Spectral boundary of complemented invariant subspaces in $L^p(\mathbb{R})$

by

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ABSTRACT. In this note we construct a compact set K of zero Hausdorff dimension which satisfies certain "arithmetic type" thickness property. The concept of this "arithmetic thickness" came from applications to Harmonic Analysis, introduced in a paper of V. Lebedev and A. Olevskiĭ. For example, there are no spectral sets whose "essential boundary" can contain the above set K.

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Introduction

We say that a Borel set $\Lambda \subseteq \mathbb{R}$ (more precisely the corresponding equivalence class) is a *spectral set* for $L^p(\mathbb{R})$, $1 < p < \infty$, if its indicator function belongs to the multiplier algebra:

$$\mathbf{1}_\Lambda \in \mathcal{M}_p(\mathbb{R}). \tag{1}$$

If Λ satisfies (1) then it yields a *translation invariant complemented* subspace E_Λ in $L^p(\mathbb{R})$:

$$E_\Lambda \stackrel{def}{=} \mathrm{Clos}\ \{f \in L^p(\mathbb{R}) \cap L^2(\mathbb{R}) : \widehat{f}|_{{}^c\Lambda} = 0\},$$

where $\widehat{f}$ denotes the Fourier transform of f.

Conversely, every invariant complemented subspace can be obtained in this way; in a natural sense Λ is called the *spectrum* of E_Λ.

For $p = 2$ every Λ is a spectral set. For $p \neq 2$ the situation is much more difficult. Some classical examples are known, like $\Lambda = \mathbb{R}^+$, which is spectral for any $p \in]1, \infty[$, or more general ones, coming from the Littlewood-Paley decomposition.

All said above is well-known; see [LO1], or [O] for the precise definitions and references.

It was recently dicovered in [LO1] that for $p \neq 2$ spectral sets *cannot have complicated structure.*

For example a Cantor set of positive Lebesgue measure is never spectral.

Theorem A. [LO1] *If Λ is spectral for some $p \neq 2$*

$$\textit{the essential boundary } \partial\Lambda \textit{ has Lebesgue measure zero,} \tag{2}$$

or, equivalently

$$\Lambda \textit{ and } {}^c\Lambda \textit{ both are (equivalent to) open sets.} \tag{3}$$

In the above theorem we used the term *essential boundary* in the sense of [LO1], that is, $\partial\Lambda = \overline{\Lambda^d} \cap \overline{({}^c\Lambda)^d}$, where Λ^d denotes the Lebesgue density points of Λ and by $\overline{A}$ we denote the closure of A.

In the sequel we will also use the concept of the (common) essential boundary of two disjoint measurable sets Λ and Λ', which is denoted by $\partial(\Lambda, \Lambda')$ and $x \in \mathbb{R}$ belongs to it iff any $U(x)$ neighborhood of x contains portions of positive measure of both sets.

This result means that every spectral set Λ can be viewed as a union of some family of disjoint open intervals accumulated to a *closed set F of measure zero* - its essential boundary.

To obtain a full characterization of such families effectievely - this problem up to date seems to be hopeless.

In this note we are interested in what can be said about the essential boundary of a spectral set.

Definition. For a given $p \neq 2$ a compact set $K \subseteq \mathbb{R}$ is called a *spectral-subboundary*, $(K \in SB)$ iff there exists a spectral set Λ such that $\partial\Lambda \supset K$.

Obviously, this notion means some kind of "thinness". It is natural to compare it to metrical "thinness". Theorem A tells that $K \in SB$ implies that $mK = 0$, where m denotes the Lebesgue measure.

Is the converse implication true?

We prove that the answer is NO, even if one replaces the Lebesgue measure with the much more sensitive Hausdorff one.

We follow the general approach of [LO1-2] which involves an analysis of possible distributions of equidistant nets through Λ.

Definition. We say that a given pair of disjoint sets $\Lambda,\ \Lambda'$ has the *universal coloring* property,

$$(\Lambda, \Lambda') \in UC \tag{4}$$

if for any $N \in \mathbb{N}$ and for any coloring of the set $\{1, ..., N\}$ by two colors (say, by *red* and *blue*) there exists $x_0, h \in \mathbb{R}$ such that

$$x_k \overset{def}{=} x_0 + kh \in \Lambda \text{ or } \Lambda' \text{ if, } k \text{ is } \textit{blue}, \text{ or } \textit{red}, \text{ respectively.} \tag{5}$$

This notion plays a crucial role in [LO1-2]. It appeared there in a slightly stronger form: x_k was required to be a density point of the corresponding set; below we will use the UC property for open sets and hence this stronger assumption will also be satisfied.

Actually, Theorem A was obtained by the following implications:

$$m(\partial(\Lambda, \Lambda')) > 0 \Rightarrow (\Lambda, \Lambda') \in UC \Rightarrow \Lambda \text{ is not a spectral set.} \tag{6}$$

Next we introduce the following property of "arithmetical thickness" ($\mathcal{A}$-thickness):

Definition. We say that a compact set K is $\mathcal{A}$-*thick*, that is $K \in \mathcal{A}$, iff for any two disjoint open sets $\Lambda,\ \Lambda'$

$$K \subseteq \partial(\Lambda, \Lambda') \Rightarrow (\Lambda, \Lambda') \in UC. \tag{7}$$

Our main result says that $\mathcal{A}$-thick compact sets may be metrically small. We remark that in [K] several different concepts of thin sets in Harmonic Analysis were considered.

Main Results

Theorem 1. *There exists a compact perfect set* $K \subseteq \mathbb{R}$ *such that:*

(i) $\dim_H K = 0$*, and*

(ii) $K \in \mathcal{A}$.

