_{u,v,w}$ to $\mathbb{T}_{r,u,v,w}$ also by $\mathcal{F}_{\text{can}}$. See [MR04, Example 1.1.2] for a definition of the *propagation* of local conditions.

We define the Selmer structure $\mathcal{F}_{\text{can}}$ on $\mathfrak{T}_{u,v}$ (and its propagation to its quotients $\mathfrak{T}_{r,u,v}$) in a similar way:

- $\Sigma(\mathcal{F}_{\text{can}}) = \{\ell : \mathfrak{T} \text{ is ramified at } \ell\} \cup \{p, \infty\}$.

- $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{u,v}) := \begin{cases} H^1(\mathbb{Q}_p, \mathfrak{T}_{u,v}) & , \text{ if } \ell = p, \\ H^1_f(\mathbb{Q}_\ell, \mathfrak{T}_{u,v}) & , \text{ if } \ell \in \Sigma(\mathcal{F}_{\text{can}}) - \{p, \infty\} \end{cases}$,

We also define a Selmer structure $\mathcal{F}_{\text{can}}$ on $\mathfrak{T}/\mathfrak{m}$ as follows:

- Set $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_p, \mathfrak{T}/\mathfrak{m}) = H^1(\mathbb{Q}_p, \mathfrak{T}/\mathfrak{m})$.
- Propagate the local conditions at $\ell \neq p$ given by $\mathcal{F}_{\text{can}}$ on $\mathfrak{T}_{1,1}$ to $\mathfrak{T}/\mathfrak{m}$.

Note in particular that the Selmer structure $\mathcal{F}_{\text{can}}$ on $\mathfrak{T}/\mathfrak{m}$ will not always be the propagation of the canonical Selmer structure on $\mathfrak{T}_{1,1}$.

2.1. **Local conditions at $\ell \neq p$.** In this section we compare various alterations of the local conditions at $\ell \neq p$. Define

$$H^1_{\text{ur}}(\mathbb{Q}_\ell, M) := \ker\left(H^1(\mathbb{Q}_\ell, M) \longrightarrow H^1(I_\ell, M)\right)$$

for any M on which $G_{\mathbb{Q}_\ell}$ acts. Using the exact sequence

$$0 \longrightarrow \mathbb{T}_{u,v,w} \longrightarrow \mathbb{T}_{u,v,w} \otimes_{\mathcal{O}} \Phi \longrightarrow \mathcal{A}_{u,v,w} \longrightarrow 0$$

define also

$$H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = \text{im}\left(H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w} \otimes_{\mathcal{O}} \Phi) \longrightarrow \mathcal{A}_{u,v,w}\right).$$

Define finally

$$H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \ker\left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}\right),$$

where the map is induced from the injection $\mathbb{T}_{r,u,v,w} \hookrightarrow \mathcal{A}_{u,v,w}$. Lemma 1.3.8(i) of [Rub00] shows that

(2.1) $$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}).$$

Similarly one defines $H^1_f(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})$ and $H^1_f(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v})$, and verifies using [Rub00, Lemma 1.3.8(i)] that

(2.2) $$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) = H^1_f(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}).$$

2.2. **Local conditions at p.**

Proposition 2.5. *Assuming* (**H.nA**),

(i) $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w}) = H^1(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w})$,
(ii) $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}) = H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v})$.

Proof. We need to check that

$$\text{coker}\left(H^1(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w})\right) = 0.$$

Note that

$$\text{coker}\left(H^1(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{T}_{r,u,v,w})\right) = H^2(\mathbb{Q}_p, \mathbb{T}_{u,v,w})[\varpi^r],$$

so it suffices to check that $H^2(\mathbb{Q}_p, \mathbb{T}_{u,v,w}) = 0$.

Now by local duality, (**H.nA**) implies that $H^2(\mathbb{Q}_p, \bar{T}) = 0$. Using the fact that the cohomological dimension of G_p is two, it follows that

$$H^2(\mathbb{Q}_p, \mathbb{T})/\mathcal{M} \cdot H^2(\mathbb{Q}_p, \mathbb{T}) = 0,$$

where $\mathcal{M} = (\varpi, X_1, X_2, X_3)$ is the maximal ideal of the ring R. By Nakayama's Lemma, we therefore see that $H^2(\mathbb{Q}_p, \mathbb{T}) = 0$. Using again the fact that the cohomological dimension of G_p is two, we conclude that

$$0 = H^2(\mathbb{Q}_p, \mathbb{T}) \twoheadrightarrow H^2(\mathbb{Q}_p, \mathbb{T}_{u,v,w})$$

and the proof of (i) follows.

The proof of (ii) is similar but more delicate as the ring $\mathfrak{R}$ is not necessarily regular. As above, we first check that

$$H^2(\mathbb{Q}_p, \mathfrak{T}) = 0. \tag{2.3}$$

Considering the G_p-cohomology induced from the exact sequence

$$0 \longrightarrow \mathfrak{T} \xrightarrow{\gamma-1} \mathfrak{T} \longrightarrow \mathcal{T} \longrightarrow 0$$

and using Nakayama's lemma, (2.3) is reduced to verifying that $H^2(\mathbb{Q}_p, \mathcal{T}) = 0$. Similarly, using the exact sequences

$$0 \longrightarrow \mathcal{T} \xrightarrow{X} \mathcal{T} \longrightarrow \mathcal{T}/X \longrightarrow 0$$

$$0 \longrightarrow \mathcal{T}/X \xrightarrow{\varpi} \mathcal{T}/X \longrightarrow \mathfrak{T}_{1,1} \longrightarrow 0$$

in turn, we reduce to checking that

$$H^2(\mathbb{Q}_p, \mathfrak{T}_{1,1,1}) = 0 \tag{2.4}$$

The assertion (2.4) is proved below. We first show that (ii) follows from (2.4).

As above,

$$\text{coker}\left(H^1(\mathbb{Q}_p, \mathfrak{T}_{u,v}) \longrightarrow H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v})\right) = H^2(\mathbb{Q}_p, \mathfrak{T}_{u,v})[\varpi^r].$$

By (2.4) and the fact that the cohomological dimension of G_p is two, it follows that

$$0 = H^2(\mathbb{Q}_p, \mathfrak{T}) \twoheadrightarrow H^2(\mathbb{Q}_p, \mathfrak{T}_{u,v}).$$

This proves that $H^2(\mathbb{Q}_p, \mathfrak{T}_{u,v}) = 0$ and it follows that

$$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}) = \text{im}\left(H^1(\mathbb{Q}_p, \mathfrak{T}_{u,v} \longrightarrow H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v})\right) = H^1(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}),$$

as desired. □

Claim. *Assuming* (**H.nA**), *we have* $H^2(\mathbb{Q}_p, \mathfrak{T}_{1,1,1}) = 0$.

Proof. The property (1.1) shows that $\mathfrak{T}_{1,1,1}[\mathfrak{m}] \cong \bar{T}$, hence that $\mathfrak{T}^*_{1,1,1}/\mathfrak{m} \cong \bar{T}^*$. Since we assumed **H.nA**, it thus follows that

$$H^0(\mathbb{Q}_p, \mathfrak{T}^*_{1,1,1}/\mathfrak{m}) = 0.$$

The module $\mathfrak{T}^*_{1,1,1}$ is free of of finite rank over the Gorenstein artinian ring $\mathcal{R}_0$, hence by [MR04, Lemma 2.1.4] we conclude that $H^0(\mathbb{Q}_p, \mathfrak{T}^*_{1,1,1}) = 0$ as well. Claim now follows by local duality. □

2.3. **Kolyvagin primes and transverse conditions.** Let $\tau \in G_{\mathbb{Q}}$ be as in the statement of the hypothesis (**H.2**).

Definition 2.6. For $\bar{\mathfrak{n}} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$, define

(i) $H_{\bar{\mathfrak{n}}} = \ker\left(G_{\mathbb{Q}} \to \mathrm{Aut}(\mathbb{T}_{r,u,v,w}) \oplus \mathrm{Aut}(\boldsymbol{\mu}_{\boldsymbol{p^r}})\right)$,

(ii) $L_{\bar{\mathfrak{n}}} = \bar{\mathbb{Q}}^{H_{\bar{\mathfrak{n}}}}$,

(iii) $\mathcal{P}_{\bar{\mathfrak{n}}} = \{\text{primes } \ell : \mathrm{Fr}_\ell \text{ is conjugate to } \tau \text{ in } \mathrm{Gal}(L_{\bar{\mathfrak{n}}}/\mathbb{Q})\}$.

The collection $\mathcal{P}_{\bar{\mathfrak{n}}}$ is called the collection of *Kolyvagin primes* for $\mathbb{T}_{r,u,v,w}$. Set $\mathcal{P} = \mathcal{P}_{(1,1,1,1)}$ and define $\mathcal{N}_{\bar{\mathfrak{n}}}$ to be the set of square free products of primes in $\mathcal{P}_{\bar{\mathfrak{n}}}$.

We similarly define for $\bar{\mathfrak{s}} = (r, u, v)$ the sollection of Kolyvagin primes $\mathcal{P}_{\bar{\mathfrak{s}}}$ for $\mathfrak{T}_{r,u,v}$ and the set $\mathcal{N}_{\bar{\mathfrak{s}}}$ of square free products of primes in $\mathcal{P}_{\bar{\mathfrak{s}}}$.

Definition 2.7. The partial order $\prec$ on the collection of quadruples $(r, u, v, w) \in (\mathbb{Z}_{>0})^4$ is defined by setting

$$\bar{\mathfrak{n}} = (r, u, v, w) \prec (r', u', v', w') = \bar{\mathfrak{n}}'$$

if $r \le r'$, $u \le u'$, $v \le v'$ and $w \le w'$.

We denote the partial order defined on triples of positive integers in an identical manner also by $\prec$.

To ease notation, set $\mathbb{T}_{\bar{\mathfrak{n}}} := \mathbb{T}_{r,u,v,w}$ and $R_{\bar{\mathfrak{n}}} := R_{r,u,v,w}$ for $\bar{\mathfrak{n}} = (r, u, v, w)$. Define similarly $\mathfrak{T}_{\bar{\mathfrak{n}}} := \mathfrak{T}_{r,u,v}$ and $\mathfrak{R}_{\bar{\mathfrak{n}}} := \mathfrak{R}_{r,u,v}$.

Remark 2.8. Suppose ℓ is a Kolyvagin prime in $\mathcal{P}_{\bar{\mathfrak{n}}}$ (resp., in $\mathcal{P}_{\bar{\mathfrak{s}}}$), where $\bar{\mathfrak{n}}$ (resp., $\bar{\mathfrak{s}}$) are as above. Then as τ acts trivially on $\boldsymbol{\mu}_{p^r}$ and Fr_ℓ is conjugate to τ in $\mathrm{Gal}(L_{\bar{\mathfrak{n}}}/\mathbb{Q})$, it follows that Fr_ℓ acts trivially on $\boldsymbol{\mu}_{p^r}$ and hence that $\ell \equiv 1 \mod p^r$. In particular

$$|\mathbb{F}_\ell^\times| \cdot \mathbb{T}_{r,u,v,w} = (\ell - 1)\mathbb{T}_{r,u,v,w} = 0$$
$$(\text{resp., } |\mathbb{F}_\ell^\times| \cdot \mathfrak{T}_{r,u,v} = 0).$$

Throughout this section, fix a Kolyvagin prime $\ell \in \mathcal{P}_{\bar{\mathfrak{n}}}$ (or in $\mathcal{P}_{\bar{\mathfrak{s}}}$, whenever we talk about quotients of $\mathfrak{T}$).

Definition 2.9. Let T be one of $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$.

(i) The submodule of $H^1(\mathbb{Q}_\ell, T)$ given by

$$H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, T) = \ker\left(H^1(\mathbb{Q}_\ell, T) \longrightarrow H^1(\mathbb{Q}_\ell(\boldsymbol{\mu}_{\boldsymbol{\ell}}), T)\right)$$

is called the *transverse submodule*.

(ii) The *singular quotient* $H^1_s(\mathbb{Q}_\ell, T)$ is defined by the exactness of the sequence

$$0 \longrightarrow H^1_f(\mathbb{Q}_\ell, T) \longrightarrow H^1(\mathbb{Q}_\ell, T) \longrightarrow H^1_s(\mathbb{Q}_\ell, T) \longrightarrow 0 \tag{2.5}$$

Definition 2.10. Let T be one of $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$ and suppose $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ (or $n \in \mathcal{N}_{\bar{\mathfrak{s}}}$ if we are talking about quotients of $\mathfrak{T}$). The *modified Selmer structure* $\mathcal{F}_{\mathrm{can}}(n)$ on T is defined with the following data:

- $\Sigma(\mathcal{F}_{\mathrm{can}}(n)) = \Sigma(\mathcal{F}_{\mathrm{can}}) \cup \{\text{primes } \ell : \ell \mid n\}$.
- If $\ell \nmid n$ then $H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}_\ell, T) = H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_\ell, T)$.
- If $\ell \mid n$ then $H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}_\ell, T) = H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, T)$.

Remark 2.11. Proposition 1.3.2 of [MR04] shows that $\mathcal{F}_{\mathrm{can}}(n)^* = \mathcal{F}^*_{\mathrm{can}}(n)$.

Lemma 2.12. *Let T be one of the rings $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$. Then the* transverse subgroup *$H^1_{\rm tr}(\mathbb{Q}_\ell, T) \subset H^1(\mathbb{Q}_\ell, T)$ projects isomorphically onto $H^1_s(\mathbb{Q}_\ell, T)$. In other words (2.5) above has a functorial splitting.*

Proof. This is [MR04, Lemma 1.2.4] which is proved for a general artinian coefficient ring. □

Proposition 2.13. *Let $\bar{\mathfrak{n}} = (r, u, v, w)$ and $\bar{\mathfrak{s}} = (r, u, v)$ be as above.*

(i) *There are canonical functorial isomorphisms*

$$H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}) \cong \mathbb{T}_{\bar{\mathfrak{n}}}\big/(\mathrm{Fr}_\ell - 1)\mathbb{T}_{\bar{\mathfrak{n}}},$$

$$H^1_s(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}) \cong (\mathbb{T}_{\bar{\mathfrak{n}}})^{\mathrm{Fr}_\ell = 1} \ .$$

(ii) *There is a canonical isomorphism (called the finite-singular comparison isomorphism)*

$$\phi_\ell^{fs} : H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}) \longrightarrow H^1_s(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}) \otimes \mathbb{F}_\ell^\times \ .$$

(iii) *The $R_{\bar{\mathfrak{n}}}$ modules $H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}})$, $H^1_s(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}})$ and $H^1_{\rm tr}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}})$ are free of rank one.*

The analogous statements hold true when $\mathbb{T}_{\bar{\mathfrak{n}}}$ is replaced by $\mathfrak{T}_{\bar{\mathfrak{s}}}$ or $\mathfrak{T}/\mathfrak{m}$ (and the ring $R_{\bar{\mathfrak{n}}}$ by $\mathfrak{R}_{\bar{\mathfrak{s}}}$ or $k = \mathfrak{R}/\mathfrak{m}$).

Proof. (i) is [MR04, Lemma 1.2.1]. The finite-singular comparison isomorphism is defined in [MR04, Definition 1.2.2] and (ii) is [MR04, Lemma 1.2.3]. (iii) follows from (i), (ii) and Lemma 2.12.

Note that all the results quoted from [MR04] apply in our setting thanks to Remark 2.8. □

3. Core vertices and deforming Kolyvagin systems

Let $\bar{\mathfrak{n}} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$ and $\bar{\mathfrak{s}} = (r, u, v) \in (\mathbb{Z}_{>0})^3$. Assume throughout this section that $\chi(\mathbb{T}) = \chi(\mathfrak{T}) = 1$.

3.1. **Core vertices.** Suppose T is one of $\mathbb{T}_{r,u,v,w}$, $\mathfrak{T}_{r,u,v}$ or $\mathfrak{T}/\mathfrak{m}$ and S is the corresponding quotient ring $R_{r,u,v,w}$, $\mathfrak{R}_{r,u,v}$ or $\Bbbk$.

Definition 3.1. The integer $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ (resp., $n \in \mathcal{N}_{\bar{\mathfrak{s}}}$) is called a *core vertex* for the Selmer structure $\mathcal{F}_{\rm can}$ on T if

(i) $H^1_{\mathcal{F}_{\rm can}(n)^*}(\mathbb{Q}, T^*) = 0$,
(ii) $H^1_{\mathcal{F}_{\rm can}(n)}(\mathbb{Q}, T)$ is a free S-module of rank one.

Suppose that the hypotheses (**H1**)-(**H4**), (**H.Tam**), and (**H.nA**) hold true. The following theorem is fundamental in proving the existence of Kolyvagin systems.

Theorem 3.2. *Let $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ (resp., $n \in \mathcal{N}_{\bar{\mathfrak{s}}}$) be a core vertex for the Selmer structure $\mathcal{F}_{\rm can}$ on the residual representation $\bar{T}$. Then n is a core vertex for the Selmer structure $\mathcal{F}_{\rm can}$ on T as well.*

Theorem 3.2 is proved in §4. We first show how Theorem 3.2 is used to prove the existence of Kolyvagin systems for the big Galois representations $\mathbb{T}$ and $\mathfrak{T}$.

