ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 6.

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To 100th birthday of
Professor A.O.Gelfond.

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§6.0. Foreword.

Let

$$|z| \geq 1, -3\pi/2 < \arg(z) \leq \pi/2, \log(z) = \ln(|z|) + i\arg(z).$$

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$.

Let

$$f^{\vee}_{l,1}(z,\nu) = f_{l,1}(z,\nu) = \sum_{k=0}^{\nu}(-1)^{(\nu+k)l}(z)^k \binom{\nu}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l}, \tag{1}$$

where $l = 0,\ 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z}$. Let

$$R(t,\nu) = \frac{\prod\limits_{j=1}^{\nu}(t-j)}{\prod\limits_{j=0}^{\nu}(t+j)}, \tag{2}$$

where $\nu \in [0, +\infty) \cap \mathbb{Z}$,

$$f^{\vee}_{l,2}(z,\nu) = f_{l,2}(z,\nu) = \sum_{t=1+\nu}^{+\infty} z^{-t}(R(t,\nu))^{2+l}, \tag{3}$$

where $l = 0,\ 1,\ 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1,\ \ldots,\ \nu$, the zeros of the order $2+l$, it follows that

$$f_{l,2}(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}(R(t,\nu))^{2+l}, \tag{4}$$

for $l = 0,\ 1,\ 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$f^{\vee}_{l,3}(z,\nu) = f_{l,3}(z,\nu) = (\log(z))f_{l,2}(z,\nu) + f_{l,4}(z,\nu), \tag{5}$$

where

$$f_{l,4}(z,\nu) = -\sum_{t=1+\nu}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu), \tag{6}$$

$l = 0,\ 1,\ 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$, and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1,\ \ldots,\ \nu$, the zeros of the order $2+l$, it follows that

$$f_{l,4}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu) \tag{7}$$

for $l = 0,\ 1,\ 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$f^{\vee}_{l,5}(z,\nu) = -i\pi f_{l,3}(z,\nu) + f_{l,5}(z,\nu), \tag{8}$$

with $l = 1,\ 2,\ \nu \in [0,+\infty) \cap \mathbb{Z}$ and

$$f_{l,5}(z,\nu) = \tag{9}$$

$$2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z))f_{l,4}(z,\nu) + f_{l,6}(z,\nu) =$$

$$= -2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z))f_{l,3}(z,\nu) + f_{l,6}(z,\nu),$$

where

$$f_{l,6}(z,\nu) = 2^{-1}\sum_{t=1+\nu}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu), \tag{10}$$

and since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ has in the points $t = 1,\ \ldots,\ \nu$, the zeros of the order $2+l$, and $l = 1,\ 2$ now, it follows that

$$f_{l,6}(z,\nu) = 2^{-1}\sum_{t=1+\nu}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu) \tag{11}$$

for $l = 1,\ 2$ and $\nu \in [0,+\infty) \cap \mathbb{Z}$. Let

$$(12)\qquad f^{\vee}_{l,7}(z,\nu)=f_{l,7}(z,\nu)+(2\pi^2/3)f_{l,3}(z,\nu).$$

with $l=2$, $\nu\in[0,+\infty)\cap\mathbb{Z}$ and

$$(13)\qquad f_{l,7}(z,\nu)=$$

$$-3^{-1}(\log(z))^3f_{l,2}(z,\nu)+2^{-1}(\log(z))^2f_{l,3}(z,\nu)+f_{l,8}(z,\nu)+$$
$$(\log(z))(f_{l,5}(z,\nu)+2^{-1}(\log(z))^2f_{l,2}(z,\nu)-(\log(z))f_{l,3}(z,\nu))=$$
$$6^{-1}(\log(z))^3f_{l,2}(z,\nu)-2^{-1}(\log(z))^2f_{l,3}(z,\nu)+(\log(z))f_{l,5}(z,\nu)+f_{l,8}(z,\nu)=$$
$$(1/6)(\log(z))^3f_{l,2}(z,\nu)+(1/2)(\log(z))^2f_{l,4}(z,\nu)+$$
$$(\log(z))f_{l,6}(z,\nu)+f_{l,8}(z,\nu),$$

where

$$(14)\qquad f_{l,8}(z,\nu)=-6^{-1}\sum_{t=\nu+1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^3(R^{2+l})\right)(t,\nu),$$

and, since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ have in the points $t=1,\ldots,\nu$, the zeros of the order $2+l$, and $l=2$ now, it follows that

$$(15)\qquad f_{l,8}(z,\nu)=-6^{-1}\sum_{t=1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^3(R^{2+l})\right)(t,\nu).$$

Let

$$\mathfrak{K}_0=\{1,\,2,\,3\},\ \mathfrak{K}_1=\{1,\,2,\,3,\,5\},\ \mathfrak{K}_2=\{1,\,2,\,3,\,5,\,7\}.$$

Let λ be a variable. We denote by $T_{n,\lambda}$ the diagonal $n\times n$-matrix, i-th diagonal element of which is equal to λ^{i-1} for $i=1,\ldots,n$. We denote by δ the operator $z\frac{d}{dz}$. Let further $l=0,\,1,\,2$, $k\in\mathfrak{K}_l$, $|z|>1,\nu\in\mathbb{N}$, and let $Y_{l,k}(z;\nu)$ be the columnn with $4+2l$ elements, i-th of which is equal to $(\nu^{-1}\delta)^{i-1}f^{\vee}_{l,k}(z,\nu)$ for $i=1,\ldots,4+2l$.

Theorem 1. *The following equalities hold*

$$(16)\qquad A^{\sim}_l(z;\nu)Y_{l,k}(z;\nu)=T_{4+2l,1-\nu^{-1}}Y_{l,k}(z;\nu-1),$$

$$(17)\qquad Y_{l,k}(z;\nu)=T_{4+2l,-1}A^{\sim}_l(z;-\nu)T_{4+2l,-1+\nu^{-1}}Y_{l,k}(z;\nu-1),$$

where $l=0,\,1,\,2$, $k\in\mathfrak{K}_l$, $|z|>1,\nu\in\mathbb{N},\nu\geq 2$,

$$(18)\qquad A^{\sim}_l(z;\nu)=S^{\sim}_l+z\sum_{i=0}^{1+l}\nu^{-i}V^{\sim *}_l(i)$$

with

$$(19)\qquad S^{\sim}_0=\begin{pmatrix}1&-4&8&-12\\0&1&-4&8\\0&0&1&-4\\0&0&0&1\end{pmatrix}$$

$$S_1^{\sim} = \begin{pmatrix} -1 & 6 & -18 & 38 & -66 & 102 \\ 0 & -1 & 6 & -18 & 38 & -66 \\ 0 & 0 & -1 & 6 & -18 & 38 \\ 0 & 0 & 0 & -1 & 6 & -18 \\ 0 & 0 & 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{20}$$

$$S_2^{\sim} = \begin{pmatrix} 1 & -8 & 32 & -88 & 192 & -360 & 608 & -952 \\ 0 & 1 & -8 & 32 & -88 & 192 & -360 & 608 \\ 0 & 0 & 1 & -8 & 32 & -88 & 192 & -360 \\ 0 & 0 & 0 & 1 & -8 & 32 & -88 & 192 \\ 0 & 0 & 0 & 0 & 1 & -8 & 32 & -88 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{21}$$

$$V_0^{\sim *}(0) = 4 \begin{pmatrix} 4 & -5 & -2 & 3 \\ -3 & 4 & 1 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{pmatrix},$$

$$V_0^{\sim *}(1) = 4 \begin{pmatrix} 3 & -6 & 3 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(0) = \begin{pmatrix} 146 & -198 & -180 & 268 & 66 & -102 \\ -102 & 146 & 108 & -180 & -38 & 66 \\ 66 & -102 & -52 & 108 & 18 & -38 \\ -38 & 66 & 12 & -52 & -6 & 18 \\ 18 & -38 & 12 & 12 & 2 & -6 \\ -6 & 18 & -20 & 12 & -6 & 2 \end{pmatrix},$$

$$V_1^{\sim *}(1) = \begin{pmatrix} 240 & -516 & 108 & 372 & -204 & 0 \\ -160 & 348 & -84 & -236 & 132 & 0 \\ 96 & -212 & 60 & 132 & -76 & 0 \\ -48 & 108 & -36 & -60 & 36 & 0 \\ 16 & -36 & 12 & 20 & -12 & 0 \\ 0 & -4 & 12 & -12 & 4 & 0 \end{pmatrix},$$

$$V_1^{\sim *}(2) = \begin{pmatrix} 102 & -306 & 306 & -102 & 0 & 0 \\ -66 & 198 & -198 & 66 & 0 & 0 \\ 38 & -114 & 114 & -38 & 0 & 0 \\ -18 & 54 & -54 & 18 & 0 & 0 \\ 6 & -18 & 18 & -6 & 0 & 0 \\ -2 & 6 & -6 & 2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(0) = 8 \begin{pmatrix} 176 & -249 & -364 & 545 & 280 & -431 & -76 & 119 \\ -119 & 176 & 227 & -364 & -169 & 280 & 45 & -76 \\ 76 & -119 & -128 & 227 & 92 & -169 & -24 & 45 \\ -45 & 76 & 61 & -128 & -43 & 92 & 11 & -24 \\ 24 & -45 & -20 & 61 & 16 & -43 & -4 & 11 \\ -11 & 24 & -1 & -20 & -5 & 16 & 1 & -4 \\ 4 & -11 & 8 & -1 & 4 & -5 & 0 & 1 \\ -1 & 4 & -7 & 8 & -7 & 4 & -1 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(1) = 8 \begin{pmatrix} 455 & -1020 & -113 & 1552 & -603 & -628 & 357 & 0 \\ -300 & 682 & 44 & -996 & 404 & 394 & -228 & 0 \\ 185 & -428 & -3 & 592 & -253 & -228 & 135 & 0 \\ -104 & 246 & -16 & -316 & 144 & 118 & -72 & 0 \\ 51 & -124 & 19 & 144 & -71 & -52 & 33 & 0 \\ -20 & 50 & -12 & -52 & 28 & 18 & -12 & 0 \\ 5 & -12 & 1 & 16 & -9 & -4 & 3 & 0 \\ 0 & -2 & 8 & -12 & 8 & -2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(2) = 8 \begin{pmatrix} 400 & -1243 & 972 & 542 & -1028 & 357 & 0 & 0 \\ -259 & 808 & -642 & -332 & 653 & -228 & 0 & 0 \\ 156 & -489 & 396 & 186 & -384 & 135 & 0 & 0 \\ -85 & 268 & -222 & -92 & 203 & -72 & 0 & 0 \\ 40 & -127 & 108 & 38 & -92 & 33 & 0 & 0 \\ -15 & 48 & -42 & -12 & 33 & -12 & 0 & 0 \\ 4 & -13 & 12 & 2 & -8 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(3) = 8 \begin{pmatrix} 119 & -476 & 714 & -476 & 119 & 0 & 0 & 0 \\ -76 & 304 & -456 & 304 & -76 & 0 & 0 & 0 \\ 45 & -180 & 270 & -180 & 45 & 0 & 0 & 0 \\ -24 & 96 & -144 & 96 & -24 & 0 & 0 & 0 \\ 11 & -44 & 66 & -44 & 11 & 0 & 0 & 0 \\ -4 & 16 & -24 & 16 & -4 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The above matices $A_l^{\sim}(z;\nu), S_l^{\sim}$ and $V_l^{\sim *}(i)$ have the following properties:

$$A_l^{\sim}(z;-\nu)T_{4+2l,-1}A_l^{\sim}(z;\nu) = T_{4+2l,-1}, \tag{22}$$

$$S_l^{\sim}T_{4+2l,-1} = (S_l^{\sim}T_{4+2l,-1})^{-1} \tag{23}$$

$$S_l^{\sim}T_{4+2l,-1}V_l^{\sim *}(i) = -(-1)^i V_l^{\sim *}(i)T_{4+2l,-1}S_l^{\sim}, \tag{24}$$

$$V_l^{\sim *}(i)T_{4+2l,-1}V_l^{\sim *}(k) = 0T_{4+2l,-1}, \tag{25}$$

where

$$l = 0,\ 1,\ 2,\ i \in [0, 1+l] \cap \mathbb{Z},\ k \in [0, 1+l] \cap \mathbb{Z}.$$

Proof. Full proof can be found in [2] – [6].

