Congruences between cusp forms and Eisenstein series of half-integral weight

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1. Introduction

Recently, several authors have investigated congruences between modular forms of half-integral weight (cf. eg. [5,8]). In [8] Maeda gave an example where a congruence modulo ℓ between two cusp forms of even integer weight 2k and level N descends via the Shimura correspondence to a congruence modulo ℓ between two cusp forms of weight $k+\frac{1}{2}$ (in Maeda's example k=4, N=52, ℓ=433), and in [5],§1. Koblitz remarked that the classical congruence modulo 691 between the discriminant function Δ and the Eisenstein series G_{12} on $SL_2(\mathbb{Z})$ (normalized to have constant term $\frac{691}{24.2730}$) descends to a corresponding congruence between a cusp form and an Eisenstein series of weight $\frac{13}{2}$ on $\Gamma_0(4)$.

The purpose of this note is to give another example for the above phenomenon where congruences between modular forms of integer weight descend to (or are induced from) congruences between modular forms of half-integer weight. Let p be a prime with $p\equiv 3 \pmod 4$ and let D be a negative fundamental discriminant. Suppose that ℓ is a prime which divides the exact numerator of $\frac{p-1}{12}$ but does not divide the class number h(D). Then Mazur ([9], cf. also Gross [4],§11.) showed that there is a non-zero cusp form of weight 2 on $\Gamma_0(p)$ with ℓ-integral coefficients which modulo ℓ is congruent to the properly normalized Eisenstein series of weight 2 on $\Gamma_0(p)$ and which, moreover, has the property that under the Hecke algebra it generates the $\mathbb{C}$-linear space spanned by those Hecke eigenforms F which satisfy $L(F,1)L(F\otimes\varepsilon_D,1)\neq 0$; here $L(F,s)$ resp. $L(F\otimes\varepsilon_D,s)$ are the Hecke L-functions associated to F resp. the twist $F\otimes\varepsilon_D$ and ε_D is the quadratic character attached to the extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$.

In the special case D=-p we shall show that Mazur's congruence is induced via a Shimura map from a congruence modulo ℓ between a cusp form $\underline{C}_p$ and the (properly normalized) Cohen-Eisenstein series of weight $\frac{3}{2}$ on $\Gamma_0(4p)$. Moreover, the Hecke module generated by $\underline{C}_p$ is spanned over $\mathbb{C}$ by those eigenforms f for which $L(F,1)c_f(p)\neq 0$ (or equivalently, by results of Waldspurger ([12]), $L(L,1)L(F\otimes\varepsilon_{-p},1)\neq 0$), where F corresponds to f by the Shimura isomorphism and $c_f(p)$ is the p^{th} Fourier coefficient of f. The method of proof essentially is the same as that in [6] where it was used to show a refined version of the theorem of Waldspurger for modular forms on $SL_2(\mathbb{Z})$.

Probably our result (and also a more general statement for higher weights) is true for general D and could be proved by the same methods as here. However, since the technical details become much more tedious in the general case (compare with [6]) we have restricted to D=-p.

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2. Modular forms of integral and half-integral weight

As before, p denotes a prime congruent to 3 modulo 4, and h(-p) is the class number of the field extension $\mathbb{Q}(\sqrt{-p})/\mathbb{Q}$. We shall assume p>3. We sometimes denote by $\varepsilon_{-p}=(\frac{-}{p})$ the Legendre symbol.

Let us begin with recollecting several facts from the theory of modular forms of integral and half-integral weight.

We denote by $M_2^-(p)$ ($S_2^-(p)$) the space of modular forms (cusp forms) of weight 2 on $\Gamma_0(p)$ on which the Atkin-Lehner involution W_p acts with eigenvalue -1 (or equivalently, on which the Hecke operator U_p which replaces the n^{th} Fourier coefficient by the pn^{th} one acts as the identity). One has

$$M_2^-(p) = \mathbb{C}G_{2,p}\oplus S_2^-(p),$$

where

$$G_{2,p}(z) = \frac{p-1}{24} + \sum_{n\geq 1} \sigma_1(n)_p q^n \qquad (q=e^{2\pi iz}, z\in\mathfrak{H}=\text{upper half-plane}, \ \sigma_1(n)_p = \sum_{d|n,\, p\nmid d} d)$$

is the normalized Eisenstein series of weight 2 on $\Gamma_0(p)$.