3.2. **Kolyvagin systems for big Galois representation.**

3.2.1. *Kolyvagin systems over Artinian rings.* Throughout this section fix $\bar{\mathfrak{n}} = (r, u, v, w)$ and $\bar{\mathfrak{i}} \in (\mathbb{Z}_{>0})^4$ (resp., $\bar{\mathfrak{s}} = (r, u, v)$ and $\bar{\mathfrak{j}} \in (\mathbb{Z}_{>0})^3$) such that $\bar{\mathfrak{n}} \prec \bar{\mathfrak{i}}$ (resp., $\bar{\mathfrak{s}} \prec \bar{\mathfrak{j}}$). Let T be one of $\mathbb{T}_{\bar{\mathfrak{n}}}$, $\mathfrak{T}_{\bar{\mathfrak{s}}}$ or $\mathfrak{T}/\mathfrak{m}$, and let S be the corresponding quotient ring $R_{\bar{\mathfrak{n}}}$, $\mathfrak{R}_{\bar{\mathfrak{s}}}$ or $\Bbbk$. Let $\mathcal{P}$ denote the collection of Kolyvagin primes $\mathcal{P}_{\bar{\mathfrak{i}}}$ (resp., $\mathcal{P}_{\bar{\mathfrak{j}}}$) and $\mathcal{N}$ denote the set of square free products of primes in $\mathcal{P}$.

Many of the definitions and arguments in this section follow closely [MR04] and [Büy11].

Definition 3.3.

(i) If X is a graph and Mod_R is the category of R-modules, a *simplicial sheaf* $\mathcal{S}$ on X with values in Mod_R is a rule assigning
- an R-module $\mathcal{S}(v)$ for every vertex v of X,
- an R-module $\mathcal{S}(e)$ for every edge e of X,
- an R-module homomorphism $\psi_v^e : \mathcal{S}(v) \to \mathcal{S}(e)$ whenever the vertex v is an endpoint of the edge e.

(ii) A *global section* of $\mathcal{S}$ is a collection $\{\kappa_v \in \mathcal{S}(v) : v \text{ is a vertex of } X\}$ such that, for every edge $e = \{v, v'\}$ of X, we have $\psi_v^e(\kappa_v) = \psi_{v'}^e(\kappa_{v'})$ in $\mathcal{S}(e)$. We write $\Gamma(\mathcal{S})$ for the R-module of global sections of $\mathcal{S}$.

Definition 3.4. (Mazur-Rubin) For the Selmer triple $(T, \mathcal{F}_{\text{can}}, \mathcal{P})$, we define a graph $\mathcal{X} = \mathcal{X}(\mathcal{P})$ by taking the set of vertices of $\mathcal{X}$ to be $\mathcal{N}$, and the edges to be $\{n, n\ell\}$ whenever $n, n\ell \in \mathcal{N}$ (with ℓ prime).

(i) The *Selmer sheaf* $\mathcal{H}$ is the simplicial sheaf on $\mathcal{X}$ given as follows. Set $G_n := \otimes_{\ell | n} \mathbb{F}_\ell^\times$. We take
- $\mathcal{H}(n) := H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, T) \otimes G_n$ for $n \in \mathcal{N}$,
- if e is the edge $\{n, n\ell\}$ then $\mathcal{H}(e) := H^1_s(\mathbb{Q}_\ell, T) \otimes G_{n\ell}$.

We define the vertex-to-edge maps to be
- $\psi_{n\ell}^e : H^1_{\mathcal{F}_{\text{can}}(n\ell)}(\mathbb{Q}, T) \otimes G_{n\ell} \to H^1_s(\mathbb{Q}_\ell, T) \otimes G_{n\ell}$ is localization followed by the projection to the singular cohomology $H^1_s(\mathbb{Q}_\ell, T)$.
- $\psi_n^e : H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, T) \otimes G_n \to H^1_s(\mathbb{Q}_\ell, T) \otimes G_{n\ell}$ is the composition of localization at ℓ with the finite-singular comparison map ϕ_ℓ^{fs}.

(ii) A *Kolyvagin system* for the triple $(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ is simply a global section of the Selmer sheaf $\mathcal{H}$.

Let $\Gamma(\mathcal{H})$ denote the S-module of global sections of $\mathcal{H}$. We set $\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) := \Gamma(\mathcal{H})$ and call it the *Kolyvagin systems for the Selmer structure* $\mathcal{F}_{\text{can}}$ *on* T. More explicitly, an element $\boldsymbol{\kappa} \in \mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ is a collection $\{\kappa_n\}$ of cohomology classes indexed by $n \in \mathcal{N}$ such that for every $n, n\ell \in \mathcal{N}$ we have:

- $\kappa_n \in H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, T) \otimes G_n$,
- $\phi_\ell^{fs}(\mathrm{loc}_\ell(\kappa_n)) = \mathrm{loc}_\ell^s(\kappa_{n\ell})$.

Here, loc_ℓ^s stands for the composite

$$H^1(\mathbb{Q}, T) \xrightarrow{\mathrm{loc}_\ell} H^1(\mathbb{Q}_\ell, T) \longrightarrow H^1_s(\mathbb{Q}_\ell, T).$$

The goal of this section is to prove the following theorem, assuming (**H1**)-(**H4**), (**H.Tam**) and (**H.nA**).

Theorem 3.5. *The S-module* $\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P})$ *is free of rank one.*

Theorem 3.5 is proved in two steps. As the first step, we prove:

Theorem 3.6. *Suppose $n \in \mathcal{N}$ is any core vertex for the Selmer structure $\mathcal{F}_{\text{can}}$ on T. Then the natural map*

$$\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \longrightarrow H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, T) \otimes G_n$$

(given by $\boldsymbol{\kappa} \mapsto \kappa_n$) is surjective.

The arguments of [Büy11, Theorem 3.11] (which in turn modifies the arguments of Howard in [MR04] appropriately so as to apply them with general artinian rings) may be used to prove Theorem 3.6. The main point is that we have Theorem 3.2 here in place of [Büy11, Theorem 2.27].

Define a subgraph $\mathcal{X}^0 = \mathcal{X}^0(\mathcal{P})$ of $\mathcal{X}$ whose vertices are the core vertices of $\mathcal{X}$ and whose edges are defined as follows: We join n and $n\ell$ by an edge in $\mathcal{X}^0$ if and only if the localization map

$$H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \bar{T}) \longrightarrow H^1_f(\mathbb{Q}_\ell, \bar{T})$$

is *non-zero*. We define the sheaf $\mathcal{H}^0$ on $\mathcal{X}^0$ as the restriction of the Selmer sheaf $\mathcal{H}$ to $\mathcal{X}^0$.

Lemma 3.7. *The graph $\mathcal{X}^0$ is connected.*

Proof. The edges of $\mathcal{X}^0$ are defined in terms of $\bar{T}$ (and not T itself) so the arguments that go into the proof of [MR04, Theorem 4.3.12] apply. □

The following Theorem, combined with Theorem 3.6 completes the proof of Theorem 3.5.

Theorem 3.8. *Suppose $n \in \mathcal{N}$ is any core vertex for the Selmer structure $\mathcal{F}_{\text{can}}$ on T. Then the natural map*

$$\mathbf{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \longrightarrow H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, T) \otimes G_n$$

is is injective.

Theorem 3.8 is proved arguing as in the proof of [Büy11, Theorem 3.12]. The essential input is the fact that the graph $\mathcal{X}^0$ is connected (Lemma 3.7).

3.2.2. *Kolyvagin systems over 'big' rings.* The goal of this section is to prove using the results of the previous section that the R-module

$$\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) := \varprojlim_{\bar{\mathfrak{n}}} \left(\varinjlim_{\bar{\mathfrak{i}}} \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{i}}}) \right)$$

(resp., the $\mathfrak{R}$-module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) := \varprojlim_{\bar{\mathfrak{s}}} \left(\varinjlim_{\bar{\mathfrak{j}}} \mathbf{KS}(\mathfrak{T}_{\bar{\mathfrak{s}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{j}}}) \right)$) is free of rank one. This will be accomplished modulo Theorem 3.2, whose proof will be given in §4.

Lemma 3.9. *Fix a quadruple $\bar{\mathfrak{n}}$ as above. For any $\bar{\mathfrak{i}} \succ \bar{\mathfrak{n}}$, the natural restriction map*

$$\mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{n}}}) \longrightarrow \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{i}}})$$

is an isomorphism.

Proof. Theorems 3.6 and 3.8 applied with a core vertex $n \in \mathcal{N}_{\bar{\mathfrak{i}}}$, we have isomorphisms

$$\mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{n}}}) \stackrel{\sim}{\longrightarrow} H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}}) \stackrel{\sim}{\longleftarrow} \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{i}}})$$

compatible with the restriction map

$$\mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{n}}}) \longrightarrow \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\text{can}}, \mathcal{P}_{\bar{\mathfrak{i}}}).$$

Note that $n \in \mathcal{N}_{\bar{\mathfrak{i}}}$ as above exists by [MR04, Corollary 4.1.9] and Theorem 3.2 above. □

Lemma 3.10. *Let $\bar{\mathfrak{n}}' \prec \bar{\mathfrak{n}}$ and let $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ be a core vertex. The map*

$$H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}}) \longrightarrow H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}'})$$

is surjective.

Proof. We verify the assertion of the Lemma for $\bar{\mathfrak{n}}' = (r, u, v, w)$ and $\bar{\mathfrak{n}} = (r+1, u, v, w)$. The proof of the general case follows by applying this argument (or where necessary, its slightly modified form) repeatedly.

We have the following commutative diagram, where the vertical isomorphism is obtained from (a slight variation of) Proposition 4.17(iv) below:

$$\begin{tikzcd}
H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{r+1,u,v,w}) \arrow[r, "\text{reduction}"] \arrow[dr, "\varpi"'] & H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{r,u,v,w}) \arrow[d, "\cong"', "{[\varpi]}"] \\
& H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, T_{r+1,u,v,w})[\varpi^r]
\end{tikzcd}$$

Since $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ is a core vertex (and therefore $H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}})$ is a free $R_{\bar{\mathfrak{n}}}$-module of rank one), the map on the diagonal is surjective. This proves that the horizontal map is surjective as well. ☐

Lemma 3.11. *The map*

$$\mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_{\bar{\mathfrak{n}}}) \longrightarrow \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}'}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_{\bar{\mathfrak{n}}})$$

is surjective for $\bar{\mathfrak{n}}' \prec \bar{\mathfrak{n}}$.

Proof. By Theorem 3.6, Theorem 3.8 and Lemma 3.9 applied with a core vertex $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ to both $\mathbb{T}_{\bar{\mathfrak{n}}}$ and $\mathbb{T}_{\bar{\mathfrak{n}}'}$, we obtain the following commutative diagram with vertical isomorphisms:

$$\begin{tikzcd}
\mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_{\bar{\mathfrak{n}}}) \arrow[r] \arrow[d, "\cong"'] & \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}'}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_{\bar{\mathfrak{n}}}) \arrow[d, "\cong"] \\
H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}}) \otimes G_n \arrow[r, two heads] & H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}'}) \otimes G_n
\end{tikzcd}$$

where the surjection in the second row is Lemma 3.10. It follows at once that the upper horizontal map in the diagram is surjective as well. ☐

Theorem 3.12. *Under the running hypotheses the following hold.*

(i) *The R-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ is free of rank one, generated by a Kolyvagin system $\boldsymbol{\kappa}$ whose image $\bar{\boldsymbol{\kappa}} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ is non-zero.*

(ii) *The $\mathfrak{R}$-module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ is free of rank one. When the ring $\mathcal{R}$ is regular, the module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ is generated by $\boldsymbol{\kappa}$ whose image $\bar{\boldsymbol{\kappa}} \in \mathbf{KS}(\bar{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ is non-zero.*

Proof. Lemma 3.9 shows that

$$\varinjlim_{\bar{\mathfrak{i}}} \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_{\bar{\mathfrak{i}}}) = \mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_{\bar{\mathfrak{n}}}).$$

The proof of (i) now follows by Theorem 3.5 and Lemma 3.11. (ii) is proved similarly, by appropriately modifying the ingredients that go into the proof of (i). ☐

Definition 3.13. Given a Kolyvagin system

$$\boldsymbol{\kappa}=\{\kappa_m(\bar{\mathfrak{n}})\}_{\bar{\mathfrak{n}},m\in\mathcal{N}_{\bar{\mathfrak{n}}}}\in\overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\text{can}},\mathcal{P})=\varprojlim_{\bar{\mathfrak{n}}}\mathbf{KS}(\mathbb{T}_{\bar{\mathfrak{n}}},\mathcal{F}_{\text{can}},\mathcal{P}_{\bar{\mathfrak{n}}})$$

(where the last equality is by Lemma 3.9), we define the leading term of $\boldsymbol{\kappa}$ to be the element

$$\kappa_1=\{\kappa_1(\bar{\mathfrak{n}})\}_{\bar{\mathfrak{n}}}\in\varprojlim_{\bar{\mathfrak{n}}}H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T}_{\bar{\mathfrak{n}}})=H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T}).$$

4. The existence of core vertices

The goal of this section is to verify the truth of Theorem 3.2.

4.1. **Cartesian properties.** Let Quot$(\mathbb{T})$ denote the collection $\{\mathbb{T}_{\bar{\mathfrak{n}}}:\bar{\mathfrak{n}}\in(\mathbb{Z}^+)^4\}$ of quotients of $\mathbb{T}$ and similarly, let Quot$(\mathfrak{T})=\{\mathfrak{T}_{\bar{\mathfrak{m}}}:\bar{\mathfrak{m}}\in(\mathbb{Z}^+)^3\}\cup\{\mathfrak{T}/\mathfrak{m}\}$.

Given $\bar{\alpha}=(\alpha_1,\cdots,\alpha_d)\in(\mathbb{Z}^+)^d$ and $1\le i\le d$, we define $\bar{\alpha}_{+,i}\in(\mathbb{Z}^+)^d$ to be the d-tuple whose jth coordinate is $\alpha_i+\delta_{ij}$. Here δ_{ij} is Kronecker's delta.

Definition 4.1. A local condition $\mathcal{F}$ at a prime ℓ is said to be *cartesian* on the collection Quot$(\mathbb{T})$ if it satisfies the following conditions:

For $\bar{\alpha}\in(\mathbb{Z}^+)^4$,

(C1) (Weak functoriality)

(C1.a) $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}})$ is the exact image of $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}_{+,1}})$ under the canonical map

$$H^1(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}_{+,1}})\longrightarrow H^1(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}}),$$

(C1.b) For $i\in\{2,3,4\}$, $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}})$ lies in the image of $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}_{+.i}})$ under the map

$$H^1(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}_{+.i}})\longrightarrow H^1(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}}).$$

(C2) (Cartesian property) For $1\le i\le 4$,

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}})=\ker\left(H^1(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}})\longrightarrow\frac{H^1(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}_{+,i}})}{H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathbb{T}_{\bar{\alpha}_{+,i}})}\right).$$

If $i>1$, the arrow here is induced from the injection $\mathbb{T}_{\bar{\alpha}}\xrightarrow{[X_i]}\mathbb{T}_{\bar{\alpha}_{+,i}}$, where $[X_i]$ stands for multiplication by X_i. When $i=1$, the arrow is induced from the injection $\mathbb{T}_{\bar{\alpha}}\xrightarrow{[\varpi]}\mathbb{T}_{\bar{\alpha}_{+,1}}$.

Definition 4.2. A local condition $\mathcal{F}$ at a prime ℓ is said to be *cartesian* on the collection Quot$(\mathfrak{T})$ if it satisfies the following conditions:

For $\bar{\alpha}\in(\mathbb{Z}^+)^3$,

(D1.a) $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}})$ is the exact image of $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}_{+,1}})$ under the canonical map

$$H^1(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}_{+,1}})\longrightarrow H^1(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}}),$$

(D1.b) For $i\in\{2,3\}$, $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}})$ lies in the image of $H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}_{+.i}})$ under the map

$$H^1(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}_{+.i}})\longrightarrow H^1(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}}).$$

(D2) For $1\le i\le 3$,

$$H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}})=\ker\left(H^1(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}})\longrightarrow\frac{H^1(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}_{+,i}})}{H^1_{\mathcal{F}}(\mathbb{Q}_\ell,\mathfrak{T}_{\bar{\alpha}_{+,i}})}\right).$$

(D3) $H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) = \ker\left(H^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \longrightarrow \dfrac{H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})}{H^1_{\mathcal{F}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})}\right)$, where the arrow is induced from the injection

$$\mathfrak{R}/\mathfrak{m} = \mathtt{k} \xrightarrow{\sim} \mathcal{R}_0[\mathfrak{m}] \hookrightarrow \mathcal{R}_0.$$

4.1.1. *Cartesian properties at* p.

Proposition 4.3. *Assuming* (**H.nA**)*, the local condition at* p *given by* $\mathcal{F}_{\text{can}}$ *on the collection* $\text{Quot}(\mathbb{T})$ *(resp., on the collection* $\text{Quot}(\mathfrak{T})$*) is cartesian.*

Proof. This is obvious thanks to Proposition 2.5. □

4.1.2. *Cartesian properties primes* $\ell \neq p$ *for the coefficient ring* R. Throughout this section the hypothesis (**H.Tam**) is in force.

