ITERATED INTEGRALS OF MODULAR FORMS AND NONCOMMUTATIVE MODULAR SYMBOLS

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REFERENCES

[Za] **D. Zagier.** *Values of zeta functions and their applications.* **First European Congress of Mathematics, vol. II, Birkhäuser Verlag, Basel, 1994, 497–512.**

[Ra] **G. Racinet.** *Doubles mélanges des polylogarithms multiples aux racines de l'unité.* **Publ. Math. IHES, 95 (2002), 185–231. e–Print math.QA/0202142**

[DeGo] **P. Deligne A. Goncharov.** *Groupes fondamentaux motiviques de Tate mixte.* **e–Print math.NT/0302267**

[Ma1] **Yu. Manin.** *Parabolic points and zeta-functions of modular curves.* **Russian: Izv. AN SSSR, ser. mat. 36:1 (1972), 19–66. English: Math. USSR Izvestija, publ. by AMS, vol. 6, No. 1 (1972), 19–64, and Selected papers, World Scientific, 1996, 202–247.**

[Ma2] **Yu. Manin.** *Periods of parabolic forms and p–adic Hecke series.* **Russian: Mat. Sbornik, 92:3 (1973), 378–401. English: Math. USSR Sbornik, 21:3 (1973), 371–393 and Selected papers, World Scientific, 1996, 268–290.**

[Ma3] **Yu. Manin.** *Iterated integrals of modular forms and noncommutative modular symbols.* **e–print math.NT/0502576**

[Me] **L. Merel.** *Quelques aspects arithmétiques et géométriques de la théorie des symboles modulaires.* **Thése de doctorat, Université Paris VI, 1993.**

[Ha] **R. Hain.** *Iterated integrals and algebraic cycles: examples and prospects.* **e–Print math.AG/0109204**

[Dr] **V. G. Drinfeld.** *On quasi–triangular quasi–Hopf algebras and some groups closely associated with Gal* $(\overline{\mathbf{Q}}/\mathbf{Q})$. **Algebra and Analysis 2:4 (1990); Leningrad Math. J. 2:4 (1991), 829–860.**

0. INTRODUCTION

• MULTIPLE ZETA VALUES

a) Definition via multiple Dirichlet series:

$$\zeta(m_1,\dots,m_k)=\sum_{0<n_1<\dots<n_k}\frac{1}{n_1^{m_1}\dots n_k^{m_k}}$$

Convergence: $m_i \geq 1$ **and** $m_k > 1$.
b) Definition via iterated integrals:

$$\zeta(m_1,\dots,m_k)=\int_0^1\frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\int_0^{t_2}\cdots\int_0^{t_{m_k-1}}\frac{dt_{m_k}}{1-t_{m_k}}\cdots$$

The sequence of differential forms in the iterated integral consists of consecutive subsequences $\frac{dt}{t},\dots,\frac{dt}{t},\frac{dt}{1-t}$ **of lengths** $m_k, m_{k-1},\dots,m_1$.

• SHUFFLE RELATIONS

Relations between iterated integrals.

Relations between multiple series.

• DRINFELD'S ASSOCIATOR, QUANTIZATION, GROTHENDIECK–TEICHMUELLER GROUP.

• **GEOMETRY ON $\mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\}$:**

$\frac{dt}{t}, \frac{dt}{1-t}$ **span the space of meromorphic differential forms with no more than logarithmic singularities at** $\{0, 1, \infty\}$.

• **LIFT TO THE UPPER HALF PLANE** $H := \{z \in \mathbf{C} \,|\, \mathrm{Im}\, z > 0\}$ **AND** $\overline{H} := H \cup \mathbf{P}^1(\mathbf{Q})$:

$$(\mathbf{P}^1(\mathbf{C}), \{0, 1, \infty\}) \cong \Gamma_0(4) \setminus (\overline{H}, cusps)$$

$\frac{dt}{t}, \frac{dt}{1-t}$ **lift to Eisenstein series of weight two for** $\Gamma_0(4) \subset SL(2, \mathbf{Z})$.

The integration path $[0, 1]$ **lifts to the geodesic connecting two cusps.**

• **GOAL OF THIS WORK: A GENERALIZATION OF THE THIS PICTURE ALONG THE FOLLOWING LINES:**

– $\Gamma_0(4)$ **is replaced by an arbitrary (congruence) subgroup** Γ **of** $SL(2, \mathbf{Z})$.

