DESINGULARIZING WEIGHTS AND THE HEAT KERNEL BOUNDS

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Abstract. We obtain sharp upper weighted bounds on the heat kernel for the parabolic operator $\partial_t = \Delta_x + V(t,x)$, $x \in \mathbb{R}^d$, $d \geq 3$, with potential V having singularities of the type $c|x|^{-2}$, with a suitably small constant c. These bounds differ from the standard ones by a multiplicative factor of the form $\psi(t-s) \cdot \phi(x) \cdot \phi(y)$ with appropriate weight functions $\phi(x)$ and the diagonal term $\psi(\tau)$ that differs from the standard case $(\tau^{-d/2})$ for the positive sign singularity. The growth rate of the weights $\phi(x)$ depends (explicitly) on the constants c. This is the first result dealing with a situation where the lower and upper bounds behave differently at $t = 0$ and/or $x = 0$. Standard bounds $(\phi \equiv 1)$ and methods (estimates in L^p-spaces without desingularizing weights) fail for singular potentials. Even in the standard case of time dependent potentials from the parabolic Kato class our result is new.

I. INTRODUCTION. We obtain sharp upper weighted bounds on the heat kernel $Z_V(t,x;s,y)$ for the parabolic operator $\partial_t - \Delta_x + V(t,x)$, $x \in R^d$, $d \geq 3$, with potentials V having singularities of the type $|x|^{-2}$. These bounds are of the form $Z_V(t,x;s,y) \leq C\psi(t-s)\phi(x)\phi(y)e^{-c|x-y|^2/(t-s)}$ for appropriate weight functions $\phi(x)$, "the diagonal term" functions $\psi(\tau)$ and constants C, c. Standard bounds $(\phi \equiv 1)$ and methods (estimates in L^p-spaces without desingularizing weights) fail for singular potentials. For example, in the case of the time independent potential satisfying the inequalities $W + \gamma V_0 \leq V \leq W + \beta V_0$, where W belongs to the Kato class, $V_0(x) \equiv (d-2/2)^2|x|^{-2}$ and γ, β are arbitrary constants such that $0 \leq \gamma \leq \beta < 1$, we establish two sided bounds on $Z_{-V}(t,x;s,y)$ with weights $\phi_i(x) = |x|^{-\sigma_i}$ for $|x| \leq 1$ and $\phi_i(x) = 1$ for $|x| \geq 1$, where $(d-2/2)(-1+\sqrt{1+\gamma}) < \sigma_1 < \sigma_2 < (d-2/2)(1-\sqrt{1-\beta})$, and $\psi_2(\tau)$ for the upper bound being the usual $\tau^{-d/2}$ while $\psi_1(\tau)$ for the lower bound equal $[\psi_2(\tau)]^{(q-1)/q}$, $2 < q < d/\sigma_1$. Even in the standard case of time dependent potentials from the parabolic Kato class our result is new.

Kato parabolic class. The following question is important. What is the "maximal" class of potentials for which a (unique) weak heat kernel exists and satisfies the upper and lower Gaussian bounds? Our answer to this question is as follows. Given a $w \in L^1_{\rm loc}(R^1_+ \times R^d)$, set

$$k(w;r) \;:=\; \operatorname*{ess\,sup}_{u\geq 0,\, x\in R^r} \int_u^{u+r} \int_{R^d} \Gamma_{t-u}(x-y)|w|(t,y)\,dy\,dt$$

where $\Gamma_t(z) := (4\pi t)^{-d/2} \exp\Big(-\frac{|z|^2}{4t}\Big)$.

We say that w belongs to the Kato parabolic class and we write $w \in K_d^P$ if and only if

$$k(w) \ := \ \lim_{r\to 0} k(w;r)$$

exists and is finite.

Theorem 1. *Let $d \geq 2$ and $V = V^+ - V^-$, $V^\pm \geq 0$. Assume that $V \in K_d^P$ and $k(V^-) < 1$. Then the unique weak heat kernel, $Z_V(t,x;s,y)$, for parabolic operator $\partial_t - \Delta_x + V(t,x)$ in R^d exists and, for any $\varepsilon \in]0, 1-k(V^-)[$ and $\delta > 0$, there are constants $c_\varepsilon^{(2)}$, $c_{\varepsilon,\delta}^{(1)} > 0$ and ω_i such that the estimates*

$$c_{\varepsilon,\delta}^{(1)} e^{(t-s)\omega_1} \Gamma_{\frac{t-s}{1+k(V^+)+\delta}}(x-y) \ \leq \ Z_V(t,x;s,y) \ \leq \ c_\varepsilon^{(2)} e^{(t-s)\omega_2} \Gamma_{\frac{t-s}{1-k(V^-)-\varepsilon}}(x-y)$$

are satisfied for all $t > s \geq 0$ and a.e. $(x,y) \in R^d \times R^d$.