We remark that $\dim_H$ denotes the usual Hausdorff dimension, and by a suitable refinement of our proof one can obtain compact sets with $\mathcal{H}_h(K) = 0$, where $\mathcal{H}_h$ is the Hausdorff measure defined by using a function h ($h(0) = 0$, $h(x) > 0$, h is monotone increasing and continuous from the right for all $t \geq 0$). In the special case when $h(x) = x^s$, that is, we use the s-dimensional Hausdorff measure we will use the notation $\mathcal{H}^s$.

Theorem 2. *There exists a compact perfect set* K*, satisfying property (i) and*

(iii) $K \notin SB$.

Finally our last result gives some additional information about $\mathcal{A}$-thickness:

Theorem 3. *If* $K \in \mathcal{A}$ *then it contains arbitrarily long arithmetical progressions.*

Reduction of Theorem 2 to Theorem 1

It is enough to show that property (ii) implies property (iii). Let $K \in \mathcal{A}$. Suppose that for some $p \neq 2$, $K \in SB$, that is, there exists a spectral set Λ such that $\partial\Lambda \supset K$. Due to Theorem A we can assume that Λ is open and ${}^c\Lambda$ is equivalent to an open set Λ'. It is clear that Λ' is disjoint from Λ. Since Λ' and ${}^c\Lambda$ are equivalent $(\Lambda')^d = ({}^c\Lambda)^d$ and hence $K \subseteq \partial\Lambda = \partial(\Lambda, \Lambda')$. Then $K \in \mathcal{A}$ implies that the pair (Λ, Λ') satisfies the UC-property. According to (6) Λ is not a spectral set.■

Proof of Theorem 1

We define K as the intersection of the nested closed sets E_k. Each E_k consists of finitely, say n_k, many closed intervals $I_{j,k}$, $j = 1, ..., n_k$.

We assume that for a fixed k the intervals $I_{j,k}$ are indexed from left to right and they are disjoint. Denote the length of $I_{j,k}$ by $\ell_{j,k}$. We choose the intervals $I_{j,k}$ in a way that $\ell_{j,k}$ is non-increasing in j, that is, $\ell_{j,k} \geq \ell_{j',k}$ for $j \leq j'$ when k is fixed. We denote by ℓ_k the length of the shortest interval $I_{j,k}$, clearly, $\ell_k = \ell_{n_k,k}$. In our construction $I_{j',k-1} \cap E_k$ will consist of $\nu_{j',k}$ many equally spaced intervals of the form $I_{j,k}$ and all these $I_{j,k}$'s will be of the same length denoted by $\eta_{j',k}$.

If $j' < j''$ holds then *the distance of the consecutive intervals* $I_{j,k}$ belonging to $I_{j'',k-1} \cap E_k$ will be *much less than the length of the* $I_{j,k}$'s belonging to $I_{j',k-1} \cap E_k$.

Next we give a formal inductive definition of the sets E_k.

Put $E_1 = I_{1,1} = [0, 1]$.

Assume that $k \geq 2$, and $E_{k-1} = \cup_{j=1}^{n_{k-1}} I_{j,k-1}$ is defined and

$$\ell_{1,k-1} \geq \ell_{2,k-1} \geq ... \geq \ell_{n_{k-1},k-1} = \ell_{k-1}.$$

We also assume that $\ell_{j,k-1}$ is an integer multiple of ℓ_{k-1} for $j = 1, ..., n_{k-1}$. Set

$$\lambda_{1,k} = \frac{\ell_{k-1}}{2k^k} \text{ and } \nu_{1,k} = \frac{\ell_{1,k-1}}{\lambda_{1,k}}.$$

Since $\ell_{1,k-1}$ is an integer multiple of ℓ_{k-1} it is easy to see that $\nu_{1,k}$ is an integer. Divide $I_{1,k-1}$ into $\nu_{1,k}$ many intervals of length $\lambda_{1,k}$ and denote these intervals by $J'_1, ..., J'_{\nu_{1,k}}$. Finally in each interval J'_j choose a small closed interval J_j which has the same midpoint as J'_j and the length of J_j equals

$$\eta_{1,k} = \frac{\lambda_{1,k}}{2^k \nu_{1,k}^{k-1}}.$$

We will define $E_k \cap I_{1,k-1}$ such that it will equal the union of the intervals J_j, that is, $J_j = I_{j,k}$ for $j = 1, ..., \nu_{1,k}$. Observe that ℓ_{k-1} is an integer multiple of $\eta_{1,k}$.

Now assume that $j' \geq 2$, the $I_{j,k}$'s are defined in the intervals $I_{1,k-1}, ..., I_{j'-1,k-1}$ and the ones in $I_{j'-1,k-1}$ are of length $\eta_{j'-1,k}$. We also assume that ℓ_{k-1} is an integer multiple of $\eta_{j'-1,k}$. Since $\ell_{j',k-1}$ is an integer multiple of ℓ_{k-1} it will also be an integer multiple of $\eta_{j'-1,k}$. Set

$$\lambda_{j',k} = \frac{\eta_{j'-1,k}}{k^k} \text{ and } \nu_{j',k} = \frac{\ell_{j',k-1}}{\lambda_{j',k}},$$

observe that $\nu_{j',k}$ is an integer. Now we subdivide $I_{j',k-1}$ like we divided $I_{1,k-1}$, that is, divide $I_{j',k-1}$ into $\nu_{j',k}$ many intervals of length $\lambda_{j',k}$ and denote them by $J'_1, ..., J'_{\nu_{j',k}}$. In each interval J'_j choose a small closed interval J_j which has the same midpoint as J'_j and the length of J_j equals

$$\eta_{j',k} = \frac{\lambda_{j',k}}{2^k \nu_{j',k}^{k-1}}.$$

Observe that $\lambda_{j',k}$ and $\eta_{j'-1,k}$ are both integer multiples of $\eta_{j',k}$. Finally, let $E_k \cap I_{j',k-1}$ equal the union of the intervals J_j.

Repeating the above steps for a fixed k we can define $E_k \cap I_{j',k-1}$ for all $j' = 1, ..., n_{k-1}$. Continuing this process define E_k for all k.