– **Eisenstein series of weight two are replaced by arbitrary modular forms (cusp form + Eisenstein series) with respect to** Γ.

NB In this talk I focus on cusp forms (absent in the classical polyzetas setting).

– **Iterated integrals are taken along geodesics connecting two cusps.**

• **MODULAR SYMBOLS.**

Integration of cusp forms along geodesics connecting two cusps = theory of classical modular symbols (of arbitrary weight).

Hence the subject of this talk can be called "iterated modular symbols".

1. FORMALISM OF ITERATED INTEGRALS OF 1–FORMS ON A RIEMANNIAN SURFACE

Here it will be shown that if one replaces a simple integral not by an individual iterated integral but by a generating series of all such integrals, then the usual properties like additivity and variable change formula reappear in a multiplicative/noncommutative version.

• **NOTATION.**

X **= a connected Riemann surface.**

$\mathcal{O}_X$ **= the structure sheaf of holomorphic functions,**

Ω^1_X **= the sheaf of holomorphic 1–forms.**

V **= a finite set indexing various families, in particular:**

$\omega_V := (\omega_v \,|\, v \in V)$ **a family of holomorphic 1–forms.**

$A_V := (A_v \,|\, v \in V)$ **= noncommuting free formal variables.**

$$\Omega := \sum_{v \in V} A_v \omega_v$$

$\gamma :\, [0,1] \to U$ **= a piecewise smooth path.**

• **Total iterated integral of Ω along γ:**

$$J_\gamma(\Omega) := 1 + \sum_{n=1}^{\infty} \int_0^1 \gamma^*(\Omega)(t_1) \int_0^{t_1} \gamma^*(\Omega)(t_2) \cdots \int_0^{t_{n-1}} \gamma^*(\Omega)(t_n) \in \mathbf{C}\langle\langle A_V \rangle\rangle$$

integral is taken over the simplex $0 < t_n < \cdots < t_1 < 1$.

γ**,** γ' **with the same ends are homotopic** $\Rightarrow J_\gamma(\Omega) = J_{\gamma'}(\Omega)$**.**

Another notation: $z_i = \gamma(t_i) \in X$**,** $a = \gamma(0)$**,** $z = \gamma(1)$**,**

$$J_a^z(\Omega) := 1 + \sum_{n=1}^{\infty} \int_a^z \Omega(z_1) \int_a^{z_1} \Omega(z_2) \cdots \int_a^{z_{n-1}} \Omega(z_n)\,.$$

If U **is connected and simply connected, this is an unambiguously defined element of** $\mathcal{O}_X(U)\langle\langle A_V\rangle\rangle$**. Otherwise it is a multivalued function of** z **in this domain.**

• **PROPOSITION. (i)** $J_a^z(\Omega)$ **as a function of** z **satisfies the equation**

$$dJ_a^z(\Omega) = \Omega(z)\, J_a^z(\Omega).$$

In other words, $J_a^z(\Omega)$ **is a horizontal (multi)section of the flat connection** $\nabla_\Omega := d - l_\Omega$ **on** $\mathcal{O}_X\langle\langle A_V\rangle\rangle$**, where** l_Ω **is the operator of left multiplication by** Ω**.**

(ii) If U **is a simply connected neighborhood of** a**,** $J_a^z(\Omega)$ **is the only horizontal section with initial condition** $J_a^a = 1$**. Any other horizontal section** K^z **can be uniquely written in the form** $CJ_a^z(\Omega)$**,** $C \in \mathbf{C}\langle\langle A_V\rangle\rangle$**. In particular, for any** $b \in U$**,**

$$J_b^z(\Omega) = J_a^z(\Omega) J_b^a(\Omega)$$

• **COROLLARY. Let** γ **be a closed oriented contractible contour in** U**,** $a_1, \dots, a_n$ **points along this contour (cyclically) ordered compatibly with orientation. Then**

$$J_{a_2}^{a_1}(\Omega) J_{a_3}^{a_2}(\Omega) \dots J_{a_n}^{a_{n-1}}(\Omega) J_{a_1}^{a_n}(\Omega) = 1. \qquad (*)$$

Formula (*) is the multiplicative version of the additivity of simple integrals with respect to the union of integration paths.