Its proof rests upon some consequences of DuHamel's principle and the Lie product formula for propagators. This theorem generalizes and strengthens (some) results in [Se1] and [Zh1-3]. For its applications see [Se2].

Singular potentials. We now consider the case of $d \geq 3$ and assume that potential V is time independent and satisfies the following constraints.

There are constants $\lambda_i, \gamma, \beta; 0 \leq \gamma \leq \beta < 1$, such that $\lambda_1 + \gamma V_0(x) \ \leq \ V(x) \ \leq \ \lambda_2 + \beta V_0(x)$ (pointwise a.e.).

Due to the Hardy inequality, we know that V is a $-\Delta$-form bounded operator in the Hilbert space $L^2 = L^2(R^d, dx)$ with relative bound β so that the form difference $H^- := -\Delta \dot{-} V$ is a well defined (self-adjoint bounded from below) operator in L^2. Due to [Se2, Th.III.5.1], $\exp(-tH^-)$ extends by continuity to a holomorphic semigroup on L^r for all $r \in I_m :=]p(\beta), p'(\beta)[$ where

$$p'(\beta) \ = \ \frac{d}{d-2} \frac{2}{1-\sqrt{1-\beta}}, \quad p(\beta) \ = \ p'(\beta)/\big(p'(\beta)-1\big) \ .$$

Moreover, for all $t > 0$, $\ell = 0, 1, \ldots$, $p(\beta) < r \leq s < p'(\beta)$, there are constants $M = M(r, s, \ell, \lambda_2)$ such that

$$\|\partial_t^\ell e^{-tH^-}\|_{r\to s} \ \leq \ M e^{\lambda_2 t} t^{-\ell - d(s-r)/2rs} \ .$$

Here and below we denote by $\|T\|_{r\to s}$ the operator norm of the map T: $L^r \to L^s$.

It is well known that for such a singular potential the heat kernel of $\exp(-tH^-)$ does not admit the standard upper Gaussian bound and the semigroup is not ultracontractive. However, we show that for a suitable weight ϕ the *weighted* semigroup $\exp(-tH_\phi^-)$: $L^2_\phi \to L^2_\phi$ is ultracontractive and derive an upper heat kernel bound of the form $e^{-tH^-}(x,y) \le ct^{-d/2}\phi(x)\phi(y)$. We also consider operator $H^+ := -\Delta \dot{+} V$, the form sum of $-\Delta$ and V, and establish an upper heat kernel bound of the form $e^{-tH^+}(x,y) \le C_t\varphi(x)\varphi(y)$ for appropriate functions φ and C_t. As a byproduct of these estimates we obtain two-sided weighted Gaussian bounds on $e^{-tH^-}(x,y)$. These bounds become sharp when $\gamma = \beta$. To formulate precisely the results let us define weight functions

$$\phi_i(x) \;:=\; |x|^{-\sigma_i} \;\text{ if }\; |x| \le 1 \;\text{ and }\; \phi_i(x) \;=\; 1 \;\text{ if }\; |x| \ge 1,\; i = 1,2,3 \;,$$

where

$$\sigma_1 \;=\; \frac{d-2}{2}\big(-1+\sqrt{1+\gamma}\big), \quad \sigma_2 \;=\; \frac{d-2}{2}\big(1-\sqrt{1-\beta}\big) \quad\text{and}\quad \sigma_3 \;=\; -\sigma_1 \;.$$

Let q be any number from the interval $]2, d/\sigma_1[$.

Theorem 2. *There are constants $c_i = c_i(q) > 0$ and ω_i such that the inequalities*

$$c_1 t^{-d(q-1)/(2q)} e^{-\omega_1 t - \frac{|x-y|^2}{2t}} \phi_1(x)\phi_1(y) \;\le\; e^{-tH^-}(x,y) \;\le\; c_2 t^{-d/2} e^{\omega_2 t}\phi_2(x)\phi_2(y) \;,$$
$$e^{-tH^+}(x,y) \;\le\; c_3 t^{-d(1+q)/(2q)} e^{\omega_1 t}\phi_3(x)\phi_3(y) \;,$$

hold for all $t > 0$ and $x, y \in R^d$.

Theorem 2 seems to be the first result dealing with a situation where the lower and upper bounds on $e^{-tH^-}(x,y)$ behave differently at $t = 0$ and/or at $x = 0$.

Combined Kato-Singular potentials.

