ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 6.

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To 100th birthday of
Professor A.O.Gelfond.

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§6.0. Foreword.

Let

$$|z| \geq 1, -3\pi/2 < \arg(z) \leq \pi/2, \log(z) = \ln(|z|) + i\arg(z).$$

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$.

Let

$$f^{\vee}_{l,1}(z, \nu) = f_{l,1}(z, \nu) = \sum_{k=0}^{\nu} (-1)^{(\nu+k)l} (z)^k \binom{\nu}{k}^{2+l} \binom{\nu+k}{\nu}^{2+l}, \tag{1}$$

where $l = 0,\ 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z}$. Let

$$R(t, \nu) = \frac{\prod_{j=1}^{\nu}(t-j)}{\prod_{j=0}^{\nu}(t+j)}, \tag{2}$$

where $\nu \in [0, +\infty) \cap \mathbb{Z}$,

$$f^{\vee}_{l,2}(z,\nu) = f_{l,2}(z,\nu) = \sum_{t=1+\nu}^{+\infty} z^{-t}(R(t,\nu))^{2+l}, \tag{3}$$

where $l = 0, 1, 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1, \ldots, \nu$, the zeros of the order $2+l$, it follows that

$$f_{l,2}(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}(R(t,\nu))^{2+l}, \tag{4}$$

for $l = 0, 1, 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$f^{\vee}_{l,3}(z,\nu) = f_{l,3}(z,\nu) = (\log(z))f_{l,2}(z,\nu) + f_{l,4}(z,\nu), \tag{5}$$

where

$$f_{l,4}(z,\nu) = -\sum_{t=1+\nu}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu), \tag{6}$$

$l = 0, 1, 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1, \ldots, \nu$, the zeros of the order $2+l$, it follows that

$$f_{l,4}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu) \tag{7}$$

for $l = 0, 1, 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$f^{\vee}_{l,5}(z,\nu) = -i\pi f_{l,3}(z,\nu) + f_{l,5}(z,\nu), \tag{8}$$

with $l = 1, 2$, $\nu \in [0,+\infty) \cap \mathbb{Z}$ and

$$f_{l,5}(z,\nu) = \tag{9}$$

$$2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z))f_{l,4}(z,\nu) + f_{l,6}(z,\nu) =$$
$$= -2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z))f_{l,3}(z,\nu) + f_{l,6}(z,\nu),$$

where

$$f_{l,6}(z,\nu) = 2^{-1}\sum_{t=1+\nu}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu), \tag{10}$$

and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1, \ldots, \nu$, the zeros of the order $2+l$, and $l = 1, 2$ now, it follows that

$$f_{l,6}(z,\nu) = 2^{-1}\sum_{t=1+\nu}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu) \tag{11}$$

for $l = 1, 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$f^{\vee}_{l,7}(z,\nu)=f_{l,7}(z,\nu)+(2\pi^2/3)f_{l,3}(z,\nu). \tag{12}$$

with $l=2$, $\nu\in[0,+\infty)\cap\mathbb{Z}$ and

$$\begin{gathered}f_{l,7}(z,\nu)= \\ -3^{-1}(\log(z))^3f_{l,2}(z,\nu)+2^{-1}(\log(z))^2f_{l,3}(z,\nu)+f_{l,8}(z,\nu)+ \\ (\log(z))(f_{l,5}(z,\nu)+2^{-1}(\log(z))^2f_{l,2}(z,\nu)-(\log(z))f_{l,3}(z,\nu))= \\ 6^{-1}(\log(z))^3f_{l,2}(z,\nu)-2^{-1}(\log(z))^2f_{l,3}(z,\nu)+(\log(z))f_{l,5}(z,\nu)+f_{l,8}(z,\nu)= \\ (1/6)(\log(z))^3f_{l,2}(z,\nu)+(1/2)(\log(z))^2f_{l,4}(z,\nu)+ \\ (\log(z))f_{l,6}(z,\nu)+f_{l,8}(z,\nu),\end{gathered} \tag{13}$$

where

$$f_{l,8}(z,\nu)=-6^{-1}\sum_{t=\nu+1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^3(R^{2+l})\right)(t,\nu), \tag{14}$$

and, since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ have in the points $t=1,\ \dots\ ,\nu$, the zeros of the order $2+l$, and $l=2$ now, it follows that

$$f_{l,8}(z,\nu)=-6^{-1}\sum_{t=1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^3(R^{2+l})\right)(t,\nu). \tag{15}$$

Let

$$\mathfrak{K}_0=\{1,\ 2,\ 3\},\ \mathfrak{K}_1=\{1,\ 2,\ 3,\ 5\},\ \mathfrak{K}_2=\{1,\ 2,\ 3,\ 5,\ 7\}.$$

Let λ be a variable. We denote by $T_{n,\lambda}$ the diagonal $n\times n$-matrix, i-th diagonal element of which is equal to λ^{i-1} for $i=1$, ..., n. We denote by δ the operator $z\frac{d}{dz}$. Let further $l=0,\ 1,\ 2,\ k\in\mathfrak{K}_l,\ |z|>1, \nu\in\mathbb{N}$, and let $Y_{l,k}(z;\nu)$ be the columnn with $4+2l$ elements, i-th of which is equal to $(\nu^{-1}\delta)^{i-1}f^{\vee}_{l,k}(z,\nu)$ for $i=1$, ..., $4+2l$.

Theorem 1. *The following equalities hold*

$$A^{\sim}_l(z;\nu)Y_{l,k}(z;\nu)=T_{4+2l,1-\nu^{-1}}Y_{l,k}(z;\nu-1), \tag{16}$$

$$Y_{l,k}(z;\nu)=T_{4+2l,-1}A^{\sim}_l(z;-\nu)T_{4+2l,-1+\nu^{-1}}Y_{l,k}(z;\nu-1), \tag{17}$$

where $l=0,\ 1,\ 2,\ k\in\mathfrak{K}_l,\ |z|>1,\nu\in\mathbb{N},\nu\geq 2,$

$$A^{\sim}_l(z;\nu)=S^{\sim}_l+z\sum_{i=0}^{1+l}\nu^{-i}V^{\sim*}_l(i) \tag{18}$$

with

$$S^{\sim}_0=\begin{pmatrix}1&-4&8&-12\\0&1&-4&8\\0&0&1&-4\\0&0&0&1\end{pmatrix} \tag{19}$$

$$S_1^{\sim}=\begin{pmatrix}-1&6&-18&38&-66&102\\0&-1&6&-18&38&-66\\0&0&-1&6&-18&38\\0&0&0&-1&6&-18\\0&0&0&0&-1&6\\0&0&0&0&0&-1\end{pmatrix},\tag{20}$$

$$S_2^{\sim}=\begin{pmatrix}1&-8&32&-88&192&-360&608&-952\\0&1&-8&32&-88&192&-360&608\\0&0&1&-8&32&-88&192&-360\\0&0&0&1&-8&32&-88&192\\0&0&0&0&1&-8&32&-88\\0&0&0&0&0&1&-8&32\\0&0&0&0&0&0&1&-8\\0&0&0&0&0&0&0&1\end{pmatrix},\tag{21}$$

$$V_0^{\sim *}(0)=4\begin{pmatrix}4&-5&-2&3\\-3&4&1&-2\\2&-3&0&1\\-1&2&-1&0\end{pmatrix},$$

$$V_0^{\sim *}(1)=4\begin{pmatrix}3&-6&3&0\\-2&4&-2&0\\1&-2&1&0\\0&0&0&0\end{pmatrix},$$

$$V_1^{\sim *}(0)=\begin{pmatrix}146&-198&-180&268&66&-102\\-102&146&108&-180&-38&66\\66&-102&-52&108&18&-38\\-38&66&12&-52&-6&18\\18&-38&12&12&2&-6\\-6&18&-20&12&-6&2\end{pmatrix},$$

$$V_1^{\sim *}(1)=\begin{pmatrix}240&-516&108&372&-204&0\\-160&348&-84&-236&132&0\\96&-212&60&132&-76&0\\-48&108&-36&-60&36&0\\16&-36&12&20&-12&0\\0&-4&12&-12&4&0\end{pmatrix},$$

$$V_1^{\sim *}(2)=\begin{pmatrix}102&-306&306&-102&0&0\\-66&198&-198&66&0&0\\38&-114&114&-38&0&0\\-18&54&-54&18&0&0\\6&-18&18&-6&0&0\\-2&6&-6&2&0&0\end{pmatrix},$$

$$V_2^{\sim *}(0)=8\begin{pmatrix}176&-249&-364&545&280&-431&-76&119\\-119&176&227&-364&-169&280&45&-76\\76&-119&-128&227&92&-169&-24&45\\-45&76&61&-128&-43&92&11&-24\\24&-45&-20&61&16&-43&-4&11\\-11&24&-1&-20&-5&16&1&-4\\4&-11&8&-1&4&-5&0&1\\-1&4&-7&8&-7&4&-1&0\end{pmatrix},$$

$$V_2^{\sim *}(1)=8\begin{pmatrix}455&-1020&-113&1552&-603&-628&357&0\\-300&682&44&-996&404&394&-228&0\\185&-428&-3&592&-253&-228&135&0\\-104&246&-16&-316&144&118&-72&0\\51&-124&19&144&-71&-52&33&0\\-20&50&-12&-52&28&18&-12&0\\5&-12&1&16&-9&-4&3&0\\0&-2&8&-12&8&-2&0&0\end{pmatrix},$$

$$V_2^{\sim *}(2)=8\begin{pmatrix}400&-1243&972&542&-1028&357&0&0\\-259&808&-642&-332&653&-228&0&0\\156&-489&396&186&-384&135&0&0\\-85&268&-222&-92&203&-72&0&0\\40&-127&108&38&-92&33&0&0\\-15&48&-42&-12&33&-12&0&0\\4&-13&12&2&-8&3&0&0\\-1&4&-6&4&-1&0&0&0\end{pmatrix},$$

$$V_2^{\sim *}(3)=8\begin{pmatrix}119&-476&714&-476&119&0&0&0\\-76&304&-456&304&-76&0&0&0\\45&-180&270&-180&45&0&0&0\\-24&96&-144&96&-24&0&0&0\\11&-44&66&-44&11&0&0&0\\-4&16&-24&16&-4&0&0&0\\1&-4&6&-4&1&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}.$$

The above matices $A_l^{\sim}(z;\nu), S_l^{\sim}$ and $V_l^{\sim *}(i)$ have the following properties:

$$A_l^{\sim}(z;-\nu)T_{4+2l,-1}A_l^{\sim}(z;\nu)=T_{4+2l,-1}, \tag{22}$$

$$S_l^{\sim}T_{4+2l,-1}=(S_l^{\sim}T_{4+2l,-1})^{-1} \tag{23}$$

$$S_l^{\sim}T_{4+2l,-1}V_l^{\sim *}(i)=-(-1)^iV_l^{\sim *}(i)T_{4+2l,-1}S_l^{\sim}, \tag{24}$$

$$V_l^{\sim *}(i)T_{4+2l,-1}V_l^{\sim *}(k)=0T_{4+2l,-1}, \tag{25}$$

where

$$l=0,\ 1,\ 2,\ i\in[0,1+l]\cap\mathbb{Z},\ k\in[0,1+l]\cap\mathbb{Z}.$$

Proof. Full proof can be found in [2] – [6].