• **PROPOSITION. Consider the comultiplication**

$$\Delta : \mathbf{C}\langle\langle A_V\rangle\rangle \to \mathbf{C}\langle\langle A_V\rangle\rangle\widehat{\otimes}_{\mathbf{C}}\mathbf{C}\langle\langle A_V\rangle\rangle,\ \Delta(A_v) = A_v\otimes 1 + 1\otimes A_v$$

and extend it to the series with coefficients $\mathbf{C}(X)$ and $\Omega^1(X)$. Then

$$\Delta\left(J_a^z(\omega_V)\right) = J_a^z(\omega_V)\widehat{\otimes}_{\mathcal{O}_X} J_a^z(\omega_V)\,. \qquad (**)$$

• **CLAIM 1. The identity (**) encodes all shuffle relations between the iterated integrals of the forms ω_v.**

• **CLAIM 2. The identity (**) is equivalent to the fact that $\log J_a^z(\omega_V)$ can be expressed as a series in commutators (of arbitrary length) of the variables A_v.**

Formula (**) is called "the group–like property of $J_a^z(\Omega)$." It is a multiplicative version of the additivity of a simple integral as a functional of the integration form.

• **FUNCTORIALITY.** $g: X\to X$ **= an automorphism such that g^* maps into itself the linear space spanned by ω_v:**

$g^*(\omega_v) = \sum_u g_{vu}\omega_u$.

Define $g_*(A_u) = \sum_v A_v g_{vu}$.

• **LEMMA.**

$$J_{ga}^{gz}(\omega_V) = g_*(J_a^z(\omega_V))\,. \qquad (***)$$

Formula (***) is a multiplicative version of the variable change formula.

2. 1–FORMS OF MODULAR TYPE ON THE UPPER HALF–PLANE

• **DEFINITION. (i) A 1–form ω on H is called a form of modular type, if it can be represented as $f(z)z^{s-1}dz$ where s is a complex number, and $f(z)$ is a modular form of some weight with respect to a finite index subgroup Γ subgroup of the modular group of the modular group $SL(2,\mathbf{Z})$.**

The modular form $f(z)$ is then well defined and called the associated modular form (to ω), and the number s is called the Mellin argument of ω.

(ii) ω is called a form of cusp modular type if the associated $f(z)$ is a cusp form.

• **REMINDER.**

(i) Action of weight k of $\gamma \in GL^+(2,\mathbf{Z})$ upon functions on H:

$$f|[\gamma]_k(z) := (\det\gamma)^{k/2} f([\gamma]z)\,(c_\gamma z + d_\gamma)^{-k}.$$

(ii) $f(z)$ is a modular form of weight k for the group Γ if $f|[\gamma]_k(z) = f(z)$ for all $\gamma \in \Gamma$ and $f(z)$ is finite at cusps.

(iii) Such a form is a cusp form if it vanishes at cusps.

3. CLASSICAL INTEGRALS OF 1–FORMS OF CUSP MODULAR TYPE

Ordinary (non–iterated) integrals from our vantage point furnish linear (in A_v) terms of the iterated theory. For 1–forms of cusp modular type, theory of ordinary integrals consists of the following parts.

- **CLASSICAL MELLIN TRANSFORM OF $f(z)$:**

$$\Lambda(f;s) := \int_{i\infty}^{0} f(z) z^{s-1} dz.$$

Assume that is Γ normalized by the involution

$$g = g_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} :$$

Denote weight of f by $k = 2r$. If $g_N^*(f(z)(dz)^r) = \varepsilon_f\, f(z)(dz)^r, \quad \varepsilon_f = \pm 1$, then

$$\Lambda(f;s) = -\varepsilon_f N^{r-s} \Lambda(f;k-s)\,.$$

Critical strip: $0 < \mathrm{Re}\, s < k$**.**

I will define the "total iterated Mellin transform" and extend the functional equation to it.

• **THE SPACE OF MODULAR SYMBOLS $MS_k(\Gamma)$.** **It is the space of R–linear functionals on the space of cusp forms $S_k(\Gamma)$ spanned by the Shimura integrals:**

$$f(z) \mapsto \int_\alpha^\beta f(z) z^{m-1} dz; \quad 1 \le m \le k-1; \quad \alpha, \beta \in \mathbf{P}^1(\mathbf{Q}).$$

Three descriptions of $MS_k(\Gamma)$ are known:

(i) Formal (Shimura – Eichler – Manin): generators and relations.