We let $M^-_{3/2}(p)$ ($S^-_{3/2}(p)$) be the space of modular forms (cusp forms) of weight $\frac{3}{2}$ on $\Gamma_0(4p)$ which have a Fourier expansion $\sum_{n\geq 0} c(n)q^n$ with $c(n)=0$ whenever $n\equiv 1,2 \pmod 4$ or $(\frac{n}{p})=-1$ ([10,7]). One has

$$M^-_{3/2}(p) = \mathbb{C}\mathcal{H}_{3/2,p} \oplus S^-_{3/2}(p),$$

where

$$\mathcal{H}_{3/2,p}(z) = \sum_{n\geq 0} H(n)_p q^n$$

is the Cohen-Eisenstein series. By definition,

$$H(n)_p = H(p^2n) - pH(n)$$

with $H(n)$ (for $n>0$) the number of classes of positive definite binary quadratic forms of discriminant $-n$ (where forms equivalent to a multiple of x^2+y^2 or x^2+xy+y^2 are counted with multiplicity $\frac{1}{2}$ or $\frac{1}{3}$, respectively), and with $H(0)=-\frac{1}{12}$.

The series $\sum_{n\geq 0} H(n)q^n$ (which is not a modular form) and its transformation law under $\Gamma_0(4)$ have been studied by Cohen ([2]) and Zagier ([13]). From the formulas given there it is easy to see that $\mathcal{H}_{3/2,p}$ is indeed a true modular form of weight $\frac{3}{2}$ on $\Gamma_0(4p)$ (cf. also [4], §12.).

For every prime ℓ there is a Hecke operator on $M^-_{3/2}(p)$ preserving the Eisenstein space $\mathbb{C}H_{3/2,p}$ and the space of cusp forms $S^-_{3/2}(p)$. The spaces $S^-_{3/2}(p)$ and $S^-_2(p)$ are isomorphic as modules over the Hecke algebra ([7]). More precisely, for every normalized Hecke eigenform $F = \sum_{n\geq 1} a(n)q$ $\in S^-_2(p)$ there is a non-zero Hecke eigenform $f = \sum_{n\geq 1} c(n)q^n$ in $S^-_{3/2}(p)$ (unique up to multiplication by non-zero scalars) which has the same eigenvalues as F, and the Fourier coefficients of F and f are related by

(1) $$c(|D|)a(n) = \sum_{d|n,\, p\nmid d} (\tfrac{D}{d})\, c(|D|n^2/d^2) \qquad (n\geq 1,\ D<0 \text{ a fundamental discriminant}).$$

We denote by $\mathcal{S}_{-p}: M^-_{3/2}(p) \to M^-_2(p)$ the Shimura lifting associated to the fundamental discriminant $-p$ and defined by

(2) $$\mathcal{S}_{-p} \sum_{n\geq 0} c(n)q^n = \tfrac{1}{2}h(-p)c(0) + \sum_{n\geq 1} \Big(\sum_{d|n} (\tfrac{d}{p})\, c(pn^2/d^2) \Big) q^n.$$

Then

(3) $\mathcal{S}_{-p}\mathcal{H}_{3/2,p} = h(-p)G_{2,p}$,

and $\mathcal{S}_{-p}$ maps cusp forms to cusp forms.

