# VÉNÉREAU POLYNOMIALS AND RELATED FIBER BUNDLES

SHULIM KALIMAN AND MIKHAIL ZAIDENBERG

Abstract. The Vénéreau polynomials

$$v_n := y + x^n(xz + y(yu + z^2)), \qquad n \geq 1,$$

on $\mathbb{A}^4_{\mathbb{C}}$ have all fibers isomorphic to the affine space $\mathbb{A}^3_{\mathbb{C}}$. Moreover, for all $n \geq 1$ the map $(v_n, x) : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ yields a flat family of affine planes over $\mathbb{A}^2_{\mathbb{C}}$. In the present note we show that over the punctured plane $\mathbb{A}^2_{\mathbb{C}} \backslash \{\bar{0}\}$, this family is a fiber bundle. This bundle is trivial if and only if $v_n$ is a variable of the ring $\mathbb{C}[x][y, z, u]$ over $\mathbb{C}[x]$.

It is an open question whether $v_1$ and $v_2$ are variables of the polynomial ring $\mathbb{C}^{[4]} = \mathbb{C}[x, y, z, u]$, whereas S. Vénéreau established that $v_n$ is indeed a variable of $\mathbb{C}[x][y, z, u]$ over $\mathbb{C}[x]$ for $n \geq 3$. In this note we give another proof of Vénéreau's result based on the above equivalence. We also discuss some other equivalent properties, as well as the relations to the Abhyankar-Sathaye Embedding Problem and to the Dolgachev-Weisfeiler Conjecture on triviality of flat families with fibers affine spaces.

In [KVZ1, KVZ2] polynomials in four variables of the form

$$p = f(x, y)u + g(x, y, z) \tag{1}$$

were studied. It was shown that $p : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ is a flat family of affine spaces (i.e., every fiber $p^*(c)$, $c \in \mathbb{A}^1_{\mathbb{C}}$, is reduced and isomorphic to $\mathbb{A}^3_{\mathbb{C}}$) provided that $p^*(0) \simeq \mathbb{A}^3_{\mathbb{C}}$ ([KVZ2, Theorem 3.21]). As for the latter condition, a criterion is given in terms of the morphism $\pi : C \to \Gamma$, where $C = \{f = g = 0\} \subseteq \mathbb{A}^3_{\mathbb{C}}$ and $\Gamma = \{f = 0\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ are affine curves and $\pi$ is the projection $(x, y, z) \longmapsto (x, y)$ [KVZ2, Theorems 2.11 and 3.21]. Based on this, S. Vénéreau [Ve] considered the following polynomials generating flat families of affine spaces:

$$v_n := y + x^n(xz + y(yu + z^2)) = x^n y^2 u + y + x^{n+1} z + x^n y z^2, \qquad n \geq 1\,. \tag{2}$$

He showed that for every $n \geq 3$, $v_n$ is a variable of the ring $\mathbb{C}^{[4]}$, or in other words, a coordinate of a polynomial automorphism of $\mathbb{A}^4_{\mathbb{C}}$. Furthermore, some sufficient conditions were found (see [KVZ2, Section 4.1]) for a polynomial $p$ of form (1) to be a variable. However we do not know whether the latter is true if these sufficient conditions are not satisfied, and this is the case for $v_1$ and $v_2$. That is, we do not know whether the hypersurfaces $v_i^{-1}(0) \simeq \mathbb{A}^3_{\mathbb{C}}$, $i = 1, 2$, in $\mathbb{A}^4_{\mathbb{C}}$ can be rectified by means of polynomial automorphisms of $\mathbb{A}^4_{\mathbb{C}}$ (this is a particular case of the Abhyankar-Sathaye Embedding Problem).

It was also shown in [Ve] that for every $n \geq 3$ the morphism

$$\Phi_n = (v_n, x) : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$$

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defines a trivial family of affine planes. This means that the Vénéreau polynomials $v_n$ ($n \geq 3$) are $x$-variables i.e., variables of the $\mathbb{C}[x]$-algebra $\mathbb{C}[x][y, z, u]$.

If a polynomial $p$ as in (1) defines a flat family of affine spaces $p : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ then $p$ also defines, along with a suitable second polynomial $q \in \mathbb{C}[x, y] \subseteq \mathbb{C}^{[4]}$, which is a variable in $\mathbb{C}[x, y]$, a flat family of affine planes $\Phi = (p, q) : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ (see [KVZ2, Theorem 3.21]). Thus the question arises whether the two remaining families of affine planes $\Phi_n = (v_n, x) : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ ($n = 1, 2$) and, more generally, all such families of affine planes $\Phi = (p, q) : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ as above, are trivial.

In the present note we address the former special question by showing in Proposition 1 that $\Phi_n$ restricts to an algebraic fiber bundle, say, $\lambda_n$ over the punctured plane:

$$\lambda_n := \Phi_n|(\mathbb{A}^4_{\mathbb{C}} \backslash \mathbb{A}^2_{\mathbb{C}}) : \mathbb{A}^4_{\mathbb{C}} \backslash \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}} \backslash \{\bar{0}\},$$

where $\mathbb{A}^2_{\mathbb{C}} \subseteq \mathbb{A}^4_{\mathbb{C}}$ is the coordinate $(z, u)$-plane. This bundle has $\mathbb{A}^2_{\mathbb{C}}$ as the typical fiber and $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ as the structure group. We show that $\lambda_n$ is trivial for every $n \geq 3$, thus recovering Vénéreau's result (see Corollary 1 below). In the cases $n = 1, 2$ we were not able to carry the computations needed by our methods, and so the question remains open.