Theorem 2′. *Assume that 1) $W \in K_d^P$ with $k(W) = 0$; 2) For some constants $0 \le \gamma \le \beta < 1$, the inequality $\gamma V_0(x) \le V(t,x) \le \beta V_0(x)$ is valid pointwise a.e. $t > 0$ and $x \in R^d$. Then there exist weak heat kernels Z_{W-V}, Z_{W+V} for parabolic operators $\partial_t - \Delta + W - V$ and $\partial_t - \Delta + W + V$ respectively and they are unique. There are constants $c_i > 0, \omega_i$ such that the estimates*

$$\begin{aligned} c_1(t-s)^{-d(q-1)/(2q)} e^{-(t-s)\omega_1 - c_0\frac{|x-y|^2}{t-s}} \tilde\phi_1(x)\tilde\phi_1(y) \;&\le\; Z_{W-V}(t,x;s,y) \\ &\le\; c_2(t-s)^{-d/2} e^{(t-s)\omega_2}\tilde\phi_2(x)\tilde\phi_2(y) \;, \\ Z_{W+V}(t,x;s,y) \;&\le\; c_3 t^{-d(1+q)/(2q)} e^{(t-s)\omega_1}\tilde\phi_3(x)\tilde\phi_3(y) \end{aligned}$$

hold for all $t > 0$, $x, y \in R^d$, where $\tilde{\phi}_i$ are defined in the same way as ϕ_i but with $\tilde{\sigma}_1 > \sigma_1, \tilde{\sigma}_2 > \sigma_2$.

To formulate an analogous result with the Gaussian factor in the upper bounds let us define, for any $\varepsilon \in \Big]0, 1 \wedge \Big(\frac{1}{\beta} - 1\Big)\Big[$,

$$\phi_{1,\varepsilon}(x) \;:=\; |x|^{-\sigma_i(\varepsilon)} \;\text{ if }\; |x| \le 1 \;\text{ and }\; \phi_{i,\varepsilon}(x) \;=\; 1 \;\text{ if }\; |x| \ge 1,\; i = 1,2,3,$$

where $\sigma_3(\varepsilon) = -\sigma_1(\varepsilon)$ and

$$\sigma_1(\varepsilon) \;=\; \frac{d-2}{2(1-\varepsilon)}\big(-1 + \sqrt{1+\gamma-\varepsilon\gamma}\big), \quad \sigma_2(\varepsilon) \;=\; \frac{d-2}{2(1+\varepsilon)}\big(1 - \sqrt{1-\beta-\varepsilon\beta}\big)\;.$$

Let q be any number from the interval $]2, d/\sigma_1(\varepsilon)[$.

Theorem 3. *There are constants $c_i = c_i(\varepsilon, q) > 0$ and $\omega_i = \omega_i(\varepsilon)$ such that the inequalities*

$$e^{-tH^-}(x,y) \;\le\; c_1 t^{-d/2}\phi_{2,\varepsilon}(x)\phi_{2,\varepsilon}(y)\;\exp\Big]\omega_1 t - c_2\frac{|x-y|^2}{4t}\Big[,$$

$$e^{-tH^+}(x,y) \;\le\; c_3 t^{-d(1+q)/(2q)}\phi_{3,\varepsilon}(x)\phi_{3,\varepsilon}(y)\;\exp\Big]\omega_2 t - c_4\frac{|x-y|^2}{4t}\Big[$$

are satisfied for all $t > 0$ and $x, y \in R^d$.

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II. AUXILIARY RESULTS. Let X be a Banach space and assume that for each t the operator $-A(t)$ generates a holomorphic semigroup in X. Assume that there is a family U of strongly continuous functions $U^{t,s}$ (propagators) defined on $0 \le s \le t < \infty$, which takes values in $L(X)$, the space of all bounded linear operators from X into X; $U^{s,s} = I$ for each $s \ge 0$; $U^{t,r}U^{r,s} = U^{t,s}$ for $0 \le s \le r \le t < \infty$; there is a dense subspace D_U such that $\partial_t U^{t,s}f + A(t)U^{t,s}f = 0$ and $\partial_t U^{t,s}f - U^{t,s}A(s)f = 0$ for all $f \in D_U$.

Let P be another such family generated by $-B(t)$ with $D_P = D = D_U$. Let us assume that $P^{t,s}D \subseteq D$ for $0 \le s \le t < \infty$ and $P^{t,s}f$ (resp. $U^{t,s}f$) are strongly differentiable for each $f \in D$ with respect to t (resp. $s \in [0,t]$). Then the following DuHamel formula

$$U^{t,s}f - P^{t,s}f \;=\; \int_s^t U^{t,\tau}\big(B(\tau) - A(\tau)\big)P^{\tau,s}f\,d\tau, \quad \forall f \in D$$

is valid. Set $W(t) := B(t) - A(t)$ and assume that $W(t) \in L(X)$. Let $\|P^{t,s}\| \le Me^{\omega(t-s)}$. Then the following Miyadera condition holds: *There are numbers $\eta > 0$ and $0 < a < 1$ such that*

$$\text{(M)} \qquad \sup_{u \ge 0} \int_u^{u+\eta} \big\|W(\tau)P^{\tau,u}f\big\|e^{-\omega\tau}\,d\tau \le a\|f\|, \quad \forall f \in D\;.$$