Lemma 4.4.

(i) $H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})$.
(ii) $H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w})$.
(iii) $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w})$.
(iv) *The following sequence is exact:*

$$0 \longrightarrow H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell,\mathcal{A}_{u,v,w})}{H^1_f(\mathbb{Q}_\ell,\mathcal{A}_{u,v,w})}$$

Proof. (iv) follows from (2.1).

By [Rub00, Lemma 1.3.5] we have the following two exact sequences:

$$0 \longrightarrow H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \longrightarrow \mathcal{W}/(\text{Fr}_\ell - 1)\mathcal{W} \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \longrightarrow H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \longrightarrow \mathcal{W}^{\text{Fr}_\ell=1} \longrightarrow 0 \tag{4.2}$$

where $\mathcal{W} = \mathcal{A}^{I_v}_{u,v,w}/(\mathcal{A}^{I_\ell}_{u,v,w})_{\text{div}}$. In Lemma 4.5, we check under the assumption (**H.Tam**) that $\mathcal{W}^{\text{Fr}_\ell=1} = 0$. Since $\mathcal{W}$ is a finite module, the exact sequence

$$0 \longrightarrow \mathcal{W}^{\text{Fr}_\ell=1} \longrightarrow \mathcal{W} \xrightarrow{\text{Fr}_\ell-1} \mathcal{W} \longrightarrow \mathcal{W}/(\text{Fr}_\ell - 1)\mathcal{W}$$

shows that $\mathcal{W}/(\text{Fr}_\ell - 1)\mathcal{W} = 0$. This proves that

$$H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \ , \ H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) = H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}). \tag{4.3}$$

This proves (i) and (ii). By (2.1) and (4.3) it now follows that

$$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \text{im}\left(H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{u,v,w}) \to H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w})\right) \subset H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{k,u,v,w})$$

$$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) = \ker\left(H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}\right) \supset H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathbb{T}_{k,u,v,w})$$

and the proof of (iii) follows. □

Lemma 4.5. $\mathcal{W}^{\text{Fr}_\ell=1} = 0$.

Proof. As we have $\mathcal{A}_{1,1,1}[\varpi] = \bar{T}$, it follows that $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}[\varpi]) = 0$ since we assume **H.Tam**, hence also that

$$H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}) = 0.$$

Using the $G_{\mathbb{Q}_\ell}$-cohomology of the exact sequences

$$0 \longrightarrow \mathcal{A}_{1,1,w} \xrightarrow{[X_3]} \mathcal{A}_{1,1,w+1} \longrightarrow \mathcal{A}_{1,1,1} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{A}_{1,v,w} \xrightarrow{[X_2]} \mathcal{A}_{1,v+1,w} \longrightarrow \mathcal{A}_{1,1,w} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{A}_{u,v,w} \xrightarrow{[X_1]} \mathcal{A}_{u+1,v,w} \longrightarrow \mathcal{A}_{1,v,w} \longrightarrow 0,$$

it follows by induction that

$$H^0(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = 0. \tag{4.4}$$

Taking the $G_{\mathbb{Q}_\ell}/I_\ell$-invariance of the short exact sequence

$$0 \longrightarrow (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}} \longrightarrow \mathcal{A}_{u,v,w}^{I_\ell} \longrightarrow \mathcal{W} \longrightarrow 0$$

we see by (4.4) that

$$\mathcal{W}^{\text{Fr}_\ell=1} \hookrightarrow H^1(G_{\mathbb{Q}_\ell}/I_\ell, (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}) \cong (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}/(\text{Fr}_\ell - 1).$$

To conclude with the proof, it therefore suffices to show that

$$(\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}/(\text{Fr}_\ell - 1) = 0.$$

For any $\alpha \in \mathbb{Z}^+$, (4.4) shows

$$H^0(G_{\mathbb{Q}_\ell}/I_\ell, (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha]) = 0. \tag{4.5}$$

The exact sequence

$$\left((\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha]\right)^{\text{Fr}_\ell=1} \to (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha] \xrightarrow{\text{Fr}_\ell-1} (\mathcal{A}_{u,v,w}^{I_\ell})_{\text{div}}[\varpi^\alpha] \to \mathcal{A}_{u,v,w}^{I_\ell}[\varpi^\alpha]/(\text{Fr}_\ell - 1) \to 0$$

and (4.5) shows that $\mathcal{A}_{u,v,w}^{I_\ell}[\varpi^\alpha]/(\text{Fr}_\ell - 1) = 0$. Passing to direct limit the Lemma follows. □

By Lemma 4.4(iv) we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u,v,w}) \arrow[r] \arrow[d] & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})} \arrow[d, "\alpha"] \\
0 \arrow[r] & H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathbb{T}_{r,u+1,v,w}) \arrow[r] & H^1(\mathbb{Q}_\ell, \mathbb{T}_{r,u+1,v,w}) \arrow[r] & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})}{H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})}
\end{tikzcd}$$

Lemma 4.6. *The map α is injective if*

$$\beta : H^1(I_\ell, \mathcal{A}_{u,v,w})^{\text{Fr}_\ell=1} \longrightarrow H^1(I_\ell, \mathcal{A}_{u+1,v,w})^{\text{Fr}_\ell=1}$$

is injective.

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})}{H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w})} \arrow[r] \arrow[d, "\alpha"'] & H^1(I_\ell, \mathcal{A}_{u,v,w})^{\text{Fr}_\ell=1} \arrow[d, "\beta"] \\
0 \arrow[r] & \frac{H^1(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})}{H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u+1,v,w})} \arrow[r] & H^1(I_\ell, \mathcal{A}_{u+1,v,w})^{\text{Fr}_\ell=1}
\end{tikzcd}$$

whose exact rows come from the Hochschild-Serre spectral sequence and the fact that

$$H^1_f(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) = H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) := \ker\left(H^1(\mathbb{Q}_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathcal{A}_{u,v,w})^{\mathrm{Fr}_\ell=1}\right),$$

where the first equality is Lemma 4.4(i). □

Consider the short exact sequence

$$0 \longrightarrow \mathcal{A}_{u,v,w} \xrightarrow{[X_1]} \mathcal{A}_{u+1,v,w} \longrightarrow \mathcal{A}_{1,v,w} \longrightarrow 0$$

The I_ℓ-cohomology of this sequence gives

$$0 \longrightarrow \mathcal{A}^{I_\ell}_{u+1,v,w}/\mathcal{A}^{I_\ell}_{u,v,w} \longrightarrow \mathcal{A}^{I_\ell}_{1,v,w} \longrightarrow H^1(I_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathcal{A}_{u+1,v,w}) \tag{4.6}$$

To ease the notation set

$$\mathcal{K}_{v,w} = \mathcal{A}^{I_\ell}_{u+1,v,w}/\mathcal{A}^{I_\ell}_{u,v,w},$$

so that the sequence (4.6) may be rewritten as

$$0 \longrightarrow \mathcal{A}^{I_\ell}_{1,v,w}/\mathcal{K}_{v,w} \longrightarrow H^1(I_\ell, \mathcal{A}_{u,v,w}) \longrightarrow H^1(I_\ell, \mathcal{A}_{u+1,v,w}) \tag{4.7}$$

Taking $G_{\mathbb{Q}_\ell}/I_\ell$-invariance in (4.7), we conclude that

Lemma 4.7. $\ker(\beta) \cong H^0(G_{\mathbb{Q}_\ell}/I_\ell, \mathcal{A}^{I_\ell}_{1,v,w}/\mathcal{K}_{v,w})$.

Lemma 4.8. *Under the assumption that* (**H.Tam**) *holds true,*

(i) $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,v,w}) = 0$,
(ii) $H^0(\mathbb{Q}_\ell, \mathcal{K}_{v,w}) = 0$.

Proof. Noting that $\bar{T} \cong \mathcal{A}_{1,1,1}$, Hypothesis (**H.Tam**) shows that

$$H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}[\varpi]) = 0$$

and also that $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,1}) = 0$. The $G_{\mathbb{Q}_\ell}$-invariance of the sequence

$$0 \longrightarrow \mathcal{A}_{1,1,w-1} \xrightarrow{[X_3]} \mathcal{A}_{1,1,w} \longrightarrow \mathcal{A}_{1,1,1} \longrightarrow 0$$

shows by induction that $H^0(\mathbb{Q}_\ell, \mathcal{A}_{1,1,w}) = 0$ for all $w \in \mathbb{Z}_{\geq 2}$. Using similarly the exact sequence

$$0 \longrightarrow \mathcal{A}_{1,v-1,w} \xrightarrow{[X_2]} \mathcal{A}_{1,v,w} \longrightarrow \mathcal{A}_{1,1,w} \longrightarrow 0$$

we conclude with the proof of (i). (ii) follows from (i) as $\mathcal{K}_{v,w}$ is a submodule of $\mathcal{A}_{1,v,w}$. □

Proposition 4.9. $\ker(\beta) = 0$.

Proof. Taking the $G_{\mathbb{Q}_\ell}/I_\ell$-invariance of the short exact sequence

$$0 \longrightarrow \mathcal{K}_{v,w} \longrightarrow \mathcal{A}^{I_\ell}_{1,v,w} \longrightarrow \mathcal{A}^{I_\ell}_{1,v,w}/\mathcal{K}_{v,w} \longrightarrow 0,$$

we conclude using Lemma 4.8 that

$$\ker(\beta) \hookrightarrow H^1(G_{\mathbb{Q}_\ell}/I_\ell, \mathcal{K}_{v,w}) \cong \mathcal{K}_{v,w}/(\mathrm{Fr}_\ell - 1)\mathcal{K}_{v,w}. \tag{4.8}$$

Lemma 4.8(ii) yields (using the fact that $\mathcal{K}_{v,w}$ is ϖ^∞-torsion) an exact sequence

$$0 \longrightarrow \mathcal{K}_{v,w}[\varpi^\alpha] \xrightarrow{\mathrm{Fr}_\ell - 1} \mathcal{K}_{v,w}[\varpi^\alpha] \longrightarrow \mathcal{K}_{v,w}[\varpi^\alpha]/(\mathrm{Fr}_\ell - 1) \longrightarrow 0$$

for every $\alpha \in \mathbb{Z}^+$. Noting that the module $\mathcal{K}_{v,w}[\varpi^\alpha]$ has finite cardinality, it follows now that

$$\mathcal{K}_{v,w}[\varpi^\alpha]/(\mathrm{Fr}_\ell - 1) = 0.$$

Passing to direct limit, Proposition follows by (4.8). □

Proposition 4.10. *The local condition at a prime $\ell \neq p$, given by $\mathcal{F}_{\text{can}}$ on the collection* $\mathrm{Quot}(\mathbb{T})$ *is cartesian.*

Proof. (**C1.a**) holds true by definition and (**C1.b**) by Lemma 4.4(iii). The parts of (**C2**) concerning the cases $2 \leq i \leq 4$ follow from Lemma 4.6 and Proposition 4.9; the part concerning the case $i = 1$ from [MR04, Lemma 3.7.1]. □

4.1.3. *Cartesian properties at $\ell \neq p$ for the coefficient ring $\mathfrak{R}$.* Assume throughout this section that (**H.Tam**) holds true. Recall that $\mathfrak{R} = \mathcal{R}[[\Gamma]]$ where $\mathcal{R}$ is a Gorenstein $\mathcal{O}$-algebra of dimension 2 with maximal ideal $\mathfrak{m}_{\mathcal{R}}$.

Lemma 4.11.

(i) $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_p, \mathfrak{T}_{r,u,v}) = H^1_{\text{ur}}(\mathbb{Q}_p, \mathfrak{T}_{r,u,v})$.

(ii) *The sequence*

$$0 \longrightarrow H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})}{H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})}$$

is exact.

Proof. The proof of Lemma 4.4 above works verbatim. □

Lemma 4.12. *In the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u,v}) \arrow[r] \arrow[d] & \frac{H^1(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})}{H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{A}_{u,v})} \arrow[d, "\alpha"] \\
0 \arrow[r] & H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{r,u+1,v}) \arrow[r] & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{r,u+1,v}) \arrow[r] & \frac{H^1(\mathbb{Q}_\ell, \mathfrak{A}_{u+1,v})}{H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{A}_{u+1,v})}
\end{tikzcd}$$

the map α is injective.

Proof. Identical to the proof of Proposition 4.9. □

Recall the ring $\mathfrak{O}$ and the module $T_{\mathfrak{O}}$ from §1.4.

Proposition 4.13.

(i) $H^1_f(\mathbb{Q}_\ell, T_{\mathfrak{O}}) = H^1_{\text{ur}}(\mathbb{Q}_\ell, T_{\mathfrak{O}})$, *where*

$$H^1_f(\mathbb{Q}_\ell, T_{\mathfrak{O}}) = \ker\left(H^1(\mathbb{Q}_\ell, T_{\mathfrak{O}}) \to H^1(I_\ell, T_{\mathfrak{O}} \otimes \mathbb{Q}_p)\right).$$

(ii) $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1})$.

(iii) $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1}) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})$.

(iv) $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \bar{T}) = \mathrm{im}\left(H^1_{\text{ur}}(\mathbb{Q}_\ell, T_{\mathfrak{O}}) \to H^1(\mathbb{Q}_\ell, \bar{T})\right) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \bar{T})$.

(v) $H^1_{\text{ur}}(\mathbb{Q}_\ell, \bar{T})$ *is the inverse image of* $H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1})[\mathfrak{m}]$ *under the map induced from* (1.1).

Proof. (i) and (ii) follows from [Rub00, Lemma 1.3.5] since we assumed (**H.Tam**), and (iii) follows mimicking the proof of Lemma 4.4(iii). We next verify (iv). By the very definition of $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \bar{T})$ (see the beginning of §2),

$$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \bar{T}) = \mathrm{im}\left(H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \to H^1(\mathbb{Q}_\ell, \bar{T})\right) = \mathrm{im}\left(H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \to H^1(\mathbb{Q}_\ell, \bar{T})\right),$$

where the second equality is thanks to (i). Thus, the assertion (iv) amounts to the statements

$$\mathrm{im}\left(H^1_{\text{ur}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \to H^1(\mathbb{Q}_\ell, \bar{T})\right) = H^1_{\text{ur}}(\mathbb{Q}_\ell, \bar{T}), \tag{4.9}$$

$$\text{(4.10)} \qquad \mathrm{im}\left(H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, T_{\mathfrak{D}}) \to H^1(\mathbb{Q}_\ell, \bar{T})\right) = H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, \bar{T}).$$

In order to verify (4.9), it suffices to check that we have a surjection

$$H^0(I_\ell, \mathfrak{T}_{1,1}) \twoheadrightarrow H^0(I_\ell, \bar{T})$$

as $G_{\mathbb{Q}_\ell}/I_\ell$ has cohomological dimension 1. Taking the I_ℓ-invariance of the exact sequence

$$\text{(4.11)} \qquad 0 \longrightarrow \mathfrak{m}_{\mathcal{R}}\mathfrak{T}_{1,1} \longrightarrow \mathfrak{T}_{1,1} \longrightarrow \bar{T} \longrightarrow 0$$

we see that

$$\mathrm{coker}\left(H^0(I_\ell, \mathfrak{T}_{1,1}) \longrightarrow H^0(I_\ell, \bar{T})\right) \hookrightarrow H^1(I_\ell, \mathfrak{m}_{\mathcal{R}}\mathfrak{T}_{1,1}).$$

As the module $H^0(I_\ell, \bar{T})$ is of finite order, the image of the injection above lands in the $\mathbb{Z}_p$-torsion submodule $H^1(I_\ell, \mathfrak{m}_{\mathcal{R}}\mathfrak{T}_{1,1})_{\mathrm{tors}}$ of $H^1(I_\ell, \mathfrak{m}_{\mathcal{R}}\mathfrak{T}_{1,1})$. On the other hand,

$$H^1(I_\ell, \mathfrak{m}_{\mathcal{R}}\mathfrak{T}_{1,1})_{\mathrm{tors}} \cong (\mathfrak{m}\mathfrak{T}_{1,1} \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{I_\ell}/\mathrm{div} = A^{I_\ell}/\mathrm{div} = 0$$

where

- M/div is short for M/M_{div};
- the second equality is obtained tensoring the exact sequence (4.11) by $\mathbb{Q}_p/\mathbb{Z}_p$ and noting that the exactness is preserved as $\mathfrak{m}_{\mathcal{R}}\mathfrak{T}_{1,1}$ is $\mathbb{Z}_p$-torsion free, and that $\bar{T}\otimes\mathbb{Q}_p/\mathbb{Z}_p = 0$;
- the last equality is (**H.Tam**).

This shows that

$$\mathrm{coker}\left(H^0(I_\ell, \mathfrak{T}_{1,1}) \longrightarrow H^0(I_\ell, \bar{T})\right) = 0$$

as desired and (4.9) is verified.

