# ITERATED INTEGRALS OF MODULAR FORMS AND NONCOMMUTATIVE MODULAR SYMBOLS

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## PLAN

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## 0. INTRODUCTION

### • MULTIPLE ZETA VALUES

**a) Definition via multiple Dirichlet series:**

$$\zeta(m_1, \ldots, m_k) = \sum_{0<n_1<\cdots<n_k} \frac{1}{n_1^{m_1} \ldots n_k^{m_k}}$$

**Convergence:** $m_i \geq 1$ **and** $m_k > 1$.
**b) Definition via iterated integrals:**

$$\zeta(m_1, \ldots, m_k) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \cdots \int_0^{t_{m_k-1}} \frac{dt_{m_k}}{1-t_{m_k}} \cdots$$

**The sequence of differential forms in the iterated integral consists of consecutive subsequences** $\frac{dt}{t}, \ldots, \frac{dt}{t}, \frac{dt}{1-t}$ **of lengths** $m_k, m_{k-1}, \ldots, m_1$.

### • SHUFFLE RELATIONS

**Relations between iterated integrals.**

**Relations between multiple series.**

### • DRINFELD'S ASSOCIATOR, QUANTIZATION, GROTHENDIECK–TEICHMUELLER GROUP.

• **GEOMETRY ON $\mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\}$:**

$\frac{dt}{t}, \frac{dt}{1-t}$ **span the space of meromorphic differential forms with no more than logarithmic singularities at $\{0, 1, \infty\}$.**

• **LIFT TO THE UPPER HALF PLANE $H := \{z \in \mathbf{C} \,|\, \mathrm{Im}\, z > 0\}$ AND $\overline{H} := H \cup \mathbf{P}^1(\mathbf{Q})$:**

$$(\mathbf{P}^1(\mathbf{C}), \{0, 1, \infty\}) \cong \Gamma_0(4) \setminus (\overline{H}, cusps)$$

$\frac{dt}{t}, \frac{dt}{1-t}$ **lift to Eisenstein series of weight two for $\Gamma_0(4) \subset SL(2, \mathbf{Z})$.**

**The integration path $[0, 1]$ lifts to the geodesic connecting two cusps.**

• **GOAL OF THIS WORK: A GENERALIZATION OF THE THIS PICTURE ALONG THE FOLLOWING LINES:**

– $\Gamma_0(4)$ **is replaced by an arbitrary (congruence) subgroup $\Gamma$ of $SL(2, \mathbf{Z})$.**

– **Eisenstein series of weight two are replaced by arbitrary modular forms (cusp form + Eisenstein series) with respect to $\Gamma$.**

**NB In this talk I focus on cusp forms (absent in the classical polyzetas setting).**

– **Iterated integrals are taken along geodesics connecting two cusps.**

• **MODULAR SYMBOLS.**

**Integration of cusp forms along geodesics connecting two cusps = theory of classical modular symbols (of arbitrary weight).**

**Hence the subject of this talk can be called "iterated modular symbols".**

## 1. FORMALISM OF ITERATED INTEGRALS OF 1–FORMS ON A RIEMANNIAN SURFACE

*Here it will be shown that if one replaces a simple integral not by an individual iterated integral but by a generating series of all such integrals, then the usual properties like additivity and variable change formula reappear in a multiplicative/noncommutative version.*

• **NOTATION.**

$X$ **= a connected Riemann surface.**

$\mathcal{O}_X$ **= the structure sheaf of holomorphic functions,**

$\Omega^1_X$ **= the sheaf of holomorphic 1–forms.**

$V$ **= a finite set indexing various families, in particular:**

$\omega_V := (\omega_v \,|\, v \in V)$ **a family of holomorphic 1–forms.**

$A_V := (A_v \,|\, v \in V)$ **= noncommuting free formal variables.**

$$\Omega := \sum_{v \in V} A_v \omega_v$$

$\gamma : [0,1] \to U$ **= a piecewise smooth path.**

• **Total iterated integral of $\Omega$ along $\gamma$:**

$$J_\gamma(\Omega) := 1 + \sum_{n=1}^{\infty} \int_0^1 \gamma^*(\Omega)(t_1) \int_0^{t_1} \gamma^*(\Omega)(t_2) \cdots \int_0^{t_{n-1}} \gamma^*(\Omega)(t_n) \in \mathbf{C}\langle\langle A_V \rangle\rangle$$