Generally speaking, we deal below with the following three categories of affine $S$-schemes $f : X \to S$ over a quasiprojective base $S$:

- The flat families of affine $m$-spaces over $S$ [1];
- The algebraic fiber bundles over $S$ with fiber $\mathbb{A}^m_{\mathbb{C}}$;
- The algebraic vector bundles of rank $m$ over $S$.

The Dolgachev-Weisfeiler Conjecture [DW, (3.8.3)] claims that the first category reduces to the second one. In turn, for an affine base $S$, the second one reduces to the third one due to the Bass-Connell-Wright Theorem [BCW].

If the Dolgachev-Weisfeiler Conjecture were answered in affirmative this would provide an affirmative answer to our general question. Indeed, in this case $\Phi$ (in particular, $\Phi_n$) would be a fiber bundle over $\mathbb{A}^2_{\mathbb{C}}$ with fiber $\mathbb{A}^2_{\mathbb{C}}$, thus a vector bundle by the Bass-Connell-Wright Theorem, hence it must be trivial due to the Quillen-Suslin Theorem.

Conversely, if any one of the bundles $\lambda_1$ or $\lambda_2$ were non-trivial this would provide a counterexample to the Dolgachev-Weisfeiler Conjecture (with $m = 2$), and, presumably, to the Abhyankar-Sathaye Embedding Problem.

Summarizing, the following equivalences hold.

**Proposition 1.** *If $v \in \mathbb{C}^{[4]}$ is a polynomial such that $\Phi = (x, v)$ yields a flat family of affine planes $\Phi : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ then the following conditions are equivalent:*

($i$) *$v$ is an $x$-variable of the ring $\mathbb{C}[x, y, z, u]$.*

($ii$) *$\Phi : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ is an algebraic fiber bundle.*

($iii$) *It is a trivial fiber bundle.*

($iv$) *$\lambda := \Phi|(\mathbb{A}^4_{\mathbb{C}} \backslash F_{\bar{0}})$, where $F_{\bar{0}} = \Phi^{-1}(\bar{0})$, is a trivial fiber bundle over $\mathbb{A}^2_{\mathbb{C}} \backslash \{\bar{0}\}$.*

*Proof.* The equivalence ($i$)$\Leftrightarrow$($iii$) is a tautology, and ($iii$)$\Leftrightarrow$($ii$) follows by combining the Bass-Connell-Wright Reduction Theorem and the Quillen-Suslin Theorem as above. The implication ($iii$)$\Rightarrow$($iv$) is evident, whereas the converse one ($iv$)$\Rightarrow$($iii$) can be easily obtained by extending the trivialization morphism

$$\mathbb{A}^4_{\mathbb{C}} \backslash F_{\bar{0}} \to (\mathbb{A}^2_{\mathbb{C}} \backslash \{\bar{0}\}) \times \mathbb{A}^2_{\mathbb{C}}$$

[1] That is, every fiber $X_s = f^*(s)$, $s \in S$, is a reduced scheme isomorphic to $\mathbb{A}^m_{\mathbb{C}}$.

and its inverse to the deleted planes $F_{\bar{0}}$ and $\{\bar{0}\} \times \mathbb{A}^2_{\mathbb{C}}$, respectively. $\square$

**Notation 1.** We use below the following polynomials from the ring $\mathbb{C}[x, y, z, u]$ :

$$w = z^2 + yu\,, \tag{3}$$

$$t = xz + yw\,, \tag{4}$$

$$s = -(2xz + yw)w\,, \tag{5}$$

$$\eta = s + x^2u = x^2u - 2xzw - yw^2\,, \tag{6}$$

$$\zeta^{(1)} = -v_1z + v_1t(v_1u + z^2 + w) + t^2(xz^2 + st)\,, \tag{7}$$

$$\zeta^{(2)} = -z + xt(v_2u + z^2 + st + w)\,, \tag{8}$$

$$\zeta^{(n)} = -z + x^{n-3}t\,(v_n\eta + x^2w)\,, \qquad n \geq 3\,, \tag{9}$$

where

$$v_n = y + x^nt \tag{10}$$

stands for the Vénéreau polynomials (2). We also consider the rational functions

$$\xi_0 = \eta/x^3, \qquad \xi_1^{(1)} = \zeta^{(1)}/v_1^3 \qquad \text{and} \qquad \xi_1^{(n)} = \zeta^{(n)}/v_n^2, \quad n \geq 2\,. \tag{11}$$

We let

$$L_0 = \mathbb{C}[x, x^{-1}], \quad L_1 = \mathbb{C}[v, v^{-1}], \quad K_0 = C[x, x^{-1}, v], \quad K_1 = \mathbb{C}[x, v, v^{-1}]\,,$$

$$R = \mathbb{C}[x, v] = K_0 \cap K_1 \qquad \text{and} \qquad M = \mathbb{C}[x, x^{-1}, v, v^{-1}]\,,$$

with the convention that $v = v_n$ whenever we consider $\Phi_n$ or $\lambda_n$. Letting $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x, v_n]$, the punctured plane $\mathbb{A}^2_{\mathbb{C}}\backslash\{\bar{0}\}$ can be covered by the Zariski open subsets

$$U_0 = \operatorname{Spec} K_0 = \mathbb{A}^2_{\mathbb{C}}\backslash\{x = 0\} \qquad \text{and} \qquad U_1 = \operatorname{Spec} K_1 = \mathbb{A}^2_{\mathbb{C}}\backslash\{v_n = 0\}\,,$$

where $U_0 \cap U_1 = \operatorname{Spec} M$.