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§6.1. Further relations.

In the Part 2, equality (12), we have delated the domain of definition of the the functions $f^{\vee}_{l,k}(z,\nu)$, where $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}_l$

from $\{\nu \in \mathbb{Z}\colon \nu \geq 0\}$ to $\{\nu \in \mathbb{Z}\colon\ \text{all}\ \nu\}$ so these functions are defined for all the $\nu \in \mathbb{Z}$. We make the same sequentially for the functions

$$f_{l,4}(z,\nu),$$

where $l = 0,\ 1,\ 2$

$$f_{l,5}(z,\nu),\ f_{l,6}(z,\nu),$$

where $l =\ 1,\ 2$ and

$$f_{l,7}(z,\nu),\ f_{l,8}(z,\nu),$$

where $l = 2$,

by means of the equality

$$f_{l,4}(z,\nu) = f^{\vee}_{l,3}(z,\nu) - (\log(z))f^{\vee}_{l,2}(z,\nu) = \tag{26}$$

$$f^{\vee}_{l,3}(z,\nu) - (\log(z))f_{l,2}(z,\nu),$$

which follows from (5),

by means of the equality

$$f_{l,5}(z,\nu) = f^{\vee}_{l,5}(z,\nu) + i\pi f^{\vee}_{l,3}(z,\nu) = \tag{27}$$

$$f^{\vee}_{l,5}(z,\nu) + i\pi f_{l,3}(z,\nu),$$

which follows from the equality (8),

by means of the equality

$$f_{l,6}(z,\nu) = f_{l,5}(z,\nu) - (\log(z))f^{\vee}_{l,4}(z,\nu) - \tag{28}$$

$$(1/2)(\log(z))^2 f^{\vee}_{l,2}(z,\nu),$$

which follows from (9),

by means of the equality

$$f_{l,7}(z,\nu) = f^{\vee}_{l,7}(z,\nu) - (2\pi^2/3)f_{l,3}(z,\nu), \tag{29}$$

which follows from (12), and by means the equality

$$f_{l,8}(z,\nu) = \tag{30}$$

$$-(6^{-1}(\log(z))^3 f_{l,2}(z,\nu) - 2^{-1}(\log(z))^2 f_{l,4}(z,\nu) -$$

$$(\log(z))f_{l,6}(z,\nu)) + f_{l,7}(z,\nu),$$

which follows from (13). Let further $\mathfrak{K}^{\wedge}_l = [1, 4+2l] \cap \mathbb{Z}$ for $l = 0\,1,\ 2,$ and let $X^{\wedge}_{l,k}(z;\nu)$, where $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}^{\wedge}_l$, $|z| > 1, \nu \in \mathbb{Z}$, be the columnn with $4+2l$ elements, i-th of which is equal to $\delta^{i-1}f_{l,k}$ for $i = 1,\ ...,\ 4+2l$.

Then, in view of (1), (3), (5),

$$X^{\wedge}_{l,k}(z;\nu) = X_{l,k}(z;\nu), \tag{31}$$

if $l = 0,\ 1,\ 2,\ k = 1,\ 2,\ 3,\ |z| > 1, \nu \in \mathbb{Z}$. Therefore, if $k \in \mathfrak{K}_l$, then together with (25), (26), (32), (33) in the part 2, we have

$$X^{\wedge}_{l,k}(z;-\nu-1) = X^{\wedge}_{l,k}(z;\nu), \tag{32}$$

$$\nu^{3+2l} X^{\wedge}_{l,k}(z;\nu-1) = A^*_l(z;\nu) X^{\wedge}_{l,k}(z;\nu), \tag{33}$$

$$(-\nu)^{3+2l} X^{\wedge}_{l,k}(z;\nu) = A^*_l(z;-\nu) X^{\wedge}_{l,k}(z;\nu-1), \tag{34}$$

where

$$A^*_l(z;\nu) = \nu^{3+2l} T_{4+2l,\nu} A^{\sim}_l(z;\nu) T_{4+2l,\nu^{-1}}, \tag{35}$$

$l = 0,\ 1,\ 2,\ k = 1,\ 2,\ 3,\ |z| > 1, \nu \in \mathbb{Z}$. In view of (8), (5),

$$X^{\wedge}_{l,5}(z,\nu) = i\pi X_{l,3}(z,\nu) + X_{l,5}(z,\nu), \tag{36}$$

for $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Therefore (35), (33) and (34) hold for $l = 1,\ 2,\ k = 5\, |z| > 1, \nu \in \mathbb{Z}$. In view of (12), (5),

$$X^{\wedge}_{l,7}(z,\nu) = X_{l,7}(z,\nu) - (2\pi^2/3) X_{l,3}(z,\nu). \tag{37}$$

Therefore (35), (33) and (34) hold for $l = 2,\ k = 7\, |z| > 1, \nu \in \mathbb{Z}$.

So, (35), (33) and (34) hold for $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}_l,\ |z| > 1, \nu \in \mathbb{Z}$.

In view of (5), (9), (13),

$$(\delta^k f_{l,3})(z,\nu) = \tag{38}$$

$$(\delta^k f_{l,4})(z,\nu) + (((\log(z))\delta^k + k\delta^{k-1}) f_{l,2})(z,\nu),$$

where $l = 0,\ 1,\ 2,\ k \in [0,\infty) \cap \mathbb{Z},\ |z| > 1, \nu \in \mathbb{Z}$,

$$(\delta^k f_{l,5})(z,\nu) = (\delta^k f_{l,6})(z,\nu) + \tag{39}$$

$$(((\log(z))\delta^k + k\delta^{k-1}) f^{\vee}_{l,4})(z,\nu) +$$

$$\left(\left(\frac{1}{2}(\log(z))^2\delta^k + k(\log(z)(\delta^{k-1} + \frac{k(k-1)}{2})\delta^{k-2}\right) f^{\vee}_{l,2}\right)(z,\nu) =$$

$$(\delta^k f_{l,6})(z,\nu) + (((\log(z))\delta^k + k\delta^{k-1}) f^{\vee}_{l,3})(z,\nu) -$$

$$\left(\left(\frac{1}{2}(\log(z))^2\delta^k + k(\log(z))\delta^{k-1} + \frac{k(k-1)}{2}\delta^{k-2}\right) f^{\vee}_{l,2}\right)(z,\nu),$$

where $l = 1,\ 2,\ k \in [0,\infty) \cap \mathbb{Z},\ |z| > 1, \nu \in \mathbb{Z}$,

$$(\delta^k f_{l,7})(z,\nu) = \tag{40}$$

$$\left(\left(\frac{1}{6}(\log(z))^3\delta^k + \frac{k}{2}(\log(z))^2\delta^{k-1}\right) f_{l,2}\right)(z,\nu) +$$

$$\left(\left(\frac{k(k-1)}{2}(\log(z))\delta^{k-2}+\frac{k(k-1)(k-2)}{6}\delta^{k-3}\right)f_{l,2}\right)(z,\nu)-$$
$$\left(\left(\frac{1}{2}(\log(z))^2\delta^k+k(\log(z))\delta^{k-1}+\frac{k(k-1)}{2}\delta^{k-2}\right)f_{l,3}\right)(z,\nu)+$$
$$(((\log(z))\delta^k+k\delta^{k-1})f_{l,5})(z,\nu)+(\delta^k f_{l,8})(z,\nu)=$$
$$\left(\left(\frac{1}{6}(\log(z))^3\delta^k+\frac{k}{2}(\log(z))^2\delta^{k-1}\right)f_{l,2}\right)(z,\nu)+$$
$$\left(\left(\frac{k(k-1)}{2}(\log(z))\delta^{k-2}+\frac{k(k-1)(k-2)}{6}\delta^{k-3}\right)f_{l,2}\right)(z,\nu)+$$
$$\left(\left(\frac{1}{2}(\log(z))^2\delta^k+k(\log(z))\delta^{k-1}+\frac{k(k-1)}{2}\delta^{k-2}\right)f_{l,4}\right)(z,\nu)+$$
$$(((\log(z))\delta^k+k\delta^{k-1})f_{l,6})(z,\nu)+(\delta^k f_{l,8})(z,\nu),$$

where $l=2$, $k\in[0,\infty)\cap\mathbb{Z}$, $|z|>1,\nu\in\mathbb{Z}$. Let

$$C_{4+2l,0}=E_{4+2l}\ \text{ for }\ l=0,\ 1,\ 2, \tag{41}$$

$$C_{4,1}=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&2&0&0\\0&0&3&0\end{pmatrix}, \tag{42}$$

$$C_{6,1}=\begin{pmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&3&0&0&0\\0&0&0&4&0&0\\0&0&0&0&5&0\end{pmatrix}, \tag{43}$$

$$C_{8,1}=\begin{pmatrix}0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&2&0&0&0&0&0&0\\0&0&3&0&0&0&0&0\\0&0&0&4&0&0&0&0\\0&0&0&0&5&0&0&0\\0&0&0&0&0&6&0&0\\0&0&0&0&0&0&7&0\end{pmatrix}, \tag{44}$$

$$C_{6,2}=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&3&0&0&0&0\\0&0&6&0&0&0\\0&0&0&10&0&0\end{pmatrix}, \tag{45}$$

$$C_{8,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 21 & 0 & 0 \end{pmatrix}, \tag{46}$$

$$C_{8,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 \end{pmatrix}, \tag{47}$$

$$C^{\vee}_{l,m}(z) = \sum_{k=0}^{m} \frac{1}{k!}(\log(z))^k C_{4+2l,m-k}, \tag{48}$$

where $l = 0,\ 1,\ 2,\ m \in [0, l+1] \cap \mathbb{Z}$. Clearly, $C^{\vee}_{l,0}(z) = E_{4+2l}$,

$$\delta C^{\vee}_{l,m}(z) = \sum_{k=0}^{m} \frac{1}{k!}(\log(z))^k C_{4+2ll,m-k} = \tag{49}$$

$$\sum_{k=1}^{m} \frac{k}{k!}(\log(z))^{k-1} C_{4+2ll,m-k} =$$

$$\sum_{k=0}^{m-1} \frac{1}{k!}(\log(z))^k C_{4+2ll,m-1-k} = C^{\vee}_{l,m-1}(z).$$