A simple iteration process based on the DuHamel formula yields

$$(*) \qquad \|U^{t,s}\| \;\leq\; \frac{M}{1-a}\,\exp\left(\left(\omega+\frac{1}{\eta}\,\log\,\frac{M}{1-a}\right)(t-s)\right)\;.$$

We will also use a product formula for quasi *contractions* R, U, T generated by $-C(t)$, $-A(t)$, $-V(t)$ respectively. If $D\big(C(t)\big) = D\big(A(t)\big) \cap D\big(V(t)\big)$ and $C(t) = A(t) + V(t)$ for all t, and $\|C(\tau)R^{\tau,r}C^{-1}(r)\| \leq \delta(t_0) < \infty$ for all $0 \leq \tau - r \leq t_0 < \infty$, then the Lie product formula

$$(**) \qquad R^{t,s} \;=\; str.\lim_{n\to\infty} F_n^{t,s}$$

holds, where $F_n^{t,s} := U^{t,t_{n-1}}T^{t,t_{n-1}}U^{t_{n-1},t_{n-2}}T^{t_{n-1},t_{n-2}}\cdot\ldots\cdot U^{t_1,s}T^{t_1,s}$, $t_i := s + \frac{i}{n}(t-s)$. One can easily prove this by mimicking the corresponding proof for semigroups in [Ne].

Let now (M,μ) be a measure space with σ-finite measures $\mu \geq 0$, $X = L^r = L^r(M,\mu)$ for some $r \in [1,\infty[$. Let U, T from $(**)$ be positivity preserving, and U ultracontractive. The latter means that $\|U^{t,s}\|_{1\to\infty} \leq S(t-s)$ for all $t-s > 0$ where function $S(\tau)$ is bounded on any compact interval from $]0,\infty[$. Suppose further that for some $p > 1$ operators $-pV(t)$ and $-A(t) - pV(t)$ generate T_p and R_p respectively, and that $(**)$ is also valid for R_p, U, T_p. If $V(t)$ is an operator of multiplication by a real valued function $V(t,x)$, then the following (pointwise a.e.) inequality

$$(***) \qquad R^{t,s}h \;\leq\; (R_p^{t,s}h^{\alpha p})^{1/p}(U^{t,s}h^{(1-\alpha)p'})^{1/p'}$$

is valid for all $0 \leq h \in L^1 \cap L^\infty$, $0 \leq \alpha \leq 1$, where $p' = p/(p-1)$. This inequality is standard in the functional integration theory. In our setting it follows via the Lie product formula (cf. [HS]). Indeed, by ultracontractivity, $U^{t,s}$ is an integral operator. Let $N(t,x;s,y)$ denote its integral kernel, which is non-negative due to positivity preserving of U. Then

$$\begin{aligned} F_n^{t,s}(V)h(x) \;&\equiv\; F_n^{t,s}h(x) \\ &= \int_M \cdots \int_M N(t,x;t_{n-1},x_{n-1})e^{-\frac{t-s}{n}V(t_{n-1},x_{n-1})}\cdot\ldots \\ &\qquad \cdot N(t_1,x_1;s,y)e^{-\frac{t-s}{n}V(s,y)}h(y)dx_{n-1}\cdot\ldots\cdot dx_1 dy \end{aligned}$$

(here we are using Hölder's inequality and the reproductive property of U)

$$\leq \big(F_n^{t,s}(pV)h^{\alpha p}(x)\big)^{1/p}\big(U^{t,s}h^{(1-\alpha)p'}(x)\big)^{1/p'}\;.$$

We now pass to the limit $n \to \infty$ in this inequality, using $(**)$, which proves $(***)$.

Proposition 1. *Let $p = m/(m-1)$ for some integer $m \geq 2$. In addition to the above conditions assume that $\|R_p^{t,s}1\|_\infty \leq \theta_\infty(t_0)$ and $\|R_p^{t,s}\|_{1\to 1} \leq \theta_1(t_0)$ for all $0 \leq t-s \leq t_0 < \infty$. Then there is a constant $c = c\big(\theta_1(t_0), \theta_\infty(t_0)\big)$ such that, for all $0 \leq t-s \leq t_0$,*

$$\|R^{t,s}\|_{1\to\infty} \ \leq \ c^{m+1} S\Big(\frac{t-s}{m}\Big) \ .$$

Proof. Applying $(***)$ with $\alpha = 0$ yields

$$\begin{aligned} \|R^{t,s}f\|_\infty \ &\leq \ \|R_p^{t,s}1\|_\infty^{1/p} \|U^{t,s}|f|^{p'}\|_\infty^{1/p'} \\ &\leq \ \|R_p^{t,s}1\|_\infty^{1/p} \|U^{t,s}\|_{1\to\infty}^{1/p'} \|f\|_m \ . \end{aligned}$$

