MODULARITY AND TOTAL ELLIPTICITY OF SOME MULTIPLE SERIES OF HYPERGEOMETRIC TYPE

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Abstract. We collect some new evidence on the validity of the conjecture that every totally elliptic hypergeometric series is modular invariant.

The following general definition of elliptic hypergeometric functions was proposed in [20].

Definition 1. *The formal (multiple) series*

$$\sum_{\lambda_1,\ldots,\lambda_n=0}^{\infty} c(\lambda_1,\ldots,\lambda_n) \quad \textit{or} \quad \sum_{0\le\lambda_1\le\ldots\le\lambda_n\le\infty} c(\lambda_1,\ldots,\lambda_n) \tag{1}$$

define elliptic hypergeometric functions if:

1) the coefficients $c(\lambda)$ are symmetric with respect to an action of the permutation group $\mathcal{S}_n$ upon the summation variables $\lambda_1,\ldots,\lambda_n$ and the free parameters entering $c(\lambda)$;

2) for all $\ell=1,\ldots,n$ the ratios

$$h_\ell(\lambda)=\frac{c(\lambda_1,\ldots,\lambda_\ell+1,\ldots,\lambda_n)}{c(\lambda_1,\ldots,\lambda_\ell,\ldots,\lambda_n)}, \tag{2}$$

are elliptic functions of λ_ℓ, $\ell=1,\ldots,n$, considered as continuous complex variables.

It is convenient to take the initial condition $c(0,\ldots,0)=1$, so that the coefficients $c(\lambda_1,\ldots,\lambda_n)$, $\lambda_j\in\mathbb{N}$, are uniquely determined from the functions $h_\ell(\lambda)$ by solving n first order recurrence relations fixed in (2).

For $n=1$, we start from a general elliptic function of order $r+1$, which can be factorized as follows [24]:

$$h(\lambda)=z\frac{\theta_1(u_0+\lambda,\ldots,u_r+\lambda;\sigma,\tau)}{\theta_1(v_0+\lambda,\ldots,v_r+\lambda;\sigma,\tau)}=z\frac{\theta(t_0q^\lambda,\ldots,t_rq^\lambda;p)}{\theta(w_0q^\lambda,\ldots,w_rq^\lambda;p)}, \tag{3}$$

where

$$\theta_1(u_0,\ldots,u_k;\sigma,\tau)=\prod_{i=0}^k\theta_1(u_i;\sigma,\tau), \qquad \theta(t_0,\ldots,t_k;p)=\prod_{i=0}^k\theta(t_i;p), \tag{4}$$

with $p=e^{2\pi i\tau}$, $\mathrm{Im}(\tau)>0$, $q=e^{2\pi i\sigma}$, and

$$\begin{aligned}\theta_1(u;\sigma,\tau) &= -i\sum_{n=-\infty}^{\infty}(-1)^n p^{(2n+1)^2/8}q^{(n+1/2)u}\\ &= p^{1/8}iq^{-u/2}\,(p;p)_\infty\,\theta(q^u;p), \quad u\in\mathbb{C},\end{aligned}$$

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being one of the standard Jacobi theta functions. The function $\theta(z;p)$ has the infinite product representation

$$\theta(z;p)=(z;p)_\infty(pz^{-1};p)_\infty, \qquad (a;p)_\infty=\prod_{n=0}^\infty(1-ap^n), \tag{5}$$

and obeys the following properties

$$\theta(pz;p)=\theta(z^{-1};p)=-z^{-1}\theta(z;p). \tag{6}$$

Sometimes it is convenient to drop for brevity bases q,p or modular parameters σ,τ in the notations for theta functions or elliptic shifted factorials. We shall use also the convenient elliptic numbers notation [3]:

$$[u;\sigma,\tau]\equiv\theta_1(u;\sigma,\tau), \quad \text{or} \quad [u]\equiv\theta_1(u).$$

The parameters u_i,v_i,t_i,w_i in (3) are connected to each other as follows

$$t_i=q^{u_i}, \qquad w_i=q^{v_i}$$

and satisfy the constraints

$$\sum_{i=0}^r(u_i-v_i)=0, \quad \text{or} \quad \prod_{i=0}^r t_i=\prod_{i=0}^r w_i, \tag{7}$$

which guarantee that the meromorphic function $h(\lambda)$ is double periodic:

$$h(\lambda+\sigma^{-1})=h(\lambda), \qquad h(\lambda+\tau\sigma^{-1})=h(\lambda).$$

Remind, that the function $[u]$ is entire, odd $[-u]=-[u]$, and doubly quasiperiodic

$$[u+\sigma^{-1}]=-[u], \qquad [u+\tau\sigma^{-1}]=-e^{-\pi i\tau-2\pi i\sigma u}[u]. \tag{8}$$

The modular $PSL(2,\mathbb{Z})$-group,

$$\tau\to\frac{a\tau+b}{c\tau+d}, \qquad \sigma\to\frac{\sigma}{c\tau+d}, \tag{9}$$

where $a,b,c,d\in\mathbb{Z}$ and $ad-bc=1$, is generated by two transformations: $\tau\to\tau+1$, $\sigma\to\sigma$, and $\tau\to-\tau^{-1}$, $\sigma\to\sigma\tau^{-1}$. In these two cases we have

$$[u;\sigma,\tau+1]=e^{\pi i/4}[u;\sigma,\tau], \tag{10}$$
$$[u;\sigma/\tau,-1/\tau]=i(-i\tau)^{1/2}e^{\pi i\sigma^2u^2/\tau}[u;\sigma,\tau], \tag{11}$$

and the sign of $(-i\tau)^{1/2}$ is fixed from the positivity of its real part condition.