Then in view of (38), (39), (40),

$$X^{\wedge}_{l,3}(z;\nu) = ((\log(z))C_{4+2l,0} + C_{4+2l,1})X^{\wedge}_{l,2}(z;\nu) + \tag{50}$$

$$X^{\wedge}_{l,4}(z;\nu) = C^{\vee}_{l,0}(z)X^{\wedge}_{l,4}(z;\nu) + C^{\vee}_{l,1}(z)X^{\wedge}_{l,2}(z;\nu),$$

where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

$$X^{\wedge}_{l,5}(z;\nu) = X^{\wedge}_{l,6}(z;\nu) + \tag{51}$$

$$((\log(z))C_{4+2l,0} + C_{4+2l,1})X^{\wedge}_{l,3}(z;\nu) -$$

$$\left(\frac{1}{2}(\log(z))^2 C_{4+2l,0} + (\log(z))C_{4+2l,1} + C_{4+2l,2}\right) X^{\wedge}_{l,2}(z;\nu) =$$

$$C^{\vee}_{l,0}(z)X^{\wedge}_{l,6}(z;\nu) + C^{\vee}_{l,1}(z)X^{\wedge}_{l,3}(z;\nu) - C^{\vee}_{l,2}(z)X^{\wedge}_{l,2}(z;\nu) =$$

$$X^{\wedge}_{l,6}(z;\nu) + ((\log(z))E_{4+2l} + C_{4+2l,1})X^{\wedge}_{l,4}(z;\nu) +$$

$$\left(\frac{1}{2}(\log(z))^2 C_{4+2l,0} + (\log(z))C_{4+2l,1} + C_{4+2l,2}\right) X^{\wedge}_{l,2}(z;\nu) =$$

$$C^{\vee}_{l,0}(z)X^{\wedge}_{l,6}(z;\nu) + C^{\vee}_{l,1}(z)X^{\wedge}_{l,4}(z;\nu) + C^{\vee}_{l,2}(z)X^{\wedge}_{l,2}(z;\nu),$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(52) $$X^{\wedge}_{l,7}(z;\nu) =$$

$$\left(\frac{1}{6}(\log(z))^3 C_{4+2l,0} + \frac{1}{2}(\log(z))^2 C_{8,1} + (\log(z))C_{4+2l,2} + C_{4+2l,3}\right) X^{\wedge}_{l,2}(z,\nu) -$$

$$\left(\frac{1}{2}(\log(z))^2 C_{4+2l,0} + (\log(z))C_{4+2l,1} + C_{4+2l,2}\right) X^{\wedge}_{l,3}(z,\nu) +$$

$$\left((\log(z))C_{4+2l,0} + C_{4+2l,1}\right) X^{\wedge}_{l,5}(z,\nu) + X^{\wedge}_{l,8}(z,\nu) =$$

$$\left(\frac{1}{6}(\log(z))^3 C_{4+2l,0} + \frac{1}{2}(\log(z))^2 C_{8,1} + (\log(z))C_{4+2l,2} + C_{4+2l,3}\right) X^{\wedge}_{l,2}(z,\nu) +$$

$$\left(\frac{1}{2}(\log(z))^2 E_{4+2l} + (\log(z))C_{4+2l,1} + C_{4,2}\right) X^{\wedge}_{l,4}(z,\nu) +$$

$$((\log(z))C_{4+2l,0} + C_{4+2l,1})X^{\wedge}_{l,6}(z,\nu) + X^{\wedge}_{l,8}(z,\nu) =$$

$$C^{\vee}_{l,0}(z)X^{\wedge}_{l,8}(z;\nu) + C^{\vee}_{l,1}(z)X^{\wedge}_{l,6}(z;\nu) + C_{l,2})^{\vee}(z)X^{\wedge}_{l,4}(z;\nu) + C^{\vee}_{l,3}(z)X^{\wedge}_{l,2}(z;\nu)$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$.

In view of (50) - (52)

(53) $$\delta X^{\wedge}_{l,3}(z;\nu) = \delta X^{\wedge}_{l,4}(z;\nu) +$$

$$C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,2}(z;\nu) + C^{\vee}_{l,0}(z)X^{\wedge}_{l,2}(z;\nu),$$

$l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(54) $$\delta X^{\wedge}_{l,5}(z;\nu) = \delta X^{\wedge}_{l,6}(z;\nu) +$$

$$C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,4}(z;\nu) + C_{l,2})\delta X^{\wedge}_{l,2}(z;\nu) + X^{\wedge}_{l,3}(z;\nu),$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

(55) $$\delta X^{\wedge}_{l,7}(z;\nu) = \delta X^{\wedge}_{l,8}(z,\nu) + C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,6}(z;\nu) +$$

$$C^{\vee}_{l,1}(z)\delta X^{\wedge}_{l,6}(z;\nu) + C_{l,2})^{v}ee(z)\delta X^{\wedge}_{l,4}(z;\nu) + C^{(}_{l,3}z)\delta X^{\wedge}_{l,2}(z;\nu) + X^{\wedge}_{l,5}(z;\nu),$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. Let

$$X^{\wedge}_{l,3,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,3}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, X^{\wedge}_{l,4,2}(z,\nu) = \begin{pmatrix} X^{\wedge}_{l,4}(z,\nu) \\ X^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$

$$C_{l,3,4}(z) = \begin{pmatrix} C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) \\ 0E_{4+2l} & C^{\vee}_{l,0}(z) \end{pmatrix},$$

where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Then, in view of (50)

(56) $$X^{\wedge}_{l,3,2}(z,\nu) = C_{l,3,4}(z)X^{\wedge}_{l,4,2}(z,\nu),$$

where $l=0,\,1,\,2,\,|z|>1, \nu\in\mathbb{Z}$. Let

$$X^{\wedge}_{l,5,3,2}(z,\nu)=\begin{pmatrix}X^{\wedge}_{l,5}(z,\nu)\\ X^{\wedge}_{l,3}(z,\nu)\\ X^{\wedge}_{l,2}(z,\nu)\end{pmatrix},X^{\wedge}_{l,6,4,2}(z,\nu)=\begin{pmatrix}X^{\wedge}_{l,6}(z,\nu)\\ X^{\wedge}_{l,4}(z,\nu)\\ X^{\wedge}_{l,2}(z,\nu)\end{pmatrix},$$

$$C_{l,5,6}(z)=\begin{pmatrix}C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) & C^{\vee}_{l,2}(z)\\ 0E_{4+2l} & C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z)\\ 0E_{4+2l} & 0C^{\vee}_{l,0}(z) & C^{\vee}_{l,0}(z)\end{pmatrix},$$

where $l=1,\,2,\,|z|>1, \nu\in\mathbb{Z}$. Then, in view of (51)

$$X^{\wedge}_{l,5,3,2}(z,\nu)=C_{l,5,6}(z)X^{\wedge}_{l,6,4,2}(z,\nu), \tag{57}$$

where $l=1,\,2,\,|z|>1, \nu\in\mathbb{Z}$. Let

$$X^{\wedge}_{l,7,5,3,2}(z,\nu)=\begin{pmatrix}X^{\wedge}_{l,7}(z,\nu)\\ X^{\wedge}_{l,5}(z,\nu)\\ X^{\wedge}_{l,3}(z,\nu)\\ X^{\wedge}_{l,2}(z,\nu)\end{pmatrix},X^{\wedge}_{l,8,6,4,2}(z,\nu)=\begin{pmatrix}X^{\wedge}_{l,8}(z,\nu)\\ X^{\wedge}_{l,6}(z,\nu)\\ X^{\wedge}_{l,4}(z,\nu)\\ X^{\wedge}_{l,2}(z,\nu)\end{pmatrix},$$

$$C_{l,7,8}=\begin{pmatrix}C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) & C^{\vee}_{l,2}(z) & C^{\vee}_{l,3}(z)\\ 0E_{4+2l} & C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z) & C^{\vee}_{l,2}(z)\\ 0E_{4+2l} & 0C^{\vee}_{l,0}(z) & C^{\vee}_{l,0}(z) & C^{\vee}_{l,1}(z)\\ 0E_{4+2l} & 0C^{\vee}_{l,0}(z) & 0C^{\vee}_{l,0}(z) & C^{\vee}_{l,0}(z)\end{pmatrix},$$

where $l=2,\,|z|>1, \nu\in\mathbb{Z}$. Then, in view of (52)

$$X^{\wedge}_{l,7,5,3,2}(z,\nu)=C_{l,7,8}X^{\wedge}_{l,8,6,4,2}(z,\nu), \tag{58}$$

Let

$$A^{*}_{l,1}(z;\nu)=A^{*}_{l}(z;\nu),$$
$$A^{\sim}_{l,1}(z;\nu)=A^{\sim}_{l}(z;\nu),$$
$$T^{*}_{l,1,\lambda}=T_{4+2l,\lambda},$$
$$A^{*}_{l,2}(z;\nu)=\begin{pmatrix}A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu)\\ 0A^{*}_{l}(z;\nu) & A^{*}_{l}(z;\nu)\end{pmatrix},$$
$$A^{\sim}_{l,2}(z;\nu)=\begin{pmatrix}A^{\sim}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu)\\ 0A^{\sim}_{l}(z;\nu) & A^{\sim}_{l}(z;\nu)\end{pmatrix},$$
$$T^{*}_{l,2,\lambda}=\begin{pmatrix}T_{4+2l,\lambda} & 0T_{4+2l,\lambda}\\ 0T_{4+2l,\lambda} & T_{4+2l,\lambda}\end{pmatrix},$$
$$A^{*}_{l,3}(z;\nu)=\begin{pmatrix}A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu)\\ 0A^{*}_{l}(z;\nu) & A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu)\\ 0A^{*}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) & A^{*}_{l}(z;\nu)\end{pmatrix},$$
$$A^{\sim}_{l,3}(z;\nu)=\begin{pmatrix}A^{\sim}_{l}(z;\nu) & 0A^{*}_{l}(z;\nu) & 0A^{\sim}_{l}(z;\nu)\\ 0A^{\sim}_{l}(z;\nu) & A^{\sim}_{l}(z;\nu) & 0A^{\sim}_{l}(z;\nu)\\ 0A^{\sim}_{l}(z;\nu) & 0A^{\sim}_{l}(z;\nu) & A^{\sim}_{l}(z;\nu)\end{pmatrix},$$
$$T^{*}_{l,3,\lambda}=\begin{pmatrix}T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda}\\ 0T_{4+2l,\lambda} & T_{4+2l,\lambda} & 0T_{4+2l,\lambda}\\ 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & T_{4+2l,\lambda}\end{pmatrix},$$

$$A^*_{l,4}(z;\nu) = \begin{pmatrix} A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) \\ 0A^*_l(z;\nu) & A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) \\ 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & A^*_l(z;\nu) & 0A^*_l(z;\nu) \\ 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & 0A^*_l(z;\nu) & A^*_l(z;\nu) \end{pmatrix},$$

$$A^{\sim}_{l,4}(z;\nu) = \begin{pmatrix} A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) \\ 0A^{\sim}_l(z;\nu) & A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) \\ 0A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) & A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) \\ 0A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) & 0A^{\sim}_l(z;\nu) & A^{\sim}_l(z;\nu) \end{pmatrix},$$

$$T^*_{l,4,\lambda} = \begin{pmatrix} T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & T_{4+2l,\lambda} & 0T_{4+2l,\lambda} \\ 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & 0T_{4+2l,\lambda} & T_{4+2l,\lambda} \end{pmatrix}.$$