**integral is taken over the simplex $0 < t_n < \cdots < t_1 < 1$.**

$\gamma$**,** $\gamma'$ **with the same ends are homotopic** $\Rightarrow J_\gamma(\Omega) = J_{\gamma'}(\Omega)$**.**

**Another notation:** $z_i = \gamma(t_i) \in X$, $a = \gamma(0)$, $z = \gamma(1)$,

$$J_a^z(\Omega) := 1 + \sum_{n=1}^{\infty} \int_a^z \Omega(z_1) \int_a^{z_1} \Omega(z_2) \cdots \int_a^{z_{n-1}} \Omega(z_n)\,.$$

**If $U$ is connected and simply connected, this is an unambiguously defined element of $\mathcal{O}_X(U)\langle\langle A_V\rangle\rangle$. Otherwise it is a multivalued function of $z$ in this domain.**

**• PROPOSITION. (i) $J_a^z(\Omega)$ as a function of $z$ satisfies the equation**

$$dJ_a^z(\Omega) = \Omega(z)\, J_a^z(\Omega).$$

**In other words, $J_a^z(\Omega)$ is a horizontal (multi)section of the flat connection $\nabla_\Omega := d - l_\Omega$ on $\mathcal{O}_X\langle\langle A_V\rangle\rangle$, where $l_\Omega$ is the operator of left multiplication by $\Omega$.**

**(ii) If $U$ is a simply connected neighborhood of $a$, $J_a^z(\Omega)$ is the only horizontal section with initial condition $J_a^a = 1$. Any other horizontal section $K^z$ can be uniquely written in the form $CJ_a^z(\Omega)$, $C \in \mathbf{C}\langle\langle A_V\rangle\rangle$. In particular, for any $b \in U$,**

$$J_b^z(\Omega) = J_a^z(\Omega) J_b^a(\Omega)$$

**• COROLLARY. Let $\gamma$ be a closed oriented contractible contour in $U$, $a_1, \dots, a_n$ points along this contour (cyclically) ordered compatibly with orientation. Then**

$$J_{a_2}^{a_1}(\Omega) J_{a_3}^{a_2}(\Omega) \dots J_{a_n}^{a_{n-1}}(\Omega) J_{a_1}^{a_n}(\Omega) = 1. \qquad (*)$$

*Formula (*) is the multiplicative version of the additivity of simple integrals with respect to the union of integration paths.*

• **PROPOSITION. Consider the comultiplication**

$$\Delta : \mathbf{C}\langle\langle A_V\rangle\rangle \to \mathbf{C}\langle\langle A_V\rangle\rangle \widehat{\otimes}_{\mathbf{C}} \mathbf{C}\langle\langle A_V\rangle\rangle,\ \Delta(A_v) = A_v \otimes 1 + 1 \otimes A_v$$

**and extend it to the series with coefficients** $\mathbf{C}(X)$ **and** $\Omega^1(X)$**. Then**

$$\Delta\left(J_a^z(\omega_V)\right) = J_a^z(\omega_V) \widehat{\otimes}_{\mathcal{O}_X} J_a^z(\omega_V)\,. \qquad (**)$$

• **CLAIM 1. The identity (**) encodes all shuffle relations between the iterated integrals of the forms** $\omega_v$**.**

• **CLAIM 2. The identity (**) is equivalent to the fact that** $\log J_a^z(\omega_V)$ **can be expressed as a series in commutators (of arbitrary length) of the variables** $A_v$**.**

*Formula (**) is called "the group–like property of* $J_a^z(\Omega)$*." It is a multiplicative version of the additivity of a simple integral as a functional of the integration form.*

• **FUNCTORIALITY.** $g : X \to X$ **= an automorphism such that** $g^*$ **maps into itself the linear space spanned by** $\omega_v$**:**

$g^*(\omega_v) = \sum_u g_{vu}\omega_u$.