Place one of the poles of $h(\lambda)$ in (3) into the point $\lambda=-1$ by setting $v_0=1$ or $w_0=q$ (this is just a convenient normalization). The resulting function $h(\lambda)$ generates the single variable elliptic hypergeometric series, which we present in the multiplicative system of notations:

$${}_{r+1}E_r\left(\begin{matrix}t_0,\dots,t_r\\ w_1,\dots,w_r\end{matrix};q,p;z\right)=\sum_{\lambda=0}^\infty\frac{\theta(t_0,t_1,\dots,t_r;p;q)_\lambda}{\theta(q,w_1,\dots,w_r;p;q)_\lambda}\,z^\lambda \tag{12}$$

with the elliptic shifted factorials defined as follows

$$\theta(t_0,\dots,t_k;p;q)_\lambda\equiv\prod_{m=0}^k\prod_{j=0}^{\lambda-1}\theta(t_mq^j;p).$$

In the additive form, elliptic shifted factorials take the form

$$[u_0,\dots,u_k]_\lambda \equiv \prod_{m=0}^{k}\prod_{j=0}^{\lambda-1}[u_m+j;\sigma,\tau].$$

Series (12) are well defined even without ellipticity constraint (7). Therefore, it is convenient to refer to the general hypergeometric type series built out from the Jacobi theta functions as *theta hypergeometric series*. In this case, the equalities (7) are called the *balancing condition*.

It is possible to extend definition (1)-(2) by relaxing the double periodicity condition to a quasiperiodicity, which is characteristic to the Jacobi theta functions (8). As a result, we get theta functions analogues of the plain and basic generalized hypergeometric series ${}_sF_r$ and ${}_s\Phi_r$ without constraints upon the parameters like (7). For further details, we refer to [20].

Theta hypergeometric series in (12) are called *totally elliptic* if the series coefficients $c(\lambda)$ are elliptic not only in the variable λ, but in all free parameters among u_i, v_i as well. As shown in [20], the requirement of total ellipticity leads to the following additional constraints upon the parameters:

$$u_0+1=u_1+v_1=\ldots=u_r+v_r, \quad \text{or} \quad qt_0=t_1w_1=\ldots=t_rw_r, \tag{13}$$

which are well-known as the *well-poisedness* conditions for plain and basic hypergeometric series. This gives a kind of explanation of the origin of this notion itself.

At the elliptic level, the *very-well-poisedness* condition differs from the ones used in the plain and basic hypergeometric cases [11]. Namely, it demands that, in addition to the constraints (13), four parameters are placed into four points of the second order of the elliptic function $h(\lambda)$ around a distinguished point:

$$\begin{aligned} &u_{r-3}=\frac{1}{2}u_0+1, \quad u_{r-2}=\frac{1}{2}u_0+1-\frac{1}{2\sigma}, \\ &u_{r-1}=\frac{1}{2}u_0+1-\frac{\tau}{2\sigma}, \quad u_r=\frac{1}{2}u_0+1+\frac{1+\tau}{2\sigma}, \end{aligned} \tag{14}$$

or, in the multiplicative form,

$$\begin{aligned} &t_{r-3}=t_0^{1/2}q, \quad t_{r-2}=-t_0^{1/2}q, \\ &t_{r-1}=t_0^{1/2}qp^{-1/2}, \quad t_r=-t_0^{1/2}qp^{1/2}. \end{aligned} \tag{15}$$

As a result, the very-well-poised theta hypergeometric series ${}_{r+1}E_r$ take the form

$${}_{r+1}E_r=\sum_{\lambda=0}^{\infty}\frac{\theta(t_0q^{2\lambda};p)}{\theta(t_0;p)}\prod_{m=0}^{r-4}\frac{\theta(t_m;p;q)_\lambda}{\theta(qt_0/t_m;p;q)_\lambda}(-qz)^\lambda. \tag{16}$$

Series of such type (with a different counting of the number of parameters and $z=-1$) have been considered for the first time by Frenkel and Turaev [9]. Their consideration was inspired by a rich set of exactly solvable statistical mechanics models built by Date et al [3]. In a different context, within a program on systematic reconsideration of special functions of one variable as “self-similar” functions (see, e.g. the review [19]), elliptic hypergeometric series have been derived by Zhedanov and the author as solutions of particular three term recurrence relation describing new families of biorthogoanl rational functions [22]. For $p\to 0$, series (16) reduce to the very-well-poised ${}_{r-1}\Phi_{r-2}$ basic hypergeometric series [11] of the argument $-qz$ (note the shift of the index r by -2).

Frenkel and Turaev [9] proved the following summation formula:

$$\sum_{\lambda=0}^{N} q^{\lambda}\frac{\theta(t_0q^{2\lambda};p)}{\theta(t_0;p)}\prod_{r=0}^{5}\frac{\theta(t_r;p;q)_\lambda}{\theta(qt_0t_r^{-1};p;q)_\lambda}
= \frac{\theta(qt_0;p;q)_N\prod_{1\leq r<s\leq 3}\theta(qt_0/t_rt_s;p;q)_N}{\theta(qt_0/t_1t_2t_3;p;q)_N\prod_{r=1}^{3}\theta(qt_0/t_r;p;q)_N}, \tag{17}$$

where $\prod_{i=1}^{5}t_i = qt_0^2$ and $t_4 = q^{-N}$, $N\in\mathbb{N}$. In the classification introduced in [20], this corresponds to a closed form expression for the terminating very-well-poised balanced ${}_{10}E_9$ theta hypergeometric series at $z=-1$.