Put $K = \cap_{k=1}^{\infty} E_k$.

It is easy to see that K is a nowhere dense non-empty compact perfect set and for any k and $j \leq n_k$ we have $K \cap I_{j,k} \neq \emptyset$. It is also clear that

$$\frac{\ell_{k-1}}{2k^k} = \lambda_{1,k} > \eta_{1,k} > ... > \eta_{n_{k-1},k} = \ell_k.$$

This implies that for any $k \geq 1$ we have

$$\sum_{j=k+1}^{\infty} \ell_j < \ell_k.$$

Since for $k = 2, 3, \ldots$ the choice of $\lambda_{j',k}$ implies that each interval $I_{j',k-1}$ contains more than four intervals belonging to E_k it is always possible to find an $x \in K$ which is in the first quarter of $I_{j',k-1}$.

Next we *verify that K is of zero Hausdorff dimension.*

We need to verify that given any $s > 0$ and $\epsilon, \delta > 0$ one can find a cover of K by sets U_i such that $d(U_i) < \delta$ and $\sum d(U_i)^s < \epsilon$.

It is enough to verify that for each $s_m = 1/m$ a suitable cover can be found. Hence assume that m is fixed.

The intervals belonging to E_k cover K for any integer k. Let

$$S_k = \sum_{j=1}^{n_k} d(I_{j,k})^{\frac{1}{m}}.$$

Claim 1. For any $k \geq m+1$ we have $S_k < S_{k-1}/2$.

We prove Claim 1 later. First we show that this claim implies $\dim_H K = 0$. Since S_m is finite, from Claim 1 it follows that $S_k \to 0$ as $k \to \infty$. Using $d(I_{j,k}) = \ell_{j,k} < \ell_{k-1}$ and $\ell_k \to 0$ as $k \to \infty$ we obtain that for any given $\epsilon, \delta > 0$ we can find k such that $d(I_{j,k}) < \delta$ for $j = 1, \ldots, n_k$ and $S_k = \sum_j d(I_{j,k})^{1/m} < \epsilon$. This implies $\mathcal{H}^{1/m}(K) = 0$. This holds for all m; hence K is of zero Hausdorff dimension.

Proof of Claim 1. Assume that $j' \leq n_{k-1}$ and $I_{j',k-1} \cap E_k$ consists of the intervals J_j, $j = 1, \ldots, \nu_{j',k}$. Then $d(J_j) = \eta_{j',k}$ for all j and

$$\sum d(J_j)^{\frac{1}{m}} = \nu_{j',k} \cdot \eta_{j',k}^{\frac{1}{m}} = \nu_{j',k} \left(\frac{\lambda_{j',k}}{2^k \nu_{j',k}^{k-1}} \right)^{\frac{1}{m}} = A$$

using $k \geq m+1$ we can continue the estimation by

$$A < \nu_{j',k} \frac{\lambda_{j',k}^{\frac{1}{m}}}{2\nu_{j',k}^{\frac{m-1}{m}}} = \frac{1}{2} \nu_{j',k}^{\frac{1}{m}} \lambda_{j',k}^{\frac{1}{m}} = \frac{1}{2} \ell_{j',k-1}^{\frac{1}{m}} = \frac{1}{2} d(I_{j',k-1})^{\frac{1}{m}}.$$

Recalling that the intervals J_j are those intervals $I_{j,k}$ which are in $I_{j',k-1}$ the above result implies that

$$\sum_{\{j : I_{j,k} \subseteq I_{j',k-1}\}} d(I_{j,k})^{\frac{1}{m}} < \frac{1}{2} d(I_{j',k-1})^{\frac{1}{m}}.$$

Summing this for all $I_{j',k-1}$ we obtain $S_k \leq S_{k-1}/2$. This proves Claim 1.

We also need to *verify that* $K \in \mathcal{A}$.

Denote by Ψ the union of the intervals contiguous to K and by s_n the set $\{1, ..., n\}$ where n is a positive integer. Assume that Λ and Λ' are disjoint open sets such that $K \subseteq \partial(\Lambda, \Lambda')$. Set $G = \Lambda \cup \Lambda'$. Then $G \subseteq \Psi$ is open and we can introduce the coloring function $\phi_1 : G \to \{blue, red\}$ by letting $\phi_1 = red$ on Λ and $\phi_2 = blue$ on Λ'. Since $K \subseteq \partial(\Lambda, \Lambda')$ the coloring ϕ_1 of G is *dense*, by this we mean that both red and blue intervals are "dense" in K, that is, $F \subseteq \overline{\{x : \phi_1(x) = red\}}$ and $K \subseteq \overline{\{x : \phi_2(x) = blue\}}$. To verify that $K \in \mathcal{A}$ we need to show that for any of the above given open sets Λ, and Λ' the universal coloring property is satisfied, that is, we need to find $x \in \mathbb{R}$ and $h^* \in \mathbb{R}$ such that $x + ih^* \in G$ for $i = 1, ..., n$ and the two colorings ϕ_1 and ϕ_2 are *compatible*, that is, $\phi_1(x + ih^*) = \phi_2(i)$ for $i = 1, ..., n$.

Choose $k_0 > 2n^2$ and a $j_0 \leq n_{k_0-1}$. Then it is easy to see that $\nu_{j_0,k} > k_0^{k_0} > n+1$ and hence $I_{j_0,k_0-1} \cap E_{k_0}$ consists of more than $n+1$ intervals of the form I_{j,k_0}. Choose a j_1 such that

$$I_{j_1,k_0}, I_{j_1+1,k_0}, ..., I_{j_1+n,k_0} \subseteq I_{j_0,k_0-1}.$$

Let $h_{k_0} = \lambda_{j_0,k_0}$. Recall that the intervals I_{j_1+i,k_0} are equally spaced in I_{j_0,k_0-1} and I_{j_1+i,k_0} can be obtained from I_{j_1,k_0} by a translation with ih_{k_0}. It is also clear that $\ell_{j_1,k_0} = \ell_{j_1+1,k_0} = ... = \ell_{j_1+n,k_0} = \eta_{j_0,k_0}$.