**Define** $g_*(A_u) = \sum_v A_v g_{vu}$ .

• **LEMMA.**

$$J_{ga}^{gz}(\omega_V) = g_*(J_a^z(\omega_V))\,. \qquad (***)$$

*Formula (***) is a multiplicative version of the variable change formula.*

## 2. 1–FORMS OF MODULAR TYPE ON THE UPPER HALF–PLANE

**• DEFINITION. (i) A 1–form $\omega$ on $H$ is called a form of modular type, if it can be represented as $f(z)z^{s-1}dz$ where $s$ is a complex number, and $f(z)$ is a modular form of some weight with respect to a finite index subgroup $\Gamma$ subgroup of the modular group of the modular group $SL(2, \mathbf{Z})$.**

**The modular form $f(z)$ is then well defined and called the associated modular form (to $\omega$), and the number $s$ is called the Mellin argument of $\omega$.**

**(ii) $\omega$ is called a form of cusp modular type if the associated $f(z)$ is a cusp form.**

**• REMINDER.**

**(i) Action of weight $k$ of $\gamma \in GL^+(2, \mathbf{Z})$ upon functions on $H$:**

$$f|[\gamma]_k(z) := (\det \gamma)^{k/2} f([\gamma]z)\,(c_\gamma z + d_\gamma)^{-k}.$$

**(ii) $f(z)$ is a modular form of weight $k$ for the group $\Gamma$ if $f|[\gamma]_k(z) = f(z)$ for all $\gamma \in \Gamma$ and $f(z)$ is finite at cusps.**

**(iii) Such a form is a cusp form if it vanishes at cusps.**

## 3. CLASSICAL INTEGRALS OF 1–FORMS OF CUSP MODULAR TYPE

*Ordinary (non–iterated) integrals from our vantage point furnish linear (in $A_v$) terms of the iterated theory. For 1–forms of cusp modular type, theory of ordinary integrals consists of the following parts.*

- **CLASSICAL MELLIN TRANSFORM OF $f(z)$:**

$$\Lambda(f;s) := \int_{i\infty}^{0} f(z) z^{s-1} dz.$$

**Assume that is $\Gamma$ normalized by the involution**

$$g = g_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} :$$

**Denote weight of $f$ by $k = 2r$. If $g_N^*(f(z)(dz)^r) = \varepsilon_f\, f(z)(dz)^r, \quad \varepsilon_f = \pm 1$, then**

$$\Lambda(f;s) = -\varepsilon_f N^{r-s} \Lambda(f;k-s)\,.$$

**Critical strip:** $0 < \operatorname{Re} s < k$**.**

*I will define the "total iterated Mellin transform" and extend the functional equation to it.*

• **THE SPACE OF MODULAR SYMBOLS $MS_k(\Gamma)$. It is the space of R–linear functionals on the space of cusp forms $S_k(\Gamma)$ spanned by the Shimura integrals:**

$$f(z) \mapsto \int_\alpha^\beta f(z) z^{m-1} dz; \quad 1 \le m \le k-1; \quad \alpha, \beta \in \mathbf{P}^1(\mathbf{Q}).$$

**Three descriptions of $MS_k(\Gamma)$ are known:**

**(i) Formal (Shimura – Eichler – Manin): generators and relations.**

**(ii) Geometric (Shokurov): (part of) middle homology of the Kuga–Sato variety $\mathcal{E}_\Gamma^{k-1}$.**

**(iii) As the dual space to the group cohomology $H^1(\Gamma, V_{k-2})$, with coefficients in the $(k-2)$–th symmetric power of the basic representation of $SL$ (Shimura).**

*I will show that the cohomological description admits a sensible iterated extension.*

• $\Lambda(f;s)$ **FOR GENERAL** $s$ **IS A FORMAL DIRICHLET SERIES CONVERGENT IN A RIGHT HALF PLANE:**

$$f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi i n z} \qquad \Longrightarrow \qquad \Lambda(f;s)=-\frac{\Gamma(s)}{(-2\pi i)^s}\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

*I will show that the components of the total Mellin transform at integral points of the critical (multidimensional) strip can be expressed as multiple Dirichlet series of a special form.*

• **IF** $\Gamma$ **IS A CONGRUENCE SUBGROUP:**

$\Lambda(f;s)$ **admits an Euler product** $\quad\Leftrightarrow\quad$ $f$ **is an eigenfuction for Hecke operators.**