3. Statement of result and proof

We let

$$G_{1,p}(z) = \tfrac{1}{2}h(-p) + \sum_{n\geq 1}\Big(\sum_{d|n}\big(\tfrac{d}{p}\big)\Big)q^n$$

be the Eisenstein series of weight 1 and Nebentypus ε_{-p} on $\Gamma_0(p)$ for the cusp $i\infty$ and put

$$C_p = \tfrac{p-1}{12}G_{1,p}^2 - \tfrac{1}{2}h(-p)^2G_{2,p}.$$

Then C_p is in $S_2^-(p)$, and if ℓ is a prime which divides the exact numerator of $\frac{p-1}{12}$ but does not divide $h(-p)$, then clearly C_p has ℓ-integral Fourier coefficients not all zero modulo ℓ and is congruent modulo ℓ to $-\frac{1}{2}h(-p)^2G_{2,p}$ (note that $\ell\neq 2$ since $p\equiv 3 \pmod 4$); moreover, if $\{F_\nu\}_{\nu=1,\dots,r}$ is the basis of normalized Hecke eigenforms for $S_2^-(p)$, then

(4) $$C_p = \frac{p-1}{12}\cdot\alpha\cdot\sqrt{p}\sum_{\nu=1}^{r}\frac{L(F_\nu,1)L(F_\nu\otimes\varepsilon_{-p},1)}{\|F_\nu\|^2}F_\nu,$$

where α is a non-zero constant not depending on p, $L(F_\nu,s)$ resp. $L(F_\nu\otimes\varepsilon_{-p},s)$ are the Hecke L-functions defined by analytic continuation of the Dirichlet series $\sum_{n\geq 1} a_\nu(n)n^{-s}$ resp. $\sum_{n\geq 1}\big(\frac{n}{p}\big)a_\nu(n)n^{-s}$ (Re $s>\frac{3}{2}$; $a_\nu(n)$ = n^{th} Fourier coefficient of F_ν) and $\|F_\nu\|^2 = \int_{\Gamma_0(p)\backslash\mathfrak{H}} |F_\nu(z)|^2dxdy$ ($x=\mathrm{Re}\, z, y=\mathrm{Im}\, z$) is the square of the Petersson norm of F_ν (cf. [4]; actually the above case is not treated explicitely in [4], but the methods, of course, work also here).

We shall show that C_p is the image under S_{-p} of a form in $S_{3/2}^-(p)$, which has properties similar to those of C_p. Let

$$\mathcal{C}_p(z) = \tfrac{p-1}{12}G_{1,p}(4z)\Theta(pz) - \tfrac{1}{2}h(-p)\mathcal{H}_{3/2,p}(z),$$

where $\Theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is the standard theta function of weight $\frac{1}{2}$ on $\Gamma_0(4)$ ([10]). Then C_p is a modular form of weight $\frac{3}{2}$ on $\Gamma_0(4p)$, and, in fact, using that $\sum_{d|n} (\frac{d}{p}) = 0$ for $(\frac{n}{p})=-1$ we see that $\mathcal{C}_p$ lies in the subspace $S^-_{3/2}(p)$.

Theorem. Let p be a prime congruent to 3 modulo 4, and let ℓ be a prime which divides the exact numerator of $\frac{p-1}{12}$, but does not divide the class number $h(-p)$. Then:

i) The function $\mathcal{C}_p$ has ℓ-integral Fourier coefficients, is non-zero modulo ℓ, and the congruence $\mathcal{C}_p \equiv -\frac{1}{2}h(-p)\mathcal{H}_{3/2,p} \pmod{\ell}$ holds.

ii) One has $\mathcal{S}_{-p}\mathcal{C}_p = C_p$, where $\mathcal{S}_{-p}$ is the Shimura map defined by (2).

iii) Let $\{f_\nu\}_{\nu=1,\dots,r}$ be a basis of Hecke eigenforms for $S^-_{3/2}(p)$, with f_ν corresponding to F_ν and $\{F_\nu\}_{\nu=1,\dots,r}$ the basis of normalized Hecke eigenforms for $S^-_2(p)$. Let $c_\nu(p)$ be the p^{th} Fourier coefficient of f_ν. Let $L(F_\nu,s)$ be the Hecke L-function attached to F_ν and defined by analytic continuation of the Dirichlet series $\sum_{n\geq 1} a_\nu(n)n^{-s}$ (Re $s>\frac{3}{2}$; $a_\nu(n)=n^{th}$ Fourier coefficient of F_ν). Then

$$(5) \qquad \mathcal{C}_p = \frac{p-1}{12}\cdot\alpha'\cdot\sum_{\nu=1}^{r} \frac{L(F_\nu,1)c_\nu(p)}{\|f_\nu\|^2} f_\nu ,$$

where α' is a non-zero constant not depending on p and $\|f_\nu\|^2 = \int_{\Gamma_0(4p)\backslash\mathfrak{H}} |f_\nu(z)|^2 y^{-1/2} dxdy$ (x=Re z, y=Im z) is the square of the Petersson norm of f_ν.