Since the coloring of G is dense and the first quarter of I_{j_1,k_0} contains points of K we can choose an $x_n \in G$ such that $\phi_1(x_n) = \phi_2(n)$ and $[x_n, x_n + \frac{\ell_{j_1,k_0}}{2}] \subseteq I_{j_1,k_0}$. Using $\ell_{k_0} \leq \ell_{j_1,k_0}$ and the translation property of the intervals I_{j_1+i,k_0} we have

$$[x_n + ih_{k_0}, x_n + ih_{k_0} + \frac{\ell_{k_0}}{2}] \subseteq I_{j_1+i,k_0} \subseteq E_{k_0} \text{ for } i = 0, ..., n.$$

Set $\ell'_k = \ell_k/2$ for $k = 1, 2, ...$

Now the following claim will imply that $K \in \mathcal{A}$.

Claim 2. Assume that n, k_0, x_n, and h_{k_0} are given such that $k_0 > 2n^2$, $x_n \in G$, the points $x_n + ih_{k_0}$, belong to different components of E_{k_0}, $\phi_1(x_n) = \phi_2(n)$, and

$$[x_n + ih_{k_0}, x_n + ih_{k_0} + \ell'_{k_0}] \subseteq E_{k_0} \text{ for } i = 1, ..., n$$

then there exists $x \in K$ and $h \in \mathbb{R}$ such that

$$|h - h_{k_0}| < n\ell'_{k_0}$$

$$|x - (x_n + nh_{k_0})| < n\ell'_{k_0}$$

and

$$\phi_1(x - ih) = \phi_2(i) \text{ for } i = 1, ..., n.$$

Proof of Claim 2 implies $K \in \mathcal{A}$. The assumptions of the claim are satisfied; hence using $x \in K$, and h from the Claim let $h^* = -h$ and observe that $\phi_1(x + ih^*) = \phi_2(i)$, $i = 1, ..., n$ shows that the coloring ϕ_1 of G and ϕ_2 are compatible.

We need the extra assumptions in the conclusion of Claim 2 to obtain a proof which is based on induction.

To prove Claim 2 we also need a subclaim:

Subclaim. Assume that $k > 2n^2$; x_n, and h_k are given such that $x_n \in G$, the points $x_n + ih_k$ for $i = 1, ..., n$ belong to different components of E_k and

$$[x_n + ih_k, x_n + ih_k + \ell'_k] \subseteq E_k \text{ for } i = 1, ..., n.$$

Then there exists h_{k+1} such that

$$h_k \leq h_{k+1} \leq h_k + n\frac{\ell'_k}{k},$$

and the points

$$x_n + ih_{k+1} \in [x_n + ih_k, x_n + ih_k + \ell'_k], \quad i = 1, ..., n$$

belong to different components of E_{k+1}; furthermore if $\overline{\ell}_{n,k+1}$ denotes the length of the interval $I_{j,k+1} \subseteq E_{k+1}$ which contains the point $x_n + nh_{k+1}$ then

$$[x_n + ih_{k+1}, x_n + ih_{k+1} + \overline{\ell}_{n,k+1}] \subseteq E_{k+1} \text{ for } i = 1, ..., n.$$

Remark. Observe that $\overline{\ell}_{n,k+1} \geq \ell_{k+1} = 2\ell'_{k+1}$ and hence the conclusion of the Subclaim implies that

$$[x_n + ih_{k+1}, x_n + ih_{k+1} + \ell'_{k+1}] \subseteq E_{k+1}$$

also holds for $i = 1, ..., n$. This implies that we can repeat the application of the subclaim.

First *we prove Claim 2* by using the Subclaim: We do induction on n.

Assume that $k_0 \geq 2$ is given. When $n = 1$ by our assumption $[x_n + h_{k_0}, x_n + h_{k_0} + \ell'_{k_0}] \subseteq E_{k_0}$. Choose j_1 such that $x_n + h_{k_0} \in I_{j_1,k_0}$. Then there exists

$$\lambda < \frac{\ell_{k_0}}{2k_0^{k_0}} = \frac{\ell'_{k_0}}{k_0^{k_0}} < \frac{\ell'_{k_0}}{2}$$

such that any subinterval of I_{j_1,k_0} with length λ contains points of E_{k_0+1}, and hence points of K as well. Thus we can choose $h \geq h_{k_0}$ such that $|h - h_{k_0}| < \ell'_{k_0}$ and $x_n + h = x \in K$. Then $|x - (x_n + h_{k_0})| = |h - h_{k_0}| < \ell'_{k_0}$ also holds.

Assume that Claim 2 is valid for $n - 1 \geq 1$ with all $k_0 > 2(n-1)^2$.

For the next step of the induction assume that $k_0 > 2n^2$ is given and the assumptions of Claim 2 are satisfied for n, k_0, x_n and h_{k_0}.

Since G is open and $x_n \in G$ there exists $\epsilon > 0$ such that $]x_n - \epsilon, x_n + \epsilon[\subseteq G$. Choose $k_1 > k_0$ such that $\epsilon / 2n^2 > \ell'_{k_1}$.