To verify (4.10), it again suffices to check that

$$\mathrm{coker}\left(H^0(I_\ell, T_{\mathfrak{D}}) \longrightarrow H^0(I_\ell, \bar{T})\right).$$

Considering the I_ℓ-invariance of the exact sequence

$$0 \longrightarrow T_{\mathfrak{D}} \xrightarrow{\pi_{\mathfrak{D}}} T_{\mathfrak{D}} \longrightarrow \bar{T} \longrightarrow 0$$

we see that

$$\mathrm{coker}\left(H^0(I_\ell, T_{\mathfrak{D}}) \longrightarrow H^0(I_\ell, \bar{T})\right) \hookrightarrow H^1(I_\ell, T_{\mathfrak{D}})_{\mathrm{tors}}.$$

As above, $H^1(I_\ell, T_{\mathfrak{D}})_{\mathrm{tors}} \cong A^{I_\ell}/\mathrm{div} = 0$ and this completes the proof of (iv).

We now prove (v). Consider the sequence

$$\text{(4.12)} \qquad 0 \longrightarrow \bar{T} \longrightarrow \mathfrak{T}_{1,1} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where the arrow $\bar{T} \to \mathfrak{T}_{1,1}$ is obtained from (1.1) and $\mathcal{Q}$ is defined by the exactness of this sequence. Taking the I_ℓ-invariance of the sequence (4.12), we obtain another exact sequence

$$0 \longrightarrow \mathcal{Q}_0 \longrightarrow H^1(I_\ell, \bar{T}) \longrightarrow H^1(I_\ell, \mathfrak{T}_{1,1})$$

where $\mathcal{Q}_0 := \mathcal{Q}^{I_\ell}\Big/\mathfrak{T}_{1,1}^{I_\ell}/\bar{T}^{I_\ell}$. Taking the $G_{\mathbb{Q}_\ell}/I_\ell$-invariance of the final exact sequence, we conclude that

$$\ker\left(H^1(I_\ell, \bar{T})^{\mathrm{Fr}_\ell=1} \longrightarrow H^1(I_\ell, \mathfrak{T}_{1,1})^{\mathrm{Fr}_\ell=1}\right) = \mathcal{Q}_0^{\mathrm{Fr}_\ell=1}$$

hence by Lemma 4.14 below that

$$\text{(4.13)} \qquad \ker\left(H^1(I_\ell, \bar{T})^{\mathrm{Fr}_\ell=1} \longrightarrow H^1(I_\ell, \mathfrak{T}_{1,1})^{\mathrm{Fr}_\ell=1}\right) = 0.$$

Consider now the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, \bar{T}) \arrow[r] \arrow[d] & H^1(\mathbb{Q}_\ell, \bar{T}) \arrow[r] \arrow[d] & H^1(I_\ell, \bar{T})^{\mathrm{Fr}_\ell=1} \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \arrow[r] & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \arrow[r] & H^1(I_\ell, \mathfrak{T}_{1,1})^{\mathrm{Fr}_\ell=1} \arrow[r] & 0 \end{tikzcd}$$

(4.13) shows that φ is injective, and a simple diagram chase yields

$$H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, \bar{T}) = \ker \left(H^1(\mathbb{Q}_\ell, \bar{T}) \longrightarrow H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1})/H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1}) \right)$$

which is a restatement of (v). □

Lemma 4.14. $\mathcal{Q}_0^{\mathrm{Fr}_\ell=1} = 0$

Proof. As $\bar{T}^{G_{\mathbb{Q}_\ell}} = 0$, it follows by the proof of [MR04, Lemma 2.1.4] that $S^{G_{\mathbb{Q}_\ell}} = 0$ for any subquotient S of $\mathfrak{T}$, in particular for $S = \mathcal{Q}_0$. □

Proposition 4.15. *The local condition at a prime $\ell \neq p$, given by $\mathcal{F}_{\mathrm{can}}$ on the collection* $\mathrm{Quot}(\mathfrak{T})$ *is cartesian.*

Proof. One verifies (**D1**) using Lemma 4.11, (**D2**) using Lemma 4.12 and [MR04, Lemma 3.7.1]. (**D3**) follows from Proposition 4.13(i) and Proposition 4.13(iv). □

4.1.4. *Cartesian properties for the transverse condition.* Recall the partial order $\prec$ from Definition 2.7 on the quadruples (resp., on the triples) of positive integers.

Proposition 4.16. *For $\bar{\mathfrak{n}}_0 = (r_0, u_0, v_0, w_0)$, suppose $\ell \in \mathcal{P}_{\bar{\mathfrak{n}}}$ is a Kolyvagin prime in the sense of Definition 2.6. Then the transverse local condition at ℓ is Cartesian on the family $\{\mathbb{T}_{\bar{\mathfrak{n}}}\}_{\bar{\mathfrak{n}} \prec \bar{\mathfrak{n}}_0}$ (resp., on the family $\{\mathfrak{T}_{\bar{\mathfrak{n}}}\}_{\bar{\mathfrak{n}} \prec \bar{\mathfrak{n}}_0} \cup \{\mathfrak{T}/\mathfrak{m}\}$).*

Proof. Suppose $\bar{\mathfrak{n}} = (r, u, v, w)$ and $\bar{\mathfrak{n}}' = (r', u', v', w')$ are such that $\bar{\mathfrak{n}} \prec \bar{\mathfrak{n}}' \prec \bar{\mathfrak{n}}_0$. Then we have the following commutative diagram whose rows are exact by Lemma 2.12:

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}'}) \arrow[r] & H^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}'}) \arrow[r] \arrow[d] & H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}'}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}) \arrow[r] & H^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}) \arrow[r] & H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}) \arrow[r] & 0 \end{tikzcd}$$

Here the vertical arrows are induced from the natural surjection $\mathbb{T}_{\bar{\mathfrak{n}}'} \twoheadrightarrow \mathbb{T}_{\bar{\mathfrak{n}}}$. This shows that $H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}'})$ is mapped into $H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}})$. Furthermore, as the $R_{\bar{\mathfrak{n}}'}$-module $\mathbb{T}_{\bar{\mathfrak{n}}'}^{\mathrm{Fr}_\ell=1}$ (resp., the $R_{\bar{\mathfrak{n}}}$-module $\mathbb{T}_{\bar{\mathfrak{n}}}^{\mathrm{Fr}_\ell=1}$) is free of rank one, it follows by Lemma 2.12 and Proposition 2.13(i) that

$$H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}'}) \twoheadrightarrow H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}}),$$

i.e., the transverse local condition on the quotients $\mathbb{T}_{\bar{\mathfrak{n}}}$ is the same as the propagation of the local condition $H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}_0})$. This verifies (even a stronger form of) (**C1**).

As the quotient

$$H^1(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}_0})/H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}_0}) \cong H^1_f(\mathbb{Q}_\ell, \mathbb{T}_{\bar{\mathfrak{n}}_0})$$

is a free $R_{\bar{\mathfrak{n}}_0}$-module of rank one, (**C2**) follows from the proof of [MR04, Lemma 3.7.1(i)], using the argument in loc.cit. for the multiplication by $[X_1]$, $[X_2]$, $[X_3]$ and $[\varpi]$ maps separately.

One verifies (**D1**) and (**D2**) for the collection $\{\mathfrak{T}_{\bar{\mathfrak{n}}}\}_{\bar{\mathfrak{n}}\prec\bar{\mathfrak{n}}_0} \cup \{\mathfrak{T}/\mathfrak{m}\}$ in an identical way. It remains to verify (**D3**). To settle that, consider the commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \arrow[r] \arrow[d, dashed] & H^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \arrow[r] \arrow[d] & H^1_f(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})[\mathfrak{m}_{\mathcal{R}}] \arrow[r] & H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})[\mathfrak{m}_{\mathcal{R}}] \arrow[r] & H^1_f(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})[\mathfrak{m}_{\mathcal{R}}] \arrow[r] & 0 \end{tikzcd}$$

As the $\mathcal{R}_0$-module $H^1_f(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})$ (resp., the $\mathtt{k}$-vector space $H^1_f(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m})$) is free of rank one (resp., is one-dimensional), it follows that the right-most arrow is injective and by chasing the diagram it follows that

$$H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) = \ker\left(H^1(\mathbb{Q}_\ell, \mathfrak{T}/\mathfrak{m}) \longrightarrow \frac{H^1(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})}{H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, \mathfrak{T}_{1,1,1})}\right),$$

which is exactly the statement of (**D4**). □

4.2. **Controlling the Selmer sheaf.** Assume throughout this section that $\chi(\mathbb{T}) = \chi(\mathfrak{T}) = 1$ in addition to the running hypotheses. Let $\bar{\mathfrak{n}} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$ and $\bar{\mathfrak{s}} = (r, u, v) \in (\mathbb{Z}_{>0})^3$. Define the quotients $R_{\bar{\mathfrak{n}}} = R/(\varpi^r, X_1^u, X_2^v, X_3^w)$ and $\mathfrak{R}_{\bar{\mathfrak{s}}} = \mathfrak{R}/(\varpi^r, X^u, (\gamma-1)^v)$.

4.2.1. *The upper bound.*

Proposition 4.17. *We have the following isomorphisms:*

(i) $H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\mathrm{can}})}/\mathbb{Q}, \bar{T}) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\mathrm{can}})}/\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}})[\mathcal{M}]$,

(ii) $H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\mathrm{can}})}/\mathbb{Q}, \mathfrak{T}/\mathfrak{m}) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\mathrm{can}})}/\mathbb{Q}, \mathfrak{T}_{1,1,1})[\mathfrak{m}]$,

(iii) $H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\mathrm{can}})}/\mathbb{Q}, \mathfrak{T}_{1,1,1}) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\mathrm{can}})}/\mathbb{Q}, \mathfrak{T}_{\bar{\mathfrak{s}}})[(\varpi, X, \gamma-1)]$,

(iv) $H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \bar{T}) \xrightarrow{\sim} H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}})[\mathcal{M}]$,

(v) $H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathfrak{T}/\mathfrak{m}) \xrightarrow{\sim} H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathfrak{T}_{\bar{\mathfrak{s}}})[\mathfrak{m}]$.

Proof. (i), (ii) and (iii) follows from the proof of [MR04, Lemma 3.5.2]; see in particular the displayed equation (7) in loc.cit. (iv) is now verified using (i) and Propositions 4.3, 4.10 and 4.16. (v) follows from (ii), (iii) and Propositions 4.3, 4.15 and 4.16. □

Corollary 4.18. *Let $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ (resp., $n \in \mathcal{N}_{\bar{\mathfrak{s}}}$) be a core vertex for the Selmer structure $\mathcal{F}_{\mathrm{can}}$ on $\bar{T}$ (in the sense of Definition 3.1). Then,*

(i) *the $R_{\bar{\mathfrak{n}}}$-module* $\mathrm{Hom}\left(H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}}), \Phi/\mathcal{O}\right)$ *and,*

(ii) *the $\mathfrak{R}_{\bar{\mathfrak{s}}}$-module* $\mathrm{Hom}\left(H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathfrak{T}_{\bar{\mathfrak{s}}}), \Phi/\mathcal{O}\right)$

are both cyclic.

Proof. By Proposition 4.17(iii), it follows that

$$\mathrm{Hom}\left(H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}}), \Phi/\mathcal{O}\right) \Big/ \mathcal{M} \cong \mathrm{Hom}\left(H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \bar{T}), \Phi/\mathcal{O}\right).$$

Since the $\mathtt{k}$-vector space $\mathrm{Hom}\left(H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \bar{T}), \Phi/\mathcal{O}\right)$ is one-dimensional (thanks to our assumption that n is a core vertex and that $\chi(\mathbb{T}) = 1$), it follows $\mathrm{Hom}\left(H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}}), \Phi/\mathcal{O}\right)$

is a cyclic $R_{\bar{\mathfrak{n}}}$-module by Nakayama's Lemma. The statement for $\mathfrak{T}_{\bar{\mathfrak{s}}}$ is proved in an identical fashion, using Proposition 4.17(iv) instead of Proposition 4.17(iii). $\square$

Remark 4.19. There exists infinitely many n as in the statement of Corollary 4.18 thanks to [MR04, §4.1].

4.2.2. *The lower bound.* As above, let $\bar{\mathfrak{n}} = (r, u, v, w) \in (\mathbb{Z}_{>0})^4$ and $\bar{\mathfrak{s}} = (r, u, v) \in (\mathbb{Z}_{>0})^3$.

Proposition 4.20. *For $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ we have,*

$$\text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}})\right) - \text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\text{can}}(n)^*}(\mathbb{Q}, \mathbb{T}^*_{\bar{\mathfrak{n}}})\right) = \text{length}_{\mathcal{O}}(R_{\bar{\mathfrak{n}}}).$$

Similarly for $n \in \mathcal{N}_{\bar{\mathfrak{s}}}$,

$$\text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \mathfrak{T}_{\bar{\mathfrak{s}}})\right) - \text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\text{can}}(n)^*}(\mathbb{Q}, \mathfrak{T}^*_{\bar{\mathfrak{s}}})\right) = \text{length}_{\mathcal{O}}(\mathfrak{R}_{\bar{\mathfrak{s}}}).$$

Proof. By [MR04, Corollary 2.3.6] it suffices to verify the assertions of the proposition only when $n = 1$.

Let $\mathbb{T}_{u,v,w}$ be as in §1.4, so that $\mathbb{T}_{u,v,w}$ is a free $\mathcal{O}$-module of rank uvw. Theorem 4.1.13 of [MR04] (applied with the $\mathcal{O}[[G_{\mathbb{Q}}]]$-representation $T = \mathbb{T}_{u,v,w}$ and its quotient $\mathbb{T}_{\bar{\mathfrak{n}}} = \mathbb{T}_{u,v,w}/\varpi^r$) shows that

$$\text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}})\right) - \text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*_{\bar{\mathfrak{n}}})\right) = ruvw \cdot \chi(\mathbb{T}) = \text{length}_{\mathcal{O}}(R_{\bar{\mathfrak{n}}}),$$

as desired. Similarly, repeating the arguments above for the free $\mathcal{O}$-module $\mathfrak{T}_{u,v}$ (of rank $uv \cdot \dim_{\Bbbk}(\mathcal{R}_0)$), we conclude with the second assertion. $\square$

Corollary 4.21. *For n as in Proposition 4.20,*

(i) $\text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}})\right) \geq \text{length}_{\mathcal{O}}(R_{\bar{\mathfrak{n}}})$,

(ii) $\text{length}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \mathfrak{T}_{\bar{\mathfrak{s}}})\right) \geq \text{length}_{\mathcal{O}}(\mathfrak{R}_{\bar{\mathfrak{s}}})$.

We are now ready to prove Theorem 3.2:

Corollary 4.22.

(i) *Let $n \in \mathcal{N}_{\bar{\mathfrak{n}}}$ be a core vertex for the Selmer structure $\mathcal{F}_{\text{can}}$ on the residual representation $\bar{T}$. Then, the $R_{\bar{\mathfrak{n}}}$-module $H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}})$ is free of rank one and $H^1_{\mathcal{F}_{\text{can}}(n)^*}(\mathbb{Q}, \mathbb{T}^*_{\bar{\mathfrak{n}}}) = 0$.*

(ii) *Let $s \in \mathcal{N}_{\bar{\mathfrak{s}}}$) be a core vertex for $\mathcal{F}_{\text{can}}$ on $\bar{T}$.Then, the $\mathfrak{R}_{\bar{\mathfrak{s}}}$-module $H^1_{\mathcal{F}_{\text{can}}(s)}(\mathbb{Q}, \mathfrak{T}_{\bar{\mathfrak{s}}})$ is free of rank one and $H^1_{\mathcal{F}_{\text{can}}(s)^*}(\mathbb{Q}, \mathfrak{T}^*_{\bar{\mathfrak{s}}}) = 0$.*

Proof. It follows from Corollaries 4.18 and 4.21 that $\text{Hom}\left(H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}, \mathbb{T}_{\bar{\mathfrak{n}}}), \Phi/\mathcal{O}\right)$ (resp., $\text{Hom}\left(H^1_{\mathcal{F}_{\text{can}}(s)}(\mathbb{Q}, \mathfrak{T}_{\bar{\mathfrak{s}}}), \Phi/\mathcal{O}\right)$) is a free $R_{\bar{\mathfrak{n}}}$-module (resp., a free $\mathfrak{R}_{\bar{\mathfrak{s}}}$-module) of rank one. The first halves of (i) and (ii) follow from the Gorenstein property of R and $\mathfrak{R}$, c.f., [Gro67, Prop. 4.9 and 4.10]. The point is that $\Phi/\mathcal{O}$ is an injective hull of $\Bbbk$ and thus a dualizing module for R and $\mathfrak{R}$.

The second halves (the vanishing statements of the dual Selmer groups) follow from the first halves and Proposition 4.20. $\square$

Remark 4.23. In this remark, we record the following simple fact which we need at various points to follow: Assuming (**H3**), R-module $H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T})$ is R-torsion-free. Indeed, we first note

$$H^0(\mathbb{Q},X)=0 \tag{4.14}$$

for every subquotient X of $\mathbb{T}$, by [MR04, Lemma 2.1.4]. Let $H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T})_{\text{tor}}$ denote the R-torsion submodule of $H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T})$. We also let F denote the field of fractions of R and set $\mathbb{V}=\mathbb{T}\otimes F$, $\mathbb{W}=\mathbb{V}/\mathbb{T}$. Observe that

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T})_{\text{tor}}=\ker\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T})\longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{V})\right)\cong H^0(\mathbb{Q},\mathbb{W}),$$

and thus we are reduced to verify the vanishing of $H^0(\mathbb{Q},\mathbb{W})$. Suppose

$$0\neq t\otimes 1/g\in H^0(\mathbb{Q},\mathbb{W})$$

(with $t\in\mathbb{T}$ and $g\in R$). We thus obtain a non-trivial element $\bar{t}\in H^0(\mathbb{Q},\mathbb{T}/g\mathbb{T})$ and contradict (4.14).