Applying $(***)$ with $\alpha = n/(n+1)$ where $1 \leq n \leq m-1$ yields

$$\begin{aligned} \|R^{t,s}f\|_{m/n} \ &\leq \ \|R_p^{t,s}|f|^{\alpha p}\|_{m/np}^{1/p} \|U^{t,s}|f|^{(1-\alpha)m}\|_\infty^{1/m} \\ &\leq \ \|R_p^{t,s}\|_{(m-1)/n\to(m-1)/n}^{1/p} \|f\|_{\alpha m/n}^{\alpha} \|U^{t,s}\|_{1\to\infty}^{1/m} \|f\|_{(1-\alpha)m}^{1-\alpha} \\ &\leq \ \|R_p^{t,s}\|_{(m-1)/n\to(m-1)/n}^{1/p} \|U^{t,s}\|_{1\to\infty}^{1/m} \|f\|_{m/(n+1)} \ . \end{aligned}$$

By interpolation, $\|R_p^{t,s}\|_{q\to q} \leq c\big(\theta_1(t_0), \theta_\infty(t_0)\big)$ for all $1 \leq q \leq \infty$ and $0 \leq t-s \leq t_0$. By the ultracontractivity of U, $\|U^{t,s}\|_{1\to\infty} \leq S(t-s)$. The desired assertion follows now from the reproductive property of $R^{t,s}$.

Proposition 1 is particularly useful if one can effectively estimate $\theta_i(t_0)$. The latter can be carried out by means of $(*)$.

III. PROOF OF THEOREM 1.

We are going now to apply the techniques developed in section II to the parabolic operator $\partial_t - \Delta + V$ in R^d, $d \geq 2$. Let V fulfill the assumptions of Theorem 1. Choose a sequence $\{V_n,\ n \geq 1\}$ of bounded smooth potentials such that $V_n^\pm \leq V^\pm$ and $k(V^\pm - V_n^\pm) \to 0$ as $n \to \infty$. Set $P^{t,s} := e^{(t-s)\Delta}$. Then it is easy to see that the Miyadera condition (M) in $L^1 = L^1(R^d, dx)$ is satisfied for V_n^- and $(p_0-1)V_n^+$ uniformly in n, where constant p_0 is determined by the condition: $0 < k(V^+)(p_0-1) < 1$. Set $A(t) := -\Delta + V_n(t)$ with domain $D\big(A(t)\big) := (1-\Delta)^{-1}L^1$. Then, due to the standard theory (see e.g. [Fr]), U, P fulfill all of the assumptions which led us to $(*)$. As a result we have for $0 \leq s \leq t < \infty$

$$(*') \qquad \|U^{t,s}\|_{1\to 1} \ = \ \|U^{t,s}\ 1\|_\infty \ \leq \ \frac{1}{1-\alpha}\exp\Big(\Big(\omega + \frac{1}{\eta}\log\frac{1}{1-\alpha}\Big)(t-s)\Big)$$

with constants α, ω and η, dependent only on $k(V^-)$.

The same estimate holds for U_p generated by $A(t) := -\Delta + pV_n(t)$ for any $p \in]1, 1/k(V^-)[$. *We will denote it by* $(*'_p)$. Of course, constants α, ω and η from $(*'_p)$ depend on $pk(V^-)$.

At this stage it is easy to see that Proposition 1 is applicable in this setting. As a consequence it follows

$$\|U^{t,s}\|_{1\to\infty} \ \le \ c^{m+1} m^{d/2} \big(4\pi(t-s)\big)^{-d/2} \qquad (0 \le s < t < \infty),$$

where $m \ge 2$ is the maximal integer satisfying the inequality $k(V^-) < (m-1)/m$, and c denotes the R.H.S. of $(*'_p)$.

Setting $p_{12} = p_1 p_2, m_{12} = \frac{m_1 m_2}{(m_1 - 1_(m_2 - 1)}$, $p_i = \frac{m_i}{m_i - 1}$ where $m_i > 2$, $m_1 m_2 k(V^-) < (m_1 - 1)(m_2 - 1)$, we immediately conclude that

$$\|U^{t,s}_{p_{12}}\|_{1\to\infty} \ \le \ c^{m_{12}+1} m_{12}^{d/2} \big(4\pi(t-s)\big)^{-d/2} \qquad (0 \le s < t < \infty) \ .$$

Using this bound and $(***)$, we have $U^{t,s}(x,y) \le \big(U^{t,s}_{p_{12}}(x,y)\big)^{1/p_{12}} \big(P^{t,s}(x,y)\big)^{1/p'_{12}}$ or

$$(1_-) \qquad Z_{-V_n^-}(t,x;s,y) \ \le \ c_2 e^{(t-s)\omega_2} \Gamma_{(t-s)p'_{12}}(x-y)$$

for all $x, y \in R^d$ and all $0 \le s < t < \infty$.