In the multivariable case, the general form of the elliptic hypergeometric series is not established yet. There are several types of such series in the literature leading to explicit summation formulae, generalizing (17). They extend the rich set of identities for series of hypergeometric type that were found first by Biedenharn, Holman, and Louck (see, e.g. [10]) and lifted further to the basic hypergeometric series level by Milne (see, e.g. [13, 14]) and Gustafson (see, e.g. [4, 12]). The first two multiple elliptic hypergeometric series were proposed by Warnaar [23], for one of which he has proved an extension of the Frenkel-Turaev sum. For the second type of series, Warnaar conjectured a similar summation formula. It was proved at the plain and q-hypergeometric series levels by van Diejen and the author in [6] (some simplified versions of this sum were proven earlier in [5]). A partial proof of the elliptic sum was given as well. It was proved completely by Rosengren with the help of a different technique in [15].

The third type of series was proposed in [7] together with an elliptic extension of the multiple Jackson sum of [4, 14], which was proved partially on the basis of modular group invariance property. In the present paper, we consider two more types of multiple series, analyze some of their properties, discuss the corresponding Jackson type summation formulae, and show their partial validity using the modularity arguments. In an independent detailed study, Rosengren [16] arrived to the latter sums as well, proved them (together with the sum proposed in [7]), and analyzed some of their consequences.

As in the single summation series example, multiple theta hypergeometric series are called totally elliptic if their coefficients $c(\lambda)$ are elliptic in the summation indices λ_ℓ together with all free parameters entering the arguments of θ_1-functions. The main purpose of the present paper consists in providing new evidence of the validity of the following conjecture.

Conjecture 2. *Every totally elliptic hypergeometric series is automatically modular invariant.*

For $n=1$ series (in the general situation) and C_n sums discussed in [6, 7, 15, 23], this statement was proved in [20]. We prove it below for the very first example of multiple elliptic hypergeometric series summation formula found by Warnaar in [23] (it was missed inadvertently in the analysis of [20]) and for two above-mentioned new examples of elliptic hypergeometric series. This note should be considered as a complement to [20].

In [23], Warnaar proved the following elliptic analogue of the Schlosser [18] multivariable C_n Jackson sum:

$$\sum_{\lambda_1,\dots,\lambda_n=0}^{N} q^{\sum_{j=1}^n j\lambda_j} \prod_{1\le i<j\le n} \frac{\theta(t_it_j^{-1}q^{\lambda_i-\lambda_j}, t_it_jq^{\lambda_i+\lambda_j};p)}{\theta(t_it_j^{-1}, t_it_j;p)}$$
$$\times \prod_{j=1}^{n} \frac{\theta(t_j^2q^{2\lambda_j};p)\theta(t_j^2, bt_j, ct_j, dt_j, et_j, q^{-N};p;q)_{\lambda_j}}{\theta(t_j^2;p)\theta(q, t_jq/b, t_jq/c, t_jq/d, t_jq/e, t_j^2q^{N+1};p;q)_{\lambda_j}} \tag{18}$$
$$= \prod_{1\le i<j\le n} \frac{\theta(t_it_jq^N;p)}{\theta(t_it_j;p)} \prod_{j=1}^{n} \frac{\theta(t_j^2q, q^{2-j}/bc, q^{2-j}/bd, q^{2-j}/cd;p;q)_N}{\theta(t_j^{-1}q^{2-n}/bcd, t_jq/b, t_jq/c, t_jq/d;p;q)_N},$$

where parameters satisfy the balancing constraint $bcde = q^{N-n+2}$.

Remark 1. The summation formulae given in [18, 23] contain one more parameter a, but it can be removed by rescaling the variables t_j, b, c, d, e.

Theorem 3. *Theta hypergeometric series appearing at the left-hand side of (18) is totally elliptic and modular invariant.*

Proof. Consider ratios of successive series coefficients in (18):

$$h_\ell(\lambda) = q^\ell \frac{\theta(t_\ell^2q^{2\lambda_\ell+2};p)}{\theta(t_\ell^2q^{2\lambda_\ell};p)} \prod_{i=1}^{\ell-1} \frac{\theta(t_it_\ell^{-1}q^{\lambda_i-\lambda_\ell-1}, t_it_\ell q^{\lambda_i+\lambda_\ell+1};p)}{\theta(t_it_\ell^{-1}q^{\lambda_i-\lambda_\ell}, t_it_\ell q^{\lambda_i+\lambda_\ell};p)}$$
$$\times \prod_{j=\ell+1}^{n} \frac{\theta(t_\ell t_j^{-1}q^{\lambda_\ell-\lambda_j+1}, t_\ell t_jq^{\lambda_\ell+\lambda_j+1};p)}{\theta(t_\ell t_j^{-1}q^{\lambda_\ell-\lambda_j}, t_\ell t_jq^{\lambda_\ell+\lambda_j};p)} \tag{19}$$
$$\times \frac{\theta(t_\ell^2q^{\lambda_\ell}, bt_\ell q^{\lambda_\ell}, ct_\ell q^{\lambda_\ell}, dt_\ell q^{\lambda_\ell}, et_\ell q^{\lambda_\ell}, q^{-N+\lambda_\ell};p)}{\theta(q^{\lambda_\ell+1}, t_\ell q^{\lambda_\ell+1}/b, t_\ell q^{\lambda_\ell+1}/c, t_\ell q^{\lambda_\ell+1}/d, t_\ell q^{\lambda_\ell+1}/e, t_\ell^2q^{N+1+\lambda_\ell};p)}.$$

Prove first that this function is elliptic in λ_k. For this, it is sufficient to replace q^{λ_k} by pq^{λ_k}, apply the rule $\theta(pz;p) = -z^{-1}\theta(z;p)$, and to check cancellation of all extra factors appearing from that. For $k < \ell$ or $k > \ell$, the situation is very simple and we skip its consideration. For $k = \ell$, the first theta functions fraction in (19) yields the factor q^{-4}, the next two products, $q^{2(1-n)}$. The last fraction involving parameters $b, \dots, e$ generates the product $q^{2N+6}(bcde)^{-2}$. As a result, the whole resulting multiplier is equal to 1 because of the balancing condition.