Keeping x_n fixed we can apply the Subclaim for $k = k_0, k_0 + 1, ..., k_1 - 1$ and we can choose h_{k_1} such that

$$h_{k_0} \leq h_{k_1} \leq h_{k_0} + n \left(\frac{\ell'_{k_0}}{k_0} + \frac{\ell'_{k_0+1}}{k_0+1} + ... + \frac{\ell'_{k_1-1}}{k_1-1} \right) < h_{k_0} + 2n \frac{\ell'_{k_0}}{k_0},$$

the points $x_n + ih_{k_1}$ belong to different components of E_{k_1}, and

$$[x_n + ih_{k_1}, x_n + ih_{k_1} + \overline{\ell}_{n,k_1}] \subseteq E_{k_1} \text{ for } i = 1, ..., n.$$

Using that $\overline{\ell}_{n,k_1} \geq 2\ell'_{k_1}$ we also have

$$[x_n + ih_{k_1}, x_n + ih_{k_1} + 2\ell'_{k_1}] \subseteq E_{k_1} \text{ for } i = 1, ..., n.$$

Choose j_1 such that $x_n + h_{k_1} \in I_{j_1,k_1}$. Then there exists $\lambda < \ell'_{k_1}/k_1^{k_1} \leq \ell'_{k_1}/4$ such that any subinterval of length λ in I_{j_1,k_1} contains points of E_{k_1+1} and points of K as well. Since we have a dense coloring of G we can choose an $x_{n-1} \in G$ such that $\phi_1(x_{n-1}) = \phi_2(n-1)$ and $x_{n-1} \in [x_n + h_{k_1}, x_n + h_{k_1} + \lambda]$. Then an easy computation shows that

$$[x_{n-1} + ih_{k_1}, x_{n-1} + ih_{k_1} + \ell'_{k_1}] \subseteq [x_n + (i+1)h_{k_1}, x_n + (i+1)h_{k_1} + 2\ell'_{k_1}] \subseteq E_{k_1}$$

holds for $i = 1, ..., n-1$ and by Claim 2 applied for $n-1, k_1, x_{n-1}$, and h_{k_1} there exists $x \in K$ and $h \in \mathbb{R}$ such that

$$|h - h_{k_1}| < (n-1)\ell'_{k_1}$$

$$|x - (x_{n-1} + (n-1)h_{k_1})| < (n-1)\ell'_{k_1}$$

and

$$\phi_1(x - ih) = \phi_2(i) \text{ for } i = 1, ..., n-1.$$

Then

$$|h - h_{k_0}| \leq |h - h_{k_1}| + |h_{k_1} - h_{k_0}| < (n-1)\ell'_{k_1} + 2n\frac{\ell'_{k_0}}{k_0} < n\ell'_{k_0},$$

furthermore

$$|x - (x_n + nh_{k_0})| \leq |x - (x_{n-1} + (n-1)h_{k_1})| + |x_{n-1} + (n-1)h_{k_1} - (x_n + nh_{k_0})| \leq$$

$$(n-1)\ell'_{k_1} + |x_{n-1} - (x_n + h_{k_0})| + (n-1)|h_{k_1} - h_{k_0}| <$$

$$< (n-1)\ell'_{k_1} + 2n\frac{\ell'_{k_0}}{k_0} + \lambda + (n-1)2n\frac{\ell'_{k_0}}{k_0} < n\ell'_{k_0},$$

where in the last estimation we used that $k_0 > 2n^2$, $\ell'_{k_1} < \ell'_{k_0}/k_0^{k_0}$ and $\lambda < \ell'_{k_1}/4$.

Finally, observe that

$$|x - nh - x_n| \leq |x - (x_{n-1} + (n-1)h_{k_1})| + |x_{n-1} - x_n - h_{k_1}| + n|h_{k_1} - h| <$$

$$(n-1)\ell'_{k_1} + \lambda + n(n-1)\ell'_{k_1} < \epsilon$$

Thus $x - nh \in (x_n - \epsilon, x_n + \epsilon)$ and hence $\phi_1(x - nh) = \phi_1(x_n) = \phi_2(n)$ this, together with the induction hypothesis implies

$$\phi_1(x - ih) = \phi_2(i) \text{ for } i = 1, ..., n.$$

This completes the proof of Claim 2.

Finally we *verify the Subclaim.* We do again induction on n.

For $n = 1$ we have $[x_n + h_k, x_n + h_k + \ell'_k] \subseteq E_k$. Choose j_1 such that $x_n + h_k \in I_{j_1,k}$. Then there exists $\lambda_1 \leq \ell'_k/(k+1)^{(k+1)}$ such that during the construction of E_{k+1} the interval $I_{j_1,k}$ is being split into intervals of length λ_1 and each such subinterval contains a component of E_{k+1}. Thus we can choose h_{k+1} such that

$$h_k \leq h_{k+1} \leq h_k + \lambda_1 < h_k + \frac{\ell'_k}{k}$$

and $x_n + h_{k+1}$ is the left endpoint of a component of E_{k+1}. Then

$$[x_n + h_{k+1}, x_n + h_{k+1} + \overline{\ell}_{n,k+1}] \subseteq E_{k+1},$$

where $\overline{\ell}_{n,k+1}$ is the length of the interval $I_{j,k+1}$ which contains $x_n + h_{k+1}$. This completes the proof when $n = 1$.

Now assume that $n \geq 2$, $k > 2n^2$, and the Subclaim is true for $n - 1$. Observe that using $x_n \in G$ and the points $x_n + ih$, $i = 1, ..., n-1$ the assumptions of the Subclaim (used for $n - 1$) are satisfied; hence by the induction hypothesis there exists h'_{k+1} such that

$$h_k \leq h'_{k+1} \leq h_k + (n-1)\frac{\ell'_k}{k},$$

and

$$x_n + ih'_{k+1} \in [x_n + ih_k, x_n + ih_k + \ell'_k] \text{ for } i = 1, ..., n-1$$

and these points belong to different components of E_{k+1}; furthermore for $i = 1, ..., n-1$ we have

$$[x_n + ih'_{k+1}, x_n + ih'_{k+1} + \overline{\ell}_{n-1,k+1}] \subseteq E_{k+1},$$

where $\overline{\ell}_{n-1,k+1}$ is the length of the interval $I_{j,k+1}$ which contains $x_n+(n-1)h'_{k+1}$. Now, by our assumptions,

$$x_n+nh_k \le x_n+nh'_{k+1} \le x_n+nh_k+n_{n-1}\frac{\ell'_k}{k} < x_n+nh_k+\frac{\ell'_k}{2}.$$