5. Applications

5.1. **Elliptic curves and interpolation of the Beilinson-Kato Kolyvagin sytem.** Suppose $E_{/\mathbb{Q}}$ is an elliptic curve without CM which has split-multiplicative reduction at all primes ℓ dividing its conductor N_E. Let $\bar{T}=E[p]$ be the p-torsion on E. Let

$$\bar{\rho}_E: G_{\mathbb{Q}}\longrightarrow \mathrm{GL}(\bar{T})=\mathrm{GL}_2(\mathbb{F}_p)$$

be the $\bmod p$ Galois representation attached to E. Suppose that the universal deformation problem for $\bar{\rho}_E$ is unobstructed; see Example 1.2.2 above for the content of this assumption. Let R be the universal deformation ring and $\boldsymbol{\rho}_E$ the universal deformation of $\bar{\rho}_E$ and the $\mathbb{T}$ the deformation space on which $G_{\mathbb{Q}}$ acts by $\boldsymbol{\rho}_E$. Since we assumed that the deformation problem is unobstructed, $R\cong\mathbb{Z}_p[[X_1,X_2,X_3]]$ and $\mathbb{T}$ is free of rank two as an R-module. The Weil-pairing shows that $\chi(\bar{T})=1$.

We further suppose that $\mathbb{T}$ satisfies the hypotheses (**H1**)-(**H4**) as well as (**H.Tam**)(i) and (**H.nA**). Before discussing the applications of Theorem 3.12 to this setting, we first explain the contents of the extra hypotheses (**H.Tam**) and (**H.nA**) in this particular setting.

Proposition 5.1. *Suppose $E_{/\mathbb{Q}}$ is an elliptic curve as above and $\ell|N_E$ is a prime. Assume that*

- *p does not divide the Tamagawa number c_ℓ at ℓ,*
- *p does not divide $\ell-1$.*

Then (**H.Tam**) *holds true for $\bar{T}=E[p]$.*

Proof. Let $T=T_p(E)$ be the p-adic Tate module of E. Under the running assumptions, there is a non-split exact sequence

$$0\longrightarrow\mathbb{Z}_p(1)\longrightarrow T\longrightarrow\mathbb{Z}_p\longrightarrow 0 \tag{5.1}$$

of $\mathbb{Z}_p[[G_\ell]]$-modules. Let $\sigma=\partial(1)\in H^1(\mathbb{Q}_\ell,\mathbb{Z}_p(1))$ where $\partial:\mathbb{Z}_p\to H^1(\mathbb{Q}_\ell,\mathbb{Z}_p(1))$ is the connecting homomorphism in the long exact sequence of the $G_{\mathbb{Q}_\ell}$-cohomology of the sequence (5.1). Kummer theory gives an isomorphism

$$\mathrm{ord}_\ell: H^1(\mathbb{Q}_\ell,\mathbb{Z}_p(1))\xrightarrow{\sim}\mathbb{Q}_\ell^{\times,\wedge}\xrightarrow{\sim}\mathbb{Z}_p. \tag{5.2}$$

According to [CE56] pp. 290 and 292, $-\sigma$ is the extension class of the sequence (5.1) inside $\mathrm{Ext}^1_{\mathbb{Z}_p[G_{\mathbb{Q}_\ell}]}(\mathbb{Z}_p, \mathbb{Z}_p(1)) = H^1(\mathbb{Q}_\ell, \mathbb{Z}_p(1))$. Hence $\mathrm{ord}_\ell(\sigma) \neq 0$ as the sequence (5.1) is non-split and by [Büy13a, Prop. 3.3] it further follows that $\mathrm{ord}_\ell(\sigma) \in \mathbb{Z}_p^\times$ and therefore ∂ is surjective.

We have the following diagram below with exact rows and commutative squares

$$\begin{tikzcd} & H^0(\mathbb{Q}_\ell, T) \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r, two heads, "\partial"] \arrow[d, two heads] & H^1(\mathbb{Q}_\ell, \mathbb{Z}_p(1)) \cong \mathbb{Z}_p \arrow[d, two heads] \\ 0 \arrow[r] & H^0(\mathbb{Q}_\ell, \boldsymbol{\mu}_p) \arrow[r] & H^0(\mathbb{Q}_\ell, \bar{T}) \arrow[r] & \mathbb{Z}/p\mathbb{Z} \arrow[r, "\bar{\partial}"] & H^1(\mathbb{Q}_\ell, \boldsymbol{\mu}_p) \cong \mathbb{Z}/p\mathbb{Z} \end{tikzcd}$$

This shows that the map $\bar{\partial}$ is surjective as well and hence

$$H^0(\mathbb{Q}_\ell, \bar{T}) \cong H^0(\mathbb{Q}_\ell, \boldsymbol{\mu}_p) = 0$$

as we assumed $p \nmid \ell - 1$. □

Remark 5.2. The assumption (**H.nA**) is to say that p is *non-anomalous* for E (in the sense of Mazur [Maz72]). Given an elliptic curve E, Mazur in loc.cit. points out that anomalous primes should be scarce.

Let f be any elliptic newform of weight $\omega \geq 2$ and let

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}_f)$$

be the Galois representation attached to f by Deligne with coefficients in the ring of integers $\mathcal{O}_f$ of a finite extension Φ_f of of $\mathbb{Q}_p$. Let T_f be the free $\mathcal{O}_f$ module of rank 2 on which $G_{\mathbb{Q}}$ acts via ρ_f. Let $\mathfrak{m}_f$ denote the maximal ideal of $\mathcal{O}_f$ and let $\bar{\rho}_f$ the residual representation of $\bar{\rho}_f$ $\bmod \mathfrak{m}_f$. Suppose that $\bar{\rho}_f \cong \bar{\rho}$ so that ρ_f is a deformation of $\bar{\rho}$ to the ring $\mathcal{O}_f$. We thus have a ring homomorphism $\varphi_f : R \to \mathcal{O}_f$ that induces and isomorphism $T_f \cong \mathbb{T} \otimes_{\varphi_f} \mathcal{O}_f$, and by functoriality a commutative diagram

$$\begin{tikzcd} \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) \arrow[rr, "\varphi_f"] \arrow[dr] & & \overline{\mathbf{KS}}(T_f, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) \arrow[dl] \\ & \mathbf{KS}(\bar{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) & \end{tikzcd} \tag{5.3}$$

Until the end of this section, we let $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ denote an arbitrary big Kolyvagin system which generates the cyclic R-module $\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ and we let $\varphi_f(\boldsymbol{\kappa})$ be its image in $\overline{\mathbf{KS}}(T_f, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$. Let $\boldsymbol{\kappa}_f^{\mathrm{Kato}} \in \overline{\mathbf{KS}}(T_f, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ be the Kolyvagin system obtained from the Beilinson-Kato Euler system attached to the modular form f (as in [MR04, Theorem 3.2.4]).

Theorem 5.3 (Interpolation)**.** *There is a $\lambda_f \in \mathcal{O}_f$ such that*

$$\lambda_f \cdot \varphi_f(\boldsymbol{\kappa}) = \boldsymbol{\kappa}_f^{\mathrm{Kato}}.$$

Remark 5.4. Theorem 5.3 states that the *improvements* (by the factors λ_f) of the Beilinson-Kato Kolyvagin systems interpolate to give rise to the big Kolyvagin system, rather than the Beilinson-Kato Kolyvagin systems themselves. Note that the Kolyvagin system $\varphi_f(\boldsymbol{\kappa})$ is called an *improvement* to $\boldsymbol{\kappa}_f^{\mathrm{Kato}}$ as the bound (on the relevant Selmer group) obtained using $\varphi_f(\boldsymbol{\kappa})$ improves that obtained using $\boldsymbol{\kappa}_f^{\mathrm{Kato}} = \lambda_f \cdot \varphi_f(\boldsymbol{\kappa})$ by a factor of λ_f. In particular, when the Kolyvagin system $\boldsymbol{\kappa}_f^{\mathrm{Kato}}$ is itself primitive (in the sense of [MR04, Definition 4.5.5], see also Corollary 5.2.13(ii) and Theorem 5.3.10(iii) in loc.cit.) we have $\lambda_f \in \mathcal{O}_f^\times$. In particular, if the elliptic curve E has good ordinary reduction at p and the newform f is p-ordinary, it indeed follows from [SU13] that the interpolation factor λ_f is a unit.

Proof of Theorem 5.3. Let $\bar{\boldsymbol{\kappa}}$ be the image of $\boldsymbol{\kappa}$ in $\mathbf{KS}(\bar{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$. By Theorem 3.12 it follows that $\bar{\boldsymbol{\kappa}} \neq 0$, so it follows by [MR04, Theorem 5.2.10(ii)] and the commutative diagram (5.3) that $\varphi_f(\boldsymbol{\kappa})$ generates the free $\mathcal{O}_f$-module $\overline{\mathbf{KS}}(T_f, \mathcal{F}_{\text{can}}, \mathcal{P})$ of rank one. $\square$

Remark 5.5. In Theorem 5.18 and Corollary 5.19 below we will explain how the big Kolyvagin system $\boldsymbol{\kappa}$ may be used to obtain bounds on certain Selmer groups. Before that, we remark that

(i) the Kolyvagin system $\boldsymbol{\kappa}_f^{\text{Kato}}$ is related to a special value of L-function attached to f, by the work of Kato [Kat04, §14];
(ii) the recent works of Kisin (unpublished lecture notes) show that the the classical modular points are Zariski dense in $\mathrm{Spec}(R)$.

Thanks to these remarks, one would hope that the bounds we shall obtain below in terms $\boldsymbol{\kappa}$ will be ultimately related to an appropriate p-adic L-function (whose existence for the time being is highly conjectural) in 3-variables. See §5.4 for an elaboration of this point.

Proposition 5.6. *We assume further that the elliptic curve $E_{/\mathbb{Q}}$ has good reduction at p. Then the leading term κ_1 of $\boldsymbol{\kappa}$ is non-zero.*

Proof. Let $T = T_p(E)$ denote the p-adic Tate module of E. By universality, we have a unique ring homomorphism
$$\varphi_\Lambda : R \longrightarrow \mathbb{Z}_p[[\Gamma]]$$
which induces a map
$$\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\text{can}}, \mathcal{P}) \xrightarrow{\varphi_\Lambda} \overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\text{can}}, \mathcal{P}).$$
Let $\boldsymbol{\kappa}^{\text{Kato},\Lambda} \in \overline{\mathbf{KS}}(T \otimes \Lambda, \mathcal{F}_{\text{can}}, \mathcal{P})$ denote the Beilinson-Kato Kolyvagin system obtained as in [MR04, §6.2]. Let $\kappa_1^{\text{Kato},\Lambda} \in H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, T \otimes \Lambda)$ denote the leading term of $\boldsymbol{\kappa}^{\text{Kato},\Lambda}$. When E has good ordinary reduction at p, it follows from [Kat04, Theorem 16.6] and a non-vanishing result of Rohrlich [Roh84] that $\kappa_1^{\text{Kato},\Lambda} \neq 0$. Also in the case when E has good supersingular reduction at p, [Kob03, Theorem 6.3] and Rohrlich's theorem shows that $\kappa_1^{\text{Kato},\Lambda} \neq 0$.

For $\boldsymbol{\kappa}$ as above, one verifies that $\lambda \cdot \varphi_\Lambda(\boldsymbol{\kappa}) = \boldsymbol{\kappa}^{\text{Kato},\Lambda}$ for some $\lambda \in \Lambda$, using the proof of Theorem 5.3 and [Büy11, Theorem 3.23]. This shows that the leading term κ_1 of $\boldsymbol{\kappa}$ is non-zero. $\square$

5.2. **Hida's nearly ordinary deformation revisited.** Suppose $E_{/\mathbb{Q}}$, is an elliptic curve without CM and $\bar{\rho}_E$ and $\bar{T} = E[p]$ are defined as in §5.1. Suppose in this section that E is p-ordinary and has split multiplicative reduction at every prime ℓ dividing its conductor N_E. Let f_E be the newform of weight 2 and level N_E associated to E by the work of Wiles and Taylor-Wiles.

Let $\Gamma^{\text{w}} = 1 + p\mathbb{Z}_p$. Identify $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$ by $\boldsymbol{\mu}_{p-1}$ via the Teichmüller character ω so that we have
$$\mathbb{Z}_p^\times \cong \Delta \times \Gamma^{\text{w}}.$$
Set $\Lambda^{\text{w}} = \mathbb{Z}_p[[\Gamma^{\text{w}}]]$. Let $\mathfrak{h}^{\text{ord}}$ Hida's universal ordinary Hecke algebra parametrizing Hida family passing through f_E, which is finite flat over Λ^{w} by [Hid86a, Theorem 1.1]. We will recall some basic properties of $\mathfrak{h}^{\text{ord}}$, for details the reader may consult [Hid86a, Hid86b] and [EPW06, §2] for a survey. The eigenform $f = f_E$ fixed as above corresponds to an *arithmetic specialization*
$$\mathfrak{s}_f : \mathfrak{h}^{\text{ord}} \longrightarrow \mathbb{Z}_p$$

Decompose $\mathfrak{h}^{\mathrm{ord}}$ into a direct sum of its completions at maximal ideals and let $\mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$ be the (unique) summand through which $\mathfrak{s}_f$ factors. The localization of $\mathfrak{h}^{\mathrm{ord}}$ at $\ker(\mathfrak{s}_f)$ is a discrete valuation ring [Nek06, §12.7.5], and hence there is a unique minimal prime $\mathfrak{a} \subset \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}$ such that $\mathfrak{s}_f$ factors through the integral domain

$$\mathcal{R} = \mathfrak{h}^{\mathrm{ord}}_{\mathfrak{m}}/\mathfrak{a}. \tag{5.4}$$

The Λ^{w}-algebra $\mathcal{R}$ is called the branch of the Hida family on which f_E lives, by duality it corresponds to a family $\mathbb{F}$ of ordinary modular forms. Hida [Hid86b] gives a construction of a big $G_{\mathbb{Q}}$-representation $\mathcal{T}$ with coefficients in $\mathcal{R}$ and the ring $\mathcal{R}$ is Gorenstein of dimension two (as in [Wil95, TW95]). Let $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$ be the cyclotomic Iwasawa algebra and set $\mathfrak{R} = \mathcal{R} \otimes_{\mathbb{Z}_p} \Lambda$ and $\mathfrak{T} = \mathcal{T} \otimes_{\mathbb{Z}_p} \Lambda$, where we allow $G_{\mathbb{Q}}$ act both on $\mathcal{T}$ and Λ. Following Ochiai, we call $\mathfrak{T}$ the universal ordinary deformation of $\bar{T}$.

Suppose that the hypotheses (**H1**)-(**H4**) as well as (**H.Tam**) and (**H.nA**) hold true. In this case, Theorem 3.12 recovers a theorem of Ochiai [Och05], which he establishes by deforming Kato's Euler system[3] along the ordinary locus of the universal deformation space. For any ordinary eigenform g that lives in the branch $\mathcal{R}$ of the Hida family, let

$$\mathfrak{s}_g : \mathcal{R} \longrightarrow \mathcal{O}_g$$

denote the corresponding arithmetic specialization and $T_g = \mathcal{T} \otimes_{\mathfrak{s}_g} \mathcal{O}_g$ the associated Galois representation, where $\mathcal{O}_g$ is the integers of a finite extension of $\mathbb{Q}_p$. Let

$$\boldsymbol{\kappa}^{\mathrm{Kato}} \in \overline{\mathbf{KS}}(T_g \otimes \Lambda, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

be the Λ-adic Kolyvagin system for to the cyclotomic deformation $T_g \otimes \Lambda$ obtained from Kato's Euler system as in [MR04, §6.2]. The proof of the following is identical to the proof of Theorem 5.3.

Theorem 5.7. *Let $\boldsymbol{\kappa}$ be a generator of the free $\mathfrak{R}$-module $\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ of rank one. Let*

$$\overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) \xrightarrow{\mathfrak{s}_g} \overline{\mathbf{KS}}(T_g \otimes \Lambda, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

be the map induced from the arithmetic specialization $\mathfrak{s}_g$ above by functoriality. There is a $\lambda_g \in \mathcal{O}_g[[\Gamma]]$ such that

$$\lambda_g \cdot \mathfrak{s}_g(\boldsymbol{\kappa}) = \boldsymbol{\kappa}^{\mathrm{Kato}}.$$

Theorem 5.7 may be applied along with the standard Kolyvagin system machinery to recover the results of [Och05].

Corollary 5.8. *Let $\boldsymbol{\kappa}$ be as above. Its leading term $\kappa_1 \in H^1(\mathbb{Q}, \mathfrak{T})$ is then non-zero.*

Proof. For g as above, Kato proves that $\kappa_1^{\mathrm{Kato}} \in H^1(\mathbb{Q}, T_g \otimes \Lambda)$ is non-zero. Corollary follows from Theorem 5.7. $\square$

5.3. **Weak Leopoldt Conjecture and Greenberg's main conjecture.** The notation and the hypotheses of Section 3 are in effect throughout. We assume until the end of this section that the ring $\mathcal{R}$ is regular (therefore, so is the ring $\mathfrak{R}$).