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§6.1. Further relations.

In the Part 2, equality (12), we have delated the domain of definition of the the functions $f^{\vee}_{l,k}(z,\nu)$, where $l = 0, 1, 2, k \in \mathfrak{K}_l$

from $\{\nu \in \mathbb{Z}\colon \nu \geq 0\}$ to $\{\nu \in \mathbb{Z}\colon \text{ all } \nu\}$ so these functions are defined for all the $\nu \in \mathbb{Z}$. We make the same sequentially for the functions

$$f_{l,4}(z,\nu),$$

where $l = 0, 1, 2$

$$f_{l,5}(z,\nu),\ f_{l,6}(z,\nu),$$

where $l =\ 1, 2$ and

$$f_{l,7}(z,\nu),\ f_{l,8}(z,\nu),$$

where $l = 2$,

by means of the equality

(26) $$f_{l,4}(z,\nu) = f^{\vee}_{l,3}(z,\nu) - (\log(z))f^{\vee}_{l,2}(z,\nu) =$$

$$f^{\vee}_{l,3}(z,\nu) - (\log(z))f_{l,2}(z,\nu),$$

which follows from (5),

by means of the equality

(27) $$f_{l,5}(z,\nu) = f^{\vee}_{l,5}(z,\nu) + i\pi f^{\vee}_{l,3}(z,\nu) =$$

$$f^{\vee}_{l,5}(z,\nu) + i\pi f_{l,3}(z,\nu),$$

which follows from the equality (8),

by means of the equality

(28) $$f_{l,6}(z,\nu) = f_{l,5}(z,\nu) - (\log(z))f^{\vee}_{l,4}(z,\nu) -$$

$$(1/2)(\log(z))^2 f^{\vee}_{l,2}(z,\nu),$$

which follows from (9),

by means of the equality

(29) $$f_{l,7}(z,\nu) = f^{\vee}_{l,7}(z,\nu) - (2\pi^2/3)f_{l,3}(z,\nu),$$

which follows from (12), and by means the equality

(30) $$f_{l,8}(z,\nu) =$$

$$-(6^{-1}(\log(z))^3 f_{l,2}(z,\nu) - 2^{-1}(\log(z))^2 f_{l,4}(z,\nu)-$$

$$(\log(z))f_{l,6}(z,\nu)) + f_{l,7}(z,\nu),$$

which follows from (13). Let further $\mathfrak{K}^{\wedge}_l = [1, 4+2l] \cap \mathbb{Z}$ for $l = 0\,1, 2$, and let $X^{\wedge}_{l,k}(z;\nu)$, where $l = 0, 1, 2, k \in \mathfrak{K}^{\wedge}_l, |z| > 1, \nu \in \mathbb{Z}$, be the columnn with $4 + 2l$ elements, i-th of which is equal to $\delta^{i-1} f_{l,k}$ for $i = 1, ..., 4 + 2l$.

Then, in view of (1), (3), (5),

$$X^{\wedge}_{l,k}(z;\nu) = X_{l,k}(z;\nu), \tag{31}$$

if $l = 0,\ 1,\ 2,\ k = 1,\ 2,\ 3,\ |z| > 1, \nu \in \mathbb{Z}$. Therefore, if $k \in \mathfrak{K}_l$, then together with (25), (26), (32), (33) in the part 2, we have

$$X^{\wedge}_{l,k}(z;-\nu-1) = X^{\wedge}_{l,k}(z;\nu), \tag{32}$$

$$\nu^{3+2l}X^{\wedge}_{l,k}(z;\nu-1) = A^{*}_{l}(z;\nu)X^{\wedge}_{l,k}(z;\nu), \tag{33}$$

$$(-\nu)^{3+2l}X^{\wedge}_{l,k}(z;\nu) = A^{*}_{l}(z;-\nu)X^{\wedge}_{l,k}(z;\nu-1), \tag{34}$$

where

$$A^{*}_{l}(z;\nu) = \nu^{3+2l}T_{4+2l,\nu}A^{\sim}_{l}(z;\nu)T_{4+2l,\nu^{-1}}, \tag{35}$$

$l = 0,\ 1,\ 2,\ k = 1,\ 2,\ 3,\ |z| > 1, \nu \in \mathbb{Z}$. In view of (8), (5),

$$X^{\wedge}_{l,5}(z,\nu) = i\pi X_{l,3}(z,\nu) + X_{l,5}(z,\nu), \tag{36}$$

for $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Therefore (35), (33) and (34) hold for $l = 1,\ 2,\ k = 5\,|z| > 1, \nu \in \mathbb{Z}$. In view of (12), (5),

$$X^{\wedge}_{l,7}(z,\nu) = X_{l,7}(z,\nu) - (2\pi^2/3)X_{l,3}(z,\nu). \tag{37}$$

Therefore (35), (33) and (34) hold for $l = 2,\ k = 7\,|z| > 1, \nu \in \mathbb{Z}$.

So, (35), (33) and (34) hold for $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}_l,\ |z| > 1, \nu \in \mathbb{Z}$.

In view of (5), (9), (13),

$$(\delta^k f_{l,3})(z,\nu) = \tag{38}$$

$$(\delta^k f_{l,4})(z,\nu) + (((\log(z))\delta^k + k\delta^{k-1})f_{l,2})(z,\nu),$$

where $l = 0,\ 1,\ 2,\ k \in [0,\infty) \cap \mathbb{Z},\ |z| > 1, \nu \in \mathbb{Z}$,

$$(\delta^k f_{l,5})(z,\nu) = (\delta^k f_{l,6})(z,\nu) + \tag{39}$$

$$(((\log(z))\delta^k + k\delta^{k-1})f^{\vee}_{l,4})(z,\nu) +$$

$$\left(\left(\frac{1}{2}(\log(z))^2\delta^k + k(\log(z)(\delta^{k-1} + \frac{k(k-1)}{2})\delta^{k-2}\right) f^{\vee}_{l,2}\right)(z,\nu) =$$

$$(\delta^k f_{l,6})(z,\nu) + (((\log(z))\delta^k + k\delta^{k-1})f^{\vee}_{l,3})(z,\nu) -$$

$$\left(\left(\frac{1}{2}(\log(z))^2\delta^k + k(\log(z))\delta^{k-1} + \frac{k(k-1)}{2}\delta^{k-2}\right) f^{\vee}_{l,2}\right)(z,\nu),$$

where $l = 1,\ 2,\ k \in [0,\infty) \cap \mathbb{Z},\ |z| > 1, \nu \in \mathbb{Z}$,

$$(\delta^k f_{l,7})(z,\nu) = \tag{40}$$

$$\left(\left(\frac{1}{6}(\log(z))^3\delta^k + \frac{k}{2}(\log(z))^2\delta^{k-1}\right) f_{l,2}\right)(z,\nu) +$$

$$\left(\left(\frac{k(k-1)}{2}(\log(z))\delta^{k-2}+\frac{k(k-1)(k-2)}{6}\delta^{k-3}\right)f_{l,2}\right)(z,\nu)-$$
$$\left(\left(\frac{1}{2}(\log(z))^2\delta^{k}+k(\log(z))\delta^{k-1}+\frac{k(k-1)}{2}\delta^{k-2}\right)f_{l,3}\right)(z,\nu)+$$
$$(((\log(z))\delta^{k}+k\delta^{k-1})f_{l,5})(z,\nu)+(\delta^{k}f_{l,8})(z,\nu)=$$
$$\left(\left(\frac{1}{6}(\log(z))^3\delta^{k}+\frac{k}{2}(\log(z))^2\delta^{k-1}\right)f_{l,2}\right)(z,\nu)+$$
$$\left(\left(\frac{k(k-1)}{2}(\log(z))\delta^{k-2}+\frac{k(k-1)(k-2)}{6}\delta^{k-3}\right)f_{l,2}\right)(z,\nu)+$$
$$\left(\left(\frac{1}{2}(\log(z))^2\delta^{k}+k(\log(z))\delta^{k-1}+\frac{k(k-1)}{2}\delta^{k-2}\right)f_{l,4}\right)(z,\nu)+$$
$$(((\log(z))\delta^{k}+k\delta^{k-1})f_{l,6})(z,\nu)+(\delta^{k}f_{l,8})(z,\nu),$$

where $l=2,\ k\in[0,\infty)\cap\mathbb{Z},\ |z|>1,\nu\in\mathbb{Z}$. Let

$$C_{4+2l,0}=E_{4+2l}\ \text{ for }\ l=0,\ 1,\ 2, \tag{41}$$

$$C_{4,1}=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&2&0&0\\0&0&3&0\end{pmatrix}, \tag{42}$$

$$C_{6,1}=\begin{pmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&3&0&0&0\\0&0&0&4&0&0\\0&0&0&0&5&0\end{pmatrix}, \tag{43}$$

$$C_{8,1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&2&0&0&0&0&0&0\\0&0&3&0&0&0&0&0\\0&0&0&4&0&0&0&0\\0&0&0&0&5&0&0&0\\0&0&0&0&0&6&0&0\\0&0&0&0&0&0&7&0\end{pmatrix}, \tag{44}$$

$$C_{6,2}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&3&0&0&0&0\\0&0&6&0&0&0\\0&0&0&10&0&0\end{pmatrix}, \tag{45}$$

(46)
$$C_{8,2}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&3&0&0&0&0&0&0\\0&0&6&0&0&0&0&0\\0&0&0&10&0&0&0&0\\0&0&0&0&15&0&0&0\\0&0&0&0&0&21&0&0\end{pmatrix},$$

(47)
$$C_{8,3}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&4&0&0&0&0&0&0\\0&0&10&0&0&0&0&0\\0&0&0&20&0&0&0&0\\0&0&0&0&35&0&0&0\end{pmatrix},$$