The same argument gives

$$(1_{-(p_0-1)}) \qquad Z_{-(p_0-1)V_n^+}(t,x;s,y) \ \le \ c_3 e^{(t-s)\omega_3} \Gamma_{(t-s)c_4}(x-y)$$

for all $x, y \in R^d$ and all $0 \le s < t < \infty$.

Finally, put $W \equiv \frac{p_0 - 1}{p_0} V_n^+$ and let $H_W(t) := -\Delta + W(t)$ denote the generator of $H_W^{t,s}$. Since $-\Delta = H_W(t) - W(t)$, we obtain, using $(***)$, $\big(e^{(t-s)\Delta} h\big)^{p_0} \le \big(H^{t,s}_{p_0 W} h\big)^{p_0 - 1} H^{t,s}_{-p_0 W} h$ so that

$$\big(\Gamma_{t-s}(x-y)\big)^{p_0} \ \le \ Z_{-(p_0-1)V_n^+}(t,x;s,y) \big(Z_{V_n^+}(t,x;s,y)\big)^{p_0-1}$$

(here we are using $(1_{-(p_0-1)})$)

$$\le \ c_3 e^{(t-s)\omega_3} \Gamma_{(t-s)c_4}(x-y) \big(Z_{V_n^+}(t,x;s,y)\big)^{p_0-1}$$

or

$$(1_+) \qquad c_4 e^{(t-s)\omega_1} \Gamma_{(t-s)c_5}(x-y) \ \le \ Z_{V_n^+}(t,x;s,y) \ .$$

As soon as (a priori) estimates (1_-) and (1_+) are established the proof of Theorem 1 can be completed using standard approximation arguments (see [Se2], [Zh1-3]).

IV. PROOF OF THEOREM 2. THE UPPER BOUND ON $e^{-tH^-}(x,y)$. Let $0 \le \phi, 1/\phi \in L^2_{\rm loc}(R^d, dx)$. We define a unitary map Φ from $L^2_\phi = L^2\big(R^d, \phi^2(x)dx\big)$ onto $L^2 = L^2(R^d, dx)$ by $\Phi f = \phi f$ and set $H^-_\phi = \Phi^{-1} H^- \Phi$. We denote $L^r\big(R^d, \phi^2(x)dx\big)$ by L^r_ϕ , $\|\cdot\|_{r,\phi}$ denotes the norm in L^r_ϕ and $\langle f, g\rangle, \langle f, g\rangle_\phi$ denote pairings between dual spaces. We write $V \in PK(\beta)$ if there are constants β and $C(\beta)$ such that

$$(*) \qquad \big|\langle Vh, h\rangle\big| \ \le \ \beta\langle \nabla h, \nabla h\rangle + C(\beta)\langle h, h\rangle$$

for all $h \in H^1(R^d) :=$ the standard Sobolev space.

Let us recall the Nash inequality

$$\langle \nabla h, \nabla h\rangle \ \ge \ C_N \langle h, h\rangle^{1+2/d} \|h\|_1^{-4/d} , \qquad \forall h \in L^1 \cap H^1(R^d) \ .$$

We will usually make no difference in writing an operator and its extension by continuity.

The following conditions imply ultracontractivity of $e^{-tH^-_\phi}$.

Proposition 2. *Let $V \in PK(\beta)$ for some $\beta < 1$. Set $H^- := -\Delta \dot{-} V$. If $\phi \ge 1$ and $e^{-tH^-_\phi} \in L(L^1_\phi)$ with*

$$(**) \qquad \|e^{-tH^-_\phi}\|_{1\to 1,\phi} \ \le \ ce^{\omega t} \text{ for all } t \ge 0 \text{ and some constants } c, \omega,$$

then $e^{-tH^-_\phi}$ is ultracontractive and, for all $t > 0$,

$$\|e^{-tH^-_\phi}\|_{1\to\infty,\phi} \ \le \ \Big(\frac{d}{2C_N(d)(1-\beta)}\Big)^{d/2} c^2 e^{\lambda t} t^{-d/2} ,$$

where $\lambda = \max\{\omega, C(\beta)\}$ and $C_N(d) = C_N$ is the (sharp) constant from the Nash inequality.