With this notation the following results hold.

**Proposition 2.** *(a) The morphism $\varphi_0 = (t, \xi_0)$, respectively, $\varphi_1^{(n)} = \left(t, \xi_1^{(n)}\right)$, yields a local trivialization for the family $\lambda_n$ over $U_0$, respectively, over $U_1$. Thus $\lambda_n$ is an algebraic fiber bundle with $\mathbb{A}^2_{\mathbb{C}}$ as the typical fiber.*

*(b) The transition function*

$$\varphi_{10}^{(n)} = \varphi_1^{(n)} \circ \varphi_0^{-1} : (t, \xi_0) \longmapsto \left(t, \xi_1^{(n)}\right)$$

*is a triangular automorphism with*

$$\xi_1^{(n)} = \xi_0 + \frac{p_n(t)}{x^kv_n^l}, \qquad \textit{where} \qquad (k, l) = \begin{cases} (3, 2), & n \geq 2 \\ (3, 3), & n = 1\,, \end{cases} \tag{12}$$

*and the polynomials $p_n \in \mathbb{C}[x, v_n, t]$ are given by*

$$p_n(t) = \begin{cases} v_nt^2 - x^2t, & n \geq 3 \\ x^2t^3 + v_2t^2 - x^2t, & n = 2 \\ x^2t^4 + xv_1t^3 + v_1^2t^2 - x^2v_1t, & n = 1\,. \end{cases} \tag{13}$$

*Proof.* (a) The Nagata automorphism $\alpha \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$ of $\mathbb{A}^3_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[y,z,u]$ is given (see [Na]) by $\alpha : (y,z,u) \to (y,t_0,\eta_0)$, where

$$t_0 = z + yw \qquad \text{and} \qquad \eta_0 = u - 2zw - yw^2$$

with $w = z^2 + yu$ as in (3) above. [2] Composing $\alpha$ with the following $L_0$-automorphisms of $L_0[y,z,u]$ (cf. [Ve]):

$$g : (y,z,u) \longmapsto (x^{-2}y, xz, x^4u) \qquad \text{and} \qquad h : (y,z,u) \longmapsto (x^2y, z, x^{-2}u)$$

we obtain:

$$w \circ g = x^2 w, \qquad t_0 \circ g = t, \qquad \eta_0 \circ g = x^2\eta$$

and

$$\beta := h \circ \alpha \circ g \in \mathrm{Aut}_{L_0}(L_0[y,z,u]), \qquad \text{where} \qquad \beta : (y,z,u) \longmapsto (y,t,\eta)\,.$$

Letting

$$\gamma_n : (y,z,u) \longmapsto (v_n = y + x^n t, t, \eta)$$

it follows that $\gamma_n \circ \beta \in \mathrm{Aut}_{L_0}(L_0[y,z,u])$. Thus

$$\Phi_n^{-1}(U_0) = \mathbb{A}^4_{\mathbb{C}} \backslash \{x=0\} = \mathrm{Spec}\,\mathbb{C}[x,x^{-1},y,z,u] = \mathrm{Spec}\ \mathbb{C}[x,x^{-1},v_n,t,\eta] \cong U_0 \times \mathbb{A}^2_{\mathbb{C}}$$

with $\mathbb{A}^4_{\mathbb{C}} = \mathrm{Spec}\ \mathbb{C}[x,y,z,u]$ and $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[t,\eta]$. Hence $\varphi_0 = (t,\xi_0) = (t,\eta/x^3)$ yields indeed a trivialization of $\lambda_n$ over $U_0$.

To show that also $\varphi_1^{(n)} = (t, \xi_1^{(n)})$ yields a trivialization of $\lambda_n$ over $U_1$, we consider separately the cases $n=1$, $n=2$ and $n \geq 3$. We will constantly exploit the relations

$$ys + t^2 = x^2z^2 \qquad \Rightarrow \qquad v_n s + t^2 = x^2z^2 + x^n st \tag{14}$$

$$\Rightarrow \quad v_n\eta + t^2 = x^2(z^2 + v_n u) + x^n st \tag{15}$$

(see (6) and (10)).