*Major unsolved problem: extend this to the iterated context.*

## 4. ITERATED MELLIN TRANSFORM

• **DEFINITION. (i) Let** $f_1, \dots, f_k$ **be a finite sequence of cusp forms with respect to** $\Gamma$**,** $\omega_j(z) := f_j(z)\, z^{s_j - 1} dz$. **The iterated Mellin transform of** $(f_j)$ **is**

$$M(f_1, \dots, f_k; s_1, \dots, s_k) := I^0_{i\infty}(\omega_1, \dots, \omega_k) =$$

$$= \int_{i\infty}^0 \omega_1(z_1) \int_{i\infty}^{z_1} \omega_2(z_2) \cdots \int_{i\infty}^{z_{n-1}} \omega_n(z_n)$$

**(ii) Let** $f_V = (f_v \,|\, v \in V)$ **be a finite family of cusp forms with respect to** $\Gamma$**,** $s_V = (s_v \,|\, v \in V)$ **a finite family of complex numbers,** $\omega_V = (\omega_v)$**, where** $\omega_v(z) := f_v(z)\, z^{s_v - 1} dz$. **The total Mellin transform of** $f_V$ **is**

$$TM(f_V; s_V) := J^0_{i\infty}(\omega_V) =$$

$$= \sum_{n=0}^{\infty} \sum_{(v_1, \dots, v_n) \in V^n} A_{v_1} \dots A_{v_n}\, M(f_{v_1}, \dots, f_{v_n}; s_{v_1}, \dots, s_{v_n})$$

• **THEOREM. Assume that the space spanned by** $f_v(z)$ **is stable wrt** $g_N$**. Let** $k_v$ **be the weight of** $f_v(z)$**, and** $k_V = (k_v)$**. Then**

$$TM(f_V; s_V) = g_*(TM(f_V; k_V - s_V))^{-1}$$

**for an appropriate linear transformation** $g_{N*}$ **of formal variables** $A_v$**.**

## 5. ITERATED SHIMURA INTEGRALS

**• REMINDER ON THE NONCOMMUTATIVE GROUP COHOMOLOGY.**

$G$ **a group,** $N$ **a group with left action of** $G$**:** $(g,n) \mapsto gn$**.**

**Cocycles:** $Z^1(G,N) := \{\, u : G \to N \,|\, u(g_1 g_2) = u(g_1)\, g_1 u(g_2) \,\}$

**Cohomological cocycles:** $u' \sim u \quad \Leftrightarrow \quad \exists n \in N \ \forall g \in G,\ u'(g) = n\, u(g)\, (gn)^{-1}$.

$H^1(G,N) := Z^1(G,N)/(\sim)$.

**Marked point: class of trivial cocycles** $u(g) = n(gn)^{-1}$

**• THE CASE OF ITERATED SHIMURA INTEGRALS.**

$G := \Gamma$ **= a subgroup of modular group**

$(\omega_v)$ **= a family of Shimura differentials** $f_v(z) z^{m_v - 1} dz$**, where** $f_v$ **form a basis of** $\oplus_i S(k_i, \Gamma)$**, and for a fixed weight,** $m_v$ **ranges over all critical integers for this weight.**

**They span a** $\Gamma$**–invariant space. Put** $\Omega := \sum_{v \in V} A_v \omega_v$.

$N = \Pi :=$ **the group of group–like and** $(-id)_*$**–invariant elements of** $(1 + \sum_{v \in V} A_v \mathbf{C}\langle\langle A_v \rangle\rangle)^*$

**Left action of** $\Gamma$ **upon** $\Pi$**: functoriality action** $g_*$**.**

**• THEOREM. (i) For any** $a \in \overline{H}$**, the map** $\Gamma \to \Pi;\ \gamma \mapsto J^a_{\gamma a}(\Omega)$ **is a noncommutative 1–cocycle** $\zeta_a$ **in** $Z^1(\Gamma, \Pi)$**.**

**(ii) The cohomology class of** $\zeta_a$ **in** $H^1(\Gamma,\Pi)$ **does not depend on the choice of** $a$ **and is called the noncommutative modular symbol.**