To consider ellipticity in the parameters and modular invariance of $h_\ell(\lambda)$, make the replacements $t_j, b, c, d, e \to q^{g_j}, q^\beta, q^\gamma, q^\delta, q^\epsilon$. Then we can rewrite (19) in terms of the elliptic numbers as follows

$$h_\ell(\lambda) = \frac{[2g_\ell + 2\lambda_\ell + 2]}{[2g_\ell + 2\lambda_\ell]} \prod_{i=1, i\ne\ell}^{n} \frac{[g_i - g_\ell + \lambda_i - \lambda_\ell - 1, g_i + g_\ell + \lambda_i + \lambda_\ell + 1]}{[g_i - g_\ell + \lambda_i - \lambda_\ell, g_i + g_\ell + \lambda_i + \lambda_\ell]}$$
$$\times \frac{[2g_\ell + \lambda_\ell, \beta + g_\ell + \lambda_\ell, \gamma + g_\ell + \lambda_\ell]}{[\lambda_\ell + 1, g_\ell + \lambda_\ell + 1 - \beta, g_\ell + \lambda_\ell + 1 - \gamma]}$$
$$\times \frac{[\delta + g_\ell + \lambda_\ell, \epsilon + g_\ell + \lambda_\ell, \lambda_\ell - N]}{[g_\ell + \lambda_\ell + 1 - \delta, g_\ell + \lambda_\ell + 1 - \epsilon, 2g_\ell + N + 1 + \lambda_\ell]}. \tag{20}$$

From this representation it follows that, if we rewrite (18) in terms of the elliptic numbers, then the first factor $q^{\sum_{j=1}^n j\lambda_j}$ disappears. As a result, simultaneous permutation of the variables λ_j, λ_k and parameters g_j, g_k does not change the series coefficients $c(\lambda)$ (there are equal numbers of sign changes in the numerator and denominator of $c(\lambda)$ stemming from the antisymmetry $[-u]=-[u]$). Thus, a symmetric group invariance is naturally implemented inside these series.

Ellipticity in g_k of (20) follows from the invariance of (19) under the shift $t_k \to pt_k$. For $k<\ell$ or $k>\ell$ this is easy to verify. For $k=\ell$, the first fraction of theta functions yields the multiplier q^{-4}, the next two products give $q^{2(1-n)}$. The rest of h_ℓ generates the multiplier $q^{2N+6}(bcde)^{-2}$, so that the total multiplication factor becomes equal to 1 due to the balancing condition. In order to check ellipticity in β (or, equivalently, in γ or δ), make replacements $b\to pb, e\to p^{-1}e$ in (19). The only contributions come from the last line expression and, since $\theta(p^{-1}z;p) = -p^{-1}z\theta(z;p)$, they are seen to cancel each other.

Turn now to modular invariance. Since there is an equal number of θ_1-functions in the numerator and denominator of h_ℓ, the numerical factors appearing from the transformations $\tau\to\tau+1, \tau\to -1/\tau$ in (10), (11) cancel out. Thus, it is sufficient to verify that the difference of squares of arguments of elliptic numbers in the numerator and denominator vanishes. For the first fraction of theta functions in (20), such a difference is equal to $4+8(g_\ell+\lambda_\ell)$. The next product yields $2(1+2(g_\ell+\lambda_\ell))(n-1)$. The rest of h_ℓ generates the difference

$$2(\beta+\gamma+\delta+\epsilon)(1+2(g_\ell+\lambda_\ell)) - 4 - 4(g_\ell+\lambda_\ell)(N+3) - 2(N+1).$$

The total sum of these expressions is equal to zero, which proves modular invariance of h_ℓ-functions. In the Eichler-Zagier terminology [8], this means that all functions $h_\ell(\lambda)$, $\ell=1,\ldots,n$, in (20) are meromorphic Jacobi forms of the weight 0 and index 0. All together they form a structure associated with the root system C_n.

Although coincidence of the left and right hand sides of (18) was proved in [23], for completeness, we have verified that the expression on the right-hand side of (18) is an elliptic function of all parameters $g_j, \beta, \gamma, \delta$ and it is modular invariant. □

During the work on the elliptic Bailey chain [21], the author came across the Bhatnagar-Schlosser paper [2], where various Bailey transformation formulae for multiple ${}_{10}\Phi_9$ series were discussed. The Bailey chain built in [21] deals with an infinite sequence of such transformations for single variable elliptic hypergeometric series. A comparison of [2, 21] has naturally lead to a conjecture that much of the corresponding results of [2] should have elliptic analogues. In particular, this conjecture assumed the validity of the following elliptic generalizations of the Milne [13], Schlosser [17], and Bhatnagar [1] multivariable analogues of the Jackson sum.