Remarks. i) By results of Waldspurger ([12]; cf. also Gross [4]) $c_\nu(p)^2$ is proportional to $L(F_\nu\otimes\varepsilon_{-p},1)$, hence it follows from assertion iii) of the Theorem that the Hecke module generated by C_p is generated over $\mathbb{C}$ by those Hecke eigenforms f_ν for which $L(F_\nu,1)L(F_\nu\otimes\varepsilon_{-p},1)\neq 0$. By (4) the same, of course, is true for C_p, with f_ν replaced by F_ν.

ii) Using the commutation rule $U_p\mathcal{S}_{-p} = \mathcal{S}_{-p}U_p^2$ and observing (3) and the fact that $\mathcal{C}_p$ is in the subspace $S^-_{3/2}(p)$ one can easily check that C_p lies in the subspace $S^-_2(p)$.

Proof. Assertion i) is obvious once noticing that $W\cdot\frac{1}{2}\mathcal{H}_{3/2,p}$ has integral coefficients, where W is the exact denominator of $\frac{p-1}{12}$ (cf. [4], §§1., 11.).

Let us now prove ii). By (3) it suffices to show that $G_1(z)^2$ is the image under $\mathcal{S}_{-p}$ of $G_1(4z)\Theta(pz)$. This follows by the same arguments as used in [6] (proof of Propos. 3, p. 186). Write c(n) resp. R(n) for the n^{th} Fourier coefficient of $G_1(4z)\Theta(pz)$ resp. $G_1(z)$. Then

$$c(n) = \sum_{\substack{r\in\mathbb{Z},\, r^2\leq\frac{n}{p} \\ n\equiv pr^2(4)}} R\left(\frac{n-pr^2}{4}\right).$$

Hence the n^{th} Fourier coefficient a(n) for n>0 of the image of $G_1(4z)\Theta(pz$ under $\mathcal{S}_{-p}$ equals

$$a(n) = \sum_{d|n} \left(\frac{d}{p}\right) c(pn^2/d^2)$$

$$= \sum_{d|n} \left(\sum_{r\leq\sqrt{\frac{n}{d}},\, r\equiv\frac{n}{d}(2)} R\left(p\frac{n^2-r^2d^2}{4d^2}\right) \right).$$

Observing that $R(pm)=R(m)$ for any $m\geq 0$ and writing $n_1=\frac{n-rd}{2}$, $n_2=\frac{n+rd}{2}$, we se that

$$a(n) = \sum_{\substack{n_1,n_2\geq 0 \\ n_1+n_2=n}} \sum_{d|(n_1,n_2)} \left(\frac{d}{p}\right) R\left(\frac{n_1 n_2}{d^2}\right).$$

By the multiplicative property of R(n) the inner sum equals $R(n_1)R(n_2)$, hence

$$a(n) = \sum_{\substack{n_1,n_2\geq 0 \\ n_1+n_2=n}} R(n_1)R(n_2),$$

which is the n^{th} Fourier coefficient of $G_1(z)^2$.