(48)
$$C^{\vee}_{l,m}(z)=\sum_{k=0}^{m}\frac{1}{k!}(\log(z))^kC_{4+2l,m-k},$$

where $l=0,\ 1,\ 2,\ m\in[0,l+1]\cap\mathbb{Z}$. Clearly, $C^{\vee}_{l,0}(z)=E_{4+2l}$,

(49)
$$\delta C^{\vee}_{l,m}(z)=\sum_{k=0}^{m}\frac{1}{k!}(\log(z))^kC_{4+2ll,m-k}=$$
$$\sum_{k=1}^{m}\frac{k}{k!}(\log(z))^{k-1}C_{4+2ll,m-k}=$$
$$\sum_{k=0}^{m-1}\frac{1}{k!}(\log(z))^kC_{4+2ll,m-1-k}=C^{\vee}_{l,m-1}(z).$$

Then in view of (38), (39), (40),

(50)
$$X^{\wedge}_{l,3}(z;\nu)=((\log(z))C_{4+2l,0}+C_{4+2l,1})X^{\wedge}_{l,2}(z;\nu)+$$
$$X^{\wedge}_{l,4}(z;\nu)=C^{\vee}_{l,0}(z)X^{\wedge}_{l,4}(z;\nu)+C^{\vee}_{l,1}(z)X^{\wedge}_{l,2}(z;\nu),$$

where $l=0,\ 1,\ 2,\ |z|>1,\nu\in\mathbb{Z}$,

(51)
$$X^{\wedge}_{l,5}(z;\nu)=X^{\wedge}_{l,6}(z;\nu)+$$
$$((\log(z))C_{4+2l,0}+C_{4+2l,1})X^{\wedge}_{l,3}(z;\nu)-$$
$$\left(\frac{1}{2}(\log(z))^2C_{4+2l,0}+(\log(z))C_{4+2l,1}+C_{4+2l,2}\right)X^{\wedge}_{l,2}(z;\nu)=$$
$$C^{\vee}_{l,0}(z)X^{\wedge}_{l,6}(z;\nu)+C^{\vee}_{l,1}(z)X^{\wedge}_{l,3}(z;\nu)-C^{\vee}_{l,2}(z)X^{\wedge}_{l,2}(z;\nu)=$$
$$X^{\wedge}_{l,6}(z;\nu)+((\log(z))E_{4+2l}+C_{4+2l,1})X^{\wedge}_{l,4}(z;\nu)+$$

$$\left(\frac{1}{2}(\log(z))^2 C_{4+2l,0} + (\log(z))C_{4+2l,1} + C_{4+2l,2}\right) X^{\wedge}_{l,2}(z;\nu) =$$

$$C^{\vee}_{l,0}(z)X^{\wedge}_{l,6}(z;\nu) + C^{\vee}_{l,1}(z)X^{\wedge}_{l,4}(z;\nu) + C^{\vee}_{l,2}(z)X^{\wedge}_{l,2}(z;\nu),$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(52) $$X^{\wedge}_{l,7}(z;\nu) =$$

$$\left(\frac{1}{6}(\log(z))^3 C_{4+2l,0} + \frac{1}{2}(\log(z))^2 C_{8,1} + (\log(z))C_{4+2l,2} + C_{4+2l,3}\right) X^{\wedge}_{l,2}(z,\nu) -$$

$$\left(\frac{1}{2}(\log(z))^2 C_{4+2l,0} + (\log(z))C_{4+2l,1} + C_{4+2l,2}\right) X^{\wedge}_{l,3}(z,\nu) +$$

$$\left((\log(z))C_{4+2l,0} + C_{4+2l,1}\right) X^{\wedge}_{l,5}(z,\nu) + X^{\wedge}_{l,8}(z,\nu) =$$

$$\left(\frac{1}{6}(\log(z))^3 C_{4+2l,0} + \frac{1}{2}(\log(z))^2 C_{8,1} + (\log(z))C_{4+2l,2} + C_{4+2l,3}\right) X^{\wedge}_{l,2}(z,\nu) +$$

$$\left(\frac{1}{2}(\log(z))^2 E_{4+2l} + (\log(z))C_{4+2l,1} + C_{4,2}\right) X^{\wedge}_{l,4}(z,\nu) +$$

$$((\log(z))C_{4+2l,0} + C_{4+2l,1})X^{\wedge}_{l,6}(z,\nu) + X^{\wedge}_{l,8}(z,\nu) =$$

$$C^{\vee}_{l,0}(z)X^{\wedge}_{l,8}(z;\nu) + C^{\vee}_{l,1}(z)X^{\wedge}_{l,6}(z;\nu) + C_{l,2)}{}^{\vee}(z)X^{\wedge}_{l,4}(z;\nu) + C^{\vee}_{l,3}(z)X^{\wedge}_{l,2}(z;\nu)$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$.

In view of (50) - (52)

(53) $$\delta X^{\wedge}_{l,3}(z;\nu) = \delta X^{\wedge}_{l,4}(z;\nu) +$$

$$C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,2}(z;\nu) + C^{\vee}_{l,0}(z)X^{\wedge}_{l,2}(z;\nu),$$

$l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(54) $$\delta X^{\wedge}_{l,5}(z;\nu) = \delta X^{\wedge}_{l,6}(z;\nu) +$$

$$C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,4}(z;\nu) + C_{l,2)}\delta X^{\wedge}_{l,2}(z;\nu) + X^{\wedge}_{l,3}(z;\nu),$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(55) $$\delta X^{\wedge}_{l,7}(z;\nu) = \delta X^{\wedge}_{l,8}(z,\nu) + C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,6}(z;\nu) +$$

$$C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,6}(z;\nu) + C_{l,2)}{}^{v}ee(z)\delta X^{\wedge}_{l,4}(z;\nu) + C^{(}_{l,3}z)\delta X^{\wedge}_{l,2}(z;\nu) + X^{\wedge}_{l,5}(z;\nu),$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. Let

$$X^{\wedge}_{l,3,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,3}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, X^{\wedge}_{l,4,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,4}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$

$$C_{l,3,4}(z) = \begin{pmatrix} C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) \\ 0E_{4+2l} & C^{\vee}_{l,0}(z) \end{pmatrix},$$

where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Then, in view of (50)

(56) $$X^{\wedge}_{l,3,2}(z,\nu) = C_{l,3,4}(z)X^{\wedge}_{l,4,2}(z,\nu),$$

where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Let

$$X^{\wedge}_{l,5,3,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,5}(z,\nu) \\ X^{\wedge}_{l,3}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, X^{\wedge}_{l,6,4,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,6}(z,\nu) \\ X^{\wedge}_{l,4}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$

$$C_{l,5,6}(z) = \begin{pmatrix} C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) & C^{\vee}_{l,2}(z) \\ 0E_{4+2l} & C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) \\ 0E_{4+2l} & 0C^{\vee}_{l,0}(z) & C^{\vee}_{l,0}(z) \end{pmatrix},$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Then, in view of (51)

$$X^{\wedge}_{l,5,3,2}(z,\nu) = C_{l,5,6}(z)X^{\wedge}_{l,6,4,2}(z,\nu), \tag{57}$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Let

$$X^{\wedge}_{l,7,5,3,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,7}(z,\nu) \\ X^{\wedge}_{l,5}(z,\nu) \\ X^{\wedge}_{l,3}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, X^{\wedge}_{l,8,6,4,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,8}(z,\nu) \\ X^{\wedge}_{l,6}(z,\nu) \\ X^{\wedge}_{l,4}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$

$$C_{l,7,8} = \begin{pmatrix} C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) & C^{\vee}_{l,2}(z) & C^{\vee}_{l,3}(z) \\ 0E_{4+2l} & C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) & C^{\vee}_{l,2}(z) \\ 0E_{4+2l} & 0C^{\vee}_{l,0}(z) & C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) \\ 0E_{4+2l} & 0C^{\vee}_{l,0}(z) & 0C^{\vee}_{l,0}(z) & C^{\vee}_{l,0}(z) \end{pmatrix},$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. Then, in view of (52)

$$X^{\wedge}_{l,7,5,3,2}(z,\nu) = C_{l,7,8}X^{\wedge}_{l,8,6,4,2}(z,\nu), \tag{58}$$

Let

$$A^{*}_{l,1}(z;\nu) = A^{*}_{l}(z;\nu),$$

$$A^{\sim}_{l,1}(z;\nu) = A^{\sim}_{l}(z;\nu),$$

$$T^{*}_{l,1,\lambda} = T_{4+2l,\lambda},$$

$$A^{*}_{l,2}(z;\nu) = \begin{pmatrix} A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) \\ 0A^{*}_{l}(z;\nu) & A^{*}_{l}(z;\nu) \end{pmatrix},$$

$$A^{\sim}_{l,2}(z;\nu) = \begin{pmatrix} A^{\sim}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) \\ 0A^{\sim}_{l}(z;\nu) & A^{\sim}_{l}(z;\nu) \end{pmatrix},$$

$$T^{*}_{l,2,\lambda} = \begin{pmatrix} T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & T_{4+2l,\lambda} \end{pmatrix},$$

$$A^{*}_{l,3}(z;\nu) = \begin{pmatrix} A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) \\ 0A^{*}_{l}(z;\nu) & A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) \\ 0A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) & A^{*}_{l}(z;\nu) \end{pmatrix},$$

$$A^{\sim}_{l,3}(z;\nu) = \begin{pmatrix} A^{\sim}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) & 0A^{\sim}_{l}(z;\nu) \\ 0A^{\sim}_{l}(z;\nu) & A^{\sim}_{l}(z;\nu) & 0A^{\sim}_{l}(z;\nu) \\ 0A^{\sim}_{l}(z;\nu) & 0A^{\sim}_{l}(z;\nu) & A^{\sim}_{l}(z;\nu) \end{pmatrix},$$

$$T^{*}_{l,3,\lambda} = \begin{pmatrix} T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & T_{4+2l,\lambda} \end{pmatrix},$$

$$A^*_{l,4}(z;\nu) = \begin{pmatrix} A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) \\ 0A^*_l(z;\nu) & A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) \\ 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & A^*_l(z;\nu) & 0A^*_l(z;\nu) \\ 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & A^*_l(z;\nu) \end{pmatrix},$$

$$A^\sim_{l,4}(z;\nu) = \begin{pmatrix} A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) \\ 0A^\sim_l(z;\nu) & A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) \\ 0A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) & A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) \\ 0A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) & 0A^\sim_l(z;\nu) & A^\sim_l(z;\nu) \end{pmatrix},$$

$$T^*_{l,4,\lambda} = \begin{pmatrix} T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & T_{4+2l,\lambda} \end{pmatrix}.$$