*Case* $n=1$. We denote

$$\zeta' := \frac{v_1\eta + t^2}{x} = x(v_1u + z^2) + st \tag{16}$$

(see (15)),

$$\begin{aligned}\zeta'' := \frac{v_1\zeta' + t^3}{x} &= v_1(v_1u + z^2) + xz^2t + st^2 \\ &= yw + xtw + xtv_1u + xz^2t + st^2\end{aligned} \tag{17}$$

(see (3) and (10)), and

$$\zeta''' := \zeta'' - t = -xz + xtw + xtv_1u + xz^2t + st^2 \tag{18}$$

[2] Let us observe in passing that $\alpha = \mu \circ \delta \circ \mu^{-1}$, where $\mu : (y,z_1,u_1) \longmapsto (y, z = yz_1, u = yu_1)$, or else $\mu : (y,t_1,\eta_1) \longmapsto (y, t_0 = yt_1, \eta_0 = y\eta_1)$ is a birational morphism, or in other words, an affine modification of $\mathbb{A}^3_{\mathbb{C}}$ with the locus $(D,\bar{0})$, where $D = \{y=0\}$, and $\delta := \delta_3 \circ \delta_2 \circ \delta_1 \in \mathrm{Aut}(\mathbb{A}^3_{\mathbb{C}})$ is a tame $\mathbb{C}[y]$-automorphism with

$$\begin{aligned}\delta_1 &: (y,z_1,u_1) \longmapsto (y,z_1,w_1)\,, \qquad w_1 := u_1 + z_1^2,\\ \delta_2 &: (y,z_1,w_1) \longmapsto (y,t_1,w_1)\,, \qquad t_1 := z_1 + y^2w_1,\\ \delta_3 &: (y,t_1,w_1) \longmapsto (y,t_1,\eta_1)\,, \qquad \eta_1 := w_1 - t_1^2\,.\end{aligned}$$

(see (4)). Now

$$\zeta^{(1)} := \frac{v_1\zeta''' + t^4}{x} = \frac{v_1^3\eta + p_1(t)}{x^3} \tag{19}$$

verifies both (7) and (12) for $n = 1$.

Further, for any point $C = (c_1, c_2) \in \mathbb{A}^2_{\mathbb{C}}$ with $c_1 \neq 0$, $c_2 \neq 0$, the functions $(t, \eta)$, hence also $(t, \zeta^{(1)})$, give global coordinates on the fiber $F_C = \Phi_1^{-1}(C) \simeq \mathbb{A}^2_{\mathbb{C}}$ over $C$. In the case $c_1 = 0$, $c_2 \neq 0$ the following hold:

$$v_1 = y = c_2, \quad w = c_2^{-1}t = c_2u + z^2, \quad \eta = s = -c_2^{-1}t^2\,, \tag{20}$$

and

$$\zeta^{(1)} = -c_2z + 2t^2 - c_2^{-1}t^5 \tag{21}$$

(see (3), (7), (10) and (14)). Therefore we obtain:

$$\mathbb{C}[z, u] = \mathbb{C}[z, t] = \mathbb{C}[t, \zeta^{(1)}] = \mathbb{C}[t, \xi_1^{(1)}]\,. \tag{22}$$

Clearly, $F_C \simeq \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[z, u]$ for $C = (0, c_2)$ with $c_2 \neq 0$. In other words, $(z, u)$ give coordinates on the fiber $F_C$. Thus by (22) for $c_2 \neq 0$ the functions $(t, \zeta^{(1)})$, and hence also $(t, \xi_1^{(1)})$, provide as well coordinates on $F_C$. Now (a) follows for $n = 1$.

*Case* $n = 2$. From (15) we obtain:

$$\zeta' := \frac{v_2\eta + t^2}{x^2} = z^2 + uv_2 + st\,. \tag{23}$$

It can be easily seen that the polynomial

$$\zeta^{(2)} := \frac{v_2\zeta' + t^3 - t}{x} = \frac{v_2^2\eta + p_2(t)}{x^3} \tag{24}$$

verifies both (8) and (12) for $n = 2$. It follows that for $C = (c_1, c_2)$ with $c_1 \neq 0$, $c_2 \neq 0$, the map $(t, \eta) \longmapsto (t, \zeta^{(2)})$ provides an isomorphism

$$F_C \simeq \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\,\mathbb{C}[t, \eta] = \operatorname{Spec}\mathbb{C}[t, \zeta^{(2)}]\,.$$

As above, for $c_1 = 0$, $c_2 \neq 0$ we get

$$\mathbb{C}[z, u] = \mathbb{C}[z, t] = \mathbb{C}[t, \zeta^{(2)}]$$

and so, the functions $(t, \zeta^{(2)})$ still give coordinates on the fiber $F_C$. This proves (a) for $n = 2$.

*Case* $n \geq 3$. Letting again

$$\zeta' := \frac{v_n\eta + t^2}{x^2} = z^2 + v_nu + x^{n-2}st \tag{25}$$

(see (15)) we can easily see that

$$\zeta^{(n)} := \frac{v_n\zeta' - t}{x} = \frac{v_n^2\eta + p_n(t)}{x^3} \tag{26}$$

verifies both (9) and (12), the map $(t, \eta) \longmapsto (t, \zeta^{(n)})$ provides an isomorphism $F_C \simeq \mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[t, \zeta^{(n)}]$ as soon as $c_1 \neq 0$ and $c_2 \neq 0$, and, moreover, the functions $(t, \zeta^{(n)})$ yield coordinates on any fiber $F_C$ with $c_2 \neq 0$. This proves (a) in the general case. Actually (b) has been established in the course of proof of (a). $\square$