In view of (35)

$$A^*_{l,i}(z;\nu) = \nu^{3+2l} T_{l,i,\nu} A^{\sim}_{l,i}(z;\nu) T_{l,i,\nu^{-1}} \tag{59}$$

for $i = 1, 2, 3, 4$. In view of (33), (34),

$$\nu^{3+2l} X^{\wedge}_{l,3,2}(z;\nu-1) = A^*_{l,2}(z;\nu) X^{\wedge}_{l,3,2}(z;\nu), \tag{60}$$

$$(-\nu)^{3+2l} X^{\wedge}_{l,3,2}(z;\nu) = A^*_{l,2}(z;-\nu) X^{\wedge}_{l,3,2}(z;\nu-1), \tag{61}$$

where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

$$\nu^{3+2l} X^{\wedge}_{l,5,3,2}(z;\nu-1) = A^*_{l,3}(z;\nu) X^{\wedge}_{l,5,3,2}(z;\nu), \tag{62}$$

$$(-\nu)^{3+2l} X^{\wedge}_{l,5,3,2}(z;\nu) = A^*_{l,3}(z;-\nu) X^{\wedge}_{l,5,3,2}(z;\nu-1), \tag{63}$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$,

$$\nu^{3+2l} X^{\wedge}_{l,7,5,3,2}(z;\nu-1) = A^*_{l,4}(z;\nu) X^{\wedge}_{l,7,5,3,2}(z;\nu), \tag{64}$$

$$(-\nu)^{3+2l} X^{\wedge}_{l,7,5,3,2}(z;\nu) = A^*_{l,4}(z;-\nu) X^{\wedge}_{l,7,5,3,2}(z;\nu-1), \tag{65}$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. In view of (56), (60), (61),

$$\nu^{3+2l} X^{\wedge}_{l,4,2}(z;\nu-1) = A^{**}_{l,2}(z;\nu) X^{\wedge}_{l,4,2}(z;\nu), \tag{66}$$

$$(-\nu)^{3+2l} X^{\wedge}_{l,4,2}(z;\nu) = A^{**}_{l,2}(z;-\nu) X^{\wedge}_{l,4,2}(z;\nu-1), \tag{67}$$

where

$$A^{**}_{l,2}(z;\nu) = (C_{l,3,4}(z))^{-1} A^*_{l,2}(z;\nu) C_{l,3,4}(z),$$

$l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$.

In view of (57), (62), (63),

$$\nu^{3+2l} X^{\wedge}_{l,6,4,2}(z;\nu-1) = A^{**}_{l,3}(z;\nu) X^{\wedge}_{l,6,4,2}(z;\nu), \tag{68}$$

$$(69) \qquad (-\nu)^{3+2l} X^{\wedge}_{l,6,4,2}(z;\nu) = A^{**}_{l,3}(z;-\nu) X^{\wedge}_{l,6,4,2}(z;\nu-1),$$

where

$$A^{**}_{l,3}(z;\nu) = (C_{l,5,6}(z))^{-1} A^{*}_{l,3}(z;\nu) C_{l,5,6}(z),$$

$l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. In view of (58), (64), (65),

$$(70) \qquad \nu^{3+2l} X^{\wedge}_{l,8,6,4,2}(z;\nu-1) = A^{**}_{l,4}(z;\nu) X^{\wedge}_{l,8,6,4,2}(z;\nu),$$

$$(71) \qquad (-\nu)^{3+2l} X^{\wedge}_{l,8,6,4,2}(z;\nu) = A^{**}_{l,3}(z;-\nu) X^{\wedge}_{l,6,4,2}(z;\nu-1),$$

where

$$A^{**}_{l,4}(z;\nu) = (C_{l,7,8}(z))^{-1} A^{*}_{l,4}(z;\nu) C_{l,7,8}(z),$$

$l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$. Since

$$Y_{l,k}(z,\nu) = (T_{4+2l,\nu})^{-1}(z,\nu) X_{l,k}(z,\nu),$$

where $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}_l,\ |z| > 1, \nu \in \mathbb{N}, \nu \geq 2$, let

$$Y^{\wedge}_{l,k}(z,\nu) = (T_{4+2l,\nu})^{-1}(z,\nu) X^{\wedge}_{l,k}(z,\nu),$$

where $l = 0,\ 1,\ 2,\ k \in \mathfrak{K}^{\wedge}_l,\ |z| > 1, \nu \in \mathbb{N}, \nu \geq 2$,

$$Y^{\wedge}_{l,3,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,3}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, Y^{\wedge}_{l,4,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,4}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$

where $l = 0,\ 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$, let

$$Y^{\wedge}_{l,5,3,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,5}(z,\nu) \\ Y^{\wedge}_{l,3}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, Y^{\wedge}_{l,6,4,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,6}(z,\nu) \\ Y^{\wedge}_{l,4}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$

where $l = 1,\ 2,\ |z| > 1, \nu \in \mathbb{Z}$, let

$$Y^{\wedge}_{l,7,5,3,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,7}(z,\nu) \\ Y^{\wedge}_{l,5}(z,\nu) \\ Y^{\wedge}_{l,3}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix}, Y^{\wedge}_{l,8,6,4,2}(z,\nu) = \begin{pmatrix} Y^{\wedge}_{l,8}(z,\nu) \\ Y^{\wedge}_{l,6}(z,\nu) \\ Y^{\wedge}_{l,4}(z,\nu) \\ Y^{\wedge}_{l,2}(z,\nu) \end{pmatrix},$$

where $l = 2,\ |z| > 1, \nu \in \mathbb{Z}$. Then

$$(72) \qquad Y^{\wedge}_{l,3,2}(z,\nu) = (T_{l,2,\nu}*)^{-1} X^{\wedge}_{l,3,2}(z,\nu),$$

$$(73) \qquad Y^{\wedge}_{l,4,2}(z,\nu) = (T^{*}_{l,2,\nu})^{-1} X^{\wedge}_{l,4,2}(z,\nu),$$

for $l = 0, 1, 2,\ |z| > 1, \nu \in \mathbb{Z}$,

$$(74) \qquad Y^{\wedge}_{l,5,3,2}(z,\nu) = (T^{*}_{l,3,\nu})^{-1} X^{\wedge}_{l,5,3,2}(z,\nu),$$

$$(75) \qquad Y^{\wedge}_{l,6,4,2}(z,\nu) = (T^{*}_{l,3,\nu})^{-1} X^{\wedge}_{l,6,4,2}(z,\nu),$$

for $l=1,2,\ |z|>1, \nu\in\mathbb{Z}$,

(76) $$Y^{\wedge}_{l,7,5,3,2}(z,\nu)=(T^{*}_{l,4,\nu})^{-1}X^{\wedge}_{l,7,5,3,2}(z,\nu),$$

(77) $$Y^{\wedge}_{l,8,6,4,2}(z,\nu)=(T^{*}_{l,4,\nu})^{-1}X^{\wedge}_{l,8,6,4,2}(z,\nu),$$

for $l=2,\ |z|>1, \nu\in\mathbb{Z}$. In view of (59) – (65), (72) – (77),

(78) $$Y^{\wedge}_{l,3,2}(z;\nu-1)=((T_{2,1-1/nu}))^{-1}A^{\sim}_{l,2}(z;\nu)Y^{\wedge}_{l,3,2}(z;\nu),$$

(79) $$Y^{\wedge}_{l,3,2}(z;\nu)=T_{2,-1}A^{\sim}_{l,2}(z;\nu)Y^{\wedge}_{l,3,2}(z;\nu-1)T_{2,-1+1/nu},$$

(80) $$Y^{\wedge}_{l,5,3,2}(z;\nu-1)=((T_{2,1-1/nu}))^{-1}A^{\sim}_{l,3}(z;\nu)Y^{\wedge}_{l,5,3,2}(z;\nu),$$

(81) $$Y^{\wedge}_{l,5,3,2}(z;\nu)=T_{3,-1}A^{\sim}_{l,3}(z;\nu)Y^{\wedge}_{l,5,3,2}(z;\nu-1)T_{3,-1+1/nu},$$

(82) $$Y^{\wedge}_{l,7,5,3,2}(z;\nu-1)=((T_{4,1-1/nu}))^{-1}A^{\sim}_{l,4}(z;\nu)Y^{\wedge}_{l,7,5,3,2}(z;\nu),$$

(83) $$Y^{\wedge}_{l,7,5,3,2}(z;\nu)=T_{4,-1}A^{\sim}_{l,4}(z;\nu)Y^{\wedge}_{l,7,5,3,2}(z;\nu-1)T_{4,-1+1/nu}.$$

(84) $$Y^{\wedge}_{l,4,2}(z;\nu-1)=((T_{2,1-1/nu}))^{-1}A^{\sim\sim}_{l,2}(z;\nu)Y^{\wedge}_{l,4,2}(z;\nu),$$

(85) $$Y^{\wedge}_{l,4,2}(z;\nu)=T_{2,-1}A^{\sim\sim}_{l,2}(z;\nu)Y^{\wedge}_{l,4,2}(z;\nu-1)T_{2,-1+1/nu},$$

(86) $$Y^{\wedge}_{l,6,4,2}(z;\nu-1)=((T_{2,1-1/nu}))^{-1}A^{\sim\sim}_{l,3}(z;\nu)Y^{\wedge}_{l,6,4,2}(z;\nu),$$

(87) $$Y^{\sim}_{l,6,4,2}(z;\nu)=T_{3,-1}A^{\sim\sim}_{l,3}(z;\nu)Y^{\wedge}_{l,6,4,2}(z;\nu-1)T_{3,-1+1/nu},$$

(88) $$Y^{\sim}_{l,8,6,4,2}(z;\nu-1)=((T_{4,1-1/nu}))^{-1}A^{\sim\sim}_{l,4}(z;\nu)Y^{\wedge}_{l,8,6,4,2}(z;\nu),$$

(89) $$Y^{\sim}_{l,8,6,4,2}(z;\nu)=T_{4,-1}A^{\sim}_{l,4}(z;\nu)Y^{\wedge}_{l,8,6,4,2}(z;\nu-1)T_{4,-1+1/nu}.$$

§6.2. Further properties of my auxiliary fuctions.