In view of (35)

$$A^*_{l,i}(z;\nu) = \nu^{3+2l} T_{l,i,\nu} A^\sim_{l,i}(z;\nu) T_{l,i,\nu^{-1}} \tag{59}$$

for $i = 1, 2, 3, 4$. In view of (33), (34),

$$\nu^{3+2l} X^\wedge_{l,3,2}(z;\nu-1) = A^*_{l,2}(z;\nu) X^\wedge_{l,3,2}(z;\nu), \tag{60}$$

$$(-\nu)^{3+2l} X^\wedge_{l,3,2}(z;\nu) = A^*_{l,2}(z;-\nu) X^\wedge_{l,3,2}(z;\nu-1), \tag{61}$$

where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

$$\nu^{3+2l} X^\wedge_{l,5,3,2}(z;\nu-1) = A^*_{l,3}(z;\nu) X^\wedge_{l,5,3,2}(z;\nu), \tag{62}$$

$$(-\nu)^{3+2l} X^\wedge_{l,5,3,2}(z;\nu) = A^*_{l,3}(z;-\nu) X^\wedge_{l,5,3,2}(z;\nu-1), \tag{63}$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

$$\nu^{3+2l} X^\wedge_{l,7,5,3,2}(z;\nu-1) = A^*_{l,4}(z;\nu) X^\wedge_{l,7,5,3,2}(z;\nu), \tag{64}$$

$$(-\nu)^{3+2l} X^\wedge_{l,7,5,3,2}(z;\nu) = A^*_{l,4}(z;-\nu) X^\wedge_{l,7,5,3,2}(z;\nu-1), \tag{65}$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. In view of (56), (60), (61),

$$\nu^{3+2l} X^\wedge_{l,4,2}(z;\nu-1) = A^{**}_{l,2}(z;\nu) X^\wedge_{l,4,2}(z;\nu), \tag{66}$$

$$(-\nu)^{3+2l} X^\wedge_{l,4,2}(z;\nu) = A^{**}_{l,2}(z;-\nu) X^\wedge_{l,4,2}(z;\nu-1), \tag{67}$$

where

$$A^{**}_{l,2}(z;\nu) = (C_{l,3,4}(z))^{-1} A^*_{l,2}(z;\nu) C_{l,3,4}(z),$$

$l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$.

In view of (57), (62), (63),

$$\nu^{3+2l} X^\wedge_{l,6,4,2}(z;\nu-1) = A^{**}_{l,3}(z;\nu) X^\wedge_{l,6,4,2}(z;\nu), \tag{68}$$

(69) $$(-\nu)^{3+2l}X^{\wedge}_{l,6,4,2}(z;\nu) = A^{**}_{l,3}(z;-\nu)X^{\wedge}_{l,6,4,2}(z;\nu-1),$$

where
$$A^{**}_{l,3}(z;\nu) = (C_{l,5,6}(z))^{-1}A^{*}_{l,3}(z;\nu)C_{l,5,6}(z),$$
$l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. In view of (58), (64), (65),

(70) $$\nu^{3+2l}X^{\wedge}_{l,8,6,4,2}(z;\nu-1) = A^{**}_{l,4}(z;\nu)X^{\wedge}_{l,8,6,4,2}(z;\nu),$$

(71) $$(-\nu)^{3+2l}X^{\wedge}_{l,8,6,4,2}(z;\nu) = A^{**}_{l,3}(z;-\nu)X^{\wedge}_{l,6,4,2}(z;\nu-1),$$

where
$$A^{**}_{l,4}(z;\nu) = (C_{l,7,8}(z))^{-1}A^{*}_{l,4}(z;\nu)C_{l,7,8}(z),$$
$l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Since
$$Y_{l,k}(z,\nu) = (T_{4+2l,\nu})^{-1}(z,\nu)X_{l,k}(z,\nu),$$
where $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}_l,\ |z| > 1, \nu \in \mathbb{N}, \nu \geq 2$, let
$$Y^{\wedge}_{l,k}(z,\nu) = (T_{4+2l,\nu})^{-1}(z,\nu)X^{\wedge}_{l,k}(z,\nu),$$
where $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}^{\wedge}_l,\ |z| > 1, \nu \in \mathbb{N}, \nu \geq 2$,
$$Y^{\wedge}_{l,3,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,3}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, Y^{\wedge}_{l,4,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,4}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$
where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$, let
$$Y^{\wedge}_{l,5,3,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,5}(z,\nu) \\ Y^{\wedge}_{l,3}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, Y^{\wedge}_{l,6,4,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,6}(z,\nu) \\ Y^{\wedge}_{l,4}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$
where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$, let
$$Y^{\wedge}_{l,7,5,3,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,7}(z,\nu) \\ Y^{\wedge}_{l,5}(z,\nu) \\ Y^{\wedge}_{l,3}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, Y^{\wedge}_{l,8,6,4,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,8}(z,\nu) \\ Y^{\wedge}_{l,6}(z,\nu) \\ Y^{\wedge}_{l,4}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$
where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. Then

(72) $$Y^{\wedge}_{l,3,2}(z,\nu) = (T_{l,2,\nu}*)^{-1}X^{\wedge}_{l,3,2}(z,\nu),$$

(73) $$Y^{\wedge}_{l,4,2}(z,\nu) = (T^{*}_{l,2,\nu})^{-1}X^{\wedge}_{l,4,2}(z,\nu),$$

for $l = 0, 1, 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(74) $$Y^{\wedge}_{l,5,3,2}(z,\nu) = (T^{*}_{l,3,\nu})^{-1}X^{\wedge}_{l,5,3,2}(z,\nu),$$

(75) $$Y^{\wedge}_{l,6,4,2}(z,\nu) = (T^{*}_{l,3,\nu})^{-1}X^{\wedge}_{l,6,4,2}(z,\nu),$$

for $l = 1, 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(76) $$Y^{\wedge}_{l,7,5,3,2}(z,\nu) = (T^{*}_{l,4,\nu})^{-1} X^{\wedge}_{l,7,5,3,2}(z,\nu),$$

(77) $$Y^{\wedge}_{l,8,6,4,2}(z,\nu) = (T^{*}_{l,4,\nu})^{-1} X^{\wedge}_{l,8,6,4,2}(z,\nu),$$

for $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. In view of (59) – (65), (72) – (77),

(78) $$Y^{\wedge}_{l,3,2}(z;\nu-1) = ((T_{2,1-1/nu}))^{-1} A^{\sim}_{l,2}(z;\nu) Y^{\wedge}_{l,3,2}(z;\nu),$$

(79) $$Y^{\wedge}_{l,3,2}(z;\nu) = T_{2,-1} A^{\sim}_{l,2}(z;\nu) Y^{\wedge}_{l,3,2}(z;\nu-1) T_{2,-1+1/nu},$$

(80) $$Y^{\wedge}_{l,5,3,2}(z;\nu-1) = ((T_{2,1-1/nu}))^{-1} A^{\sim}_{l,3}(z;\nu) Y^{\wedge}_{l,5,3,2}(z;\nu),$$

(81) $$Y^{\wedge}_{l,5,3,2}(z;\nu) = T_{3,-1} A^{\sim}_{l,3}(z;\nu) Y^{\wedge}_{l,5,3,2}(z;\nu-1) T_{3,-1+1/nu},$$

(82) $$Y^{\wedge}_{l,7,5,3,2}(z;\nu-1) = ((T_{4,1-1/nu}))^{-1} A^{\sim}_{l,4}(z;\nu) Y^{\wedge}_{l,7,5,3,2}(z;\nu),$$

(83) $$Y^{\wedge}_{l,7,5,3,2}(z;\nu) = T_{4,-1} A^{\sim}_{l,4}(z;\nu) Y^{\wedge}_{l,7,5,3,2}(z;\nu-1) T_{4,-1+1/nu}.$$

(84) $$Y^{\wedge}_{l,4,2}(z;\nu-1) = ((T_{2,1-1/nu}))^{-1} A^{\sim\sim}_{l,2}(z;\nu) Y^{\wedge}_{l,4,2}(z;\nu),$$

(85) $$Y^{\wedge}_{l,4,2}(z;\nu) = T_{2,-1} A^{\sim\sim}_{l,2}(z;\nu) Y^{\wedge}_{l,4,2}(z;\nu-1) T_{2,-1+1/nu},$$

(86) $$Y^{\wedge}_{l,6,4,2}(z;\nu-1) = ((T_{2,1-1/nu}))^{-1} A^{\sim\sim}_{l,3}(z;\nu) Y^{\wedge}_{l,6,4,2}(z;\nu),$$

(87) $$Y^{\sim}_{l,6,4,2}(z;\nu) = T_{3,-1} A^{\sim\sim}_{l,3}(z;\nu) Y^{\wedge}_{l,6,4,2}(z;\nu-1) T_{3,-1+1/nu},$$

(88) $$Y^{\sim}_{l,8,6,4,2}(z;\nu-1) = ((T_{4,1-1/nu}))^{-1} A^{\sim\sim}_{l,4}(z;\nu) Y^{\wedge}_{l,8,6,4,2}(z;\nu),$$

(89) $$Y^{\sim}_{l,8,6,4,2}(z;\nu) = T_{4,-1} A^{\sim}_{l,4}(z;\nu) Y^{\wedge}_{l,8,6,4,2}(z;\nu-1) T_{4,-1+1/nu}.$$

§6.2. Further properties of my auxiliary fuctions.