An elliptic Milne A_n sum:

$$\begin{aligned}
&\sum_{\substack{0\le\lambda_j\le N_j\\ j=1,\ldots,N_j}} q^{\sum_{j=1}^n j\lambda_j} \prod_{j=1}^n \frac{\theta(t_jq^{\lambda_j+|\lambda|};p)}{\theta(t_j;p)} \prod_{1\le i<j\le n} \frac{\theta(t_it_j^{-1}q^{\lambda_i-\lambda_j};p)}{\theta(t_it_j^{-1};p)} \\
&\quad\times \prod_{i,j=1}^n \frac{\theta(t_it_j^{-1}q^{-N_j};p;q)_{\lambda_i}}{\theta(qt_it_j^{-1};p;q)_{\lambda_i}} \prod_{j=1}^n \frac{\theta(t_j;p;q)_{|\lambda|}}{\theta(t_jq^{1+N_j};p;q)_{|\lambda|}} \\
&\quad\times \frac{\theta(b,c;p;q)_{|\lambda|}}{\theta(q/d,q/e;p;q)_{|\lambda|}} \prod_{j=1}^n \frac{\theta(dt_j,et_j;p;q)_{\lambda_j}}{\theta(t_jq/b,t_jq/c;p;q)_{\lambda_j}} \\
&\quad= \frac{\theta(q/bd,q/cd;p;q)_{|N|}}{\theta(q/d,q/bcd;p;q)_{|N|}} \prod_{j=1}^n \frac{\theta(t_jq,t_jq/bc;p;q)_{N_j}}{\theta(t_jq/b,t_jq/c;p;q)_{N_j}},
\end{aligned} \tag{21}$$

where $|\lambda|=\lambda_1+\cdots+\lambda_n$, $|N|=N_1+\cdots+N_n$, and $bcde=q^{1+|N|}$.

An elliptic Bhatnagar-Schlosser D_n sum (as stated in [2], the D_n sums discussed in [1, 17] are essentially equivalent to each other):

$$\sum_{\substack{0\leq\lambda_j\leq N_j\\ j=1,\dots,N_j}} q^{\sum_{j=1}^n j\lambda_j}\prod_{j=1}^n\frac{\theta(t_jq^{\lambda_j+|\lambda|};p)}{\theta(t_j;p)}\prod_{1\leq i<j\leq n}\frac{\theta(t_it_j^{-1}q^{\lambda_i-\lambda_j};p)}{\theta(t_it_j^{-1};p)}$$
$$\times\prod_{1\leq i<j\leq n}\theta(t_it_jq/bcd;p;q)^{-1}_{\lambda_i+\lambda_j}\prod_{i,j=1}^n\theta(t_it_jq^{1+N_j}/bcd;p;q)_{\lambda_i}$$
$$\times\prod_{i,j=1}^n\frac{\theta(t_it_j^{-1}q^{-N_j};p;q)_{\lambda_i}}{\theta(qt_it_j^{-1};p;q)_{\lambda_i}}\prod_{j=1}^n\frac{\theta(t_j;p;q)_{|\lambda|}\theta(bcdt_j^{-1};p;q)_{|\lambda|-\lambda_j}}{\theta(t_jq^{1+N_j},bcdt_j^{-1}q^{-N_j};p;q)_{|\lambda|}}$$
$$\times\frac{\theta(b,c,d;p;q)_{|\lambda|}}{\prod_{j=1}^n\theta(t_jq/b,t_jq/c,t_jq/d;p;q)_{\lambda_j}}$$
$$=\prod_{j=1}^n\frac{\theta(t_jq,t_jq/bc,t_jq/bd,t_jq/cd;p;q)_{N_j}}{\theta(t_jq/bcd,t_jq/b,t_jq/c,t_jq/d;p;q)_{N_j}}. \tag{22}$$

Remark 2. In the presentation of Milne A_n sum given in the appendix of [2] there is an extra parameter a, but it can be removed by rescaling the variables t_j, d, and e. Similarly, in the Schlosser version of the D_n sum given there, the extra parameter a can be removed by rescaling t_j (the same is true for the Bhatnagar version of this sum).

Theorem 4. *The A_n and D_n theta hypergeometric series standing on the left-hand sides of the equalities (21) and (22) are totally elliptic and modular invariant. The corresponding right-hand side expressions obey the same properties.*

Proof. Consider the functions $h_\ell(\lambda)$ in the A_n case:

$$h_\ell(\lambda)=q^\ell\prod_{\substack{j=1\\ j\neq\ell}}^n\frac{\theta(t_jq^{\lambda_j+|\lambda|+1};p)}{\theta(t_jq^{\lambda_j+|\lambda|};p)}\frac{\theta(t_\ell q^{\lambda_\ell+|\lambda|+2};p)}{\theta(t_\ell q^{\lambda_\ell+|\lambda|};p)}$$
$$\times\prod_{i=1}^{\ell-1}\frac{\theta(t_it_\ell^{-1}q^{\lambda_i-\lambda_\ell-1};p)}{\theta(t_it_\ell^{-1}q^{\lambda_i-\lambda_\ell};p)}\prod_{j=\ell+1}^n\frac{\theta(t_\ell t_j^{-1}q^{\lambda_\ell-\lambda_j+1};p)}{\theta(t_\ell t_j^{-1}q^{\lambda_\ell-\lambda_j};p)}$$
$$\times\prod_{j=1}^n\frac{\theta(t_\ell t_j^{-1}q^{\lambda_\ell-N_j},t_jq^{|\lambda|};p)}{\theta(t_\ell t_j^{-1}q^{\lambda_\ell+1},t_jq^{1+N_j+|\lambda|};p)}$$
$$\times\frac{\theta(bq^{|\lambda|},cq^{|\lambda|},dt_\ell q^{\lambda_\ell},et_\ell q^{\lambda_\ell};p)}{\theta(q^{|\lambda|+1}/d,q^{|\lambda|+1}/e,t_\ell q^{\lambda_\ell+1}/b,t_\ell q^{\lambda_\ell+1}/c;p)}. \tag{23}$$

Prove first ellipticity of $h_\ell(\lambda)$ in λ_k. For $k<\ell$ or $k>\ell$, the shift $q^{\lambda_k}\to pq^{\lambda_k}$ in h_ℓ gives the following extra multipliers: q^{-n} and q^{-2} from the first two fractions of theta functions, respectively; the next two products yield q; from the product $\prod_{j=1}^n$ on the third line we get the contribution $q^{n+|N|}$; and the rest of h_ℓ generates the factor $q^{1-|N|}$. The total multiplier is equal to 1.