(ii) Geometric (Shokurov): (part of) middle homology of the Kuga–Sato variety $\mathcal{E}_\Gamma^{k-1}$.

(iii) As the dual space to the group cohomology $H^1(\Gamma, V_{k-2})$, with coefficients in the $(k-2)$–th symmetric power of the basic representation of SL (Shimura).

I will show that the cohomological description admits a sensible iterated extension.

• $\Lambda(f;s)$ **FOR GENERAL** s **IS A FORMAL DIRICHLET SERIES CONVERGENT IN A RIGHT HALF PLANE:**

$$f(z)=\sum_{n=1}^{\infty} a_n e^{2\pi i n z} \qquad \Longrightarrow \qquad \Lambda(f;s)=-\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

I will show that the components of the total Mellin transform at integral points of the critical (multidimensional) strip can be expressed as multiple Dirichlet series of a special form.

• **IF** Γ **IS A CONGRUENCE SUBGROUP:**

$\Lambda(f;s)$ **admits an Euler product** $\quad\Leftrightarrow\quad$ f **is an eigenfuction for Hecke operators.**

Major unsolved problem: extend this to the iterated context.

4. ITERATED MELLIN TRANSFORM

• **DEFINITION. (i) Let** $f_1, \dots, f_k$ **be a finite sequence of cusp forms with respect to** Γ**,** $\omega_j(z) := f_j(z)\, z^{s_j-1}dz$. **The iterated Mellin transform of** (f_j) **is**

$$M(f_1, \dots, f_k; s_1, \dots, s_k) := I^0_{i\infty}(\omega_1, \dots, \omega_k) =$$

$$= \int_{i\infty}^0 \omega_1(z_1) \int_{i\infty}^{z_1} \omega_2(z_2) \cdots \int_{i\infty}^{z_{n-1}} \omega_n(z_n)$$

(ii) Let $f_V = (f_v \,|\, v \in V)$ **be a finite family of cusp forms with respect to** Γ**,** $s_V = (s_v \,|\, v \in V)$ **a finite family of complex numbers,** $\omega_V = (\omega_v)$**, where** $\omega_v(z) := f_v(z)\, z^{s_v-1}dz$. **The total Mellin transform of** f_V **is**

$$TM(f_V; s_V) := J^0_{i\infty}(\omega_V) =$$

$$= \sum_{n=0}^{\infty} \sum_{(v_1,\dots,v_n)\in V^n} A_{v_1} \dots A_{v_n} M(f_{v_1}, \dots, f_{v_n}; s_{v_1}, \dots, s_{v_n})$$

• **THEOREM. Assume that the space spanned by** $f_v(z)$ **is stable wrt** g_N**. Let** k_v **be the weight of** $f_v(z)$**, and** $k_V = (k_v)$**. Then**

$$TM(f_V; s_V) = g_*(TM(f_V; k_V - s_V))^{-1}$$

for an appropriate linear transformation g_{N*} **of formal variables** A_v**.**

5. ITERATED SHIMURA INTEGRALS

• REMINDER ON THE NONCOMMUTATIVE GROUP COHOMOLOGY.

G a group, N a group with left action of G: $(g, n) \mapsto gn$.

Cocycles: $Z^1(G, N) := \{\, u : G \to N \mid u(g_1 g_2) = u(g_1)\, g_1 u(g_2) \,\}$

Cohomological cocycles: $u' \sim u \quad \Leftrightarrow \quad \exists n \in N \;\forall g \in G,\; u'(g) = n\, u(g)\, (gn)^{-1}$.

$H^1(G, N) := Z^1(G, N)/(\sim)$.

Marked point: class of trivial cocycles $u(g) = n(gn)^{-1}$

• THE CASE OF ITERATED SHIMURA INTEGRALS.

$G := \Gamma$ **= a subgroup of modular group**

(ω_v) **= a family of Shimura differentials** $f_v(z) z^{m_v - 1} dz$, **where** f_v **form a basis of** $\oplus_i S(k_i, \Gamma)$**, and for a fixed weight,** m_v **ranges over all critical integers for this weight.**
They span a Γ**–invariant space. Put** $\Omega := \sum_{v \in V} A_v \omega_v$.