**Remark 1.** By Proposition 2(b) the transition function $\varphi_{10}^{(n)}$ of $\lambda^{(n)}$ as in (12) takes values in the subgroup of plane triangular automorphisms of the form

$$(t,\xi)\longmapsto\left(t,\xi+\frac{q(t)}{x^n v^m}\right)\qquad\text{with}\qquad q\in R[t],\quad n,m\in\mathbb{N}\cup\{0\}\,.$$

However, within this subgroup, the fiber bundles $\lambda_n$ are non-trivial. Indeed, triviality of $\lambda_n$ is equivalent to the existence of a decomposition

$$\frac{p_n(t)}{x^k v^l}=\frac{a_n(t)}{x^{k+\alpha}}-\frac{b_n(t)}{v^{l+\beta}}\qquad\text{with}\qquad a_n,\,b_n\in R[t],\quad \alpha,\beta\ge 0\,,$$

and hence is equivalent to: $p_n\in(x^k,v^l)R[t]$, which is not the case (see (13)).

The following results will be useful for establishing the triviality of bundles $\lambda_n$.

**Proposition 3.** *Let $\lambda$ be an algebraic fiber bundle over $\mathbb{A}^2_{\mathbb{C}}\backslash\{\bar{0}\}$, where $\mathbb{A}^2_{\mathbb{C}}=\operatorname{Spec}\mathbb{C}[x,v]$, with $\mathbb{A}^2_{\mathbb{C}}=\operatorname{Spec}\mathbb{C}[t,\xi]$ as the typical fiber. We suppose that the restrictions $\lambda|U_i$ ($i=0,1$) are trivial. If the transition function*

$$\varphi_{10}:U_0\cap U_1\to\operatorname{Aut}(\mathbb{C}[t,\xi])$$

*has the form:*

$$\varphi_{10}:(t,\xi)\longmapsto\left(t,\xi+\frac{p(t)}{x^k v^l}\right)\tag{27}$$

*with $p\in R[t]$, where $R=\mathbb{C}[x,v]$ and $k,l\in\mathbb{N}$, then the following hold.*

*(a) $\lambda$ is trivial if and only if*

$$\varphi_{10}=\tau_1\circ\tau_0^{-1}\qquad\textit{with}\qquad \tau_i\in\operatorname{Aut}_{K_i}K_i[t,\xi],\ i=0,1\,.\tag{28}$$

*If (28) is fulfilled then necessarily $\tau_0$ and $\tau_1$ have the form:*

$$\tau_0:(t,\xi)\longmapsto\left(a,\ \frac{b_0}{x^k}\right)\qquad\textit{and}\qquad \tau_1:(t,\xi)\longmapsto\left(a,\ \frac{b_1}{v^l}\right),\tag{29}$$

*where $a,\,b_0,\,b_1\in R[t,\xi]$ satisfy the cocycle relation*

$$x^k b_1-v^l b_0=p(a)\,.\tag{30}$$

*Furthermore, up to multiplying $b_0,\,b_1$ by a constant we may assume that the following Jacobian relations hold:*

$$\operatorname{jac}(\tau_0)=1\Leftrightarrow\operatorname{jac}(a,b_0)=x^k,\qquad \operatorname{jac}(\tau_1)=1\Leftrightarrow\operatorname{jac}(a,b_1)=v^l\,,\tag{31}$$

*where $\operatorname{jac}(*,*)$ stands for the jacobian in $t,\xi$.*

*(b) Any solution $(a,\,b_0,\,b_1)$ in $(R[t,\xi])^3$ of (30) satisfies*

$$\frac{\operatorname{jac}(a,b_0)}{x^k}=\frac{\operatorname{jac}(a,b_1)}{v^l}=:d\in R\tag{32}$$

*and*

$$\operatorname{jac}(b_0,b_1)=-d\cdot p'(a)\,.\tag{33}$$

*Such a solution also verifies (31) if and only if $d=1$. In the latter case*

$$\operatorname{jac}(b_0,b_1)=-p'_t(a)\,.\tag{34}$$

*Proof.* Although (a) is well known, we remind the proof. We have: $\varphi_{10} = \varphi_1 \circ \varphi_0^{-1}$, where $\varphi_i$ is a trivialization of $\lambda|U_i$ $(i = 0, 1)$. If (28) holds then $\psi|U_i := \tau_i^{-1}\varphi_i$ $(i = 0, 1)$ defines a global trivialization of $\lambda$. Indeed, $\psi$ is a well defined regular map over $U_0 \cup U_1 = \mathbb{A}^2_{\mathbb{C}} \backslash \{\bar{0}\}$, since by (28), $\tau_0^{-1}\varphi_0 = \tau_1^{-1}\varphi_1$ in $U_0 \cap U_1$. Conversely, if $\psi$ is a global trivialization of $\lambda$ then $\tau_i := \varphi_i \circ \psi^{-1} \in \mathrm{Aut}_{K_i} K_i[t, \xi]$ $(i = 0, 1)$ satisfy (28), as required.