In view of (2), (6), (12), (17) and (18) in the Part 1,

(90)
$$f_{l,1}(z,\nu)=f^{\vee}_{l,1}(z,\nu)=\sum_{k=0}^{\nu}(-1)^{(\nu+k)l}(z)^{k}\binom{\nu}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l},$$

where $l=0,\ 1,\ 2,\ \nu\in[0,+\infty)\cap\mathbb{Z}$,

(91) $$f_{l,2+2j}(z,\nu)=\frac{(-1)^{j}}{j!}\times$$

$$\sum_{t=1}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^{j}(R^{2+l})\right)(t,\nu),$$

where $l = 0,\ 1,\ 2,\ j \in [0, l+1]\cap\mathbb{Z},\ \nu \in [0,+\infty)\cap\mathbb{Z}, |z| > 1$. Expanding function $(R(t;\nu))^{2+l}$ into partial fractions, we obtain

$$(R(t;\nu))^{2+l} = \sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}(t+k)^{-i}\right) \tag{92}$$

where $l = 0,\ 1,\ 2,\ \nu \in [0,+\infty)\cap\mathbb{Z}$,

$$\alpha_{l,2+l-j,k,\nu} = \frac{1}{j!}\lim_{t\to -k}\left(\frac{\partial}{\partial t}\right)^{j}(R(t;\nu)(t+k))^{2+l}, \tag{93}$$

where $j = 0,\ ...\ 2+l-1$. Let

$$S_{i,k}(\nu) = -\left(\sum_{\kappa=k+1}^{\nu+k} 1/\kappa^{i}\right) - \left(\sum_{\kappa=1}^{\nu-k} 1/\kappa^{i}\right) + \sum_{\kappa=1}^{k} 1/\kappa^{i}, \tag{94}$$

where $\nu \in \mathbb{Z}\cap[0,+\infty),\ i \in \mathbb{N}, k \in \mathbb{Z}\cap[0,+\nu)$. In view of (93), (94),

$$\alpha_{l,2+l,k,\nu} = (-1)^{l(\nu+k)}((\nu+k)!/((k!)^2(\nu-k)!))^{2+l} = \tag{95}$$

$$(-1)^{l(\nu+k)}\binom{\nu}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l},$$

$$\alpha_{l,2+l-1,k,\nu} = \alpha_{l,2+l,k,\nu}(2+l)S_{1,k}(\nu) \tag{96}$$

for $l = 0,\ 1,\ 2,\ \nu \in [0,+\infty)\cap\mathbb{Z}$,

$$2\alpha_{l,2+l-2,k,\nu} = \alpha_{l,2+l,k,\nu}((2+l)^2(S_{1,k}(\nu))^2 - (2+l)S_{2,k}(\nu)) \tag{97}$$

for $l = 1,\ 2,\ \nu \in [0,+\infty)\cap\mathbb{Z}$,

$$6\alpha_{l,2+l-3,k,\nu} = \alpha_{l,2+l,k,\nu}\times \tag{98}$$

$$((2+l)^3(S_{1,k}(\nu))^3 - 3(2+l)^2(S_{1,k}(\nu))(S_{2,k}(\nu)) + 2(2+l)S_{3,k}(\nu))$$

for $l = 2,\ \nu \in [0,+\infty)\cap\mathbb{Z}$. In view of (91), (92)

$$f_{l,2+2j}(z,\nu) = \tag{99}$$

$$\sum_{i=1}^{2+l}\left(\sum_{t=1}^{\infty}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^{k}z^{-t-k}(t+k)^{-i-j}\right)\right) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^{k}\left(\sum_{t=1}^{\infty}z^{-t-k}(t+k)^{-i-j}\right)\right) =$$

$$\sum_{i=1}^{2+l}\left(\sum_{k=0}^{\nu}\alpha_{l,i,k,\nu}z^{k}\left(L_{i+j}(1/z) - \sum_{t=1}^{k}z^{-t}(t)^{-i-j}\right)\right) =$$

$$\left(\sum_{i=1}^{2+l} \alpha^*_{l,i}(z;\nu) L_{i+j}(1/z)\right) - \beta^*_{l,j}(z;\nu) =$$

$$\left(\sum_{i=-\infty}^{\infty} \alpha^*_{l,i}(z;\nu) L_{i+j}(1/z)\right) - \beta^*_{l,j}(z;\nu) =$$

where

$$\alpha^*_{l,i}(z;\nu) = 0, \tag{100}$$

if $(i-2-l)(i-1) > 0$,

$$\alpha^*_{l,i}(z;\nu) = \sum_{k=0}^{\nu} \alpha_{l,i,k,\nu} z^k, \tag{101}$$

for $l = 0,\ 1,\ 2,\ i \in [1, l+2] \cap \mathbb{Z},\ j \in [0, l+1] \cap \mathbb{Z},\ \nu \in [0, +\infty) \cap \mathbb{Z},\ |z| > 1,$

$$\beta^*_{l,j}(z;\nu) = \tag{102}$$

$$\sum_{i=1}^{2+l} \left(\sum_{k=0}^{\nu} \alpha_{l,i,k,\nu} \left(\sum_{t=1}^{k} z^{k-t} (t)^{-i-j} \right) \right)$$

for $l = 0,\ 1,\ 2,\ i \in [1, l+2] \cap \mathbb{Z},\ j \in [0, l+1] \cap \mathbb{Z},\ \nu \in [0, +\infty) \cap \mathbb{Z},\ |z| > 1,$

$$L_n(1/z) = \sum_{t=1}^{+\infty} z^{-t} t^{-n}, \tag{103}$$

where $n \in \mathbb{Z},\ |z| > 1.$

Clearly,

$$(-\delta)^k (L_n(1/z)) = L_{n-k}(1/z), \tag{104}$$

where $k \in [0, +\infty) \cap \mathbb{Z},\ n \in \mathbb{Z},\ |z| > 1,$

$$L_1(1/z)) = -\log(1 - 1/z), \tag{105}$$

$$-\delta(L_1(1/z)) = L_0(1/z) = 1 + 1/(z-1), \tag{106}$$

$$\delta^2(L_1(1/z)) = L_{-1}(1/z) = 1/(z-1) + 1/(z-1)^2, \tag{107}$$

$$-\delta^3(L_1(1/z)) = L_{-2}(1/z) = \tag{108}$$

$$1/(z-1) + 3/(z-1)^2 + 2/(z-1)^3,$$

$$\delta^4(L_1(1/z)) = L_{-3}(1/z) = \frac{1}{z-1} + \tag{109}$$

$$\frac{7}{(z-1)^2} + \frac{12}{(z-1)^3} + \frac{6}{(z-1)^4},$$

$$(110)\qquad -\delta^5(L_1(1/z)) = L_{-4}(1/z) = \frac{1}{z-1} + \frac{15}{(z-1)^2} + \frac{50}{(z-1)^3} + \frac{60}{(z-1)^4} + \frac{24}{(z-1)^5},$$

$$(111)\qquad \delta^6(L_1(1/z)) = L_{-5}(1/z) = \frac{1}{z-1} + \frac{31}{(z-1)^2} + \frac{180}{(z-1)^3} + \frac{390}{(z-1)^4} + \frac{360}{(z-1)^5} + \frac{120}{(z-1)^6},$$

$$(112)\qquad -\delta^7(L_1(1/z)) = L_{-6}(1/z) = \frac{1}{z-1} + \frac{63}{(z-1)^2} + \frac{602}{(z-1)^3} + \frac{2100}{(z-1)^4} + \frac{2360}{(z-1)^5} + \frac{2520}{(z-1)^6} + \frac{720}{(z-1)^7},$$

$$(113)\qquad \delta^n(L_1(1/z)) = (-1)^n L_{1-n}(1/z) = (-1)^n \sum_{k=1}^{n} a(n,k) 1/(z-1)^k,$$

where

$$(114)\qquad a(1,0) = a(1,1) = 1,\ a(n+1,k) = ka(n,k) + (k-1)a(n,k-1)$$

for $k \in [0,n] \cap \mathbb{Z}, a(n,-1) = a(n,n+1) = 0$, and $n \in \mathbb{N}$.

According to (99), (104) and Leibnitz formula,

$$(115)\qquad (\delta)^m f_{l,2+2j}(z,\nu) = \sum_{k=0}^{m} \binom{m}{k} \left(\sum_{i=-\infty}^{+\infty} ((-1)^k (\delta^{m-k} \alpha^*_{l,i}(z;\nu)) L_{j+i-k}(1/z) \right) - \delta^m \beta^*_{l,j}(z;\nu) = \left(\sum_{\kappa=1-m}^{2+l} \left(\sum_{k=\max(0,1-\kappa)}^{min(m,2+l-\kappa)} \binom{m}{k} (-1)^k \delta^{m-k} \alpha^*_{l,k+\kappa}(z;\nu) \right) L_{j+\kappa}(1/z) \right) - \delta^m \beta^*_{l,j}(z;\nu) = \left(\sum_{\kappa=1-m}^{2+l} \alpha^{*\vee}_{m,l,\kappa}(z;\nu) L_{j+\kappa}(1/z) \right) - \delta^m \beta^*_{l,j}(z;\nu),$$

where

$$(116)\qquad \alpha^{*\vee}_{m,l,\kappa}(z;\nu) =$$

$$\sum_{k=\max(0,1-\kappa)}^{min(m,2+l-\kappa)} \binom{m}{k} (-1)^k \delta^{m-k} \alpha^*_{l,k+\kappa}(z;\nu),$$

where $m \in \mathbb{N}, l = 0,\ 1,\ 2,\ i \in [1, l+2] \cap \mathbb{Z},\ j \in [0, l+1] \cap \mathbb{Z},\ \nu \in [0, +\infty) \cap \mathbb{Z},\ |z| > 1$. In view of (91),

$$\delta^m f_{l,2+2j}(z,\nu) = \frac{(-1)^j}{j!} \times \tag{117}$$

$$\sum_{t=1}^{\infty} z^{-t} (-t)^m \left(\left(\frac{\partial}{\partial t} \right)^j (R^{2+l}) \right) (t,\nu),$$

where $l = 0,\ 1,\ 2,\ \{m,\nu\} \subset [0,\infty) \cap \mathbb{Z},\ j \in [0, l+1] \cap \mathbb{Z},\ |z| > 1$. Therefore, in view of (2), if

$$m \le l + j,$$

then

$$\frac{(-1)^j}{j!} Res_{t=\infty} \left((-t)^m \left(\frac{\partial}{\partial t} \right)^j (R^{2+l}) \right) (t,\nu) = 0,$$