Similarly, replacing q^{λ_ℓ} by pq^{λ_ℓ}, we get successively the multipliers q^{1-n}, q^{-4}, $q^{1-\ell}$, $q^{\ell-n}$, $q^{2n+2|N|}$, and $q^{2-2|N|}$, whose total product equals 1. Thus the coefficients $c(\lambda)$ in the left-hand side of (21) are elliptic functions of all λ_k.

In order to see that there exists a natural action of the symmetric group, we make substitutions $t_j, b, c, d, e \to q^{g_j}, q^{\beta}, q^{\gamma}, q^{\delta}, q^{\epsilon}$, and rewrite sum (21) in terms of the elliptic numbers:

$$\sum_{\substack{0\leq\lambda_j\leq N_j\\ j=1,\dots,N_j}} \prod_{j=1}^{n} \frac{[g_j+\lambda_j+|\lambda|]}{[g_j]} \prod_{1\leq i<j\leq n} \frac{[g_i-g_j+\lambda_i-\lambda_j]}{[g_i-g_j]}$$
$$\times \prod_{i,j=1}^{n} \frac{[g_i-g_j-N_j]_{\lambda_i}}{[1+g_i-g_j]_{\lambda_i}} \prod_{j=1}^{n} \frac{[g_j]_{|\lambda|}}{[g_j+1+N_j]_{|\lambda|}}$$
$$\times \frac{[\beta,\gamma]_{|\lambda|}}{[1-\delta,1-\epsilon]_{|\lambda|}} \prod_{j=1}^{n} \frac{[\delta+g_j,\epsilon+g_j]_{\lambda_j}}{[g_j+1-\beta,g_j+1-\gamma]_{\lambda_j}} \tag{24}$$
$$= \frac{[1-\beta-\delta,1-\gamma-\delta]_{|N|}}{[1-\delta,1-\beta-\gamma-\delta]_{|N|}} \prod_{j=1}^{n} \frac{[1+g_j,1+g_j-\beta-\gamma]_{N_j}}{[g_j+1-\beta,g_j+1-\gamma]_{N_j}}.$$

Consider the permutation of (λ_k, g_k, N_k) with (λ_m, g_m, N_m) for some fixed k and m. There are only two factors which require some consideration. The first one, given by the product $\prod_{1\leq i<j\leq n}$ appears to be symmetric due to an equal number of sign changes in the numerator and denominator (in the same way as it was for the sum (18)). The product $\prod_{i,j=1}^{n}$ appears to be symmetric because it is taken over all possible values of both indices i and j. Thus, according the definition given at the beginning, we are dealing with particular elliptic hypergeometric series.

Prove now the total ellipticity of functions h_ℓ. Ellipticity in β (or γ), or the invariance of (23) under the shifts $b \to pb, e \to p^{-1}e$, follows from the cancellation of multipliers $-b^{-1}q^{-|\lambda|}$, $-p^{-1}et_\ell q^{\lambda_\ell}$ and $-q^{|\lambda|+1}e^{-1}$, $-pbt_\ell^{-1}q^{-\lambda_\ell-1}$ coming from the numerator and denominator of the last line expression, respectively. Similar situation holds for the shifts $d \to pd, e \to p^{-1}e$, which proves ellipticity in δ.

An essentially more complicated computation is needed in order to see, that $c(\lambda)$ in (24) is modular invariant. Consider for this the difference of squares of arguments of elliptic numbers in the numerator and denominator of h_ℓ. The first ratio of theta functions gives $n-1+2(|g|-g_\ell+n|\lambda|-\lambda_\ell)$, where $|g| = g_1+\ldots+g_n$. The second ratio yields $4(1+|\lambda|+\lambda_\ell+g_\ell)$. The next two products generate the difference $n-1+2n(g_\ell+\lambda_\ell)-2|g|-2|\lambda|$. The product $\prod_{j=1}^{n}$ of ratios of theta functions contributes $-2(n+|N|)(1+g_\ell+\lambda_\ell+|\lambda|)$. For the rest of h_ℓ, we get

$$(\beta+|\lambda|)^2+(\gamma+|\lambda|)^2+(\delta+g_\ell+\lambda_\ell)^2+(\epsilon+g_\ell+\lambda_\ell)^2-(|\lambda|+1-\delta)^2$$
$$-(|\lambda|+1-\epsilon)^2-(g_\ell+\lambda_\ell+1-\beta)^2-(g_\ell+\lambda_\ell+1-\gamma)^2.$$

The total sum of all these cumbersome expressions is equal to zero, which means that h_ℓ is invariant under transformations (10), (11) generating the full modular group.