In view of (2), (6), (12), (17) and (18) in the Part 1,

(90)
$$f_{l,1}(z,\nu) = f^{\vee}_{l,1}(z,\nu) = \sum_{k=0}^{\nu} (-1)^{(\nu+k)l} (z)^k \binom{\nu}{k}^{2+l} \binom{\nu+k}{\nu}^{2+l},$$

where $l = 0,\ 1,\ 2,\ \nu \in [0,+\infty) \cap \mathbb{Z}$,

(91) $$f_{l,2+2j}(z,\nu) = \frac{(-1)^j}{j!} \times$$

$$\sum_{t=1}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^{j}(R^{2+l})\right)(t,\nu),$$

where $l = 0, 1, 2, j \in [0, l+1] \cap \mathbb{Z}, \nu \in [0, +\infty) \cap \mathbb{Z}, |z| > 1$. Expanding function $(R(t; \nu))^{2+l}$ into partial fractions, we obtain

$$(92) \qquad (R(t;\nu))^{2+l} = \sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu} \alpha_{l,i,k,\nu}(t+k)^{-i}\right)$$

where $l = 0, 1, 2, \nu \in [0, +\infty) \cap \mathbb{Z}$,

$$(93) \qquad \alpha_{l,2+l-j,k,\nu} = \frac{1}{j!}\lim_{t\to -k}\left(\frac{\partial}{\partial t}\right)^{j}(R(t;\nu)(t+k))^{2+l},$$

where $j = 0, \ldots 2 + l - 1$. Let

$$(94) \qquad S_{i,k}(\nu) = -\left(\sum_{\kappa=k+1}^{\nu+k} 1/\kappa^{i}\right) - \left(\sum_{\kappa=1}^{\nu-k} 1/\kappa^{i}\right) + \sum_{\kappa=1}^{k} 1/\kappa^{i},$$

where $\nu \in \mathbb{Z} \cap [0, +\infty),\ i \in \mathbb{N}, k \in \mathbb{Z} \cap [0, +\nu)$. In view of (93), (94),

$$(95) \qquad \alpha_{l,2+l,k,\nu} = (-1)^{l(\nu+k)}((\nu+k)!/((k!)^2(\nu-k)!))^{2+l} =$$

$$(-1)^{l(\nu+k)}\binom{\nu}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l},$$

$$(96) \qquad \alpha_{l,2+l-1,k,\nu} = \alpha_{l,2+l,k,\nu}(2+l)S_{1,k}(\nu)$$

for $l = 0, 1, 2, \nu \in [0, +\infty) \cap \mathbb{Z}$,

$$(97) \qquad 2\alpha_{l,2+l-2,k,\nu} = \alpha_{l,2+l,k,\nu}((2+l)^2(S_{1,k}(\nu))^2 - (2+l)S_{2,k}(\nu))$$

for $l = 1, 2, \nu \in [0, +\infty) \cap \mathbb{Z}$,

$$(98) \qquad 6\alpha_{l,2+l-3,k,\nu} = \alpha_{l,2+l,k,\nu} \times$$

$$((2+l)^3(S_{1,k}(\nu))^3 - 3(2+l)^2(S_{1,k}(\nu))(S_{2,k}(\nu)) + 2(2+l)S_{3,k}(\nu))$$

for $l = 2, \nu \in [0, +\infty) \cap \mathbb{Z}$. In view of (91), (92)

$$(99) \qquad f_{l,2+2j}(z,\nu) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{t=1}^{\infty}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^{k}z^{-t-k}(t+k)^{-i-j}\right)\right) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^{k}\left(\sum_{t=1}^{\infty}z^{-t-k}(t+k)^{-i-j}\right)\right) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^{k}\left(L_{i+j}(1/z) - \sum_{t=1}^{k}z^{-t}(t)^{-i-j}\right)\right) =$$

$$\left(\sum_{i=1}^{2+l}\alpha^*_{l,i}(z;\nu)L_{i+j}(1/z)\right)-\beta^*_{l,j}(z;\nu)=$$

$$\left(\sum_{i=-\infty}^{\infty}\alpha^*_{l,i}(z;\nu)L_{i+j}(1/z)\right)-\beta^*_{l,j}(z;\nu)=$$

where

$$\alpha^*_{l,i}(z;\nu)=0, \tag{100}$$

if $(i-2-l)(i-1)>0$,

$$\alpha^*_{l,i}(z;\nu)=\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^k, \tag{101}$$

for $l=0,\,1,\,2,\;i\in[1,l+2]\cap\mathbb{Z},\;j\in[0,l+1]\cap\mathbb{Z},\;\nu\in[0,+\infty)\cap\mathbb{Z},\;|z|>1,$

$$\beta^*_{l,j}(z;\nu)= \tag{102}$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}\left(\sum_{t=1}^{k}z^{k-t}(t)^{-i-j}\right)\right)$$

for $l=0,\,1,\,2,\;i\in[1,l+2]\cap\mathbb{Z},\;j\in[0,l+1]\cap\mathbb{Z},\;\nu\in[0,+\infty)\cap\mathbb{Z},\;|z|>1,$

$$L_n(1/z)=\sum_{t=1}^{+\infty}z^{-t}t^{-n}, \tag{103}$$

where $n\in\mathbb{Z},\;|z|>1.$

Clearly,

$$(-\delta)^k(L_n(1/z))=L_{n-k}(1/z), \tag{104}$$

where $k\in[0,+\infty)\cap\mathbb{Z},\;n\in\mathbb{Z},\;|z|>1,$

$$L_1(1/z))=-\log(1-1/z), \tag{105}$$

$$-\delta(L_1(1/z))=L_0(1/z)=1+1/(z-1), \tag{106}$$

$$\delta^2(L_1(1/z))=L_{-1}(1/z)=1/(z-1)+1/(z-1)^2, \tag{107}$$

$$-\delta^3(L_1(1/z))=L_{-2}(1/z)= \tag{108}$$

$$1/(z-1)+3/(z-1)^2+2/(z-1)^3,$$

$$\delta^4(L_1(1/z))=L_{-3}(1/z)=\frac{1}{z-1}+ \tag{109}$$

$$\frac{7}{(z-1)^2}+\frac{12}{(z-1)^3}+\frac{6}{(z-1)^4},$$

(110)
$$-\delta^5(L_1(1/z)) = L_{-4}(1/z) = \frac{1}{z-1} + \frac{15}{(z-1)^2} +$$
$$\frac{50}{(z-1)^3} + \frac{60}{(z-1)^4} + \frac{24}{(z-1)^5},$$

(111)
$$\delta^6(L_1(1/z)) = L_{-5}(1/z) = \frac{1}{z-1} + \frac{31}{(z-1)^2} + \frac{180}{(z-1)^3} + \frac{390}{(z-1)^4} +$$
$$\frac{360}{(z-1)^5} + \frac{120}{(z-1)^6},$$

(112)
$$-\delta^7(L_1(1/z)) = L_{-6}(1/z) = \frac{1}{z-1} + \frac{63}{(z-1)^2} + \frac{602}{(z-1)^3} +$$
$$\frac{2100}{(z-1)^4} + \frac{2360}{(z-1)^5} + \frac{2520}{(z-1)^6} + \frac{720}{(z-1)^7},$$

(113)
$$\delta^n(L_1(1/z)) = (-1)^n L_{1-n}(1/z) =$$
$$(-1)^n \sum_{k=1}^{n} a(n,k)1/(z-1)^k,$$

where

(114)
$$a(1,0) = a(1,1) = 1,\ a(n+1,k) = ka(n,k) + (k-1)a(n,k-1)$$

for $k \in [0,n] \cap \mathbb{Z}, a(n,-1) = a(n,n+1) = 0$, and $n \in \mathbb{N}$.

According to (99), (104) and Leibnitz formula,

(115)
$$(\delta)^m f_{l,2+2j}(z,\nu) =$$
$$\sum_{k=0}^{m} \binom{m}{k} \left(\sum_{i=-\infty}^{+\infty} ((-1)^k (\delta^{m-k} \alpha^*_{l,i}(z;\nu)) L_{j+i-k}(1/z) \right) -$$
$$\delta^m \beta^*_{l,j}(z;\nu) =$$
$$\left(\sum_{\kappa=1-m}^{2+l} \left(\sum_{k=\max(0,1-\kappa)}^{min(m,2+l-\kappa)} \binom{m}{k} (-1)^k \delta^{m-k} \alpha^*_{l,k+\kappa}(z;\nu) \right) L_{j+\kappa}(1/z) \right) -$$
$$\delta^m \beta^*_{l,j}(z;\nu) =$$
$$\left(\sum_{\kappa=1-m}^{2+l} \alpha^{*\vee}_{m,l,\kappa}(z;\nu) L_{j+\kappa}(1/z) \right) -$$
$$\delta^m \beta^*_{l,j}(z;\nu),$$

where

(116)
$$\alpha^{*\vee}_{m,l,\kappa}(z;\nu) =$$

$$\sum_{k=\max(0,1-\kappa)}^{min(m,2+l-\kappa)}\binom{m}{k}(-1)^k\delta^{m-k}\alpha^*_{l,k+\kappa}(z;\nu),$$

where $m\in\mathbb{N}, l=0,\ 1,\ 2,\ i\in[1,l+2]\cap\mathbb{Z},\ j\in[0,l+1]\cap\mathbb{Z},\ \nu\in[0,+\infty)\cap\mathbb{Z},\ |z|>1$. In view of (91),

(117)
$$\delta^m f_{l,2+2j}(z,\nu)=\frac{(-1)^j}{j!}\times$$

$$\sum_{t=1}^{\infty}z^{-t}(-t)^m\left(\left(\frac{\partial}{\partial t}\right)^j\left(R^{2+l}\right)\right)(t,\nu),$$

where $l=0,\ 1,\ 2,\ \{m,\nu\}\subset[0,\infty)\cap\mathbb{Z},\ j\in[0,l+1]\cap\mathbb{Z},\ |z|>1$. Therefore, in view of (2), if

$$m\le l+j,$$

then

$$\frac{(-1)^j}{j!}Res_{t=\infty}\left((-t)^m\left(\frac{\partial}{\partial t}\right)^j\left(R^{2+l}\right)\right)(t,\nu)=0,$$

and, consequently, in view of (116),

(118)
$$\alpha^{*\vee}_{m,l,1-j}(z,\nu)=$$

$$\sum_{k=j}^{m}\binom{m}{k}(-1)^k(\delta^{m-k}\alpha^*_{l,k+1-j})(1;\nu)=0,$$

$$\sum_{i=0}^{m-j}\binom{m}{i+j}(-1)^{i+j}(\delta^{m-j-i}\alpha^*_{l,i+1})(1;\nu)=0,$$

(119)
$$\lim_{z\to1}\alpha^{*\vee}_{m,l,1-j}(z;\nu)/(z-1)=(\delta\alpha^{*\vee}_{m,l,1})(1;\nu)$$

moreover, there exists

(120)
$$\lim_{z\to1}\delta^m f_{l,2+2j}(z;\nu)$$

in this case. If $m=l+j+1$, then in view of (117),

(121)
$$\delta^{l+j+1}f_{l,2+2j}(z,\nu)=\frac{(-1)^j}{j!}\times$$

$$\sum_{t=1}^{+\infty}z^{-t}(-t)^{l+j+1}\left(\left(\frac{\partial}{\partial t}\right)^j\left(R^{2+l}\right)\right)(t,\nu)=O(-log(1-1/|z|)$$

for fixed $\nu\in[0,\infty)\cap\mathbb{Z}$ and $|z|\to1+0$, where $l=0,\ 1,\ 2,\ j\in[0,l+1]\cap\mathbb{Z}$; on the other hand, in view of (115)