$N = \Pi :=$ **the group of group–like and** $(-id)_*$**–invariant elements of** $(1 + \sum_{v \in V} A_v \mathbf{C}\langle\langle A_v \rangle\rangle)^*$

Left action of Γ **upon** Π**: functoriality action** g_*.

• THEOREM. (i) For any $a \in \overline{H}$**, the map** $\Gamma \to \Pi;\ \gamma \mapsto J^a_{\gamma a}(\Omega)$ **is a noncommutative 1–cocycle** ζ_a **in** $Z^1(\Gamma, \Pi)$**.**

(ii) The cohomology class of ζ_a **in** $H^1(\Gamma, \Pi)$ **does not depend on the choice of** a **and is called the noncommutative modular symbol.**

(iii) This cohomology class belongs to the cuspidal subset $H^1(\Gamma, \Pi)_{cusp}$ **consisting of those cohomology classes whose restriction on all stabilizers of** Γ**–cusps is trivial.**

REDUCTION TO THE COHOMOLOGY OF $SL(2,\mathbf{Z})$

• **NONCOMMUTATIVE SHAPIRO LEMMA. Let $G \subset H$ be a subgroup, N a left G–group, $N_H := \mathrm{Map}_G\,(N, H)$ with pointwise multiplication and left action of G, $(g_*\phi)(h) := \phi(hg)$. There is a canonical isomorphism of pointed sets:**

$$H^1(G, N) = H^1(H, N_H).$$

• **APPLICATION TO THE ITERATED SHIMURA INTEGRALS.**

$G := \Gamma$, $\quad H := SL(2,\mathbf{Z})$, $\quad N := \Pi$, $\quad \Pi^0 := N_H$.

Generators of $H = SL(2,\mathbf{Z})$:

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

• **THEOREM. (i) An iterated Shimura cocycle restricted to (σ, τ) belongs to the set**

$$\{\,(X, Y) \in \Pi^0 \times \Pi^0 \,|\, X \cdot \sigma_* X = 1,\; Y \cdot \tau_* Y \cdot \tau_*^2 Y = 1\,\}.$$

(ii) The cohomology relation between cocycles translates as

$$(X, Y) \sim (m^{-1} X \sigma_*(m), m^{-1} Y \tau_* Y).$$

(iii) Cuspidal part of the cohomology is generated by the pairs

$$\{\,(X, Y) \,|\, \exists Z,\; X \cdot (\sigma\tau)_* Y = Z^{-1} (\sigma\tau)_* Z\,\}$$

6. ITERATED SHIMURA INTEGRALS AS MULTIPLE DIRICHLET SERIES

• **NOTATION. Start with the family of 1–forms on** H**:**

$$\omega_v(z)=\sum_{n=1}^{\infty}c_{v,n}e^{2\pi inz}z^{m_v-1}dz,\quad c_{v,n}\in\mathbf{C}\,,\quad m_v\in\mathbf{Z},\, m_v\geq 1;\; c_{v,n}=O(n^C).$$

Put

$$L(z;\omega_{v_k},\dots,\omega_{v_1};j_k,\dots,j_1):=$$

$$=(2\pi iz)^{j_k}\sum_{n_1,\dots,n_k\geq 1}\frac{c_{v_1,n_1}\dots c_{v_k,n_k}e^{2\pi i(n_1+\dots+n_k)z}}{n_1^{m_{v_1}+j_0-j_1}(n_1+n_2)^{m_{v_2}+j_1-j_2}\dots(n_1+\dots+n_k)^{m_{v_k}+j_{k-1}-j_k}}\,.$$

Exponentials ensure absolute convergence for any z **with** $\mathrm{Im}\,z>0$**.**