As far as the total ellipticity and modularity of the expression standing on the right-hand side of (24) are concerned, they are easily checked. Indeed, the invariance of the right-hand side of (21) under the shifts $t_j \to pt_j$ or $d \to pd$ (i.e., ellipticity in g_j or δ) is obvious. The shift $b \to pb$ generates from the elliptic shifted factorials $\theta(\ldots)_{|N|}$ the factor $c^{|N|}$, which is cancelled by the extra multipliers coming from the elliptic shifted factorials $\theta(\ldots)_{N_j}$. By symmetry, this means ellipticity in both β and γ.

As to the difference of squares of the arguments of elliptic numbers in the numerator and denominator of the right-hand side of (24), the ratio of $[\dots]_{|N|}$ elliptic shifted factorials gives (after simplifications) $-2|N|\beta\gamma$. The product of ratios of $[\dots]_{N_j}$ yields $\sum_{j=1}^n N_j((1+g_j)^2+(1+g_j-\beta-\gamma)^2-(1+g_j-\beta)^2-(1+g_j-\gamma)^2)$, which is seen to cancel the previous expression. This proves the claimed modularity property.

Turn now to the D_n sum (22). Ratios of the series coefficients in the left-hand side look as follows

$$\begin{aligned} h_\ell(\lambda) = q^\ell \prod_{\substack{j=1\\ j\neq \ell}}^n \frac{\theta(t_jq^{\lambda_j+|\lambda|+1}, bcdt_j^{-1}q^{|\lambda|-\lambda_j};p)}{\theta(t_jq^{\lambda_j+|\lambda|}, t_\ell t_j q^{\lambda_j+\lambda_\ell+1}/bcd;p)} \frac{\theta(t_\ell q^{\lambda_\ell+|\lambda|+2};p)}{\theta(t_\ell q^{\lambda_\ell+|\lambda|};p)} \\ \times \prod_{i=1}^{\ell-1} \frac{\theta(t_it_\ell^{-1}q^{\lambda_i-\lambda_\ell-1};p)}{\theta(t_it_\ell^{-1}q^{\lambda_i-\lambda_\ell};p)} \prod_{j=\ell+1}^n \frac{\theta(t_\ell t_j^{-1}q^{\lambda_\ell-\lambda_j+1};p)}{\theta(t_\ell t_j^{-1}q^{\lambda_\ell-\lambda_j};p)} \\ \times \prod_{j=1}^n \frac{\theta(t_\ell t_j q^{1+N_j+\lambda_\ell}/bcd, t_\ell t_j^{-1}q^{-N_j+\lambda_\ell}, t_jq^{|\lambda|};p)}{\theta(t_\ell t_j^{-1}q^{\lambda_\ell+1}, t_jq^{1+N_j+|\lambda|}, bcdt_j^{-1}q^{|\lambda|-N_j};p)} \\ \times \frac{\theta(bq^{|\lambda|}, cq^{|\lambda|}, dq^{|\lambda|};p)}{\theta(t_\ell q^{\lambda_\ell+1}/b, t_\ell q^{\lambda_\ell+1}/c, t_\ell q^{\lambda_\ell+1}/d;p)}. \end{aligned} \tag{25}$$

The shift $q^{\lambda_k} \to pq^{\lambda_k}$ does not change these functions, that is, they are elliptic in the λ_k variables. For $k<\ell$ or $k>\ell$, it follows from the cancellation of quasiperiodicity multipliers

$$q^{-n}, \qquad -\frac{t_kt_\ell}{bcd}q^{1+\lambda_k+\lambda_\ell}, \qquad \prod_{\substack{j=1\\ j\neq k,\ell}} \left(-\frac{t_j}{bcd}q^{\lambda_j-|\lambda|}\right),$$

appearing from the product $\prod_{j=1,j\neq\ell}^n$, the factor q^{-2} coming from the next ratio of theta functions, q (from the next two products), and $\prod_{j=1}^n(-bcdt_j^{-1}q^{|\lambda|+1})$, $-q^{-3|\lambda|}/bcd$ (from the last two lines expressions, respectively). For $k=\ell$ we have multipliers q^{1-n} (from the first ratio), $\prod_{j=1,j\neq\ell}^n(t_j^2t_\ell q^{2\lambda_j+\lambda_\ell-|\lambda|+1}/b^2c^2d^2)$ (from the second ratio), q^{-4} (third ratio), q^{1-n} (next two products), and

$$\prod_{j=1}^n \frac{b^2c^2d^2}{t_\ell t_j^2}q^{1+|\lambda|-\lambda_\ell}, \qquad \frac{t_\ell^3}{b^2c^2d^2}q^{3(1+\lambda_\ell-|\lambda|)}$$

(from the last two lines, respectively). Their product equals 1, that is, the functions h_ℓ are elliptic in all λ_k.

After rewriting sum (22) in terms of the elliptic numbers

$$\sum_{\substack{0\le\lambda_j\le N_j\\ j=1,\dots,N_j}} \prod_{j=1}^n \frac{[g_j+\lambda_j+|\lambda|]}{[g_j]} \prod_{1\le i<j\le n} \frac{[g_i-g_j+\lambda_i-\lambda_j]}{[g_i-g_j]}$$
$$\times \frac{\prod_{i,j=1}^n [g_i+g_j+1+N_j-\beta-\gamma-\delta]_{\lambda_i}}{\prod_{1\le i<j\le n}[g_i+g_j+1-\beta-\gamma-\delta]_{\lambda_i+\lambda_j}}$$
$$\times \prod_{i,j=1}^n \frac{[g_i-g_j-N_j]_{\lambda_i}}{[1+g_i-g_j]_{\lambda_i}} \prod_{j=1}^n \frac{[g_j]_{|\lambda|}[\beta+\gamma+\delta-g_j]_{|\lambda|-\lambda_j}}{[g_j+1+N_j,\beta+\gamma+\delta-g_j-N_j]_{|\lambda|}}$$
$$\times \frac{[\beta,\gamma,\delta]_{|\lambda|}}{\prod_{j=1}^n [g_j+1-\beta, g_j+1-\gamma, g_j+1-\delta]_{\lambda_j}}$$
$$= \prod_{j=1}^n \frac{[g_j+1, g_j+1-\beta-\gamma, g_j+1-\beta-\delta, g_j+1-\gamma-\delta]_{N_j}}{[g_j+1-\beta-\gamma-\delta, g_j+1-\beta, g_j+1-\gamma, g_j+1-\delta]_{N_j}}, \qquad (26)$$