(122)
$$\delta^{l+1+j}f_{l,2+j}(z,\nu)=$$

$$\left(\sum_{\kappa=-l-j}^{2+l}\alpha^{*\vee}_{l+j+1,l,\kappa}(z;\nu)L_\kappa(1/z)\right)-$$

$$\delta^{l+j+1}\beta^*_{l,0}(z;\nu);$$

in view of (113), (114),

$$\lim_{|z|>1,\, z\to 1,} L_{1-n}(1/z)(z-1)^n \neq 0,$$

where $n \in \mathbb{N}$. Multiplying sequentially (121) by $(z-1)^n$ for $n = l+j+1$, ..., 1 and turning $z \in (1, +\infty)$ to 1, we obtain equalities

$$\alpha^{*\vee}_{l+j+1,l,\kappa}(1;\nu) = 0$$

for $\kappa = -l-j$, ..., 0; if $-l-j \leq \kappa \leq -j$, then $m = l+j+1 < 2+l-\kappa$ and, in view of (116),

$$0 = \alpha^{*\vee}_{l+1+j,l,\kappa}(z;\nu) = \tag{123}$$

$$\sum_{k=1-\kappa}^{1+l+j} \binom{m}{k} (-1)^k \delta^{m-k} \alpha^*_{l,k+\kappa}(1;\nu)$$

for $\kappa = -l-j$, ..., $-j$; in particular, (123) holds for $j = 0$; if $j \geq 1$ and $1-j \leq \kappa \leq 0$, then $m = l+j+1 \geq 2+l-\kappa$ and

$$0 = \alpha^{*\vee}_{l+1+j,l,\kappa}(z;\nu) = \tag{124}$$

$$\sum_{k=1-\kappa}^{2+l-\kappa} \binom{m}{k} (-1)^k \delta^{m-k} \alpha^*_{l,k+\kappa}(1;\nu)$$

for $\kappa = 1-j$, ..., 0.

Lemma 2.1. *Let*

$$s \in \mathbb{N},\ k = 1,\ ...,\ s,\ a_{0,k} \in \mathbb{C},\ \alpha_{0,k} \in \mathbb{Z},$$

$$\sigma = \sum_{k=1}^{s} \alpha_{0,k} \neq 0 \tag{125}$$

$$f(z) = \prod_{k=1}^{s} (z - a_{0,k})^{\alpha_{0,k}},\ \delta = z\frac{\partial}{\partial z}.$$

Then for $i = 1$, ..., $2s$ *there exist* $a_{1,i} \in \mathbb{C}$, $\alpha_{1,i} \in \mathbb{Z}$ *such that*

$$\sum_{i=1}^{2s} \alpha_{1,i} = \sigma,$$

$$f_1(z) = (\delta f)(z) = \sigma \prod_{i=1}^{2s} (t - a_{1,i})^{\alpha_{1,i}}$$

Proof. Clearly,

$$f_1(z) = f(z) \left(\sum_{k=1}^{s} \frac{z\alpha_{0,k}}{z - a_{0,k}} \right)$$

and, in view of (125),

$$f(z) = \sigma \prod_{k=1}^{s} (z - a_{0,k})^{\alpha_{0,k}-1}(z - a_{1,k}),$$

for some $a_{1,k} \in \mathbb{C}$, where $k = 1, ..., s$. So, we can take $\alpha_{1,k} = 1, \alpha_{1,s+k} = \alpha_{0,k} - 1$, $a1, s + k = a_{0,k}$ for $k = 1, ..., s$. Clearly,

$$\sum i = 1^{2s} \alpha_{1,i} = \sum k = 1^s \alpha_{0,k} = \sigma.$$

■

Corollary 1. *If condition of the Lemma 2.1 are fulflled, and* $n \in \mathbb{N}$, *then for* $i = 1, ..., 2^n s$ *there exist* $a_{n,i} \in \mathbb{C}$, $\alpha_{n,i} \in \mathbb{Z}$ *such that*

$$\sum_{i=1}^{2^n s} \alpha_{n,i} = \sigma,$$

(126) $$f_n(z) = (\delta^n f)(z) = (\sigma)^n \prod_{i=1}^{2^n s} (z - a_{n,i})^{\alpha_{n,i}},$$

and

(127) $$\lim_{z \to \infty} z^{-\sigma} (\delta^n f)(z) = (\sigma)^n$$

Proof. The equality (126) can be obtained by sequential applying of the Lemma 2.1. The equality (127) directly follows from (126). ■

Corollary 2. *If condition of the Lemma 2.1 are fulflled and* $\sigma = -1$, *then*

$$Res_{z=\infty}(((-\delta)^n f)(z)) = -1$$

for any $n \in [0, +\infty \cap \mathbb{Z}$. **Proof.** The assertion easy follows from Corollary 1. ■

Corollary 3. *If condition of the Lemma 2.1 are fulflled, and* $n \in \mathbb{N}$, *then*

$$\lim_{z=\infty} \left((z)^{n-\sigma} \left(\left(\frac{\partial}{\partial z} \right)^n f \right) (z) \right) = (-1)^n (-\sigma)_n = \prod_{i=1}^{n} (\sigma + 1 - i).$$

for any $n \in [0, +\infty) \cap \mathbb{Z}$.

Proof. For $n = 0$ and $n = 1$ the assertion follows from the Corollary 2. Let $\mathfrak{m}_z$ be operator, which transforms a function $g(z)$ into $zg(z)$. Then

$$\delta(\mathfrak{m}_z)^n \left(\frac{\partial}{\partial z} \right)^n = n(\mathfrak{m}_z)^n \left(\frac{\partial}{\partial z} \right)^n + (\mathfrak{m}_z)^{n+1} \left(\frac{\partial}{\partial t} \right)^{n+1},$$

$$(\mathfrak{m}_z)^{n+1} \left(\frac{\partial}{\partial t} \right)^{n+1} = (\delta - n)(\mathfrak{m}_z)^n \left(\frac{\partial}{\partial z} \right)^n.$$

Therefore

$$(\mathfrak{m}_z)^n \left(\frac{\partial}{\partial z} \right)^n =$$

$$\prod_{i=1}^{n}(\delta+1-i)=P_n(\delta),$$

where

$$P_n(z)=\prod_{i=1}^{n}(z-1+i)=(-1)^n(-z)_n=\sum_{k=1}^{n}c_kz^k,$$

and $(-z)_n$ is the Pochhammer symbol. Therefore, in view of (127),

$$\lim_{z=\infty}\left((z)^{n-\sigma}\left(\left(\frac{\partial}{\partial z}\right)^n f\right)(z)\right)=$$

$$\sum_{k=1}^{n}c_k\lim_{z=\infty}z^{-\sigma}(\delta^k f)(z)=$$

$$\sum_{k=1}^{n}c_k\sigma^k=(\sigma)_k=\prod_{i=1}^{n}(\sigma+1-i)=(-1)^n(-\sigma)_n.$$

■

Corollary 4. *If* $\sigma=-1$, *then*

$$Res_{z=\infty}\left((-z)^n\left(\left(\frac{\partial}{\partial z}\right)^n f\right)(z)\right)=-n!$$

for any $n\in[0,+\infty)\cap\mathbb{Z}$.

Proof. The assertion direct follows from Corollary 3. ■

The above Corollary 2 easy follows from the followoing obvious

Lemma 2.2. *Let* $R>0$. *Let further a function* $f(z)$ *is regular in the domain* $\{z\in\mathbb{C}\colon R<|z|\}$. *Then*

(128) $$Res_{z=\infty}(((-\delta)^n f)(z))=Res_{z=\infty}(f(z))$$

for any $n\in[0,+\infty\cap\mathbb{Z}$. ■

Lemma 2.3. *If condition of the Lemma 2.1 are fulfilled,* $\sigma=0$, *and*

(129) $$\sigma_1=\sum_{k=1}^{s}a_k\alpha_k,$$

then

(130) $$Res_{z=\infty}(((-\delta)^n f)(z))=\sigma_1$$

for any $n\in[0,+infty\cap\mathbb{Z}$.

Proof. Clearly,

$$f(z)=\prod_{k=1}^{s}(1-a_{0,k}/z)^{\alpha_{0,k}}$$

in this case. Therefore all the conditions of the Lemma 2.2 are fulfilled. ■

Let again $m=l+j+1$; clearly, if $j=0$, then

$$Res_{t=\infty}(-t)^{1+l}R^{2+l}(t,\nu)=(-1)^l,$$

and therefore, in view of (116),

(131) $$\sum_{k=0}^{l+1}\binom{l+1}{k}(-1)^k(\delta^{l+1-k}\alpha^*_{l,1})(1;\nu)=$$

$$\alpha^{*\vee}_{l+1,l,1}(1;\nu)=(-1)^{l+1}.$$

If $m=l+j+s$, $s\geq 2$, then in view of (117),

(132) $$\delta^{l+j+2}f_{l,2+2j}(z,\nu)=\frac{(-1)^j}{j!}\times$$

$$\sum_{t=1}^{+\infty}z^{-t}(-t)^{l+j+s}\left(\left(\frac{\partial}{\partial t}\right)^j(R^{2+l})\right)(t,\nu)=O(1/(|z|-1)^{s-1})$$

for fixed $\nu\in[0,\infty)\cap\mathbb{Z}$ and $|z|\to 1+0$, where $l=0,\ 1,\ 2,\ j\in[0,l+1]\cap\mathbb{Z}$. In particular, if $j=0$, $m=l+s, s\geq 2$, $l=0,1,2$, then, in view of (117) and (132),

(133) $$\delta^{l+s}f_{l,2}(z,\nu)=O(1/(|z|-1)^{s-1}),\ (|z|\to 1+0);$$

on the other hand, in view of (115),

(134) $$\delta^{l+s}f_{l,2}(z,\nu)=$$

$$\left(\sum_{\kappa=-l+1-s}^{2+l}\alpha^{*\vee}_{l+1,l,\kappa}L_\kappa(1/z)\right)-$$

$$\delta^{l+s}\beta^*_{l,0}(z;\nu);$$

in view of (113), (114),

$$\lim_{|z|>1,\,z\to 1,}L_{1-n}(1/z)(z-1)^n\neq 0,$$

where $n\in\mathbb{N}$. Multiplying sequentially (133) by $(z-1)^n$ for $n=l+s$, ..., s and turning $z\in(1,+\infty)$ to 1, we obtain equalities

$$\alpha^{*\vee}_{l+s,l,\kappa}=0$$

for $\kappa=1-l-s$, ..., $1-s$; in this case $2+l-\kappa\geq 2+l-(1-s)>l+s=m$, $1-\kappa>0$; therefore, in view of (116),

(135) $$\sum_{k=1-\kappa}^{l+s}\binom{l+s}{k}(-1)^k\delta^{l+s-k}\alpha^*_{l,k+\kappa}(1;\nu)=0$$

for $\kappa=1-l-s$, ..., $1-s$.