Formal substitution $z=0$ **may lead to divergence.**

• **THEOREM. For any** $k\geq 1$**,** $(v_1,\dots,v_k)\in V^k$**, and** $\mathrm{Im}\,z>0$ **we have**

$$(2\pi i)^{m_{v_1}+\dots+m_{v_k}}I_{i\infty}^{z}(\omega_{v_k},\dots,\omega_{v_1})=$$

$$=(-1)^{\sum_{i=1}^{k}(m_{v_i}-1)}\sum_{j_1=0}^{m_{v_1}-1}\sum_{j_2=0}^{m_{v_2}-1+j_1}\cdots\sum_{j_k=0}^{m_{v_k}-1+j_{k-1}}(-1)^{j_k}\times$$

$$\times\frac{(m_{v_1}-1)!(m_{v_2}-1+j_1)!\dots(m_{v_k}-1+j_{k-1})!}{j_1!j_2!\dots j_k!}\,L(z;\omega_{v_k},\dots,\omega_{v_1};j_k,\dots,j_1)\,.$$

• PROPOSITION. Assume that ω_V as above is a basis of a space of 1-forms invariant with respect to g_N. Then

$$J^0_{i\infty}(\omega_V) = (g_{N*}(J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V)))^{-1} J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V)\,. \qquad (***)$$

REMARK 1. We can mix different weights.

REMARK 2. Replacing the coefficients of the formal series at the r.h.s of (***) by their (convergent) representations via multiple Dirichlet series with exponents we get such representations for $I^0_{i\infty}(\omega_{v_k}, \ldots, \omega_{v_1})$ and avoid divergences at $z = 0$.

7. SHUFFLE RELATIONS BETWEEN MULTIPLE DIRICHLET SERIES

• **DEFINITION. (i) Coefficients data C of depth k is a family of numbers $c_{n,m}^{(j,i)}$ indexed by two pairs of integers satisfying $j > i \geq 0$, $j \leq k$, and $n > m \geq 0$.**

(ii) The multiple Dirichlet series associated with C and arguments $s_1, \dots, s_k$ is

$$L_C(s_1, \dots, s_k) := \sum_{0=u_0<u_1<\dots<u_k\in\mathbf{Z}} \frac{\prod_{k\geq j>i\geq 0} c_{u_j,u_i}^{(j,i)}}{u_1^{s_1}u_2^{s_2}\dots u_k^{s_k}}$$

• **EXAMPLES. (a) Assume that $c_{n,m}^{(j,i)} = 1$ if $m > 0$ or $i > 0$ and put $c_{n,0}^{(j,0)} = a_n^{(j)}$. Then**

$$L_C(s_1, \dots, s_k) = \sum_{0<u_1<\dots<u_k\in\mathbf{Z}} \frac{a_{u_1}^{(1)}a_{u_1}^{(2)}\dots a_{u_k}^{(k)}}{u_1^{s_1}u_2^{s_2}\dots u_k^{s_k}}$$

is an ordinary multiple Dirichlet series.

(b) Define $c_{v,n}$ as in 3.1, and choose $v_1, \dots, v_k \in V$. Construct the coefficients data C

$$c^{(j,j-1)}_{n,m} := c_{v_j, n-m},$$

and $c^{(j,i)}_{n,m} = 1$ otherwise. Then $L_C(m_{v_1} + j_0 - j_1, \dots, m_{v_k} + j_{k-1} - j_k)$ becomes the formal series

$$(2\pi i z)^{-j_k}\, L(z; \omega_{v_k}, \dots, \omega_{v_1}; j_k, \dots, j_1)\,|_{z=0} =$$

$$\sum_{n_1,\dots,n_k \geq 1} \frac{c_{v_1,n_1} \dots c_{v_k,n_k}}{n_1^{m_{v_1}+j_0-j_1} (n_1+n_2)^{m_{v_2}+j_1-j_2} \dots (n_1+\cdots+n_k)^{m_{v_k}+j_{k-1}-j_k}}\,.$$

if we redenote $u_j = n_1 + \cdots + n_j$.

• SHUFFLES AND A COMPOSITION OF THE COEFFICIENTS DATA.

A (k,l,p)–shuffle with repetitions is a pair of strictly increasing maps $\sigma = (\sigma_1, \sigma_2)$,

$$\sigma_1 : [0,k] \to [0,p], \quad \sigma_2 : [0,l] \to [0,p]$$

satisfying the following conditions:

$$\sigma_1(0) = \sigma_2(0) = 0,\ \sigma_1([0,k]) \cup \sigma_2([0,l]) = [0,p],\ \max(k,l) \leq p \leq k+l.$$

• CLAIM. Let $C = (c_{n,m}^{(j,i)})$ and $D = (d_{n,m}^{(j,i)})$ be two coefficients data of depths k and l respectively; $s := (s_1, \ldots, s_k)$ and $t := (t_1, \ldots, t_l)$ the Dirichlet arguments for the data C and D, and σ a (k, l, p)–shuffle with repetitions.