Letting $\tau_i = (\hat{a}_i, \hat{b}_i)$ with $\hat{a}_i, \hat{b}_i \in K_i[t, \xi]$, by (28) we obtain:

$$(\hat{a}_1, \hat{b}_1) = \varphi_{10} \circ (\hat{a}_0, \hat{b}_0) = \left( \hat{a}_0, \hat{b}_0 + \frac{p(\hat{a}_0)}{x^k v^l} \right) . \tag{35}$$

Therefore $\hat{a}_0 = \hat{a}_1 =: a \in R[t, \xi]$, and

$$\hat{b}_0 = \frac{b_0}{x^{k+\alpha}}, \qquad \hat{b}_1 = \frac{b_1}{v^{l+\beta}} , \tag{36}$$

where $b_0, b_1 \in R[t, \xi]$ and $\alpha, \beta \geq 0$. Now (36) yields:

$$x^{k+\alpha} b_1 - v^{l+\beta} b_0 = x^\alpha v^\beta p(a) , \tag{37}$$

hence $x^\alpha | b_0$ and $v^\beta | b_1$. Thus without loss of generality we can suppose that $\alpha = \beta = 0$, and so (30) follows.

As $\mathrm{jac}\,(\tau_i)$ is a unit in $K_i[t, \xi]$ $(i = 0, 1)$ we have

$$\mathrm{jac}\,(\tau_0) = c_0 x^n \quad \text{and} \quad \mathrm{jac}\,(\tau_1) = c_0 v^m \qquad \text{for some} \qquad n, m \in \mathbb{Z} \quad \text{and} \quad c_0, c_1 \in \mathbb{C} \backslash \{0\} .$$

Since $\mathrm{jac}\,(\varphi_{10}) = 1$ it follows from (28) that

$$\mathrm{jac}\,(\tau_0) = \mathrm{jac}\,(\tau_1) \in K_0 \cap K_1 = R$$

is a unit in $R$. Hence $n = m = 0$, and we may suppose that $c_0 = c_1 = 1$, which yields (31).

(b) Applying to (30) the derivations $* \longmapsto \mathrm{jac}\,(a, *)$ and $* \longmapsto \mathrm{jac}\,(*, b_1)$, respectively, gives

$$x^k \mathrm{jac}\,(a, b_1) = v^l \mathrm{jac}\,(a, b_0) \qquad \text{and} \qquad p'(a) \mathrm{jac}\,(a, b_1) = -v^l \mathrm{jac}\,(b_0, b_1) . \tag{38}$$

Now (32) and (33) follow. □

**Remarks 2.** 1. If $(a, b_0, b_1)$ is a solution of (30) then so is $(a, b_0 + x^k c, b_1 + v^l c)$ for any $c \in R[t, \xi]$. This new solution verifies (31) if and only if $\mathrm{jac}\,(a, c) = 1 - d$ with $d$ as in (32).

Similarly, $(a + A, b_0 + B_0, b_1 + B_1)$ with $A, B_0, B_1 \in R[t, \xi]$ gives a new solution of (30) if and only if

$$\sum_{j=1}^{\deg p} \frac{p^{(j)}(a)}{j!} A^j = x^k B_1 - v^l B_0 . \tag{39}$$

2. By virtue of (32), the cocycle relation (30) and one of the Jacobian relations (31) imply the other one.

3. Modulo the plane Jacobian Conjecture the maps $\tau_0$ and $\tau_1$ as in (29) are invertible if and only if the Jacobian relations (31) hold. Presumably, in our particular setting the degrees of the coordinate polynomials can be limited to the range where the plane Jacobian Conjecture is known to be true. If so then the triviality of the bundles $\lambda_n$ reduces to the existence of a solution $(a, b_0, b_1) \in (R[t, \xi])^3$ of both (30) and (31).

**Lemma 1.** *Suppose that $p = p_n$ in (30) and $(k,l) = (3,2)$ if $n \ge 2$, $(k,l) = (3,3)$ if $n = 1$. Then there exists a solution $(a, b_0, b_1) \in (R[t,\xi])^3$ of (30) with*

$$a = \sum_{i,j\ge 0} a_{ij} x^i v^j, \qquad \text{where} \qquad a_{ij} \in S := \mathbb{C}[t,\xi]\,, \tag{40}$$

*if and only if*

$$a_{00} = 0 \qquad \text{and} \qquad a_{01} = a_{10}^2\,. \tag{41}$$

*Proof.* Clearly, (30) has a solution if and only if

$$p(a) = p_n(a) \in (x^k, v^l), \qquad \text{where} \qquad (k,l) = \begin{cases} (3,2), & n \ge 2 \\ (3,3), & n = 1\,. \end{cases} \tag{42}$$

We have

$$p_n(a) \equiv \begin{cases} a_{00}^2 v \mod (x, v^2), & n \ge 2 \\ a_{00}^3 xv \mod (x^2, v^2), & n = 1\,. \end{cases} \tag{43}$$

In any case

$$p_n(a) \in (x^k, v^l) \Rightarrow a_{00} = 0\,, \tag{44}$$

and then

$$p_n(a) \equiv \begin{cases} (a_{10}^2 - a_{01}) x^2 v \mod (x^3, v^2), & n \ge 2 \\ (a_{10}^2 - a_{01}) x^2 v^2 \mod (x^3, v^3), & n = 1\,. \end{cases} \tag{45}$$