Proof. Let $U \in L^2_\phi \cap L^1_\phi$. Set $u_t := e^{-t(\lambda + H^-_\phi)}u$. Since $e^{-tH^-_\phi} = \Phi^{-1}e^{-tH^-}\Phi$ and $e^{-tH^-}L^2 \subset D(H^-)$ $\forall t > 0$, we have $\phi u_t \in D(H^-)$. By the definition of H^-, $D(H^-) \subset H^1(R^d)$ and $\langle H^- g, h\rangle = \langle \nabla g, \nabla h\rangle - \langle Vg, h\rangle$, $g \in D(H^-)$, $h \in H^1(R^d)$. Thus, using the equation $\partial_t u_t + (\lambda + H^-_\phi)u_t = 0$, we have

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\langle u_t, u_t\rangle_\phi &= \big\langle (\lambda + H^-_\phi)u_t, u_t\big\rangle_\phi \\
&= \lambda\langle u_t, u_t\rangle_\phi + \big\langle H^-(\phi u_t), \phi u_t\big\rangle \\
&\ge (1-\beta)\big\langle \nabla(\phi u_t)\nabla(\phi u_t)\big\rangle \\
&\ge (1-\beta)C_N\langle u_t, u_t\rangle_\phi^{1+2/d}\|\phi u_t\|_1^{-4/d}
\end{aligned}$$

or, setting $v = \langle u_t, u_t\rangle_\phi$,

$$\frac{d}{dt}(v^{-2/d}) \;\geq\; \frac{4(1-\beta)}{d} C_N \|\phi u_t\|_1^{-4/d} \; .$$

Noting that $\phi \geq 1$ implies $\|\phi u_t\|_1 \leq \|u_t\|_{1,\phi}$, one has, using $(**)$,

$$\frac{d}{dt}(v^{-2/d}) \;\geq\; \frac{4(1-\beta)}{d} C_N c^{-4/d} \|u\|_{1,\phi}^{-4/d} \; .$$

Integrating this inequality over the interval $[0,t]$ yields

$$\|e^{-tH_\phi^-}\|_{1\to 2,\phi} \;\leq\; \Big(\frac{d}{4(1-\beta)C_N}\Big)^{d/4} ct^{-d/4} e^{t\lambda} \; .$$

By duality,

$$\|e^{-tH_\phi^-}\|_{1\to \infty,\phi} \;\leq\; \Big(\frac{d}{2(1-\beta)C_N}\Big)^{d/2} c^2 t^{-d/2} e^{t\lambda} \; ,$$

which completes the proof.

Remark. There is an alternative proof of Proposition 2, based on the ideas from preceding sections.

Proposition 3. *Let $V \in PK(\beta)$ for some $\beta < 1$. Suppose that*

$$\phi, |\nabla\phi|, \Big|\frac{\nabla\phi}{\phi}\Big|, \frac{\Delta\phi}{\phi} \in L^1_{\mathrm{loc}}(R^d, dx) \text{ and } -\frac{\Delta\phi}{\phi} - V \;\geq\; -\omega \text{ for some } \omega < \infty \; .$$

Then

$$\|e^{-tH_\phi^-}\|_{1\to 1,\phi} \;\leq\; e^{\omega t} \; .$$

Proof. Since $C_0^\infty(R^d)$ is a form core of H^-, $\Phi^{-1}C_0^\infty(R^d)$ is a form core of H_ϕ^-. The quadratic form h_ϕ associated with H_ϕ^- on $\Phi^{-1}C_0^\infty(R^d)$ has the form

$$h_\phi[u] \;=\; \langle\nabla u, \nabla u\rangle_\phi - \Big\langle\Big(\frac{\Delta\phi}{\phi} + V\Big)u, u\Big\rangle_\phi$$

which clearly shows that the quadratic form $h_\phi + \omega$ is Markovian. Proposition 3 is proved.

Now we construct ϕ satisfying the conditions of Propositions 2 and 3 for $V(x) \leq \lambda_2 + \beta V_0(x)$ (see the Introduction). Let $\phi(x) = |x|^{-\sigma_2}$ if $|x| \leq \frac{1}{2}$, $\phi(x) = 1$ if $|x| \geq 1$ and $1 \leq \phi \in C^2\big(R^d \backslash B_{1/2}(0)\big)$. We have $-\frac{\Delta\phi}{\phi} - V \geq -\lambda_2$ on the ball $B_{1/2}(0)$ so ϕ does fulfill all the assumptions. The required part of Theorem 2 is proved.

V. PROOF OF THEOREM 2. THE UPPER BOUND ON $e^{-tH^+}(x,y)$. We use the same idea of passing to a *weighted* space. Denoting a new weight function by φ we construct H^+_φ in L^2_φ by means of the unitary transform Φ_1: $L^2_\varphi \mapsto L^2$ defined by $\Phi_1 f = \varphi f$ so that $H^+_\varphi = \Phi_1^{-1} H^+ \Phi_1$.

First let us formulate a proposition analogous to Proposition 2.