§6.3 Return from ν to (τ,μ) in the case $l=0$.

According to the Theorem 1,

$$A_0^{\sim}(z;\nu) = S_0^{\sim} + z\sum_{i=0}^{1}\nu^{-i}V_0^{\sim *}(i) \tag{136}$$

with

$$S_0^{\sim} = \begin{pmatrix} 1 & -4 & 8 & -12 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{137}$$

$$V_0^{\sim *}(0) = 4\begin{pmatrix} 4 & -5 & -2 & 3 \\ -3 & 4 & 1 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{pmatrix}, \tag{138}$$

$$V_0^{\sim *}(1) = 4\begin{pmatrix} 3 & -6 & 3 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{139}$$

In view of (3) in Part 4,

$$A_0^{*}(z;\nu) = \nu^3 T_{4,\nu)}A_0^{\sim}(z;\nu)(T_{4,\nu)})^{-1} = \tag{140}$$

$$\begin{pmatrix} \nu^3 & -4\nu^2 & 8\nu & -12 \\ 0 & \nu^3 & -4\nu^2 & 8\nu \\ 0 & 0 & \nu^3 & -4\nu^2 \\ 0 & 0 & 0 & \nu^3 \end{pmatrix} +$$

$$z4\begin{pmatrix} 4\nu^3 & -5\nu^2 & -2\nu & 3 \\ -3\nu^4 & 4\nu^3 & \nu^2 & -2\nu \\ 2\nu^5 & -3\nu^4 & 0 & \nu^2 \\ -\nu^6 & 2\nu^5 & -\nu^4 & 0 \end{pmatrix} +$$

$$4z\begin{pmatrix} 3\nu^2 & -6\nu & 3 & 0 \\ -2\nu^3 & 4\nu^2 & -2\nu & 0 \\ \nu^4 & -2\nu^3 & \nu^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$(z-1)\begin{pmatrix} 16\nu^3+12\nu^2 & -20\nu^2-24\nu & -8\nu+12 & 12 \\ -12\nu^4-8\nu^3 & 16\nu^3+16\nu^2 & 4\nu^2-8\nu & -8\nu \\ 8\nu^5+4\nu^4 & -12\nu^4-8\nu^3 & 4\nu^2 & 4\nu^2 \\ -4\nu^6 & 8\nu^5 & -4\nu^4 & 0 \end{pmatrix} +$$

$$\begin{pmatrix} 17\nu^3+12\nu^2 & -24\nu^2-24\nu & 12 & 0 \\ -12\nu^4-8\nu^3 & 17\nu^3+16\nu^2 & -8\nu & 0 \\ 8\nu^5+4\nu^4 & -12\nu^4-8\nu^3 & \nu^3+4\nu^2 & 0 \\ -4\nu^6 & 8\nu^5 & -4\nu^4 & \nu^3 \end{pmatrix}.$$

Clearly,

(141) $$\nu = -1/2 + \tau,\ \nu^2 = \mu + 1/2 - \tau,$$

(142) $$\nu^3 = (\mu + 1)(\tau - 3/2) + 1 = -3\mu/2 - 1/2 + (\mu + 1)\tau,$$

(143) $$\nu^4 = \mu^2 + 2\mu + 1/2 - (2\mu + 1)\tau,$$

(144) $$\nu^5 = (\mu - \nu)\nu^3 = -3\mu^2/2 - \mu/2 + (\mu^2 + \mu)\tau -$$
$$(\mu^2 + 2\mu + 1/2 - (2\mu + 1)\tau) =$$
$$-5\mu^2/2 - 5\mu/2 - 1/2 + (\mu^2 + 3\mu + 1)\tau,$$

(145) $$\nu^6 = (\mu - \nu)\nu^4 = \mu^3 + 2\mu^2 + \mu/2 - (2\mu^2 + \mu)\tau -$$
$$(-5\mu^2/2 - 5\mu/2 - 1/2 + (\mu^2 + 3\mu + 1)\tau) =$$
$$\mu^3 + 9\mu^2/2 + 3\mu + 1/2 - (3\mu^2 + 4\mu + 1)\tau.$$

Therefore

(146) $$16\nu^3 + 12\nu^2 = -12\mu - 2 + (16\mu + 4)\tau$$

(147)
$$-20\nu^2 - 24\nu = -20\mu - 4\nu = -20\mu + 2 - 4\tau,\ -8\nu + 12 = -8\tau + 16,$$

(148) $$-12\nu^4 - 8\nu^3 = -12\mu^2 - 12\mu - 2 + (16\mu + 4)\tau,$$

(149) $$16\nu^3 + 16\nu^2 = 16\mu(\tau - 1/2) = -8\mu + 16\mu\tau,$$

(150) $$4\nu^2 - 8\nu = 4\mu + 6 - 12\tau,\ -8\nu = -8\tau + 4,$$

(151) $$8\nu^5 + 4\nu^4 = -16\mu^2 - 12\mu - 2 + (8\mu^2 + 16\mu + 4)\tau$$

(152) $$4\nu^2 = 4\mu + 2 - 4\tau.$$

In view of (143) and (143),

(153) $$-4\nu^6 = -4\mu^3 - 18\mu^2 - 12\mu - 2 + (12\mu^2 + 16\mu + 4)\tau$$

(154) $$8\nu^5 = -20\mu^2 - 20\mu - 4 + (8\mu^2 + 24\mu + 8)\tau,$$

(155) $$-4\nu^4 = -4\mu^2 - 8\mu - 2 + (8\mu + 4)\tau,$$

(156) $$17\nu^3 + 12\nu^2 = -27\mu/2 - 5/2 + (17\mu + 5)\tau,$$

$$-24\nu^2 - 24\nu = -24\mu, \tag{157}$$

$$-8\nu + 12 = 16 - 8\tau, \tag{158}$$

$$17\nu^3 + 16\nu^2 = -19\mu/2 - 1/2 + (17\mu + 1)\tau \tag{159}$$

$$\nu^3 + 4\nu^2 = 5\mu/2 + 3/2 + (\mu - 3)\tau. \tag{160}$$

In view of (140) – (160)

$$A_0^*(z;\nu) = \tag{161}$$

$$\frac{1}{2}\times$$
$$\begin{pmatrix} -27\mu-5 & -48\mu & 24 & 0 \\ -24\mu^2-24\mu-4 & -19\mu-1 & 8 & 0 \\ -32\mu^2-24\mu-4 & -24\mu^2-24\mu-4 & 5\mu+3 & 0 \\ -4(2\mu^3+(3\mu+1)^2) & -8(5\mu^2+5\mu+1) & -8\mu^2-16\mu-4 & -3\mu-1 \end{pmatrix} +$$
$$\tau\times$$
$$\begin{pmatrix} 17\mu+5 & 0 & 0 & 0 \\ 16\mu+4 & 17\mu+1 & -8 & 0 \\ 8\mu^2+16\mu+4 & 16\mu+4 & \mu-3 & 0 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & \mu+1 \end{pmatrix} +$$
$$\frac{z-1}{2}\times$$
$$\begin{pmatrix} -24\mu-4 & -40\mu+4 & 32 & 24 \\ -24\mu^2-24\mu-4 & -16\mu & 8\mu+12 & 8 \\ -32\mu^2-24\mu-4 & -24\mu^2-24\mu-4 & 8\mu+4 & 8\mu+4 \\ -4(2\mu^3+(3\mu+1)^2) & -8(5\mu^2+5\mu+1) & -8\mu^2-16\mu-4 & 0 \end{pmatrix} +$$
$$(z-1)\tau\times$$
$$\begin{pmatrix} 16\mu+4 & -4 & -8 & 0 \\ 16\mu+4 & 16\mu & -12 & -8 \\ 8\mu^2+16\mu+4 & 16\mu+4 & -4 & -4 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & 0 \end{pmatrix}.$$

Let

(162)
$$R_0^{\vee\vee}(0) = \begin{pmatrix} -5 & 0 & 24 & 0 \\ -4 & -1 & 8 & 0 \\ -4 & -4 & 3 & 0 \\ -4 & -8 & -4 & -1 \end{pmatrix}, \quad V_0^{\vee\vee}(0) = \begin{pmatrix} -4 & 4 & 32 & 24 \\ -4 & 0 & 12 & 8 \\ -4 & -4 & 4 & 4 \\ -4 & -4 & -4 & 0 \end{pmatrix},$$

(163)
$$R_0^{\vee\vee}(1) = \begin{pmatrix} -27 & -48 & 0 & 0 \\ -24 & -19 & 0 & 0 \\ -24 & -24 & 5 & 0 \\ -24 & -40 & -16 & -3 \end{pmatrix}, \quad V_0^{\vee\vee}(1) = \begin{pmatrix} -24 & -40 & 0 & 0 \\ -24 & -16 & 8 & 0 \\ -24 & -24 & 8 & 8 \\ -24 & -40 & -16 & 0 \end{pmatrix},$$

(164)
$$R_0^{\vee\vee}(2)=\begin{pmatrix}0&0&0&0\\-24&0&0&0\\-32&-24&0&0\\-36&-40&-8&0\end{pmatrix},\ V_0^{\vee\vee}(2)=\begin{pmatrix}0&0&0&0\\-24&0&0&0\\-32&-24&0&0\\-36&-40&-8&0\end{pmatrix},$$

(165)
$$R_0^{\vee\vee}(3)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-8&0&0&0\end{pmatrix},\ V_0^{\vee\vee}(3)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-8&0&0&0\end{pmatrix},$$

(166)
$$R_0^{\wedge\wedge}(0)=\begin{pmatrix}5&0&0&0\\4&1&-8&0\\4&4&-3&0\\4&8&4&1\end{pmatrix},\ V_0^{\wedge\wedge}(0)=\begin{pmatrix}4&-4&-8&0\\4&0&-12&-8\\4&4&-4&-4\\4&8&4&0\end{pmatrix},$$