**(iii) This cohomology class belongs to the cuspidal subset** $H^1(\Gamma,\Pi)_{cusp}$ **consisting of those cohomology classes whose restriction on all stabilizers of** $\Gamma$**–cusps is trivial.**

## REDUCTION TO THE COHOMOLOGY OF $SL(2,\mathbf{Z})$

• **NONCOMMUTATIVE SHAPIRO LEMMA. Let $G\subset H$ be a subgroup, $N$ a left $G$–group, $N_H := \mathrm{Map}_G\,(N,H)$ with pointwise multiplication and left action of $G$, $(g_*\phi)(h) := \phi(hg)$. There is a canonical isomorphism of pointed sets:**

$$H^1(G,N)=H^1(H,N_H).$$

• **APPLICATION TO THE ITERATED SHIMURA INTEGRALS.**

$G:=\Gamma$, $\quad H:=SL(2,\mathbf{Z})$, $\quad N:=\Pi$, $\quad \Pi^0:=N_H$.

**Generators of $H=SL(2,\mathbf{Z})$:**

$$\sigma=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \tau=\begin{pmatrix}0&-1\\1&-1\end{pmatrix},\quad J=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

• **THEOREM. (i) An iterated Shimura cocycle restricted to $(\sigma,\tau)$ belongs to the set**

$$\{\,(X,Y)\in\Pi^0\times\Pi^0\,|\,X\cdot\sigma_*X=1,\;Y\cdot\tau_*Y\cdot\tau_*^2Y=1\,\}.$$

**(ii) The cohomology relation between cocycles translates as**

$$(X,Y)\sim(m^{-1}X\sigma_*(m),m^{-1}Y\tau_*Y).$$

**(iii) Cuspidal part of the cohomology is generated by the pairs**

$$\{\,(X,Y)\,|\,\exists Z,\;X\cdot(\sigma\tau)_*Y=Z^{-1}(\sigma\tau)_*Z\,\}$$

## 6. ITERATED SHIMURA INTEGRALS AS MULTIPLE DIRICHLET SERIES

• **NOTATION. Start with the family of 1–forms on $H$:**

$$\omega_v(z) = \sum_{n=1}^{\infty} c_{v,n} e^{2\pi i n z} z^{m_v - 1} dz, \quad c_{v,n} \in \mathbf{C}, \quad m_v \in \mathbf{Z},\, m_v \geq 1;\ c_{v,n} = O(n^C).$$

**Put**

$$L(z; \omega_{v_k}, \dots, \omega_{v_1}; j_k, \dots, j_1) :=$$

$$= (2\pi i z)^{j_k} \sum_{n_1,\dots,n_k \geq 1} \frac{c_{v_1,n_1} \dots c_{v_k,n_k} e^{2\pi i (n_1 + \dots + n_k) z}}{n_1^{m_{v_1} + j_0 - j_1} (n_1 + n_2)^{m_{v_2} + j_1 - j_2} \dots (n_1 + \dots + n_k)^{m_{v_k} + j_{k-1} - j_k}}\,.$$

**Exponentials ensure absolute convergence for any $z$ with $\operatorname{Im} z > 0$.**

**Formal substitution $z = 0$ may lead to divergence.**

• **THEOREM. For any $k \geq 1$, $(v_1, \dots, v_k) \in V^k$, and $\operatorname{Im} z > 0$ we have**

$$(2\pi i)^{m_{v_1} + \dots + m_{v_k}} I_{i\infty}^{z}(\omega_{v_k}, \dots, \omega_{v_1}) =$$

$$= (-1)^{\sum_{i=1}^{k}(m_{v_i} - 1)} \sum_{j_1=0}^{m_{v_1}-1} \sum_{j_2=0}^{m_{v_2}-1+j_1} \cdots \sum_{j_k=0}^{m_{v_k}-1+j_{k-1}} (-1)^{j_k} \times$$

$$\times \frac{(m_{v_1} - 1)!(m_{v_2} - 1 + j_1)! \dots (m_{v_k} - 1 + j_{k-1})!}{j_1! j_2! \dots j_k!} L(z; \omega_{v_k}, \dots, \omega_{v_1}; j_k, \dots, j_1)\,.$$