and, consequently, in view of (116),

$$\alpha^{*\vee}_{m,l,1-j}(z,\nu) = \tag{118}$$

$$\sum_{k=j}^{m} \binom{m}{k} (-1)^k (\delta^{m-k} \alpha^*_{l,k+1-j})(1;\nu) = 0,$$

$$\sum_{i=0}^{m-j} \binom{m}{i+j} (-1)^{i+j} (\delta^{m-j-i} \alpha^*_{l,i+1})(1;\nu) = 0,$$

$$\lim_{z\to 1} \alpha^{*\vee}_{m,l,1-j}(z;\nu)/(z-1) = (\delta \alpha^{*\vee}_{m,l,1})(1;\nu) \tag{119}$$

moreover, there exists

$$\lim_{z\to 1} \delta^m f_{l,2+2j}(z;\nu) \tag{120}$$

in this case. If $m = l + j + 1$, then in view of (117),

$$\delta^{l+j+1} f_{l,2+2j}(z,\nu) = \frac{(-1)^j}{j!} \times \tag{121}$$

$$\sum_{t=1}^{+\infty} z^{-t} (-t)^{l+j+1} \left(\left(\frac{\partial}{\partial t} \right)^j (R^{2+l}) \right) (t,\nu) = O(-log(1 - 1/|z|))$$

for fixed $\nu \in [0,\infty) \cap \mathbb{Z}$ and $|z| \to 1 + 0$, where $l = 0,\ 1,\ 2,\ j \in [0, l+1] \cap \mathbb{Z}$; on the other hand, in view of (115)

$$\delta^{l+1+j} f_{l,2+j}(z,\nu) = \tag{122}$$

$$\left(\sum_{\kappa=-l-j}^{2+l} \alpha^{*\vee}_{l+j+1,l,\kappa}(z;\nu) L_\kappa(1/z) \right) -$$

$$\delta^{l+j+1}\beta^{*}_{l,0}(z;\nu);$$

in view of (113), (114),

$$\lim_{|z|>1,\, z\to 1,} L_{1-n}(1/z)(z-1)^n \neq 0,$$

where $n \in \mathbb{N}$. Multiplying sequentially (121) by $(z-1)^n$ for $n = l+j+1$, ..., 1 and turning $z \in (1, +\infty)$ to 1, we obtain equalities

$$\alpha^{*\vee}_{l+j+1,l,\kappa}(1;\nu) = 0$$

for $\kappa = -l-j$, ..., 0; if $-l-j \leq \kappa \leq -j$, then $m = l+j+1 < 2+l-\kappa$ and, in view of (116),

$$0 = \alpha^{*\vee}_{l+1+j,l,\kappa}(z;\nu) = \tag{123}$$

$$\sum_{k=1-\kappa}^{1+l+j} \binom{m}{k} (-1)^k \delta^{m-k} \alpha^{*}_{l,k+\kappa}(1;\nu)$$

for $\kappa = -l-j$, ..., $-j$; in particular, (123) holds for $j = 0$; if $j \geq 1$ and $1-j \leq \kappa \leq 0$, then $m = l+j+1 \geq 2+l-\kappa$ and

$$0 = \alpha^{*\vee}_{l+1+j,l,\kappa}(z;\nu) = \tag{124}$$

$$\sum_{k=1-\kappa}^{2+l-\kappa} \binom{m}{k} (-1)^k \delta^{m-k} \alpha^{*}_{l,k+\kappa}(1;\nu)$$

for $\kappa = 1-j$, ..., 0.

Lemma 2.1. *Let*

$$s \in \mathbb{N},\ k = 1,\ ...,\ s,\ a_{0,k} \in \mathbb{C},\ \alpha_{0,k} \in \mathbb{Z},$$

$$\sigma = \sum_{k=1}^{s} \alpha_{0,k} \neq 0 \tag{125}$$

$$f(z) = \prod_{k=1}^{s} (z - a_{0,k})^{\alpha_{0,k}},\ \delta = z\frac{\partial}{\partial z}.$$

Then for $i = 1$, ..., $2s$ there exist $a_{1,i} \in \mathbb{C}$, $\alpha_{1,i} \in \mathbb{Z}$ such that

$$\sum_{i=1}^{2s} \alpha_{1,i} = \sigma,$$

$$f_1(z) = (\delta f)(z) = \sigma \prod_{i=1}^{2s} (t - a_{1,i})^{\alpha_{1,i}}$$

Proof. Clearly,

$$f_1(z) = f(z) \left(\sum_{k=1}^{s} \frac{z\alpha_{0,k}}{z - a_{0,k}} \right)$$

and, in view of (125),

$$f(z)=\sigma\prod_{k=1}^{s}(z-a_{0,k})^{\alpha_{0,k}-1}(z-a_{1,k}),$$

for some $a_{1,k}\in\mathbb{C}$, where $k=1$, ..., s. So, we can take $\alpha_{1,k}=1,\alpha_{1,s+k}=\alpha_{0,k}-1$, $a1,s+k=a_{0,k}$ for $k=1$, ..., s. Clearly,

$$\sum i=1^{2s}\alpha_{1,i}=\sum k=1^{s}\alpha_{0,k}=\sigma.$$

■

Corollary 1. *If condition of the Lemma 2.1 are fulflled, and* $n\in\mathbb{N}$, *then for* $i=1$, ..., $2^n s$ *there exist* $a_{n,i}\in\mathbb{C}$, $\alpha_{n,i}\in\mathbb{Z}$ *such that*

$$\sum_{i=1}^{2^n s}\alpha_{n,i}=\sigma,$$

$$f_n(z)=(\delta^n f)(z)=(\sigma)^n\prod_{i=1}^{2^n s}(z-a_{n,i})^{\alpha_{n,i}}, \tag{126}$$

and

$$\lim_{z\to\infty}z^{-\sigma}(\delta^n f)(z)=(\sigma)^n \tag{127}$$

Proof. The equality (126) can be obtained by sequential applying of the Lemma 2.1. The equality (127) directly follows from (126). ■

Corollary 2. *If condition of the Lemma 2.1 are fulflled and* $\sigma=-1$, *then*

$$Res_{z=\infty}(((-\delta)^n f)(z))=-1$$

for any $n\in[0,+\infty\cap\mathbb{Z}$. **Proof.** The assertion easy follows from Corollary 1. ■

Corollary 3. *If condition of the Lemma 2.1 are fulflled, and* $n\in\mathbb{N}$, *then*

$$\lim_{z=\infty}\left((z)^{n-\sigma}\left(\left(\frac{\partial}{\partial z}\right)^n f\right)(z)\right)=(-1)^n(-\sigma)_n=\prod_{i=1}^{n}(\sigma+1-i).$$

for any $n\in[0,+\infty)\cap\mathbb{Z}$.

Proof. For $n=0$ and $n=1$ the assertion follows from the Corollary 2. Let $\mathfrak{m}_z$ be operator, which transforms a function $g(z)$ into $zg(z)$. Then

$$\delta(\mathfrak{m}_z)^n\left(\frac{\partial}{\partial z}\right)^n=n(\mathfrak{m}_z)^n\left(\frac{\partial}{\partial z}\right)^n+(\mathfrak{m}_z)^{n+1}\left(\frac{\partial}{\partial t}\right)^{n+1},$$

$$(\mathfrak{m}_z)^{n+1}\left(\frac{\partial}{\partial t}\right)^{n+1}=(\delta-n)(\mathfrak{m}_z)^n\left(\frac{\partial}{\partial z}\right)^n.$$

Therefore

$$(\mathfrak{m}_z)^n\left(\frac{\partial}{\partial z}\right)^n=$$

$$\prod_{i=1}^{n}(\delta+1-i)=P_n(\delta),$$

where

$$P_n(z)=\prod_{i=1}^{n}(z-1+i)=(-1)^n(-z)_n=\sum_{k=1}^{n}c_kz^k,$$

and $(-z)_n$ is the Pochhammer symbol. Therefore, in view of (127),

$$\lim_{z=\infty}\left((z)^{n-\sigma}\left(\left(\frac{\partial}{\partial z}\right)^n f\right)(z)\right)=$$

$$\sum_{k=1}^{n}c_k\lim_{z=\infty}z^{-\sigma}(\delta^k f)(z)=$$

$$\sum_{k=1}^{n}c_k\sigma^k=(\sigma)_k=\prod_{i=1}^{n}(\sigma+1-i)=(-1)^n(-\sigma)_n.$$

■

Corollary 4. *If* $\sigma=-1$, *then*

$$Res_{z=\infty}\left((-z)^n\left(\left(\frac{\partial}{\partial z}\right)^n f\right)(z)\right)=-n!$$

for any $n\in[0,+\infty)\cap\mathbb{Z}$.

Proof. The assertion direct follows from Corollary 3. ■

The above Corollary 2 easy follows from the followoing obvious

Lemma 2.2. *Let* $R>0$. *Let further a function* $f(z)$ *is regular in the domain* $\{z\in\mathbb{C}\colon R<|z|\}$. *Then*

$$Res_{z=\infty}(((-\delta)^n f)(z))=Res_{z=\infty}(f(z)) \tag{128}$$

for any $n\in[0,+\infty\cap\mathbb{Z}$. ■

Lemma 2.3. *If condition of the Lemma 2.1 are fulfilled,* $\sigma=0$, *and*

$$\sigma_1=\sum_{k=1}^{s}a_k\alpha_k, \tag{129}$$

then

$$Res_{z=\infty}(((-\delta)^n f)(z))=\sigma_1 \tag{130}$$

for any $n\in[0,+infty\cap\mathbb{Z}$.

Proof. Clearly,

$$f(z)=\prod_{k=1}^{s}(1-a_{0,k}/z)^{\alpha_{0,k}}$$

in this case. Therefore all the conditions of the Lemma 2.2 are fulfilled. ■

Let again $m=l+j+1$; clearly, if $j=0$, then

$$Res_{t=\infty}(-t)^{1+l}R^{2+l}(t,\nu)=(-1)^l,$$

and therefore, in view of (116),

$$\sum_{k=0}^{l+1} \binom{l+1}{k} (-1)^k (\delta^{l+1-k}\alpha^*_{l,1})(1;\nu) = \tag{131}$$

$$\alpha^{*\vee}_{l+1,l,1}(1;\nu) = (-1)^{l+1}.$$

If $m = l+j+s$, $s \geq 2$, then in view of (117),

$$\delta^{l+j+2} f_{l,2+2j}(z,\nu) = \frac{(-1)^j}{j!} \times \tag{132}$$

$$\sum_{t=1}^{+\infty} z^{-t}(-t)^{l+j+s}\left(\left(\frac{\partial}{\partial t}\right)^j (R^{2+l})\right)(t,\nu) = O(1/(|z|-1)^{s-1})$$

for fixed $\nu \in [0,\infty)\cap\mathbb{Z}$ and $|z| \to 1+0$, where $l = 0,\ 1,\ 2$, $j \in [0, l+1]\cap\mathbb{Z}$. In particular, if $j = 0$, $m = l+s$, $s \geq 2$, $l = 0, 1, 2$, then, in view of (117) and (132),

$$\delta^{l+s} f_{l,2}(z,\nu) = O(1/(|z|-1)^{s-1}),\ (|z| \to 1+0); \tag{133}$$

on the other hand, in view of (115),

$$\delta^{l+s} f_{l,2}(z,\nu) = \tag{134}$$

$$\left(\sum_{\kappa=-l+1-s}^{2+l} \alpha^{*\vee}_{l+1,l,\kappa} L_\kappa(1/z)\right) -$$

$$\delta^{l+s}\beta^*_{l,0}(z;\nu);$$

in view of (113), (114),

$$\lim_{|z|>1,\, z\to 1,} L_{1-n}(1/z)(z-1)^n \neq 0,$$

where $n \in \mathbb{N}$. Multiplying sequentially (133) by $(z-1)^n$ for $n = l+s$, ..., s and turning $z \in (1,+\infty)$ to 1, we obtain equalities

$$\alpha^{*\vee}_{l+s,l,\kappa} = 0$$

for $\kappa = 1-l-s$, ..., $1-s$; in this case $2+l-\kappa \geq 2+l-(1-s) > l+s = m$, $1-\kappa > 0$; therefore, in view of (116),

$$\sum_{k=1-\kappa}^{l+s} \binom{l+s}{k} (-1)^k \delta^{l+s-k}\alpha^*_{l,k+\kappa}(1;\nu) = 0 \tag{135}$$

for $\kappa = 1-l-s$, ..., $1-s$.