Clearly, $\overline{\ell}_{n-1,k+1} < \ell'_k/k^k$. Hence

$$[x_n+nh'_{k+1}, x_n+nh'_{k+1}+\frac{\ell'_k}{2}] \subseteq [x_n+nh_k, x_n+nh_k+\ell'_k] \subseteq E_k. \tag{8}$$

Since

$$x_n+(n-1)h'_{k+1} < x_n+nh'_{k+1}$$

belong to different components of E_k there are $j_{n-1}<j_n$ such that

$$x_n+(n-1)h'_{k+1} \in I_{j_{n-1},k} \text{ and } x_n+nh'_{k+1} \in I_{j_n,k};$$

hence by (8)

$$[x_n+nh'_{k+1}, x_n+nh'_{k+1}+\frac{\ell'_k}{2}] \subseteq I_{j_n,k} \subseteq E_k. \tag{9}$$

The definition of E_{k+1} implies that there exists λ_n and $\eta_{j_n,k+1}=\overline{\ell}_{n,k+1}$ such that $I_{j_n,k}$ is divided into subintervals of length λ_n, each such subinterval contains a component of E_{k+1}, and this component is of length $\overline{\ell}_{n,k+1}$. From $j_{n-1}<j_n$ and from our construction it follows that $\lambda_n < \overline{\ell}_{n-1,k+1}/(k+1)^{(k+1)}$; hence we can choose h_{k+1} such that

$$h'_{k+1} \le h_{k+1} \le h'_{k+1}+\lambda_n < h'_{k+1}+\frac{\overline{\ell}_{n-1,k+1}}{(k+1)^{(k+1)}} < h'_{k+1}+\frac{\ell'_k}{k} \le h_k+n\frac{\ell'_k}{k}$$

and, using (9), we can also assume that x_n+nh_{k+1} is the left endpoint of a component of $E_{k+1}\cap I_{j_n,k}$. Then

$$[x_n+nh_{k+1}, x_n+nh_{k+1}+\overline{\ell}_{n,k+1}] \subseteq E_{k+1}.$$

¿From

$$h'_{k+1} \le h_{k+1} \le h'_{k+1}+\frac{\overline{\ell}_{n-1,k+1}}{(k+1)^{(k+1)}},$$

and

$$\overline{\ell}_{n,k+1} < \lambda_n < \frac{\overline{\ell}_{n-1,k+1}}{(k+1)^{(k+1)}}$$

it follows that

$$x_n+ih_{k+1}+\overline{\ell}_{n,k+1} < x_n+ih'_{k+1}+\frac{(i+1)\overline{\ell}_{n-1,k+1}}{(k+1)^{(k+1)}} < x_n+ih'_{k+1}+\overline{\ell}_{n-1,k+1}$$

for $i = 1, ..., n-1$. Thus $x_n + ih'_{k+1} \le x_n + ih_{k+1}$ and

$$[x_n + ih'_{k+1}, x_n + ih'_{k+1} + \overline{\ell}_{n-1,k+1}] \subseteq E_{k+1}$$

implies

$$[x_n + ih_{k+1}, x_n + ih_{k+1} + \overline{\ell}_{n,k+1}] \subseteq E_{k+1}$$

for $i = 1, ..., n-1$. Observe that above we have already established the last property for $i = n$ as well. This completes the proof of the Subclaim and hence the proof of Theorem 1 is also complete.■

Proof of Theorem 3

Given $\ell \in \mathbb{N}$ we denote by $\phi_{2,\ell}$ the coloring of $s_{4\ell+2} = \{1, ..., 4\ell+2\}$ for which $\phi_{2,\ell}(2i-1) = red$, $\phi_{2,\ell}(2i) = blue$ for $i \in \{1, ..., 2\ell+1\}$.

We can assume that K is non-empty, compact and nowhere dense.

First we also assume that K is perfect and does not contain $\ell \ge 3$ long arithmetic progressions. We need to show that $K \notin \mathcal{A}$.

We denote by $\mathcal{C}K$ the set consisting of the intervals contiguous to K. If $x \in \Psi = \mathbb{R} \setminus K$ then $I_K(x)$ denotes the $\mathcal{C}K$ interval containing x.

Since the number ℓ is fixed, for ease of notation, we will just write ϕ_2 instead of $\phi_{2,\ell}$. Our goal is to find open sets Λ and Λ' such that $K \subseteq \partial(\Lambda, \Lambda')$, but $(\Lambda, \Lambda') \notin UC$. We will choose Λ and Λ' by defining a coloring $\phi_1 : \Psi \to \{blue, red\}$. This coloring will be constant on each $\mathcal{C}K$ interval. We will let $\Lambda = \{x \in \Psi : \phi_1(x) = red\}$, and $\Lambda' = \{x \in \Psi : \phi_1(x) = blue\}$. It is clear that if we assume that ϕ_1 is a dense coloring, that is, $K \subseteq \overline{\Lambda} \cap \overline{\Lambda'}$ then $K \subseteq \partial(\Lambda, \Lambda')$.

Since there are countably many intervals in $\mathcal{C}K$ we can choose an onto mapping $\psi_K : \Psi \to \mathbb{N}$ such that $\psi_K(x) = \psi_K(y)$ iff $I_K(x) = I_K(y)$.

We say that a pair $(x; h) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ is admissible if the terms of the sequence $\{x + ih : i \in s_{4\ell+2}\}$ belong to Ψ and $I_K(x+2ih) \ne I_K(x+2jh)$ for $i \ne j$, $i, j \in \{1, ..., 2\ell+1\}$. The set of admissible pairs is denoted by $\mathcal{P}$.

Observe that if ℓ and ϕ_2 are fixed as above, ϕ_1 is a coloring of Ψ, and $\phi_1(x+ih) = \phi_2(i)$ for all $i \in s_{4\ell+2}$ then $(x; h) \in \mathcal{P}$.