Remark 5.9. In the setting of §5.2 above, namely when $\mathcal{R}$ is Hida's universal ordinary Hecke algebra as in §5.2, the ring $\mathcal{R}$ is often regular as explained in [FO12, Lemma 2.7].

The goal of this section is to prove Theorems 5.10 and 5.11 below.

[3]Whereas we carry this out in the level of Kolyvagin systems.

Theorem 5.10. *If the $\mathfrak{R}$-module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathfrak{T}^*)^\vee$ is torsion, then there is a Kolyvagin system $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathfrak{T},\mathcal{F}_{\text{can}},\mathcal{P})$ whose leading term κ_1 is non-zero.*

Similar statement holds true when $\mathfrak{R}$ is replaced by R and $\mathfrak{T}$ by $\mathbb{T}$.

Theorem 5.11. *Suppose that there is a Kolyvagin system $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathfrak{T},\mathcal{F}_{\text{can}},\mathcal{P})$ whose leading term varies that $\kappa_1 \neq 0$. Then the $\mathfrak{R}$-module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathfrak{T}^*)^\vee$ is torsion.*

Similar statement holds true when $\mathfrak{R}$ is replaced by R and $\mathfrak{T}$ by $\mathbb{T}$.

Remark 5.12. As we have observed in Remark 4.23[4], our running assumptions guarantee for $X = R$ or $\mathfrak{R}$ (and resp., $Y = \mathbb{T}$ or $\mathfrak{T}$) that the X-module $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, Y)$ is torsion-free. In this case, the following conditions on an element $c \in H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, Y)$ are equivalent:

(1) $c \neq 0$.
(2) c is not X-torsion.
(3) $c \notin \mathfrak{p}H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, Y)$ for infinitely many height one primes $\mathfrak{p} \subset X$.
(4) $c \notin \mathfrak{p}H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, Y)$ for some height one prime $\mathfrak{p} \subset X$.

The only non-trivial step is to verify the implication $(2) \implies (3)$ and this may carried out following the proof of [How07, Lemma 2.1.7].

Remark 5.13. The conclusion of Theorem 5.11 that the $\mathfrak{R}$-module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathfrak{T}^*)^\vee$ is torsion is a form of Leopoldt's conjecture.

Before we give the proofs of Theorems 5.10 and 5.11, we prove two general preparatory lemmas from commutative algebra. Let $\mathcal{H}_0$ be a local integral domain which is a complete, regular $\mathcal{O}$-algebra that has relative dimension *one* over $\mathcal{O}$. Set $\mathcal{H} = \mathcal{H}_0[[X]]$ and suppose $\mathbf{T}$ is a finite $\mathcal{H}$-module endowed with a continuous action of $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$.

In applications, $\mathcal{H}$ will either be a certain quotient of $R = \mathcal{O}[[X_1, X_2, X_3]]$ (and $\mathbf{T} = \mathbb{T} \otimes_R \mathcal{H}$) or else $\mathcal{H}_0$ will be the ring $\mathcal{R}$. In case of the latter, we will set $\mathcal{H} = \mathfrak{R} := \mathcal{R}[[\Gamma]]$ and $\mathbf{T}$ will be the representation $\mathfrak{T} = \mathcal{T} \otimes \Lambda$ that was introduced in the previous section. Note that in either case, $\mathbf{T}$ satisfies the hypotheses (**H1**)-(**H4**), (**H.Tam**) and (**H.nA**). Suppose further until the end that $\chi(\bar{T}) = 1$.

Lemma 5.14. *Suppose M is an $\mathcal{H}$-module. Assume for a height one prime $\wp$ of $\mathcal{H}_0$ and an integer N, the quotient $M/(\wp, X + p^N)M$ is of finite order. Then M is a finitely generated $\mathcal{H}$-torsion module.*

Proof. We first give a proof assuming that $p \notin \wp$. To ease notation, write $X_N = X + p^N$. One can find an integer s so that

$$p^s \cdot (M/(\wp, X_N)M) = 0. \tag{5.5}$$

By Nakayama's Lemma M is finitely generated as an $\mathcal{H}$-module, say by $m_1, \ldots, m_r \in M$. It follows from (5.5) that

$$p^s m_i = \sum_{j=1}^{r} a_j^{(i)} m_j,$$

[4]The argument presented in Remark 4.23 applies verbatim for the regular ring $\mathfrak{R}$ and the representation $\mathfrak{T}$.

where $a_j^{(i)} \in (\wp, X_N)$. Setting $A = [a_j^{(i)}]$ and $B = A - p^s \cdot I_{r\times r}$, we conclude by (5.5) that

$$\sum_{j=1}^{i-1} a_j^{(i)} m_j + (a_i^{(i)} - p^s) m_i + \sum_{j=i+1}^{r} a_j^{(i)} m_j = 0$$

$$\implies (A - p^s \cdot I_{r\times r}) \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = B \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = 0$$

$$\implies \mathrm{adj}(B) \cdot B \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix} = 0 \implies \det(B) \cdot M = 0.$$

To conclude with the proof of the lemma, we check that the element $\det(B) \in \mathcal{H}$ is non-zero. Observe that

$$\begin{aligned} \det(B) = \det(A - p^s \cdot I_{r\times r}) &\equiv (-1)^r p^{sr} \mod (\wp, X_N) \\ &\not\equiv 0 \mod (\wp, X_N), \end{aligned}$$

as the ring $\mathcal{H}/(\wp, X_N) \cong \mathcal{H}_0/\wp$ is an integral domain of characteristic zero, as we have assumed $p \notin \wp$.

In case $p \in \wp$, note that the given condition on M translates into the statement that the module $M/(\wp, X)M$ has finite cardinality. If M were not a torsion module, $M/(\wp, X)M$ would contain a submodule isomorphic to $\mathcal{H}_0/\wp$. The latter ring, however, has infinite cardinality as it at least contains a ring isomorphic to a formal power series ring in one variable over $\mathbb{F}_p$. □

Lemma 5.15. *If the R-module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)^\vee$ is torsion, then for almost all height one primes $\wp \subset R$, the $R/\wp$-module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)^\vee / \wp H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)^\vee$ is torsion.*

Proof. Let M be any finitely generated torsion R-module, with generators $m_1, \cdots, m_r$. Since M is torsion, it follows that for all but finitely many height one primes $\wp$ of R, we have $M_\wp = 0$. This in particular means for every $1 \le i \le r$, there is $s_i \in R - \wp$ such that $s_i m_i = 0$. Set $s = s_1 \cdots s_r$ and let $\bar{s} \in R/\wp$ denote the homomorphic image of s. Note that $\bar{s} \neq 0$ and that $\bar{s} \cdot M/\wp M = 0$. This shows that the $R/\wp$-module $M/\wp M$ is torsion for almost all height one primes $\wp$. This is exactly the assertion of the Lemma with $M = H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, \mathbb{T}^*)^\vee$.

□

Proof of Theorem 5.10. We first give a proof for the statement concerning the ring $\mathfrak{R}$ and later use this result to deduce the statement for the ring R.

For any ideal I of $\mathfrak{R}$ and any subquotient M of $\mathfrak{T}$, we have by [MR04, 3.5.2] and the proof of [MR04, Lemma 3.5.3] (both of which apply thanks to our running hypothesis (**H3**)) that

$$H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, M^*[I]) \xrightarrow{\sim} H^1(\mathbb{Q}_{\Sigma(\mathcal{F}_{\text{can}})}/\mathbb{Q}, M^*)[I]$$

and hence also an injection

$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, M^*[I]) \hookrightarrow H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, M^*)[I],$$

where $\mathcal{F}^*_{\text{can}}$ on M^* is induced from the Selmer structure $\mathcal{F}_{\text{can}}$ on $\mathfrak{T}^*$ by propagation. Passing to Pontryagin duals, we thus obtain a surjection

$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, M^*)^\vee \otimes \mathfrak{R}/I \twoheadrightarrow H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}, M^*[I])^\vee. \tag{5.6}$$

By the assumption that $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathfrak{T}^*)^\vee$ is $\mathfrak{R}$-torsion, one may choose by [Mat89, Theorem 6.5] a specialization

$$\mathfrak{s}_\wp : \mathcal{R} \longrightarrow S_\wp$$

into the ring of integers $S_\wp$ of a finite extension $\Phi_\wp$ of $\mathbb{Q}_p$, whose kernel $\wp$ is a height one prime $\wp \subset \mathcal{R}$ and satisfies with the following properties:

- $S_\wp$ is integral closure of the integral domain $\mathcal{O}_\wp := \mathcal{R}/\wp$ in $\text{Frac}(\mathcal{O}_\wp) = \Phi_\wp$,
- $\wp \notin \text{Supp}_{\mathfrak{R}}(H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathfrak{T}^*)^\vee)$, where $\wp$ here denotes by slight abuse the height one prime which is the kernel of the induced map

$$\mathfrak{R} = \mathcal{R}[[\Gamma]] \xrightarrow{\mathfrak{s}_\wp} S_\wp[[\Gamma]].$$

We denote the induced ring homomorphism $\mathcal{O}_\wp \hookrightarrow S_\wp$ also by $\mathfrak{s}_\wp$.

For $\wp$ chosen as above, it follows that the module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathfrak{T}^*)^\vee/\wp$ is $\mathcal{O}_\wp[[\Gamma]]$-torsion. By (5.6) this implies that the module

$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathfrak{T}^*[\wp])^\vee \cong H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},(\mathcal{T}/\wp\mathcal{T}\otimes\Lambda)^*)^\vee$$

is $\mathcal{O}_p[[\Gamma]]$-torsion as well. It is therefore possible (using Hensel's Lemma and [Mat89, Theorem 6.5]) to choose an $N >> 0$ such that

- $\mathcal{O}_\wp[[\Gamma]]/(\gamma-1+p^N) \cong \mathcal{O}_\wp$,
- $\gamma - 1 + p^N \notin \text{Supp}_{\mathcal{O}_\wp[[\Gamma]]}\Big(H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},(\mathcal{T}/\wp\mathcal{T}\otimes\Lambda)^*)^\vee\Big)$.

For N chosen as above, we therefore have that the module

$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},(\mathcal{T}/\wp\mathcal{T}\otimes\Lambda)^*)^\vee/(\gamma-1+p^N)$$

is $\mathcal{O}_\wp$-torsion. Setting $T(\wp,N) := \mathcal{T}/\wp\mathcal{T}\otimes\Lambda/(\gamma-1+p^N)$ and applying (5.6) again, we conclude that the module

$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},(\mathcal{T}/\wp\mathcal{T}\otimes\Lambda/(\gamma-1+p^N))^*)^\vee \cong H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},T(\wp,N)^*)^\vee$$

is $\mathcal{O}_\wp$-torsion, hence finite.

When we do not vary N, we write $T(\wp)$ in place of $T(\wp,N)$ to ease notation. Let $T = T(\wp)\otimes_{\mathfrak{s}_\wp} S_\wp$ and define the Selmer structure $\mathcal{F}_T$ by setting

- $\Sigma(\mathcal{F}_T) = \Sigma(\mathcal{F}_{\text{can}}) =: \Sigma$,
- $H^1_{\mathcal{F}_T}(\mathbb{Q}_p,T) = H^1(\mathbb{Q}_p,T)$,
- $H^1_{\mathcal{F}_T}(\mathbb{Q}_\ell,T) = \ker\big(H^1(\mathbb{Q}_\ell,T)\longrightarrow H^1(I_\ell, T\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\big)$, for $\ell \neq p$.

Note that $\mathcal{F}_T$ is exactly what Mazur and Rubin call the *canonical Selmer structure* on T. Let ι denote the injection $T(\wp)\hookrightarrow T$. Then ι induces maps

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q},T^*) \longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q},T(\wp)^*)$$

$$H^1(\mathbb{Q}_\ell,T(\wp)) \longrightarrow H^1(\mathbb{Q}_\ell,T)$$

$$H^1(\mathbb{Q}_\ell,T^*) \longrightarrow H^1(\mathbb{Q}_\ell,T(\wp)^*)$$

for every prime ℓ. It is easy to see that the image of $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_\ell,T(\wp))$ lands in $H^1_{\mathcal{F}_T}(\mathbb{Q}_\ell,T)$ for every ℓ (and by local duality, the image of $H^1_{\mathcal{F}^*_T}(\mathbb{Q}_\ell,T^*)$ therefore lands in $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_\ell,T(\wp)^*)$). We thence obtain a map

$$H^1_{\mathcal{F}^*_T}(\mathbb{Q},T^*) \longrightarrow H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},T(\wp,N)^*). \tag{5.7}$$

In Lemma 5.16 below we check that the kernel and the cokernel of this map is finite for all sufficiently large N. This shows that $H^1_{\mathcal{F}_T^*}(\mathbb{Q},T^*)$ is of finite order for $N >> 0$, as we have already verified above that $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},T(\wp,N)^*)$ is finite.

Let $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathfrak{T},\mathcal{F}_{\text{can}},\mathcal{P})$ be a generator so that its image $\bar{\boldsymbol{\kappa}} \in \mathbf{KS}(\bar{T},\mathcal{F}_{\text{can}},\mathcal{P})$ is non-zero by Theorem 3.12. Hence, the image $\boldsymbol{\kappa}^{(\wp)}$ of $\boldsymbol{\kappa}$ in $\overline{\mathbf{KS}}(T,\mathcal{F}_T,\mathcal{P})$ is non-zero as well. Corollary 5.2.13 of [MR04] applies thanks to our running hypotheses and it follows that the $\kappa_1^{(\wp)} \neq 0$ and hence $\kappa_1 \neq 0$.

As for the assertion for $\mathbb{T}$, we first use Lemma 5.15 to find a height one prime of the form $\wp = (X_3 + p^M)R$ that verifies the conclusion of Lemma 5.15. For the chosen height one prime $\wp$ set $\mathcal{H} = R/\wp \cong \mathcal{O}[[X_1,X_2]]$ and $\mathbf{T} = \mathbb{T} \otimes_R \mathcal{H}$. Observe (using (5.6)) that

$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathbf{T}^*) \cong H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathbb{T}^*)[\wp] \tag{5.8}$$

and upon taking Pontryagin duals

$$H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathbf{T}^*)^\vee \cong H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathbb{T}^*)^\vee/\wp H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathbb{T}^*)^\vee.$$

Let $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\text{can}},\mathcal{P})$ be any generator and let

$$\varphi : \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\text{can}},\mathcal{P}) \longrightarrow \overline{\mathbf{KS}}(\mathbf{T},\mathcal{F}_{\text{can}},\mathcal{P})$$

denote the map induced from $R \to \mathcal{H}$. The commutativity of the diagram

$$\begin{tikzcd} \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\text{can}},\mathcal{P}) \arrow[rr, "\varphi"] \arrow[dr] & & \overline{\mathbf{KS}}(\mathbf{T},\mathcal{F}_{\text{can}},\mathcal{P}) \arrow[dl] \\ & \mathbf{KS}(\bar{T},\mathcal{F}_{\text{can}},\mathcal{P}) & \end{tikzcd} \tag{5.9}$$

shows that $\varphi(\boldsymbol{\kappa})$ generates the free $\mathcal{H}$-module $\overline{\mathbf{KS}}(\mathbf{T},\mathcal{F}_{\text{can}},\mathcal{P})$ of rank one. Furthermore, by our choice of the prime ideal $\wp$ and (5.8), the $\mathcal{H}$-module $H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},\mathbf{T}^*)^\vee$ is torsion. Using the proof above for the ring $\mathcal{H}$ in place of $\mathfrak{R}$, we conclude that

$$\varphi(\kappa_1) = \varphi(\boldsymbol{\kappa})_1 \neq 0,$$

in particular that $\kappa_1 \neq 0$.

This completes the proof of Theorem 5.10 modulo Lemma 5.16 below. □

Let T and $T(\wp,N)$ be as in the proof of Theorem 5.10.

Lemma 5.16. *When the positive integer N is sufficiently large, both the kernel and the cokernel of the map*

$$H^1_{\mathcal{F}_T^*}(\mathbb{Q},T^*) \longrightarrow H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q},T(\wp,N)^*)$$

are finite.

Proof. We first verify that the kernels and the cokernels of the maps

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q},T^*) \longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q},T(\wp,N)^*) \tag{5.10}$$

$$H^1_{\mathcal{F}_T^*}(\mathbb{Q}_\ell,T^*) \longrightarrow H^1_{\mathcal{F}^*_{\text{can}}}(\mathbb{Q}_\ell,T(\wp,N)^*) \tag{5.11}$$

have finite order for every prime ℓ. When we do not vary N, we will denote $T(\wp,N)$ simply by $T(\wp)$.

Observe that the kernel of the map (5.10) lives in $H^0(\mathbb{Q}_\Sigma/\mathbb{Q},(T/T(\wp))^*)$ and its cokernel in $H^1(\mathbb{Q}_\Sigma/\mathbb{Q},(T/T(\wp))^*)$, which are both finite.