Assertion iii) will follow from Rankin's trick, which was already used in a similar context as here in [6] and which is also the main tool in proving formula (4) ([4]). Set

$$E_{1,4p}(z) = \beta_p^{-1} \left(G_{1,p}(4z) - \frac{1}{2}\left(\frac{2}{p}\right)G_{1,p}(2z) \right),$$

where

$$\beta_p = \left(1-\frac{1}{2}\left(\frac{2}{p}\right)\right)\cdot\frac{1}{2}h(-p).$$

Then $E_{1,4p}$ is the Eisenstein series of weight 1 on $\Gamma_0(4p)$ and Nebentypus

ε_{-p} for the cusp $i\infty$ normalized to have constant term 1 (compare with [6], p.185). According to Rankin's trick ([14],p.145) the Petersson product of $E_{1,4p}(z)\cdot\Theta(pz)$ against f_ν equals (up to a simple factor) the value at $s=\frac{1}{2}$ of the convolution of the L-series of $f_\nu(z)$ and $\Theta(pz)$ (there is a slight problem of convergence, since our Eisenstein series has weight 1; but recall that $E_{1,4p}(z)$ is the holomorphic continuation to s=0 of the non-holomorphic Eisenstein series of weight 1

$$E_{1,4p}(z;s) = \frac{1}{2}\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(4p)} \frac{\varepsilon_{-p}(d)}{cz+d}\left(\frac{\operatorname{Im} z}{|cz+d|^2}\right)^s$$

$$(z\in\mathfrak{H};s\in\mathbb{C},\operatorname{Re}\ s>\tfrac{1}{2};\Gamma_\infty=\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)|\,n\in\mathbb{Z}\})$$

(Hecke's trick)). By (1) this convolution is equal to $\dfrac{2c_\nu(p)}{p^s}\,\dfrac{L(F_\nu,2s)}{L(\varepsilon_{-p},2s)}$

($L(\varepsilon_{-p},s)$=Dirichlet series attached to ε_{-p}).

On the other hand, if pr denotes the projection from the space of modular forms of weight $\frac{3}{2}$ on $\Gamma_0(4p)$ onto the subspace of forms having an expansion $\sum_{n\geq 0,n\equiv 0,3(4)} c(n)q^n$ ([7], §2.), then an easy calculation (similar to that in [6],p.195) shows that $\frac{3}{4}h(-p)\mathrm{pr}E_{1,4p}(z) = G_{1,p}(4z)\Theta(pz)$. Observing that pr is hermitean on cusp forms and maps Eisenstein series to Eisenstein series we thus deduce (5). This concludes the proof of the Theorem.

We end up with discussing the example p=11, ℓ=5 (cf. also [4],p.67f. The spaces $S_2^-(11)$ and $S_{3/2}^-(11)$ are 1-dimensional with generators $F(z) = \eta(z)^2\eta(11z)^2$ and $f(z) = \frac{1}{2}U_4 g(z)$, where $g(z) = \eta(2z)\eta(22z)\Theta(11z)$, respectively (for the latter cf. [11],p.123); here $\eta(z)$ is the Dedekind eta function and U_4 is the operator defined by $U_4\sum a(n)q^n=\sum a(4n)q^n$. The functions F and f have Fourier coefficients in $\mathbb{Z}$, and the first few coefficients of f are given in [4],p.68. Comparing the coefficients at q^3 and using c(3)=1 we find $f=\frac{3}{2}\mathcal{E}_5$, hence from the congruence $\mathcal{E}_5\equiv-\frac{1}{2}\mathcal{H}_{3/2,11}$ (mod5) we obtain $f\equiv 3\mathcal{H}_{3/2,11}$(mod5). In particular, if D<0 is a fundamental discriminant, then we conclude from $H(D)_{11}=(1-(\frac{D}{11}))h(D)$ and the above congruence that

(6) $5 | c(|D|) \Longleftrightarrow 5 | h(D)$ (if $(\frac{D}{11}) \neq 1$).

Using descend-theoretic arguments, assertion (6) can be used to prove that (for $(\frac{D}{11}) \neq 1$) $c(|D|)$ is divisible by 5 if and only if the 5-Selmer group $S^{(5)}(E^{(D)}, \mathbb{Q})$ of $E^{(D)}$ over $\mathbb{Q}$ is non-trivial, where $E^{(D)}$ is the elliptic curve $X_0(11)/\mathbb{Q}$ twisted with D (cf. [1] and [3], where also other examples are discussed; cf. also [4], §14.). This is in accordance with the conjectures of Birch and Swinnerton-Dyer.

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