§6.3 Return from ν to (τ,μ) in the case $l = 0$.

According to the Theorem 1,

$$A_0^{\sim}(z;\nu) = S_0^{\sim} + z\sum_{i=0}^{1} \nu^{-i} V_0^{\sim *}(i) \tag{136}$$

with

$$S_0^{\sim} = \begin{pmatrix} 1 & -4 & 8 & -12 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{137}$$

$$V_0^{\sim *}(0) = 4\begin{pmatrix} 4 & -5 & -2 & 3 \\ -3 & 4 & 1 & -2 \\ 2 & -3 & 0 & 1 \\ -1 & 2 & -1 & 0 \end{pmatrix}, \tag{138}$$

$$V_0^{\sim *}(1) = 4\begin{pmatrix} 3 & -6 & 3 & 0 \\ -2 & 4 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{139}$$

In view of (3) in Part 4,

$$A_0^{*}(z;\nu) = \nu^3 T_{4,\nu)} A_0^{\sim}(z;\nu)(T_{4,\nu)})^{-1} = \tag{140}$$

$$\begin{pmatrix} \nu^3 & -4\nu^2 & 8\nu & -12 \\ 0 & \nu^3 & -4\nu^2 & 8\nu \\ 0 & 0 & \nu^3 & -4\nu^2 \\ 0 & 0 & 0 & \nu^3 \end{pmatrix} +$$

$$z4\begin{pmatrix} 4\nu^3 & -5\nu^2 & -2\nu & 3 \\ -3\nu^4 & 4\nu^3 & \nu^2 & -2\nu \\ 2\nu^5 & -3\nu^4 & 0 & \nu^2 \\ -\nu^6 & 2\nu^5 & -\nu^4 & 0 \end{pmatrix} +$$

$$4z\begin{pmatrix} 3\nu^2 & -6\nu & 3 & 0 \\ -2\nu^3 & 4\nu^2 & -2\nu & 0 \\ \nu^4 & -2\nu^3 & \nu^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$(z-1)\begin{pmatrix} 16\nu^3+12\nu^2 & -20\nu^2-24\nu & -8\nu+12 & 12 \\ -12\nu^4-8\nu^3 & 16\nu^3+16\nu^2 & 4\nu^2-8\nu & -8\nu \\ 8\nu^5+4\nu^4 & -12\nu^4-8\nu^3 & 4\nu^2 & 4\nu^2 \\ -4\nu^6 & 8\nu^5 & -4\nu^4 & 0 \end{pmatrix} +$$

$$\begin{pmatrix} 17\nu^3+12\nu^2 & -24\nu^2-24\nu & 12 & 0 \\ -12\nu^4-8\nu^3 & 17\nu^3+16\nu^2 & -8\nu & 0 \\ 8\nu^5+4\nu^4 & -12\nu^4-8\nu^3 & \nu^3+4\nu^2 & 0 \\ -4\nu^6 & 8\nu^5 & -4\nu^4 & \nu^3 \end{pmatrix}.$$

Clearly,

$$\nu = -1/2 + \tau,\ \nu^2 = \mu + 1/2 - \tau, \tag{141}$$

$$\nu^3 = (\mu+1)(\tau - 3/2) + 1 = -3\mu/2 - 1/2 + (\mu+1)\tau, \tag{142}$$

$$\nu^4 = \mu^2 + 2\mu + 1/2 - (2\mu+1)\tau, \tag{143}$$

$$\nu^5 = (\mu - \nu)\nu^3 = -3\mu^2/2 - \mu/2 + (\mu^2+\mu)\tau - \\ (\mu^2 + 2\mu + 1/2 - (2\mu+1)\tau) = \\ -5\mu^2/2 - 5\mu/2 - 1/2 + (\mu^2 + 3\mu + 1)\tau, \tag{144}$$

$$\nu^6 = (\mu - \nu)\nu^4 = \mu^3 + 2\mu^2 + \mu/2 - (2\mu^2+\mu)\tau - \\ (-5\mu^2/2 - 5\mu/2 - 1/2 + (\mu^2 + 3\mu + 1)\tau) = \\ \mu^3 + 9\mu^2/2 + 3\mu + 1/2 - (3\mu^2 + 4\mu + 1)\tau. \tag{145}$$

Therefore

$$16\nu^3 + 12\nu^2 = -12\mu - 2 + (16\mu + 4)\tau \tag{146}$$

(147)
$$-20\nu^2 - 24\nu = -20\mu - 4\nu = -20\mu + 2 - 4\tau,\ -8\nu + 12 = -8\tau + 16,$$

$$-12\nu^4 - 8\nu^3 = -12\mu^2 - 12\mu - 2 + (16\mu+4)\tau, \tag{148}$$

$$16\nu^3 + 16\nu^2 = 16\mu(\tau - 1/2) = -8\mu + 16\mu\tau, \tag{149}$$

$$4\nu^2 - 8\nu = 4\mu + 6 - 12\tau,\ -8\nu = -8\tau + 4, \tag{150}$$

$$8\nu^5 + 4\nu^4 = -16\mu^2 - 12\mu - 2 + (8\mu^2 + 16\mu + 4)\tau \tag{151}$$

$$4\nu^2 = 4\mu + 2 - 4\tau. \tag{152}$$

In view of (143) and (143),

$$-4\nu^6 = -4\mu^3 - 18\mu^2 - 12\mu - 2 + (12\mu^2 + 16\mu + 4)\tau \tag{153}$$

$$8\nu^5 = -20\mu^2 - 20\mu - 4 + (8\mu^2 + 24\mu + 8)\tau, \tag{154}$$

$$-4\nu^4 = -4\mu^2 - 8\mu - 2 + (8\mu + 4)\tau, \tag{155}$$

$$17\nu^3 + 12\nu^2 = -27\mu/2 - 5/2 + (17\mu + 5)\tau, \tag{156}$$

$$-24\nu^2 - 24\nu = -24\mu, \tag{157}$$

$$-8\nu + 12 = 16 - 8\tau, \tag{158}$$

$$17\nu^3 + 16\nu^2 = -19\mu/2 - 1/2 + (17\mu + 1)\tau \tag{159}$$

$$\nu^3 + 4\nu^2 = 5\mu/2 + 3/2 + (\mu - 3)\tau. \tag{160}$$

In view of (140) – (160)

$$A_0^*(z;\nu) = \tag{161}$$

$$\frac{1}{2}\times$$
$$\begin{pmatrix} -27\mu-5 & -48\mu & 24 & 0 \\ -24\mu^2-24\mu-4 & -19\mu-1 & 8 & 0 \\ -32\mu^2-24\mu-4 & -24\mu^2-24\mu-4 & 5\mu+3 & 0 \\ -4(2\mu^3+(3\mu+1)^2) & -8(5\mu^2+5\mu+1) & -8\mu^2-16\mu-4 & -3\mu-1 \end{pmatrix} +$$
$$\tau\times$$
$$\begin{pmatrix} 17\mu+5 & 0 & 0 & 0 \\ 16\mu+4 & 17\mu+1 & -8 & 0 \\ 8\mu^2+16\mu+4 & 16\mu+4 & \mu-3 & 0 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & \mu+1 \end{pmatrix} +$$
$$\frac{z-1}{2}\times$$
$$\begin{pmatrix} -24\mu-4 & -40\mu+4 & 32 & 24 \\ -24\mu^2-24\mu-4 & -16\mu & 8\mu+12 & 8 \\ -32\mu^2-24\mu-4 & -24\mu^2-24\mu-4 & 8\mu+4 & 8\mu+4 \\ -4(2\mu^3+(3\mu+1)^2) & -8(5\mu^2+5\mu+1) & -8\mu^2-16\mu-4 & 0 \end{pmatrix} +$$
$$(z-1)\tau\times$$
$$\begin{pmatrix} 16\mu+4 & -4 & -8 & 0 \\ 16\mu+4 & 16\mu & -12 & -8 \\ 8\mu^2+16\mu+4 & 16\mu+4 & -4 & -4 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & 0 \end{pmatrix}.$$

Let

(162)
$$R_0^{\vee\vee}(0) = \begin{pmatrix} -5 & 0 & 24 & 0 \\ -4 & -1 & 8 & 0 \\ -4 & -4 & 3 & 0 \\ -4 & -8 & -4 & -1 \end{pmatrix},\ V_0^{\vee\vee}(0) = \begin{pmatrix} -4 & 4 & 32 & 24 \\ -4 & 0 & 12 & 8 \\ -4 & -4 & 4 & 4 \\ -4 & -4 & -4 & 0 \end{pmatrix},$$

(163)
$$R_0^{\vee\vee}(1) = \begin{pmatrix} -27 & -48 & 0 & 0 \\ -24 & -19 & 0 & 0 \\ -24 & -24 & 5 & 0 \\ -24 & -40 & -16 & -3 \end{pmatrix},\ V_0^{\vee\vee}(1) = \begin{pmatrix} -24 & -40 & 0 & 0 \\ -24 & -16 & 8 & 0 \\ -24 & -24 & 8 & 8 \\ -24 & -40 & -16 & 0 \end{pmatrix},$$

(164)
$$R_0^{\vee\vee}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \\ -32 & -24 & 0 & 0 \\ -36 & -40 & -8 & 0 \end{pmatrix}, \; V_0^{\vee\vee}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \\ -32 & -24 & 0 & 0 \\ -36 & -40 & -8 & 0 \end{pmatrix},$$

(165)
$$R_0^{\vee\vee}(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \end{pmatrix}, \; V_0^{\vee\vee}(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \end{pmatrix},$$