Now we turn to the definition of a suitable coloring ϕ_1.

Given $\eta > 0$ we denote by $\mathcal{P}_\eta$ the set of those $(x; h) \in \mathcal{P}$ for which $|h| \ge \eta$. Clearly, $\mathcal{P}_{\eta_0} = \emptyset$ for a sufficiently large η_0.

For $(x; h) \in \mathcal{P}$ we call the rank of $(x; h)$ the number

$$R(x; h) = \min\{\psi_K(x + 2ih) : i = 1, ..., 2\ell+1\}.$$

For any $\eta > 0$ we can consider the set $R_\eta = \{R(x; h) : (x; h) \in \mathcal{P}_\eta\}$.

Claim. For any $\eta > 0$ the set R_η is bounded from above.

Proof of the Claim. If R_η is not bounded from above then using the compactness of K we can choose a sequence $(x_n; h_n) \in \mathcal{P}_\eta$ such that $\eta_0 > |h_n| \geq \eta > 0$, $x_n \to x \in \mathbb{R}$, $h_n \to h \neq 0$ and $\psi_K(x_n + 2ih_n) \to \infty$ for any $i \in \{1, ..., 2\ell + 1\}$. Thus if i is fixed and $]a_n, b_n[= I_K(x_n + 2ih_n)$ then $b_n - a_n \to 0$. Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_n + 2ih_n = x + 2ih$. Since $a_n \in K$ we have $x + 2ih \in K$. This is true for all $i \in \{1, ..., 2\ell+1\}$, that is, K contains an arithmetic progression of length longer than ℓ, contradicting our assumption. This proves the Claim.

Now we define by induction the coloring ϕ_1 on Ψ. In each step first we color by red finitely many $\mathcal{C}K$ intervals in order to "take care" of "blocking" sequences $\{x + ih\}$ with $(x; h) \in \mathcal{P}_{\eta_k}$ with a suitable $\eta_k > 0$. Then to obtain a dense coloring we color some $\mathcal{C}K$ intervals by blue.

Step 1 of the coloring.
Choose $\eta_1 > 0$ such that

$$\eta_1 < \frac{\text{diameter }(K)}{4(4\ell + 2)}.$$

Take an $M_1 \in \mathbb{R}$ such that $R_{\eta_1} < M_1$. There are only finitely many intervals on which $\psi_K(x) \leq M_1$. Denote the union of these intervals by U_1. Without limiting generality we can assume that U_1 contains the two unbounded components of Ψ. Then $\psi_K(x) > M_1$ for $x \in \mathbb{R} \setminus (K \cup U_1)$.

For $x \in U_1$ let $\phi_1(x) = red$.

Whenever $(x; h) \in \mathcal{P}_{\eta_1}$ we have $R(x; h) \leq M_1$; and hence there exists an $i \in \{1, ..., 2\ell + 1\}$ such that $x + 2ih \in U_1$. Thus $\phi_1(x + 2ih) \neq \phi_2(2i)$ and hence the coloring ϕ_2 of $s_{4\ell+2}$ is not compatible with the coloring of $\phi_1(x + 2ih)$.

Next, choose finitely many $\mathcal{C}K$ intervals which are disjoint from U_1 such that, denoting by V_1 their union, we have the following two properties satisfied:

α_1) If $I_1, I_2 \subseteq V_1$ are disjoint $\mathcal{C}K$ intervals then $\operatorname{dist}(I_1, I_2) > 2(4\ell + 2)\eta_1$;

β_1) If $x \in K$ then $(x - 3(4\ell + 2)\eta_1, x + 3(4\ell + 2)\eta_1) \cap V_1 \neq \emptyset$.

It is easy to see that the choice of V_1 can be made satisfying α_1) and β_1). For $x \in V_1$ let $\phi_1(x) = blue$.

Clearly α_1) says that blue intervals are not too close and β_1) says that they are not too far either.

The general steps of the coloring.

Assume now that we have already accomplished Step k of our definition. Thus we have the number $\eta_k > 0$ and the sets U_k, V_k such that $U_k \cup V_k \subseteq \Psi$, $\phi_1(U_k) = \{red\}$, $\phi_1(V_k) = \{blue\}$, and the sets U_k, V_k consist of the union of finitely many $\mathcal{C}K$ intervals. If $(x; h) \in \mathcal{P}_{\eta_k}$ then there exists $i \in \{1, ..., 2\ell + 1\}$ such that $x + 2ih \in U_k$. Finally the following two properties are satisfied.

α_k) If $I_1, I_2 \subseteq V_k$ are disjoint $\mathcal{C}K$ intervals then $\operatorname{dist}(I_1, I_2) > 2(4\ell + 2)\eta_k$;

β_k) If $x \in F$ then $(x - 3(4\ell + 2)\eta_k, x + 3(4\ell + 2)\eta_k) \cap V_k \neq \emptyset$.

Step $(k+1)$ *of the coloring.*

Let $\eta_{k+1} = \eta_k/2$.

If $h \in [\eta_{k+1}, \eta_k)$ and $(x;h) \in \mathcal{P}$, that is, $(x;h) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}$ then we define the k-rank of $(x;h)$, denoted by $R_k(x;h)$, by letting

$$R_k(x;h) = \min\{\psi_K(x+2ih) : i = 1, ..., 2\ell+1, \text{ and } x+2ih \notin V_k\}.$$

Observe that assumption α_k) implies that for each $(x;h) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}$ there is at most one $i \in \{1, ..., 2\ell+1\}$ such that $x + 2ih \in V_k$ and hence $R_k(x;h)$ is well defined and finite for each $(x;h) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}$.