As for the map (5.11) when $\ell = p$, our running hypothesis (**H.nA**) along with the fact that the ideal $\wp$ is principal[5] (being a height-one prime of the regular ring $\mathcal{R}$) show that

$$H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_p, T(\wp)) := \mathrm{im}\left(H^1(\mathbb{Q}_p, \mathfrak{T}) \longrightarrow H^1(\mathbb{Q}_p, T(\wp))\right) = H^1(\mathbb{Q}_p, T(\wp)),$$

hence we have that

$$H^1_{\mathcal{F}_T^*}(\mathbb{Q}_\ell, T^*) = 0 = H^1_{\mathcal{F}_{\mathrm{can}}^*}(\mathbb{Q}_\ell, T(\wp)^*)$$

so the kernel and cokernel of (5.11) are trivial. It remains to control the kernel and the cokernel of (5.11) when $\ell \neq p$. The kernel of

$$H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_\ell, T(\wp)) \longrightarrow H^1_{\mathcal{F}_T}(\mathbb{Q}_\ell, T) \tag{5.12}$$

is controlled by

$$\ker\left(H^1(\mathbb{Q}_\ell, T(\wp)) \longrightarrow H^1(\mathbb{Q}_\ell, T)\right) = \mathrm{im}\left(H^0(\mathbb{Q}_\ell, T/T(\wp)) \longrightarrow H^1(\mathbb{Q}_\ell, T(\wp))\right)$$

which is finite.

We finally prove that the cokernel of (5.12) is finite. Consider now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, T(\wp)) & \longrightarrow & H^1(\mathbb{Q}, T(\wp)) & \longrightarrow & H^1(I_\ell, T(\wp))^{\mathrm{Fr}_\ell = 1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, T) & \longrightarrow & H^1(\mathbb{Q}, T) & \longrightarrow & H^1(I_\ell, T)^{\mathrm{Fr}_\ell = 1} & \longrightarrow & 0 \end{array}$$

The cokernel of the vertical map in the middle is controlled by $H^2(\mathbb{Q}_\ell, T/T(\wp))$ hence it is finite. Also, the kernel of the the rightmost is finite for a similar reason. This shows by snake lemma that the cokernel of the leftmost vertical map is also finite. As the index of $H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, T)$ in $H^1_f(\mathbb{Q}_\ell, T) = H^1_{\mathcal{F}_T}(\mathbb{Q}_\ell, T)$ is finite as well, we therefore proved that

$$\text{the cokernel of the map } H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, T(\wp)) \longrightarrow H^1_{\mathcal{F}_T}(\mathbb{Q}_\ell, T) \text{ is finite.} \tag{5.13}$$

Furthermore, it is not hard to see that the Λ-module

$$H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, \mathcal{T}/\wp\mathcal{T} \otimes \Lambda) = H^1(G_{\mathbb{Q}_\ell}/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda)$$

is Λ-torsion. (Note in the equality above we use the fact that I_ℓ acts trivially on Λ.) Choosing the positive integer $N >> 0$ above so that $\gamma - 1 + p^N$ does not divide the characteristic ideal of this module, we obtain a finite quotient $H^1(G_{\mathbb{Q}_\ell}/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda)/(\gamma - 1 + p^N)$. Since the cohomological dimension of $G_{\mathbb{Q}_\ell}/I_\ell$ is one, we have

$$\begin{aligned} H^1(G_{\mathbb{Q}_\ell}/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda)/(\gamma - 1 + p^N) &\xrightarrow{\sim} H^1(G_{\mathbb{Q}_\ell}/I_\ell, (\mathcal{T}/\wp\mathcal{T})^{I_\ell} \otimes \Lambda/(\gamma - 1 + p^N)) \\ &\cong H^1(G_{\mathbb{Q}_\ell}/I_\ell, (\mathcal{T}/\wp\mathcal{T} \otimes \Lambda/(\gamma - 1 + p^N))^{I_\ell}) \\ &\cong H^1(G_{\mathbb{Q}_\ell}/I_\ell, T(\wp, N)^{I_\ell}) = H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, T(\wp, N)), \end{aligned}$$

where $T(\wp, N) = \mathcal{T}/\wp\mathcal{T} \otimes \Lambda/(\gamma - 1 + p^N)$ as above. In particular, the index of $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}, T(\wp, N))$ in $H^1_{\mathrm{ur}}(\mathbb{Q}_\ell, T(\wp, N))$ is finite for $N >> 0$. This, together with (5.13) shows that the kernel and cokernel of the map (5.12), and by local duality, also the kernel and the cokernel of the map (5.11) are finite for $N >> 0$.

Using the fact that the kernels and cokernels of the maps (5.10) and (5.11) are both finite the proof of the lemma follows at once. □

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[5] This is the only point in the proof of Theorems 5.10 and 5.11 where we use that the ring $\mathcal{R}$ is regular in an essential way.

Proof of Theorem 5.11. As the proof of this Theorem in fact follows from a more general statement due to Ochiai [Och05] (see the proof Theorem 2.4 and Remark 2.5 of loc.cit.), we only give a sketch of the proof and only in the situation concerning the ring $\mathfrak{R}$ and the representation $\mathfrak{T}$. We use the notation from the proof of Theorem 5.10.

Let $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathfrak{T}, \mathcal{F}_{\text{can}}, \mathcal{P})$ be a given generator. Since κ_1 is non-torsion, it follows that there is a height one prime $\wp$ of $\mathcal{R}$ as in the proof of Theorem 5.10 and a positive integer N (chosen in way that the conclusion of Lemma 5.16 holds true) such that the image

$$\text{red}_{\wp,N}(\kappa_1) \in H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, T(\wp, N))$$

of κ_1 is non-zero. Fix such $\wp$ and N; define T (and the Selmer structure $\mathcal{F}_T$) as in the proof of Theorem 5.10. We let $\boldsymbol{\kappa}^{(\wp)} \in \overline{\mathbf{KS}}(T, \mathcal{F}_T, \mathcal{P})$ be the image of $\boldsymbol{\kappa}$. By (**H3**), the map

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T(\wp, N)) \longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, T)$$

is injective. In particular, the image $\kappa_1^{(\wp)}$ of $\text{red}_{\wp,N}(\kappa_1)$ inside $H^1_{\mathcal{F}_T}(\mathbb{Q}, T)$ is non-zero. Let $\boldsymbol{\kappa}^{(\wp)} \in \overline{\mathbf{KS}}(T, \mathcal{F}_T, \mathcal{P})$ be the image of $\boldsymbol{\kappa}$. We therefore showed the existence of a Kolyvagin system $\boldsymbol{\kappa}^{(\wp)} \in \overline{\mathbf{KS}}(T, \mathcal{F}_T, \mathcal{P})$ whose leading term verifies that $\kappa_1^{(\wp)} \neq 0$. This shows that $H^1_{\mathcal{F}_T^*}(\mathbb{Q}, T^*)$ is finite.

By Lemma 5.16, we have a map

$$H^1_{\mathcal{F}_T^*}(\mathbb{Q}, T^*) \longrightarrow H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, T(\wp, N)^*)$$

with finite kernel and cokernel. Hence $H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, T(\wp)^*)$ is finite as well. We conclude by (5.6) that

$$H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, \mathfrak{T}^*)^\vee/(\wp, \gamma - 1 + p^N) \cong H^1_{\mathcal{F}_T^*}(\mathbb{Q}, T^*)^\vee$$

is also finite. It follows from Lemma 5.14 that $H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is $\mathfrak{R}$-torsion, as desired. □

Theorem 5.17. *Suppose $H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, \mathfrak{T}^*)^\vee$ is $\mathfrak{R}$-torsion. Under the running hypotheses, the $\mathfrak{R}$-module $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathfrak{T})$ is free of rank one.*

Similar statement holds true when $\mathfrak{R}$ is replaced by R and $\mathfrak{T}$ by $\mathbb{T}$.

Proof. To simplify the arguments we suppose in addition that the ring $\mathcal{R}$ is the power series ring $\mathcal{O}[[X]]$; the general case when $\mathcal{R}$ is a general regular $\mathcal{O}$-algebra of dimension two may be treated after minor alterations. As above, choose a positive integer $N >> 0$ so that

- $\mathcal{O}[[X]]/(X + p^N) \cong \mathcal{O}$,
- $H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, \mathfrak{T}^*)^\vee/(X + p^N)$ is Λ-torsion.

By setting $T := \mathcal{T}/(X + p^N)\mathcal{T}$, we conclude using (5.6) that the module $H^1(\mathbb{Q}, (T \otimes \Lambda)^*)^\vee$ is Λ-torsion. Similarly, choose a positive integer $M >> 0$ such that

$$H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, (T \otimes \Lambda)^*)^\vee/(\gamma - 1 + p^M) \cong H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, \dot{T})^\vee$$

is finite. Here, $\dot{T}$ is the free $\mathcal{O}$-module $T \otimes \Lambda/(\gamma - 1 + p^M)$. By [MR04, Corollary 5.2.6], it follows that $\text{rank}_{\mathcal{O}}(H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \dot{T})) = \chi(\bar{T}) = 1$. Furthermore, the $\mathcal{O}$-module $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \dot{T})$ is torsion-free as since we assumed (**H3**), hence we conclude that $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \dot{T})$ is a free $\mathcal{O}$-module of rank one.

Set $X_1 = X + p^N$ and $X_2 = \gamma - 1 + p^M$ for M, N as above and define $\mathfrak{R}_{u,v} = \mathfrak{R}/(X_1^u, X_2^v)$, $\mathfrak{R}_{r,u,v} = \mathfrak{R}/(\varpi^r, X_1^u, X_2^v)$, $\mathfrak{T}_{u,v} = \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{u,v}$ and $\mathfrak{T}_{r,u,v} = \mathfrak{T} \otimes_{\mathfrak{R}} \mathfrak{R}_{r,u,v}$. Note that $\mathfrak{T}_{1,1} = \dot{T}$. As $H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathfrak{T}_{u,v}) = \varprojlim_r H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathfrak{T}_{r,u,v})$, it follows by the proof of Proposition 4.17 that

$$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathfrak{T}_{1,1}) \xrightarrow{\sim} H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}, \mathfrak{T}_{u,v})[X_1^{u-1}, X_2^{v-1}].$$

This shows that the module

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{R}_{u,v}}(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{u,v}),\mathfrak{R}_{u,v})/(X_1^{u-1},X_2^{v-1}) &\cong \mathrm{Hom}_{\mathfrak{R}_{u,v}}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{u,v})[X_1^{u-1},X_2^{v-1}],\mathfrak{R}_{u,v}\right)\\ &\cong \mathrm{Hom}_{\mathcal{O}}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{1,1}),\mathcal{O}\right),\end{aligned}$$

is cyclic, hence by Nakayama's Lemma (along with the fact that the $\mathcal{O}$-module $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{1,1}) = H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\dot{T})$ is free of rank one) we conclude that the module $\mathrm{Hom}_{\mathfrak{R}_{u,v}}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{u,v}),\mathfrak{R}_{u,v}\right)$ is cyclic as well.

On the other hand, (**H3**) shows that the module $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{u,v})$ is $\mathcal{O}$-torsion free and the proof of Proposition 4.20 shows $\mathrm{rank}_{\mathcal{O}}(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{u,v})) \geq uv$. This shows that the cyclic $\mathfrak{R}_{u,v}$-module $\mathrm{Hom}_{\mathfrak{R}_{u,v}}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{u,v}),\mathfrak{R}_{u,v}\right)$ is indeed free of rank one, hence the module $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathfrak{T}_{u,v})$ itself is free of rank one as an $\mathfrak{R}_{u,v}$-module. Passing to limit we conclude with the proof of the Theorem when the coefficient ring is $\mathfrak{R}$. In the situation when the coefficient ring is R, one easily reduces to the case discussed above using Lemma 5.15. □

Let $R = \mathcal{O}[[X_1,X_2,X_3]]$ be as above and suppose M is a finitely generated R-module. Since R is regular, localizations of R at height one primes $\mathfrak{p}$ are discrete valuation rings. If M is torsion, define

$$\mathrm{char}(M) = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{length}_{R_\mathfrak{p}}(M_\mathfrak{p})}.$$

If M is not torsion, we set $\mathrm{char}(M) = 0$.

The following theorem may be proved (under the running assumptions of §5.3) following the arguments of [Och05], see particularly the proof of Theorem 2.4 in loc.cit. As above, we let $\boldsymbol{\kappa} \in \overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathrm{can}},\mathcal{P})$ be a fixed generator and let $\kappa_1 \in H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})$ denote its leading term.

Theorem 5.18. $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q},\mathbb{T}^*)^\vee\right) \mid \mathrm{char}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})/R\cdot\kappa_1\right)$.

Using the descent formalism of Nekovář [Nek06], we are able to prove that the divisibility in this theorem is indeed sharp:

Corollary 5.19. $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q},\mathbb{T}^*)^\vee\right) = \mathrm{char}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})/R\cdot\kappa_1\right)$.

Proof. Assume without loss of generality that $\kappa_1 \neq 0$ (as otherwise, Theorem 5.10 shows that both sides of the claimed equality are 0 and Corollary holds true for trivial reasons). In this case, let f be a generator of the ideal $\mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q},\mathbb{T}^*)^\vee\right)$ and g be that of the ideal $\mathrm{char}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})/R\cdot\kappa_1\right)$. Observe that $fg \neq 0$ by Theorems 5.11 and 5.17. Choose by Theorem 5.18 an element $0 \neq h \in R$ so that $g = fh$. We contend to prove that $h \in R^\times$ and conclude the proof the Theorem.

Since the R-module $H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q},\mathbb{T}^*)^\vee$ is torsion, one may choose (as in the proof of Theorem 5.17, using (5.6)) positive integers $\alpha_1,\alpha_2,\alpha_3$ such that

- $R/(X_1+p^{\alpha_1},X_2+p^{\alpha_2},X_3+p^{\alpha_3}) \cong \mathcal{O}$,
- $H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q},\dot{T}^*)$ is finite, where $\dot{T} := \mathbb{T}/(X_1+p^{\alpha_1},X_2+p^{\alpha_2},X_3+p^{\alpha_3})$.

To ease notation set $T_i + p^{\alpha_i} = X_i$ (for $i = 1,2,3$) and $I = (X_1,X_2,X_3)$. Let $\bar{\mathfrak{r}}$ denote the image of of $\mathfrak{r} \in R$ under $R \to \dot{R} := R/I$. Lemma 5.22 below shows that $\bar{h}$ is a unit in R/I which in turn implies that $h \in R^\times$, as desired. □

The rest of this section is devoted to proving Lemma 5.22, for which we need to appeal to Nekovář's formalism of Selmer complexes. Let $\dot{\boldsymbol{\kappa}}$ be the image of the generator $\boldsymbol{\kappa}$ under the map

$$\overline{\mathbf{KS}}(\mathbb{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P}) \longrightarrow \overline{\mathbf{KS}}(\dot{T}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$$

induced from $R \to \dot{R}$. The diagram (5.9) (with T replaced by $\dot{T}$) shows that the Kolyvagin system $\dot{\boldsymbol{\kappa}}$ is primitive.

For a commutative ring X and a finitely generated X-module Y, let $\mathrm{Fitt}(Y) = \mathrm{Fitt}_X(Y)$ denote its initial Fitting ideal (calculated in the category of X-modules).

Lemma 5.20. *Fitt* $\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, \dot{T}^*)^\vee\right) \supset \dot{R}\bar{f}$.

Proof. For $X = \mathbb{T}$ or $\mathbb{T}^*$, Nekovář's Selmer complex $C^\bullet_f(\mathbb{Q}, X)$ is the following complex of (co-)finite type R-modules:

$$C^\bullet_f(\mathbb{Q}, X) = \mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(\mathbb{Q}_\Sigma/\mathbb{Q}, X) \oplus \bigoplus_{\ell\in\Sigma} C^\bullet_f(\mathbb{Q}_\ell, X) \longrightarrow \bigoplus_{\ell\in\Sigma} C^\bullet_f(\mathbb{Q}_\ell, X)\right)[-1]$$

where

$$C^\bullet_f(\mathbb{Q}, X) = \begin{cases} C^\bullet_{\mathrm{cont}}(\mathbb{Q}_p, U^+_p(X)) & \text{if } \ell = p, \\ \\ C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_\ell}/I_\ell, X^{I_\ell}) & \text{if } \ell \neq p, \end{cases}$$

and $U^+_p(\mathbb{T}) = \mathbb{T}$, $U^+_p(\mathbb{T}^*) = 0$. Let $R\Gamma_f(\mathbb{Q}, X)$ denote the corresponding object in the derived category and let $\widetilde{H}^i_f(\mathbb{Q}, X)$ be the ith cohomology group of $R\Gamma_f(\mathbb{Q}, X)$. Observe that:

(1) $\widetilde{H}^1_f(\mathbb{Q}, \mathbb{T}^*) = H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, \mathbb{T}^*)$ by [Nek06, Lemma 9.6.3] and our hypothesis (**H.nA**).

(2) There is an isomorphism $\widetilde{H}^1_f(\mathbb{Q}, \mathbb{T}^*)^\vee \cong \widetilde{H}^2_f(\mathbb{Q}, \mathbb{T})$ induced by Matlis duality (c.f., [Nek06, 8.9.6.1]). We note that since the residue field of R is finite, Matlis duality functor coincides with the Pontryagin duality functor.