(166)
$$R_0^{\wedge\wedge}(0) = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 4 & 1 & -8 & 0 \\ 4 & 4 & -3 & 0 \\ 4 & 8 & 4 & 1 \end{pmatrix}, \; V_0^{\wedge\wedge}(0) = \begin{pmatrix} 4 & -4 & -8 & 0 \\ 4 & 0 & -12 & -8 \\ 4 & 4 & -4 & -4 \\ 4 & 8 & 4 & 0 \end{pmatrix},$$

(167)
$$R_0^{\wedge\wedge}(1) = \begin{pmatrix} 17 & 0 & 0 & 0 \\ 16 & 17 & 0 & 0 \\ 16 & 16 & 1 & 0 \\ 16 & 24 & 8 & 1 \end{pmatrix}, \; V_0^{\wedge\wedge}(1) = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 16 & 16 & 0 & 0 \\ 16 & 16 & 0 & 0 \\ 16 & 24 & 8 & 0 \end{pmatrix},$$

(168)
$$R_0^{\wedge\wedge}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ 12 & 8 & 0 & 0 \end{pmatrix}, \; V_0^{\wedge\wedge}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ 12 & 8 & 0 & 0 \end{pmatrix}.$$

In view of (161) – (168),

(169)
$$A_0^*(z;\nu) =$$
$$\frac{1}{2}\left(\sum_{k=0}^{3} \mu^k (R_k^{\vee\vee} + (z-1)V_k^{\vee\vee})\right) +$$
$$\tau\left(\sum_{k=0}^{2} \mu^k (R_k^{\wedge\wedge} + (z-1)V_k^{\wedge\wedge})\right).$$

Moreover,
$$R_k^{\vee\vee} = R_k^{\vee},\; V_k^{\vee\vee} = V_k^{\vee},$$
for $k = 0,\ 1,\ 2,\ 3$,
$$R_k^{\wedge\wedge} = R_k^{\wedge},\; V_k^{\wedge\wedge} = V_k^{\wedge},$$
for $k = 0,\ 1,\ 2$, where
$$R_k^{\vee},\; V_k^{\vee},\ \text{for}\ k = 0,\ 1,\ 2,\ 3,$$

and

$$R_k^{\wedge},\ V_k^{\wedge},\ \text{for}\ k = 0,\ 1,\ 2$$

are pointed in the Part 2 and in section 3.2 of Part 3.

According to (32) and (33) of the Part 2,

$$\nu^3 X_{0,k}(z;\nu-1) = A_0^*(z;\nu)X_{0,k}(z;\nu), \tag{170}$$

$$(-\nu)^3 X_{0,k}(z;\nu) = A_0^*(z;-\nu)X_{0,k}(z;\nu-1), \tag{171}$$

where $k = 1,\ 2,\ 3,\ |z| > 1, \nu \in \mathbb{Z}$,

$$A_0^*(z;\nu) = (1/2)U_0^{\vee}(z;\mu) + U_0^{\wedge}(z;\mu)\tau, \tag{172}$$

$$\mu = \nu(\nu+1),\ \tau = \nu + \frac{1}{2}, \tag{173}$$

$$U_0^{\vee}(z,\mu) = \sum_{k=0}^{3} \mu^k (R_0^{\vee}(k)) + (z-1)V_0^{\vee}(k)), \tag{174}$$

$$U_0^{\wedge}(z,\mu) = \sum_{k=0}^{2} \mu^k (R_0^{\wedge}(k)) + (z-1)V_0^{\wedge}(k)). \tag{175}$$

So, (169) coincides with (172)

§6.4. Short derive of Apéry equation.

After replacing in (170) ν by $-\nu-1$ we obtain the equality

$$(-\nu-1)^3 X_{0,k}(z;-\nu-2) = A_0^*(z;-\nu-1)X_{0,k}(z;-\nu-1), \tag{176}$$

where $k = 1, 2, 3,\ \nu \in \mathbb{Z},\ |z| > 1$. Since

$$X_{0,k}(z;-\nu-1) = X_{0,k}(z;\nu),\ X_{0,k}(z;-\nu-2) = X_{0,k}(z;\nu+1)$$

for $k = 1, 2, 3,\ \nu \in \mathbb{Z},\ |z| > 1$, it follows that the equality (176) is equivalent to the equallity

$$-(\nu+1)^3 X_{0,k}(z;\nu+1) = A_0^*(z;-\nu-1)X_{0,k}(z;\nu), \tag{177}$$

for $k = 1, 2, 3,\ \nu \in \mathbb{Z},\ |z| > 1$. After substraction of the equality (177) from (170) we obtain the equality

$$(\nu+1)^3 X_{0,k}(z;\nu+1) + \nu^3 X_{0,k}(z;\nu-1) = \tag{178}$$

$$(A_0^*(z;\nu) - A_0^*(z;-\nu-1))X_{0,k}(z;\nu),$$

where $k = 1, 2, 3, |z| > 1, \nu \in \mathbb{Z}$. In view of (172), the equality (178) take the form

(179)
$$(\nu+1)^3 X_{0,k}(z;\nu+1) + \nu^3 X_{0,k}(z;\nu-1) = U_0^{\wedge}(z;\mu)(2\nu+1)X_{0,k}(z;\nu),$$

where $k = 1, 2, 3, |z| > 1, \nu \in \mathbb{Z}$. In view of (161) and (172),

(180)
$$U_0^{\wedge}(z;\mu) =$$
$$\begin{pmatrix} 17\mu+5 & 0 & 0 & 0 \\ 16\mu+4 & 17\mu+1 & -8 & 0 \\ 8\mu^2+16\mu+4 & 16\mu+4 & \mu-3 & 0 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & \mu+1 \end{pmatrix} +$$
$$(z-1)\times$$
$$\begin{pmatrix} 16\mu+4 & -4 & -8 & 0 \\ 16\mu+4 & 16\mu & -12 & -8 \\ 8\mu^2+16\mu+4 & 16\mu+4 & -4 & -4 \\ 12\mu^2+16\mu+4 & 8\mu^2+24\mu+8 & 8\mu+4 & 0 \end{pmatrix}.$$

Therefore, in view (179), the first component of the column $X_{0,k}(z;\nu)$ satisfies to the following equality :

(181)
$$(\nu+1)^3 f_{0,k}(z;\nu+1) + \nu^3 f_{0,k}(z;\nu-1) =$$
$$(17\mu+5)(2\nu+1)f_{0,k}(z;\nu)+$$
$$(z-1)(16\mu+4)(2\nu+1)f_{0,k}(z;\nu)+$$
$$(z-1)(-4)(2\nu+1)(\delta f_{0,k})(z;\nu)+$$
$$(z-1)(-8)(2\nu+1)(\delta^2 f_{0,k})(z;\nu),$$

where $k = 1, 2, 3, |z| > 1, \nu \in \mathbb{Z}$. Now we turn $z \in (1, +\infty)$ to 1 for $k = 2, 3$. Since

$$\lim_{z\to 1}(z-1)(\delta f_{0,2})(z;\nu) = 0,$$
$$\lim_{z\to 1}(z-1)(\delta^2 f_{0,2})(z;\nu) = 1,$$

it follows that

(182)
$$(\nu+1)^3 f_{0,2}(1;\nu+1) + \nu^3 f_{0,2}(1;\nu-1) =$$
$$(17\mu+5)(2\nu+1)f_{0,2}(1;\nu) - 8(2\nu+1).$$

In view of (5) and (38),

$$f_{l,3}(1,\nu) = f_{l,4}(1,\nu) \text{ for } l = 1, 2, 3,\ \nu \in \mathbb{Z},$$
$$\lim_{z\to 1}(z-1)(\delta^k f_{0,3})(z;\nu) = 0$$

for $k = 1, 2$. Therefore

(183)
$$(\nu+1)^3 f_{0,4}(1;\nu+1) + \nu^3 f_{0,4}(1;\nu-1) =$$
$$(17\mu+5)(2\nu+1)f_{0,4}(1;\nu).$$

So $y(\nu) = f_{0,4}(1;\nu)$ satisfies the equation

$$(\nu+1)^3 y(1;\nu+1) + \nu^3 y\nu - 1) = (17\mu+5)(2\nu+1)y(\nu).$$

Clearly, this equation can be represented in the form

(184)
$$(\nu+1)^3 y(1;\nu+1) + \nu^3 y\nu - 1) = (17\nu^3+51\nu^2+27\nu+5)y(\nu)(34\nu^3+85\nu^2).$$

Equation (184) is the famous Apéry equation.

§6.5. Corrections in the previous parts of this paper.

Polynomial $a^{\vee}_{2,3,3}(z;\nu)$ in the Part 2 must have a form

$$a^{\vee}_{2,3,3}(z;\nu) = (1/2)(-9-183\mu+2586\mu^2+7113\mu^3)+$$
$$(1/2)(z-1)(-8-176\mu+2600\mu^2+7120\mu^3)$$

instead of

$$a^{\vee}_{2,3,3}(z;\nu) = (1/2)(-9-183\mu+2856\mu^2+7113\mu^3)+$$
$$(1/2)(z-1)(-8-176\mu+2600\mu^2+7120\mu^3).$$

Polynomial $a^{\vee}_{2,4,4}(z;\nu)$ in the Part 2 must have a form

$$a^{\vee}_{2,4,4}(z;\nu) = (1/2)(-1-1255\mu-3214\mu^2+2105\mu^3)+$$
$$(1/2)(z-1)(-1248\mu-3200\mu^2+2112\mu^3),$$

instead of

$$a^{\vee}_{2,4,4}(z;\nu) = (1/2)(-1-1225\mu-3214\mu^2+2105\mu^3)+$$
$$(1/2)(z-1)(-1248\mu-3200\mu^2+2112\mu^3).$$

Polynomial $a^{\wedge}_{2,8,7}(z;\nu)$ in the Part 2 must have a form

$$a^{\wedge}_{2,8,7}(z;\nu) = 8+48\mu+80\mu^2+32\mu^3+$$
$$(z-1)(8+48\mu+80\mu^2+32\mu^3)$$

instead of

$$a^{\wedge}_{2,8,7}(z;\nu) = 8+48\mu+80\mu^2+33\mu^3+$$
$$(z-1)(8+48\mu+80\mu^2+32\mu^3).$$

On the intersection of 5-th row and 7-th column of the matrix $R^{\vee}_2(1)$ in the Part 2 must stand the number 1056 instead of 1256. On the intersection of 4-th row and 4-th column of the matrix $R^{\vee}_2(1)$ in the Part 2 must stand the number -1255 instead of -1225. On the intersection of 2-th row and 2-th column of the matrix $R^{\vee}_2(2)$ in the Part 2 must stand the number 1922 instead of 1992. On the intersection of 3-th row and 3-th column of the matrix $R^{\vee}_2(2)$ in the Part 2 must stand the number 2586 instead of 2856. On the intersection of 3-th row and 4-th column of the matrix $V^{\vee}_2(2)$ in the Part 2 must stand the number 3616 instead of 3613.

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