One can define a new coefficients data $E = C *_\sigma D$ of depth p and the Dirichlet arguments for it $s +_\sigma t$ in such a way that

$$L_C(s) \cdot L_D(t) = \sum_\sigma L_{C *_\sigma D}(s +_\sigma t)$$

where the summation is taken over all (k, l, p)–shuffles with repetitions (p variable).

DIFFERENTIALS OF THE THIRD KIND, EISENSTEIN SERIES, AND THE GENERALIZED ASSOCIATORS

We will now assume, as in the initial Drinfeld setting, that the integration limits of the iterated integral are logarithmic singularities of the form Ω. Generally, they diverge and must be regularized. The dependence on the regularization is a version of Deligne's "base point at infinity".

• NOTATION.

$a =$ **a fixed point of the Riemann surface,** z **a variable point.**

$r_{v,a} := \mathrm{res}_a\, \omega_v, \quad R_a := \mathrm{res}_a\, \Omega = \sum_v r_{v,a} A_v\,.$

t_a **:= a local parameter at** a**,** $\log t_a$ **a local branch of logarithm real on** $t_a \in \mathbf{R}_+$**.**

$t_a^{R_a} := e^{R_a \log t_a}$**.**

• DEFINITION. A local solution to $dJ^z = \Omega(z) J^z$ **is called normalized at** a **(with respect to a choice of** t_a**) if it is of the form** $J = K \cdot t_a^{R_a}$**, where** K **is holomorphic section in a neighborhood of** a **and** $K(a) = 1$**.**

• CLAIM. (i) The normalized solution exists and is unique.

(ii) It depends only on the tangent vector $\partial/\partial t_a\,|_a$**.**

(iii) If $J'_a = K'(t'_a)^{R_a}$ **is normalized with respect** t'_a**, and** $\tau_a := dt'_a/dt_a\,|_a$**, then** $J'_a = J_a \cdot \tau_a^{R_a}$**.**

• **SCATTERING OPERATORS. Given:**

– $(a, t_a), (b, t_b)$,

– $\Omega = \sum A_v \omega_v$ **with at most logarithmic singularities at** a**,** b**,**

– **a (homotopy class of) path(s) from** a **to** b **avoiding other singularities of** Ω**.**

We construct

(i) the normalized solutions J_a**,** J_b **analytically continued along** γ **and THE SCATTERING OPERATOR**

$$\widetilde{J}_b^a = J_a^{-1} J_b \in \mathbf{C}\langle\langle A_V \rangle\rangle \,.$$

Its coefficients := REGULARIZED ITERATED INTEGRALS.

• **EXAMPLE: DRINFELD'S ASSOCIATOR. Let** $X = \mathbf{P}^1(\mathbf{C})$**,** $V = \{0, 1\}$**,**

$$\omega_0 = \frac{1}{2\pi i} \frac{dz}{z}, \quad \omega_1 = \frac{1}{2\pi i} \frac{dz}{z-1} \,.$$

Then

$$\Omega = A_0 \omega_0 + A_1 \omega_1$$

has poles at $0, 1, \infty$ **with residues** $A_0/2\pi i$**,** $A_1/2\pi i$**,** $-(A_0 + A_1)/2\pi i$ **respectively. Put** $t_0 = z, t_1 = 1 - z$**. Then** $\widetilde{J}_0^1$ **in our notation is the Drinfeld associator** $\phi_{KZ}(A_0, A_1)$**.**

• GENERALIZED ASSOCIATORS AND ITERATED INTEGRALS OF EISENSTEIN SERIES OF WEIGHT TWO.

Γ **:= a congruence subgroup of the modular group**

f_v **:= a basis of Eisenstein series of weight 2 wrt** Γ

$\{\omega_v =$ **push forward of** $f_v(z)dz\}$ **: 1–forms with logarithmic singularities at cusps on** X_Γ**.**

Regularized iterated integrals of Eisenstein series of weight two between cusps provide a modular generalization of multiple zeta values.