(167)
$$R_0^{\wedge\wedge}(1)=\begin{pmatrix}17&0&0&0\\16&17&0&0\\16&16&1&0\\16&24&8&1\end{pmatrix},\ V_0^{\wedge\wedge}(1)=\begin{pmatrix}16&0&0&0\\16&16&0&0\\16&16&0&0\\16&24&8&0\end{pmatrix},$$

(168)
$$R_0^{\wedge\wedge}(2)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\8&0&0&0\\12&8&0&0\end{pmatrix},\ V_0^{\wedge\wedge}(2)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\8&0&0&0\\12&8&0&0\end{pmatrix}.$$

In view of (161) – (168),

(169)
$$A_0^*(z;\nu)=$$
$$\frac{1}{2}\left(\sum_{k=0}^{3}\mu^k(R_k^{\vee\vee}+(z-1)V_k^{\vee\vee})\right)+$$
$$\tau\left(\sum_{k=0}^{2}\mu^k(R_k^{\wedge\wedge}+(z-1)V_k^{\wedge\wedge})\right).$$

Moreover,
$$R_k^{\vee\vee}=R_k^{\vee},\ V_k^{\vee\vee}=V_k^{\vee},$$
for $k=0,\ 1,\ 2,\ 3$,
$$R_k^{\wedge\wedge}=R_k^{\wedge},\ V_k^{\wedge\wedge}=V_k^{\wedge},$$
for $k=0,\ 1,\ 2$, where
$$R_k^{\vee},\ V_k^{\vee},\ \text{for}\ k=0,\ 1,\ 2,\ 3,$$

and

$$R_k^{\wedge},\ V_k^{\wedge},\ \text{for}\, k=0,\ 1,\ 2$$

are pointed in the Part 2 and in section 3.2 of Part 3.

According to (32) and (33) of the Part 2,

(170) $$\nu^3 X_{0,k}(z;\nu-1)=A_0^*(z;\nu)X_{0,k}(z;\nu),$$

(171) $$(-\nu)^3 X_{0,k}(z;\nu)=A_0^*(z;-\nu)X_{0,k}(z;\nu-1),$$

where $k=1,\ 2,\ 3,\ |z|>1, \nu\in\mathbb{Z}$,

(172) $$A_0^*(z;\nu)=(1/2)U_0^{\vee}(z;\mu)+U_0^{\wedge}(z;\mu)\tau,$$

(173) $$\mu=\nu(\nu+1),\ \tau=\nu+\frac{1}{2},$$

(174) $$U_0^{\vee}(z,\mu)=\sum_{k=0}^{3}\mu^k(R_0^{\vee}(k))+(z-1)V_0^{\vee}(k)),$$

(175) $$U_0^{\wedge}(z,\mu)=\sum_{k=0}^{2}\mu^k(R_0^{\wedge}(k))+(z-1)V_0^{\wedge}(k)).$$

So, (169) coincides with (172)

§6.4. Short derive of Apéry equation.

After replacing in (170) ν by $-\nu-1$ we obtain the equality

(176)
$$(-\nu-1)^3X_{0,k}(z;-\nu-2)=A_0^*(z;-\nu-1)X_{0,k}(z;-\nu-1),$$

where $k=1,2,3,\ \nu\in\mathbb{Z},\ |z|>1$. Since

$$X_{0,k}(z;-\nu-1)=X_{0,k}(z;\nu),\ X_{0,k}(z;-\nu-2)=X_{0,k}(z;\nu+1)$$

for $k=1,2,3,\ \nu\in\mathbb{Z},\ |z|>1$, it follows that the equality (176) is equivalent to the equallity

(177) $$-(\nu+1)^3X_{0,k}(z;\nu+1)=A_0^*(z;-\nu-1)X_{0,k}(z;\nu),$$

for $k=1,2,3,\ \nu\in\mathbb{Z},\ |z|>1$. After substraction of the equality (177) from (170) we obtain the equality

(178) $$(\nu+1)^3X_{0,k}(z;\nu+1)+\nu^3X_{0,k}(z;\nu-1)=$$

$$(A_0^*(z;\nu)-A_0^*(z;-\nu-1))X_{0,k}(z;\nu),$$

where $k = 1, 2, 3, |z| > 1, \nu \in \mathbb{Z}$. In view of (172), the equality (178) take the form

(179)
$$(\nu+1)^3 X_{0,k}(z;\nu+1) + \nu^3 X_{0,k}(z;\nu-1) = U_0^{\wedge}(z;\mu)(2\nu+1)X_{0,k}(z;\nu),$$

where $k = 1, 2, 3, |z| > 1, \nu \in \mathbb{Z}$. In view of (161) and (172),

(180)
$$U_0^{\wedge}(z;\mu) =$$
$$\begin{pmatrix} 17\mu+5 & 0 & 0 & 0 \\ 16\mu+4 & 17\mu+1 & -8 & 0 \\ 8\mu^2+16\mu+4 & 16\mu+4 & \mu-3 & 0 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & \mu+1 \end{pmatrix} +$$
$$(z-1)\times$$
$$\begin{pmatrix} 16\mu+4 & -4 & -8 & 0 \\ 16\mu+4 & 16\mu & -12 & -8 \\ 8\mu^2+16\mu+4 & 16\mu+4 & -4 & -4 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & 0 \end{pmatrix}.$$

Therefore, in view (179), the first component of the column $X_{0,k}(z;\nu)$ satisfies to the following equality :

(181)
$$(\nu+1)^3 f_{0,k}(z;\nu+1) + \nu^3 f_{0,k}(z;\nu-1) =$$
$$(17\mu+5)(2\nu+1)f_{0,k}(z;\nu)+$$
$$(z-1)(16\mu+4)(2\nu+1)f_{0,k}(z;\nu)+$$
$$(z-1)(-4)(2\nu+1)(\delta f_{0,k})(z;\nu)+$$
$$(z-1)(-8)(2\nu+1)(\delta^2 f_{0,k})(z;\nu),$$

where $k = 1, 2, 3, |z| > 1, \nu \in \mathbb{Z}$. Now we turn $z \in (1, +\infty)$ to 1 for $k = 2, 3$. Since

$$\lim_{z\to 1}(z-1)(\delta f_{0,2})(z;\nu) = 0,$$
$$\lim_{z\to 1}(z-1)(\delta^2 f_{0,2})(z;\nu) = 1,$$

it follows that

(182)
$$(\nu+1)^3 f_{0,2}(1;\nu+1) + \nu^3 f_{0,2}(1;\nu-1) =$$
$$(17\mu+5)(2\nu+1)f_{0,2}(1;\nu) - 8(2\nu+1).$$

In view of (5) and (38),

$$f_{l,3}(1,\nu) = f_{l,4}(1,\nu) \text{ for } l = 1, 2, 3,\ \nu \in \mathbb{Z},$$
$$\lim_{z\to 1}(z-1)(\delta^k f_{0,3})(z;\nu) = 0$$

for $k = 1, 2$. Therefore

(183)
$$(\nu+1)^3 f_{0,4}(1;\nu+1) + \nu^3 f_{0,4}(1;\nu-1) =$$
$$(17\mu+5)(2\nu+1)f_{0,4}(1;\nu).$$

So $y(\nu) = f_{0,4}(1;\nu)$ satisfies the equation

$$(\nu+1)^3 y(1;\nu+1) + \nu^3 y\nu - 1) = (17\mu+5)(2\nu+1)y(\nu).$$

Clearly, this equation can be represented in the form

(184)
$$(\nu+1)^3 y(1;\nu+1) + \nu^3 y\nu - 1) = (17\nu^3 + 51\nu^2 + 27\nu + 5)y(\nu)(34\nu^3 + 85\nu^2).$$

Equation (184) is the famous Apéry equation.

§6.5. Corrections in the previous parts of this paper.

Polynomial $a^{\vee}_{2,3,3}(z;\nu)$ in the Part 2 must have a form

$$a^{\vee}_{2,3,3}(z;\nu) = (1/2)(-9 - 183\mu + 2586\mu^2 + 7113\mu^3) +$$
$$(1/2)(z-1)(-8 - 176\mu + 2600\mu^2 + 7120\mu^3)$$

instead of

$$a^{\vee}_{2,3,3}(z;\nu) = (1/2)(-9 - 183\mu + 2856\mu^2 + 7113\mu^3) +$$
$$(1/2)(z-1)(-8 - 176\mu + 2600\mu^2 + 7120\mu^3).$$

Polynomial $a^{\vee}_{2,4,4}(z;\nu)$ in the Part 2 must have a form

$$a^{\vee}_{2,4,4}(z;\nu) = (1/2)(-1 - 1255\mu - 3214\mu^2 + 2105\mu^3) +$$
$$(1/2)(z-1)(-1248\mu - 3200\mu^2 + 2112\mu^3),$$

instead of

$$a^{\vee}_{2,4,4}(z;\nu) = (1/2)(-1 - 1225\mu - 3214\mu^2 + 2105\mu^3) +$$
$$(1/2)(z-1)(-1248\mu - 3200\mu^2 + 2112\mu^3).$$

Polynomial $a^{\wedge}_{2,8,7}(z;\nu)$ in the Part 2 must have a form

$$a^{\wedge}_{2,8,7}(z;\nu) = 8 + 48\mu + 80\mu^2 + 32\mu^3 +$$
$$(z-1)(8 + 48\mu + 80\mu^2 + 32\mu^3)$$

instead of

$$a^{\wedge}_{2,8,7}(z;\nu) = 8 + 48\mu + 80\mu^2 + 33\mu^3 +$$
$$(z-1)(8 + 48\mu + 80\mu^2 + 32\mu^3).$$

On the intersection of 5-th row and 7-th column of the matrix $R^{\vee}_2(1)$ in the Part 2 must stand the number 1056 instead of 1256. On the intersection of 4-th row and 4-th column of the matrix $R^{\vee}_2(1)$ in the Part 2 must stand the number -1255 instead of -1225. On the intersection of 2-th row and 2-th column of the matrix $R^{\vee}_2(2)$ in the Part 2 must stand the number 1922 instead of 1992. On the intersection of 3-th row and 3-th column of the matrix $R^{\vee}_2(2)$ in the Part 2 must stand the number 2586 instead of 2856. On the intersection of 3-th row and 4-th column of the matrix $V^{\vee}_2(2)$ in the Part 2 must stand the number 3616 instead of 3613.

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