Assume that the set $R_{\eta_{k+1}} = \{R_k(x;h) : (x;h) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}\}$ is not bounded from above. Again, like in an earlier similar argument, we can choose a sequence $(x_n;h_n) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}$ such that x_n converges to an $x \in \mathbb{R}$, $h_n \to h$ with $\eta_{k+1} \geq |h| \geq \eta_k$, $\psi_K(x_n + 2ih_n) \to \infty$ and $x_n + 2ih_n \notin V_k$ for any $i \in \{1, ..., 2\ell+1\} \setminus \{i^*\}$ where i^* is a possible exception. Assume $i \neq i^*$ and put $]a_n, b_n[= I_K(x_n + 2ih_n)$. Again, $\lim_{n\to\infty} b_n - a_n = 0$. Therefore, $\{x+2ih : i = 1, ..., 2\ell+1,\ i \neq i^*\} \subseteq K$, but then K would contain an arithmetic progression of length at least ℓ, contradicting our assumption.

Thus $R_{\eta_{k+1}} < M_{k+1}$ for a suitable constant M_{k+1}. We can also assume that M_{k+1} is so large that $M_{k+1} > k+1$ and, whenever $x \in K$, then there exists an interval $]a,b[\in \mathcal{C}K$ such that $]a,b[\cap]x-\eta_k, x+\eta_k[\neq \emptyset$, $]a,b[\cap V_k = \emptyset$, and $\psi_K(]a,b[)$ is bounded by M_{k+1}.

Denote by U^o_{k+1} the set of points x for which $x \in \Psi$, $\psi_K(x) \leq M_{k+1}$, $x \notin U_k \cup V_k$ (that is we have not defined ϕ_1 at x).

For $x \in U^o_{k+1}$ put $\phi_1(x) = red$, and let $U_{k+1} = U_k \cup U^o_{k+1}$. Observe that $U_{k+1} \cup V_k$ contains all points x in Ψ for which $\psi_K(x) \leq M_{k+1}$.

Now take $(x;h) \in \mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k}$. Then $R_k(x;h) \leq M_{k+1}$ and hence there exists $i \in \{1, ..., 2\ell+1\}$ such that $x+2ih \notin V_k$ and $\psi_K(x+2ih) \leq M_{k+1}$. This implies $x+2ih \in U_{k+1}$ and hence $\phi_1(x+2ih) = red \neq \phi_2(2i)$, that is, the coloring ϕ_1 of $x+2ih$ is incompatible with the coloring ϕ_2 of $s_{4\ell+2}$.

Next, choose finitely many $\mathcal{C}K$ intervals which are disjoint from U_{k+1} such that denoting by V_{k+1} their union we have $V_k \subseteq V_{k+1}$ and propositions α_{k+1}), and β_{k+1}) are satisfied (that is, we have propositions α_k) and β_k) satisfied with k replaced by $k+1$). It is an easy exercise to show that we can make a suitable choice of V_{k+1}. For $x \in V_{k+1}$ put $\phi_1(x) = blue$.

By letting $k \to \infty$ we can define a coloring ϕ_1 of Ψ. If one takes $x \in K$, $\delta > 0$ then it is easy to find k such that $U_k \cap]x-\delta, x+\delta[\neq \emptyset$ and $V_k \cap]x-\delta, x+\delta[\neq \emptyset$ and hence ϕ_1 is dense.

Since $\mathcal{P} = \mathcal{P}_{\eta_0} \bigcup \cup_{k=1}^{\infty} (\mathcal{P}_{\eta_{k+1}} \setminus \mathcal{P}_{\eta_k})$ our construction implies that for any $(x;h) \in \mathcal{P}$ there exists $i \in \{1, ..., 2\ell+1\}$ such that

$$\phi_1(x+2ih) = red \neq \phi_2(2i). \tag{10}$$

However, our observation after the definition of $\mathcal{P}$ implies that for $(x;h) \notin \mathcal{P}$ there always exists $i \in \{1, ..., 2\ell+1\}$ such that (10) holds. This implies that the open sets Λ, and Λ'

defined by ϕ_1 do not have the universal coloring property; hence $K \notin \mathcal{A}$. This concludes our proof for perfect K's.

Denote by $\mathcal{N}_\ell$ the set of those non-empty compact sets which does not contain ℓ long arithmetic progressions. If K is not perfect then we can use the following

Lemma. *If the compact set $K \in \mathcal{N}_\ell$ then there exists a perfect compact set $K' \in \mathcal{N}_\ell$ such that $K \subseteq K'$.*

First we show that using this lemma we can complete the proof of Theorem 3. Indeed, if $K \in \mathcal{N}_\ell$ then choose K' according to the Lemma. By using our previous argument we can find open sets Λ and Λ' such that $K \subseteq K' \subseteq \partial(\Lambda, \Lambda')$ and $(\Lambda, \Lambda') \notin UC$. This implies that $K \notin \mathcal{A}$.■

Proof of the Lemma. For each isolated point y_n of K we will choose a suitable neighborhood $I_n =]\, y_n - r_n, y_n + r_n[$ and denote by G the union of these neighborhoods. By choosing the r_n's sufficiently small we can assume that $I_n \cap K = \{y_n\}$. Since $K \in \mathcal{N}_\ell$, using induction on n, and at each step a compactness argument, we can assume that the radii, r_n, are so small that $K \cup G$ does not contain an ℓ long arithmetic progression such that each component of G contains at most one of its terms.

Choose a compact perfect set K_0 for which $0 \in K_0 \subseteq [0,1]$ and K_0 does not contain a 3 long arithmetic progression. Set

$$K' = K \cup \bigcup_n \left(y_n + \frac{r_n}{3} K_0 \right).$$

It is not difficult to see that K' satisfies the conditions of the Lemma, we leave the details to the reader.■

Open Problems

1. Suppose that K is an independent compact set (see [K]). Is it true that $K \in SB$? Is it true that there exists a spectral set Λ such that $\partial\Lambda = K$?
2. Does the classical Cantor ternary set belong to SB?
3. Are the classes SB different for different p's? (This question is motivated by 6.2.5 of [S].)

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