Now the assertion follows. □

**Notation 2.** Letting in Proposition 3

$$p(t) = \sum_{j=0}^{\deg p} r_j t^j, \qquad \text{where} \qquad r_j \in R = \mathbb{C}[x,v]\,,$$

and supposing that $r_0 = 0$, we can define the successive approximations

$$\varphi_{10}^{(m)} : (t,\xi) \longmapsto \left( t, \xi + \frac{\sum_{j=0}^m r_j t^j}{x^k v^l} \right), \qquad m = 1, \dots, \deg p, \tag{46}$$

to our transition function $\varphi_{10}$. These $\varphi_{10}^{(m)}$ may be considered as the transition functions of a succession of algebraic fiber bundles $\lambda^{(m)}$ over the punctured plane $\mathbb{A}^2_{\mathbb{C}} \backslash \{\bar{0}\}$ with $\mathbb{A}^2_{\mathbb{C}}$ as the typical fiber. In particular, the linear part $\varphi_{10}^{(1)}$ defines a rank 2 vector bundle $\lambda^{(1)}$ over the punctured plane.

With these notation the following holds.

**Lemma 2.** *If $r_1 = -x^{k'} v^{l'}$, where $\alpha := k - k' \ge 0$ and $\beta := l - l' \ge 0$, then the vector bundle $\lambda^{(1)}$ is trivial. In particular, $\lambda_n^{(1)}$ is trivial for every $n \ge 1$.*

*Proof.* Similarly as in Proposition 3(a), $\lambda^{(1)}$ is trivial if and only if

$$\varphi_{10}^{(1)} = \tau_1^{(1)} \circ \left(\tau_0^{(1)}\right)^{-1} \qquad \text{with} \qquad \tau_i^{(1)} \in \mathrm{GL}(2, K_i),\ i = 0, 1\,. \tag{47}$$

By our assumption the linear part $\varphi_{10}^{(1)}$ of $\varphi_{10}$ is:

$$\varphi_{10}^{(1)} : (t,\xi) \longmapsto \left(t,\, \xi - x^{-\alpha}v^{-\beta}t\right) .$$

It can be represented by the following matrix:

$$g = \begin{pmatrix} 1 & 0 \\ -x^{-\alpha}v^{-\beta} & 1 \end{pmatrix} \in \mathrm{SL}(2,M)\,. \tag{48}$$

Letting e.g.,

$$\tau_0^{(1)} = \begin{pmatrix} v^{\beta} & -x^{\alpha} \\ x^{-\alpha} & 0 \end{pmatrix} \in \mathrm{SL}(2,K_0) \quad \text{and} \quad \tau_1^{(1)} = \begin{pmatrix} v^{\beta} & -x^{\alpha} \\ 0 & v^{-\beta} \end{pmatrix} \in \mathrm{SL}(2,K_1) \tag{49}$$

we obtain $g \circ \tau_0^{(1)} = \tau_1^{(1)}$, which shows that $\lambda^{(1)}$ is indeed trivial (cf. (28)). According to (13) in Proposition 2, $\lambda = \lambda_n$ corresponds to the particular case where $(\alpha,\beta) = (1,2)$, hence $\lambda_n^{(1)}$ is also trivial. □

The similar triviality result holds also for the second order approximations.

**Proposition 4.** *The fiber bundle $\lambda_n^{(2)}$ is trivial for every $n \geq 1$.*

*Proof.* By virtue of (13) all the fiber bundles $\lambda_n^{(2)}$, $n \geq 1$, actually have the same transition function

$$\varphi_{10}^{(2)} : (t,\xi) \longmapsto \left(t,\; \xi - \frac{t}{xv^2} + \frac{t^2}{x^3v}\right) . \tag{50}$$

If a decomposition

$$\varphi_{10}^{(2)} = \tau_1^{(2)} \circ \left(\tau_0^{(2)}\right)^{-1} \qquad \text{with} \qquad \tau_i^{(2)} \in \mathrm{Aut}_{K_i} K_i[t,\xi],\ i = 0,1\,, \tag{51}$$

as in (29) does exist then clearly there should also exist such a decomposition with the linear parts $\tau_i^{(1)}$ of $\tau_i^{(2)}$ $(i = 0,1)$ as in (49), where $(\alpha,\beta) = (1,2)$, i.e., with

$$a = v^2t - x\xi + O(2), \qquad b_0 = x^2t + O(2) \quad \text{and} \quad b_1 = \begin{cases} \xi + O(2), & n \geq 2 \\ v\xi + O(2), & n = 1\,, \end{cases} \tag{52}$$

where $O(2)$ stands for the terms of order $\geq 2$ in $t,\xi$. By Lemma 1, $a_{00} = 0$ and $a_{01} = a_{10}^2$. This indicates that the power series expansion of the polynomial $a \in \mathbb{C}[x,v,t,\xi]$ can start e.g., with:

$$\tilde{a} = v^2t - x\xi + v\xi^2 = v\delta - x\xi\,, \tag{53}$$

where

$$\delta := vt + \xi^2, \qquad a_{10} = -\xi \quad \text{and} \quad a_{01} = a_{10}^2 = \xi^2\,. \tag{54}$$