• **PROPOSITION. Assume that $\omega_V$ as above is a basis of a space of 1-forms invariant with respect to $g_N$. Then**

$$J^0_{i\infty}(\omega_V) = (g_{N*}(J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V)))^{-1} J^{\frac{i}{\sqrt{N}}}_{i\infty}(\omega_V)\,. \qquad (***)$$

**REMARK 1. We can mix different weights.**

**REMARK 2. Replacing the coefficients of the formal series at the r.h.s of (***) by their (convergent) representations via multiple Dirichlet series with exponents we get such representations for $I^0_{i\infty}(\omega_{v_k}, \ldots, \omega_{v_1})$ and avoid divergences at $z = 0$.**

## 7. SHUFFLE RELATIONS BETWEEN MULTIPLE DIRICHLET SERIES

• **DEFINITION. (i) Coefficients data** $C$ **of depth** $k$ **is a family of numbers** $c_{n,m}^{(j,i)}$ **indexed by two pairs of integers satisfying** $j > i \geq 0$, $j \leq k$**, and** $n > m \geq 0$**.**

**(ii) The multiple Dirichlet series associated with** $C$ **and arguments** $s_1, \ldots, s_k$ **is**

$$L_C(s_1, \ldots, s_k) := \sum_{0=u_0<u_1<\cdots<u_k\in\mathbf{Z}} \frac{\prod_{k\geq j>i\geq 0} c_{u_j,u_i}^{(j,i)}}{u_1^{s_1} u_2^{s_2} \ldots u_k^{s_k}}$$

• **EXAMPLES. (a) Assume that** $c_{n,m}^{(j,i)} = 1$ **if** $m > 0$ **or** $i > 0$ **and put** $c_{n,0}^{(j,0)} = a_n^{(j)}$. **Then**

$$L_C(s_1, \ldots, s_k) = \sum_{0<u_1<\cdots<u_k\in\mathbf{Z}} \frac{a_{u_1}^{(1)} a_{u_1}^{(2)} \ldots a_{u_k}^{(k)}}{u_1^{s_1} u_2^{s_2} \ldots u_k^{s_k}}$$

**is an ordinary multiple Dirichlet series.**

**(b) Define $c_{v,n}$ as in 3.1, and choose $v_1, \ldots, v_k \in V$. Construct the coefficients data $C$**

$$c^{(j,j-1)}_{n,m} := c_{v_j, n-m},$$

**and $c^{(j,i)}_{n,m} = 1$ otherwise. Then $L_C(m_{v_1} + j_0 - j_1, \ldots, m_{v_k} + j_{k-1} - j_k)$ becomes the formal series**

$$(2\pi i z)^{-j_k}\, L(z; \omega_{v_k}, \ldots, \omega_{v_1}; j_k, \ldots, j_1)\,|_{z=0} =$$

$$\sum_{n_1,\ldots,n_k \geq 1} \frac{c_{v_1,n_1} \ldots c_{v_k,n_k}}{n_1^{m_{v_1}+j_0-j_1}(n_1+n_2)^{m_{v_2}+j_1-j_2} \ldots (n_1+\cdots+n_k)^{m_{v_k}+j_{k-1}-j_k}}\,.$$

**if we redenote $u_j = n_1 + \cdots + n_j$.**

**• SHUFFLES AND A COMPOSITION OF THE COEFFICIENTS DATA.**

**A $(k,l,p)$–shuffle with repetitions is a pair of strictly increasing maps $\sigma = (\sigma_1, \sigma_2)$,**

$$\sigma_1 :\ [0,k] \to [0,p], \quad \sigma_2 :\ [0,l] \to [0,p]$$

**satisfying the following conditions:**

$$\sigma_1(0) = \sigma_2(0) = 0,\ \sigma_1([0,k]) \cup \sigma_2([0,l]) = [0,p],\ \max(k,l) \leq p \leq k+l.$$

• **CLAIM. Let $C = (c_{n,m}^{(j,i)})$ and $D = (d_{n,m}^{(j,i)})$ be two coefficients data of depths $k$ and $l$ respectively; $s := (s_1, \dots, s_k)$ and $t := (t_1, \dots, t_l)$ the Dirichlet arguments for the data $C$ and $D$, and $\sigma$ a $(k, l, p)$–shuffle with repetitions.**