permutational symmetry between (λ_k, g_k, N_k) and (λ_m, g_m, N_m), for fixed k and m, becomes evident.

Ellipticity in the variables g_k, $k\neq\ell$, follows from the cancellation of the following sequence of multipliers in (25), appearing after the shift $t_k\to pt_k$: q^{-1}, $-t_kt_\ell q^{1+\lambda_k+\lambda_\ell}/bcd$, $-p^{-1}bcdq^{|\lambda|-\lambda_k}t_k^{-1}$, q, $pq^{-1-|\lambda|-\lambda_\ell}t_\ell^{-1}$. Similarly, for g_ℓ we get the multipliers (in the order of appearance) $\prod_{j=1,j\neq\ell}^n(-t_jt_\ell q^{1+\lambda_j+\lambda_\ell}/bcd)$, q^{-2}, q^{1-n}, $\prod_{j=1,j\neq\ell}^n(-bcdq^{-\lambda_\ell}/t_jt_\ell)$, $-bcdq^{-1-|\lambda|-2\lambda_\ell}/t_\ell^3$, $-t_\ell^3q^{3+3\lambda_\ell}/bcd$, which total in 1. Ellipticity in β (or γ,δ) is confirmed by analogous calculation, which we skip for brevity.

The proof of the modularity of h_ℓ requires essentially longer computations. Skipping the original expression for the difference of squares of arguments of elliptic numbers in h_ℓ, the first chain of obvious simplifications in it results in the sum

$$2(1+|\lambda|+g_\ell+\lambda_\ell)(\lambda_\ell-|\lambda|-N_\ell+1)+(2g_\ell-\beta-\gamma-\delta+N_\ell+1+\lambda_\ell)^2$$
$$-(g_\ell-\beta-\gamma-\delta+N_\ell-|\lambda|)^2+\sum_{\rho=\beta,\gamma,\delta}\left((\rho+|\lambda|)^2-(g_\ell+\lambda_\ell+1-\rho)^2\right),$$

which is equal to zero.

It is more easy to work with the right-hand side of (22). Its ellipticity in parameters follows from the fact that the ratio

$$\frac{[g_j, g_j-\beta-\gamma, g_j-\beta-\delta, g_j-\gamma-\delta]}{[g_j-\beta-\gamma-\delta, g_j-\beta, g_j-\gamma, g_j-\delta]}$$

is elliptic in all variables. The modular invariance is guaranteed by the following equality

$$g_j^2+(g_j-\beta-\gamma)^2+(g_j-\beta-\delta)^2+(g_j-\gamma-\delta)^2$$
$$=(g_j-\beta-\gamma-\delta)^2+(g_j-\beta)^2+(g_j-\gamma)^2+(g_j-\delta)^2.$$

Thus, we have proved total ellipticity and modularity of both sides of both A_n and D_n sums. □

Corollary 5. *The Warnaar C_n, elliptic Milne A_n, and elliptic Bhatnagar-Schlosser D_n summation formulae (18), (21), and (22), respectively, are valid in the small σ expansion up to the order σ^{12}.*

Proof. For a sufficiently small absolute value of the parameter σ, both sides of the equalities (18), (21), (22) are holomorphic on the upper half plane of the modular parameter τ, $\mathrm{Im}(\tau) > 0$. Being $PSL(2,\mathbb{Z})$ invariant, they have σ-expansions of the form $\sum_{k=0}^{\infty} m_k(\tau)\sigma^{2k}$, where $m_k(\tau)$ are modular forms of the weight $2k$. For $\mathrm{Im}(\tau) \to \infty$ (i.e. $p \to 0$), the differences of expressions on the left-hand and right-hand sides of these equalities vanish. Therefore in the expansions of these differences in σ, only cusp forms may appear. Since there are no modular cusp forms of the weight smaller than 12, the C_n, A_n, D_n summation formulae under consideration are valid in the σ-expansion up to the terms of the order σ^{12}. These arguments are repetive to the ones, which were used already in [6, 7]. □

In addition to providing substantial evidence in favor of the main conjecture stated at the beginning, the results of this note show that the properties of total ellipticity and modular invariance give a quite simple tool for conjecturing elliptic generalizations of certain basic hypergeometric series identities. The complete proofs of the C_n sum of [7] and A_n, D_n sums (21), (22), reached by Rosengren in [16], strengthen the latter statement. It is natural to pose here a question on the existence of similar identities for other root systems.

We would like to conclude by a remark that the approach of [20] to elliptic hypergeometric series can be generalized to some extent by employing multidimensional Riemann theta functions. For this, it is necessary to demand that the functions $h_\ell(\lambda)$ in (2) are Abelian functions (i.e. multiple periodic functions living on a g-dimensional complex torus) or rational functions on a compact Riemann surface of arbitrary genus. We hope to consider elsewhere these possibilities in detail.

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