For the fiber bundles $\lambda_n^{(2)}$ the cocycle relation (30) becomes (see (13)):

$$-x^2a + va^2 = x^3b_1 - v^2b_0, \qquad n \geq 2 \tag{55}$$

$$-x^2va + v^2a^2 = x^3b_1 - v^3b_0, \qquad n = 1\,. \tag{56}$$

It is satisfied e.g., by the triple $(a,b_0,b_1)$ with

$$a = \tilde{a}, \qquad b_0 = x^2t - v\delta^2 + 2x\delta\xi \qquad \text{and} \qquad b_1 = \begin{cases} \xi, & n \geq 2 \\ v\xi, & n = 1\,, \end{cases} \tag{57}$$

cf. (52) (recall that $(k,l)=(3,2)$ if $n\ge 2$, $(k,l)=(3,3)$ if $n=1$). Thus for any $n\ge 1$ one can take e.g.,

$$\tau_0^{(2)}:(t,\xi)\longmapsto\left(a,\frac{b_0}{x^k}\right)=\left(v\delta-x\xi,\ \frac{1}{x}t-\frac{v}{x^3}\delta^2+\frac{2}{x^2}\delta\xi\right) \tag{58}$$

and

$$\tau_1^{(2)}:(t,\xi)\longmapsto\left(a,\frac{b_1}{v^l}\right)=\left(v\delta-x\xi,\ \frac{1}{v^2}\xi\right) \tag{59}$$

with $\delta=vt+\xi^2$. It is easily seen that jac $\left(\tau_i^{(2)}\right)=1$, $i=0,1$. We have checked with MAPLE that indeed $\tau_i^{(2)}\in\mathrm{Aut}_{K_i}K_i[t,\xi]$, $i=0,1$, with the inverse map $\left(\tau_1^{(2)}\right)^{-1}$ given by:

$$\left(\tau_1^{(2)}\right)^{-1}:(t,\xi)\longmapsto\left(\frac{1}{v^2}t+x\xi-v^3\xi^2,\ v^2\xi\right)\,. \tag{60}$$

In view of (50) and (60), from $\left(\tau_0^{(2)}\right)^{-1}=\left(\tau_1^{(2)}\right)^{-1}\circ\varphi_{10}^{(2)}$ we obtain:

$$\left(\tau_0^{(2)}\right)^{-1}:(t,\xi)\longmapsto \tag{61}$$

$$\left(x\xi-v^3\xi^2+\frac{2v}{x}t\xi-\frac{2v^2}{x^3}t^2\xi+\frac{2}{x^4}t^3-\frac{v}{x^6}t^4,\quad v^2\xi-\frac{1}{x}t+\frac{v}{x^3}t^2\right)\,.$$

Thus by Proposition 3(a), $\lambda_n^{(2)}$ is trivial for any $n\ge 1$, as stated. ☐

As a corollary of Propositions 1 and 4 we recover the result of Vénéreau [Ve]:

**Corollary 1.** *For every $n\ge 3$, the polynomial $v_n$ is an $x$-variable of the polynomial ring $\mathbb{C}[x,y,z,u]$. More precisely, the map*

$$\alpha_n:\mathbb{A}^4_{\mathbb{C}}\to\mathbb{A}^4_{\mathbb{C}},\qquad (x,y,z,u)\longmapsto\left(x,v_n,\zeta^{(n)},\theta^{(n)}\right)\,,$$

*is an $x$-automorphism of the ring $\mathbb{C}[x,y,z,u]$, where*

$$\theta^{(n)}:=u-x^{n-3}t\left(xw^2+\eta(\zeta^{(n)}-z+x^{n-1}tw)\right)\,. \tag{62}$$

*Proof.* Indeed, in view of (13), for $n\ge 3$ we have $\varphi_{10}=\varphi_{10}^{(2)}$ and so, $\lambda_n=\lambda_n^{(2)}$ is trivial. Furthermore, by virtue of (60) and Proposition 2(a) the trivialization $\psi_n$ of $\lambda_n$:

$$\psi_n:=\left(\tau_1^{(2)}\right)^{-1}\circ\varphi_1^{(n)}=\left(\frac{t+x\zeta^{(n)}-v_n(\zeta^{(n)})^2}{v_n^2},\ \zeta^{(n)}\right)=:\left(\theta^{(n)},\ \zeta^{(n)}\right) \tag{63}$$

extends to a morphism

$$\Psi_n:\mathbb{A}^4_{\mathbb{C}}=\mathrm{Spec}\,\mathbb{C}[x,y,z,u]\to\mathbb{A}^2_{\mathbb{C}}=\mathrm{Spec}\,\mathbb{C}[\theta^{(n)},\ \zeta^{(n)}]$$

providing an automorphism $\alpha_n:=(\Phi_n,\ \Psi_n)\in\mathrm{Aut}(\mathbb{A}^4_{\mathbb{C}})$ (see the proof of Proposition 3). The explicit expression (62) for $\theta^{(n)}$ was found by applying formulas (3)-(6), (9)-(10), (26) and (63); it was checked with MAPLE. ☐

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Department of Mathematics, University of Miami, Coral Gables, FL 33124, U.S.A.
*E-mail address*: Shulim.Kaliman@math.miami.edu

Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 St. Martin d'Hères cedex, France
*E-mail address*: Mikhail.Zaidenberg@ujf-grenoble.fr