(3) The complex $R\Gamma_f(\mathbb{Q}, \mathbb{T})$ is a perfect complex (in the sense of [Gro71, Exp. I, Cor. 5.8.1]) of R-modules concentrated in degrees 1 and 2. Indeed, the perfectness follows from the fact our ring R is regular, $\mathbb{T}$ is a free R-module and using a result of Serre and Auslander-Buchsbaum. Due to p-cohomological dimension considerations, it is easy to see that the complex $R\Gamma_f(\mathbb{Q}, \mathbb{T})$ is concentrated in degrees $[0, 3]$. By (**H3**) $\widetilde{H}^0_f(\mathbb{Q}, \mathbb{T})$ vanishes. By Matlis duality $\widetilde{H}^3_f(\mathbb{Q}, \mathbb{T}) = \widetilde{H}^0_f(\mathbb{Q}, \mathbb{T}^*)^\vee$ and the cohomology of $R\Gamma_f(\mathbb{Q}, \mathbb{T})$ in degree 3 vanishes thanks to (**H3**) as well.

Since $\widetilde{H}^2_f(\mathbb{Q}, \mathbb{T}) \cong H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, \mathbb{T}^*)^\vee$ is R-torsion, it follows that the complex $R\Gamma_f(\mathbb{Q}, \mathbb{T})$ may be represented by a complex of the form

$$\mathrm{Cone}\left(M \xrightarrow{u} M\right)[-2]$$

where M is a projective (hence free) R-module of finite rank and u is injective. Setting $\dot{M} = M \otimes_R \dot{R}$ and $\dot{u}$ to be the map induced from u, we conclude that

$$\begin{aligned} \dot{R}\bar{f} = \mathrm{char}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, \mathbb{T}^*)^\vee\right) \otimes_R \dot{R} &= \mathrm{Fitt}\,(M/uM) \otimes_R \dot{R} \\ &= \mathrm{Fitt}\left(\dot{M}/\dot{u}\dot{M}\right) \\ &\subset \mathrm{Fitt}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, \dot{T}^*)^\vee\right), \end{aligned} \tag{5.14}$$

where the final containment follows from (5.6). Lemma follows. □

Lemma 5.21. *For some $r \in \dot{R}$, we have*

$$\mathrm{Fitt}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\dot{T})/\dot{R}\cdot\dot{\kappa}_1\right) = \dot{R}\cdot\bar{g}r.$$

Proof. Using the fact that the R-module $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})$ (as well as its submodule $R\cdot\kappa_1$) is free of rank one (Theorem 5.17), it follows at once that

$$\begin{aligned}\dot{R}\bar{g} = \mathrm{char}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})/R\cdot\kappa_1\right)\otimes_R\dot{R} &= \mathrm{Fitt}\left(\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})\otimes_R\dot{R}\right)/\dot{R}\cdot\dot{\kappa}_1\right)\\ &\supset \mathrm{Fitt}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\dot{T})/\dot{R}\cdot\dot{\kappa}_1\right),\end{aligned}$$

where the final containment is because the image of $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})$ lands in $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\dot{T})$ under the map

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\mathbb{T}) \longrightarrow H^1(\mathbb{Q}_\Sigma/\mathbb{Q},\dot{T})$$

induced from the map $R\to\dot{R}$. The proof of the Lemma follows. □

Lemma 5.22. *$\bar{h}$ is a unit in the ring $\dot{R}$.*

Proof. We have the following relation of ideals of $\dot{R}$:

$$\dot{R}\bar{f} \subset \mathrm{Fitt}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q},\dot{T}^*)\right) = \mathrm{Fitt}\left(H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\dot{T})/\dot{R}\cdot\dot{\kappa}_1\right) = \dot{R}\bar{f}\bar{h}r\ , \tag{5.15}$$

where the first containment is Lemma 5.20, the first equality is thanks to [MR04, Theorem 5.2.14] the fact that the Kolyvagin system $\dot{\boldsymbol{\kappa}}$ is primitive, and the final equality is Lemma 5.21. It now follows from (5.15) that $\bar{h}\in\dot{R}^\times$, as desired. □

5.4. **The Pottharst local conditions and Kolyvagin constructed p-adic L-function.** We finish with a potential application of our results towards a main conjecture for the eigencurve, conditional on the extension of a construction due to Pottharst [Pot12a] (see also [Bel12b] for work in this direction) of a 'big' Bloch-Kato local condition on the Mazur-Coleman eigencurve. Our goal is to prove Theorem D in Section 1.3, which is labelled by Theorem 5.24 below.

Throughout this subsection, let $\bar{\rho}: G_{\mathbb{Q},\Sigma}\to \mathrm{GL}_2(\mathrm{k})$ be a *modular* Galois representation which is absolutely irreducible and for which the hypothesis (**H.nOb**) holds true, so that $R := R(\bar{\rho}) \cong \mathcal{O}[[T_1,T_2,T_3]]$. Until the end of this article, all hypotheses of Section 5.1 on $\bar{T}$ and the notation from Section 1.3 is in effect. The reader is advised to go over the definitions in Section 1.3 before going further in this section.

Assume the truth of Conjecture 1 (on the existence of a big Pottharst local condition on $\mathbb{T}^\dagger$) until the end of this article. Define $H^1_s(\mathbb{Q}_p,\mathbb{T}^\dagger) = H^1(\mathbb{Q}_p,\mathbb{T}^\dagger)/H^1_{\mathrm{Pot},\lambda}(\mathbb{Q}_p,\mathbb{T}^\dagger)$ where λ is as in the statement of Conjecture 1.

By Theorem 1.9 of [Pot12a] there is an isomorphism

$$\iota^\dagger : H^*(\mathbb{Q}_p,\mathbb{T})\otimes R^\dagger \xrightarrow{\sim} H^*(\mathbb{Q}_p,\mathbb{T}^\dagger).$$

We define $\mathrm{loc}_p^s : H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q},\mathbb{T})\to H^1_s(\mathbb{Q}_p,\mathbb{T}^\dagger)$ to be the compositum of the localization map loc_p and $\iota^\dagger$ followed by the projection $H^1(\mathbb{Q}_p,\mathbb{T}^\dagger)\to H^1_s(\mathbb{Q}_p,\mathbb{T}^\dagger)$.

Definition 5.23. For any big Kolyvagin system $\boldsymbol{\kappa}$ that generates the module $\overline{\mathbf{KS}}(\mathbb{T},\mathcal{F}_{\mathrm{can}},\mathcal{P})$, the initial Fitting ideal

$$\mathcal{L}(\boldsymbol{\kappa}) := \mathrm{Fitt}\left(H^1_s(\mathbb{Q}_p,\mathbb{T}^\dagger)/R^\dagger\cdot\mathrm{loc}_p^s(\kappa_1)\right)\subset R^\dagger$$

is called the *Kolyvagin constructed p-adic L-function*.

The ideal $\mathcal{L}(\boldsymbol{\kappa})$ should be thought of as a generalization of Perrin-Riou's [PR95] module of algebraic p-adic L-function, whose definition she gives for the cyclotomic deformation of a motive. The following interpolation property justifies that $\mathcal{L}(\boldsymbol{\kappa})$ indeed deserves to be called a "p-adic L-function".

Theorem 5.24. *For every $(\psi^\dagger, \lambda(\psi^\dagger)) \in \mathcal{C}_{\text{cl-fs}}(\bar{\rho})$ (where λ is as in Conjecture 1) let $f_{\psi^\dagger}$ denote the corresponding (classical) eigenform. The following equality of ideals of $\Lambda_E^\dagger$ holds true:*

$$\psi_\Lambda^\dagger\left(\mathcal{L}(\boldsymbol{\kappa})\right) = \delta_{\psi^\dagger}^{-1} \cdot \Lambda_E^\dagger L_p(f_{\psi^\dagger}, \lambda)$$

for some $\delta_{\psi^\dagger} \in \Lambda_E^\dagger$, where $L_p(f_{\psi^\dagger}, \lambda)$ is the p-adic L-function attached to the modular form $f_{\psi^\dagger}$ and the p-stabilization determined by $\lambda(\psi^\dagger)$.

Proof. For $\psi : R \to \mathcal{O}_E$ as in Remark 1.8, let $\widetilde{T}_\psi := \mathbb{T} \otimes_\psi \Lambda_E$ and let $\boldsymbol{\kappa}^\psi \in \overline{\mathbf{KS}}(\widetilde{T}_\psi, \mathcal{F}_{\text{can}}, \mathcal{P})$ be the image of $\boldsymbol{\kappa}$. Let $\boldsymbol{\kappa}^{\text{Kato},\psi} \in \overline{\mathbf{KS}}(\widetilde{T}_\psi, \mathcal{F}_{\text{can}}, \mathcal{P})$ be Kato's Kolyvagin system attached to the modular form f_ψ. As the Λ-module $\overline{\mathbf{KS}}(\widetilde{T}_\psi, \mathcal{F}_{\text{can}}, \mathcal{P})$ is free of rank one and generated by $\boldsymbol{\kappa}^\psi$, there is $\delta_\psi \in \Lambda$ such that $\boldsymbol{\kappa}^{\text{Kato},\psi} = \delta_\psi \cdot \boldsymbol{\kappa}^\psi$.

Denote the image of a class $c \in H^1(\mathbb{Q}, \widetilde{T}_\psi)$ in $H^1(\mathbb{Q}, \widetilde{V}_{\psi^\dagger})$ by ${}^\dagger c$. Set $H^1_s(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger}) := H^1(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger})/H^1_{\text{Pot}}(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger})$ and denote the map $H^1(\mathbb{Q}, \widetilde{V}_{\psi^\dagger}) \to H^1_s(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger})$ by loc_p^s. Under our running hypotheses, Pottharst has shown[6] in [Pot12b, §5] that

$$\text{Fitt}\left(H^1_s(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger})/\Lambda_E^\dagger \cdot \text{loc}_p^s({}^\dagger\kappa_1^{\text{Kato},\psi})\right) = \Lambda_E^\dagger L_p(f_\psi).$$

This shows that

$$\text{Fitt}\left(H^1_s(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger})/\Lambda_E^\dagger \cdot \text{loc}_p^s({}^\dagger\kappa_1^{\psi})\right) = \delta_\psi^{-1} \cdot \Lambda_E^\dagger L_p(f_\psi).$$

We therefore conclude that

$$\begin{aligned}\psi_\Lambda^\dagger\left(\mathcal{L}(\boldsymbol{\kappa})\right) &= \text{Fitt}\left(H^1_s(\mathbb{Q}_p, \mathbb{T}^\dagger)/R^\dagger \cdot \text{loc}_p^s(\kappa_1)\right) \otimes \Lambda_E^\dagger \\ &= \text{Fitt}\left(H^1_s(\mathbb{Q}_p, \widetilde{V}_{\psi^\dagger})/\Lambda_E^\dagger \cdot \text{loc}_p^s({}^\dagger\kappa_1^{\psi})\right) \\ &= \delta_\psi^{-1} \cdot \Lambda_E^\dagger L_p(f_\psi)\end{aligned}$$

as desired. □

References

[Bel12a] Joël Bellaïche. Critical p-adic L-functions. *Invent. Math.*, 189(1):1–60, 2012.

[Bel12b] Joël Bellaïche. Ranks of Selmer groups in an analytic family. *Trans. Amer. Math. Soc.*, 364(9):4735–4761, 2012.

[Büy11] Kâzım Büyükboduk. Λ-adic Kolyvagin systems. *IMRN*, 2011(14):3141–3206, 2011.

[Büy13a] Kâzım Büyükboduk. Big Heegner point Kolyvagin system for a family of modular forms, 2013. *Selecta Math.*, to appear.

[Büy13b] Kâzım Büyükboduk. Main conjectures for CM fields and a Yager-type theorem for Rubin-Stark elements, 2013. *IMRN*, to appear. `http://imrn.oxfordjournals.org/content/early/2013/07/10/imrn.rnt140`.

[CE56] Henri Cartan and Samuel Eilenberg. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[6]We note that Pottharst in loc.cit. relies on an unpublished work of Kato-Kurihara-Tsuji on the *expanded logarithm* for de Rham representations, extending Perrin-Riou's work [PR94] in the crystalline case.

[dSL97] Bart de Smit and Hendrik W. Lenstra, Jr. Explicit construction of universal deformation rings. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 313–326. Springer, New York, 1997.

[EPW06] Matthew Emerton, Robert Pollack, and Tom Weston. Variation of Iwasawa invariants in Hida families. *Invent. Math.*, 163(3):523–580, 2006.

[Fla92] Matthias Flach. A finiteness theorem for the symmetric square of an elliptic curve. *Invent. Math.*, 109(2):307–327, 1992.

[FO12] Olivier Fouquet and Tadashi Ochiai. Control theorems for Selmer groups of nearly ordinary deformations. *J. Reine Angew. Math.*, 666:163–187, 2012.

[FPR94] Jean-Marc Fontaine and Bernadette Perrin-Riou. About the Bloch and Kato conjectures: Galois cohomology and values of L-functions. (Autour des conjectures de Bloch et Kato: Cohomologie galoisienne et valeurs de fonctions L.). Providence, RI: American Mathematical Society, 1994.

[Gou01] Fernando Q. Gouvêa. Deformations of Galois representations. In *Arithmetic algebraic geometry (Park City, UT, 1999)*, volume 9 of *IAS/Park City Math. Ser.*, pages 233–406. Amer. Math. Soc., Providence, RI, 2001. Appendix 1 by Mark Dickinson, Appendix 2 by Tom Weston and Appendix 3 by Matthew Emerton.

[Gre94] Ralph Greenberg. Iwasawa theory and p-adic deformations of motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 193–223. Amer. Math. Soc., Providence, RI, 1994.

[Gro67] A. Grothendieck. *Local cohomology. A seminar given by A. Grothendieck, Harvard University, Fall 1961. Notes by R. Hartshorne.* , 1967.

[Gro71] A. Grothendieck. *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.

[Hid86a] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.

[Hid86b] Haruzo Hida. Iwasawa modules attached to congruences of cusp forms. *Ann. Sci. Éc. Norm. Supér. (4)*, 19(2):231–273, 1986.

[How07] Benjamin Howard. Variation of Heegner points in Hida families. *Invent. Math.*, 167(1):91–128, 2007.

[Kat93] Kazuya Kato. Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. I. In *Arithmetic algebraic geometry (Trento, 1991)*, volume 1553 of *Lecture Notes in Math.*, pages 50–163. Springer, Berlin, 1993.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p-adiques et applications arithmétiques. III.

[Kis09a] Mark Kisin. Modularity of 2-adic Barsotti-Tate representations. *Invent. Math.*, 178(3):587–634, 2009.

[Kis09b] Mark Kisin. The Fontaine-Mazur conjecture for GL_2. *J. Am. Math. Soc.*, 22(3):641–690, 2009.

[Kob03] Shin-ichi Kobayashi. Iwasawa theory for elliptic curves at supersingular primes. *Invent. Math.*, 152(1):1–36, 2003.

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture. I. *Invent. Math.*, 178(3):485–504, 2009.

[Mat89] Hideyuki Matsumura. *Commutative ring theory. Transl. from the Japanese by M. Reid. Paperback ed.* Cambridge etc.: Cambridge University Press, 1989.

[Maz72] Barry Mazur. Rational points of abelian varieties with values in towers of number fields. *Invent. Math.*, 18:183–266, 1972.

[Maz89] B. Mazur. Deforming Galois representations. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989.

[MR04] Barry Mazur and Karl Rubin. Kolyvagin systems. *Mem. Amer. Math. Soc.*, 168(799):viii+96, 2004.

[Nek06] Jan Nekovář. Selmer complexes. *Astérisque*, (310):viii+559, 2006.

[Och05] Tadashi Ochiai. Euler system for Galois deformations. *Ann. Inst. Fourier (Grenoble)*, 55(1):113–146, 2005.

[Pot12a] Jonathan Pottharst. Analytic families of finite-slope Selmer groups, 2012. Preprint, 37pp.

[Pot12b] Jonathan Pottharst. Cyclotomic Iwasawa theory of motives, 2012. Preprint, 25pp.

[PR94] Bernadette Perrin-Riou. Théorie d'Iwasawa des représentations p-adiques sur un corps local. *Invent. Math.*, 115(1):81–161, 1994. With an appendix by Jean-Marc Fontaine.

[PR95] Bernadette Perrin-Riou. Fonctions L p-adiques des représentations p-adiques. *Astérisque*, (229):198, 1995.

[Roh84] David E. Rohrlich. On L-functions of elliptic curves and cyclotomic towers. *Invent. Math.*, 75(3):409–423, 1984.

[Rub00] Karl Rubin. *Euler systems*, volume 147 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000. Hermann Weyl Lectures. The Institute for Advanced Study.

[Ser87] Jean-Pierre Serre. On the modular representations of degree two of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. (Sur les représentations modulaires de degré 2 de $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.). *Duke Math. J.*, 54:179–230, 1987.

[SU13] Christopher Skinner and Eric Urban. The Iwasawa Main Conjectures for GL_2. *Invent. Math., to appear.*, 2013.

[Til97] Jacques Tilouine. Hecke algebras and the Gorenstein property. New York, NY: Springer, 1997.

[TW95] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. Math. (2)*, 141(3):553–572, 1995.

[Wes04] Tom Weston. Unobstructed modular deformation problems. *Amer. J. Math.*, 126(6):1237–1252, 2004.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

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