**One can define a new coefficients data $E = C *_\sigma D$ of depth $p$ and the Dirichlet arguments for it $s +_\sigma t$ in such a way that**

$$L_C(s) \cdot L_D(t) = \sum_\sigma L_{C *_\sigma D}(s +_\sigma t)$$

**where the summation is taken over all $(k, l, p)$–shuffles with repetitions ($p$ variable).**

# DIFFERENTIALS OF THE THIRD KIND, EISENSTEIN SERIES, AND THE GENERALIZED ASSOCIATORS

*We will now assume, as in the initial Drinfeld setting, that the integration limits of the iterated integral are logarithmic singularities of the form $\Omega$. Generally, they diverge and must be regularized. The dependence on the regularization is a version of Deligne's "base point at infinity".*

• **NOTATION.**

$a$ = **a fixed point of the Riemann surface,** $z$ **a variable point.**

$r_{v,a} := \mathrm{res}_a\, \omega_v, \quad R_a := \mathrm{res}_a\, \Omega = \sum_v r_{v,a} A_v\,.$

$t_a$ **:= a local parameter at** $a$**,** $\log t_a$ **a local branch of logarithm real on** $t_a \in \mathbf{R}_+$**.**

$t_a^{R_a} := e^{R_a \log t_a}$**.**

• **DEFINITION. A local solution to** $dJ^z = \Omega(z)J^z$ **is called normalized at** $a$ **(with respect to a choice of** $t_a$**) if it is of the form** $J = K \cdot t_a^{R_a}$**, where** $K$ **is holomorphic section in a neighborhood of** $a$ **and** $K(a) = 1$**.**

• **CLAIM. (i) The normalized solution exists and is unique.**

**(ii) It depends only on the tangent vector** $\partial/\partial t_a\,|_a$**.**

**(iii) If** $J'_a = K'(t'_a)^{R_a}$ **is normalized with respect** $t'_a$**, and** $\tau_a := dt'_a/dt_a\,|_a$**, then** $J'_a = J_a \cdot \tau_a^{R_a}$**.**

• **SCATTERING OPERATORS. Given:**

– $(a, t_a), (b, t_b)$,

– $\Omega = \sum A_v \omega_v$ **with at most logarithmic singularities at** $a$**,** $b$**,**

– **a (homotopy class of) path(s) from** $a$ **to** $b$ **avoiding other singularities of** $\Omega$**.**

**We construct**

**(i) the normalized solutions** $J_a$**,** $J_b$ **analytically continued along** $\gamma$ **and THE SCATTERING OPERATOR**

$$\widetilde{J}_b^a = J_a^{-1} J_b \in \mathbf{C}\langle\langle A_V \rangle\rangle \, .$$

**Its coefficients := REGULARIZED ITERATED INTEGRALS.**

• **EXAMPLE: DRINFELD'S ASSOCIATOR. Let** $X = \mathbf{P}^1(\mathbf{C})$**,** $V = \{0, 1\}$**,**

$$\omega_0 = \frac{1}{2\pi i} \, \frac{dz}{z}, \quad \omega_1 = \frac{1}{2\pi i} \, \frac{dz}{z-1} \, .$$

**Then**

$$\Omega = A_0 \omega_0 + A_1 \omega_1$$

**has poles at** $0, 1, \infty$ **with residues** $A_0/2\pi i$**,** $A_1/2\pi i$**,** $-(A_0 + A_1)/2\pi i$ **respectively. Put** $t_0 = z, t_1 = 1 - z$**. Then** $\widetilde{J}_0^1$ **in our notation is the Drinfeld associator** $\phi_{KZ}(A_0, A_1)$**.**

**• GENERALIZED ASSOCIATORS AND ITERATED INTEGRALS OF EISENSTEIN SERIES OF WEIGHT TWO.**

$\Gamma$ **:= a congruence subgroup of the modular group**

$f_v$ **:= a basis of Eisenstein series of weight 2 wrt** $\Gamma$

$\{\omega_v =$ **push forward of** $f_v(z)dz\}$ **: 1–forms with logarithmic singularities at cusps on** $X_\Gamma$**.**

*Regularized iterated integrals of Eisenstein series of weight two between cusps provide a modular generalization of multiple zeta values.*