Proposition 4. *Let $V \in PK(\beta)$ for some $\beta < 1$. Let $H := -\Delta \dot{+} V$. Suppose that*

$$(*) \qquad \|e^{-tH}\varphi^{-1}g\|_1 \ \leq \hat{c}e^{\hat{\omega}t}t^{-d/(2q)}\|g\|_1$$

for all g from a dense subspace of L^1. Then there are constants $C = C(d,\beta,\hat{c})$ and λ such that the inequality

$$\|e^{-tH_\varphi}\|_{1\to\infty,\varphi} \ \leq \ Ce^{\lambda t}t^{-d(1+q)/(2q)}$$

is valid for all $t > 0$.

We skip a proof of Proposition 4. In contrast to ϕ from Proposition 2, we now take a weight φ which has the form $|x|^\ell$, $\ell > 0$, in $B_{1/2}(0)$, and this means that $(*)$ is fairly non-trivial.

Let us also suppose that $1 \geq \varphi(x)$, if $|x| \geq 1$. Using evident inequalities $\varphi^{-1} \leq 1 + \varphi^{-1}1_{|x|\leq 1}$, $\|e^{-tH}\varphi^{-1}g\|_1 \leq \|\varphi e^{-tH}\varphi^{-1}g\|_1 + \|\varphi^{-1}1_{|x|\leq 1}\|_q\|\varphi e^{-tH}\varphi^{-1}g\|_{q'}$, we reduce the problem to (L^1,L^1) and $(L^1,L^{q'})$ estimates for $\varphi e^{-tH}\varphi^{-1}$ acting in ordinary L^r-spaces. Now equality $\varphi e^{-tH}\varphi^{-1} = e^{-t\varphi H\varphi^{-1}}$ (in $\big(L(L^2_{\varphi^{-1}})\big)$ is no use. Following [Se1] we will construct $-H(\varphi)$, the generator of C_0-semigroup in L^2, such that $\varphi e^{-tH}\varphi^{-1}g = e^{-tH(\varphi)}g$ for all g from a dense subset of L^2. We stress that $H(\varphi)$ only formally coincides with $\varphi H\varphi^{-1}$ of domain $\{\varphi u;\ u \in D(H)\}$. We then show that $e^{-tH(\varphi)}$ for $V \geq \lambda_1 + \gamma V_0$ is indeed (quasi) contractive and $(L^1,L^{q'})$ smooth (with $\|e^{-tH(\varphi)}\|_{1\to q'} \leq ct^{-d/(2q)}e^{t\omega}$).

Let ρ: $R^d \mapsto R_+ = [0,\infty[$ and let $H := -\Delta \dot{+} V \geq 0$ be such that $H \geq -c_2\Delta - c_3$ (in the sense of the quadratic forms in L^2) for some constants $c_3, 0 < c_2$. We say that a weight ρ and the operator H are compatible if

1) $\rho, \frac{1}{\rho}, |\nabla\rho|, \left|\frac{\nabla\rho}{\rho}\right| \in L^2_{\mathrm{loc}}(R^d)$;

2) $\left(\frac{\nabla\rho}{\rho}\right)^2 \leq c_0 H + c_1$ (in the sense of the quadratic forms in L^2)

for some constants $c_0 < 1$, $c_1 < \infty$.

Proposition 5. *Let a weight ρ and the operator H be compatible. In the complex Hilbert space L^2 with the inner product $\langle u,v\rangle$, linear with respect to u, define the quadratic forms t, ℓ, ℓ_i, $i = 1,2,3$, by*

$$t[u,v] = \langle H^{1/2}u, H^{1/2}v\rangle - \ell[u,v], \quad \ell = \ell_1 + \ell_2 + \ell_3, \quad D(t) = D(\ell_i) = Q(H) \times Q(H),$$

$$\ell[u,v] := \ell(\rho)[u,v], \quad \ell_i[u,v] := \ell_i(\rho)[u,v], \quad \ell_1(\rho)[u,v] := \langle u, \rho^{-1}\nabla\rho\cdot\nabla v\rangle,$$

$$\ell_2(\rho)[u,v] := -\overline{\ell_1(\rho)[v,u]}, \quad \ell_3(\rho)[u,v] := \big\langle u\rho^{-2}(\nabla\rho)^2 v\big\rangle.$$

Then i) the form t is m-sectorial,

$$t[u,v] \;=\; \big\langle H(\rho)u,v\big\rangle, \quad u\in D\big(H(\rho)\big)\;\subset\;Q(H), \quad v\in Q(H),$$

where $H(\rho)$ is the quasi m-accretive operator associated with t.

ii) For all $\lambda\geq\lambda_0$, where $\lambda_0>0\vee\frac{c_1}{c_0}\vee\frac{c_3}{c_2}$, and all $g\in L^\infty_{\rm com}(R^d)$,

$$\begin{aligned}\rho(\lambda+H)^{-1}\rho^{-1}g \;&=\; \big(\lambda+H(\rho)\big)^{-1}g\ ,\\ \rho e^{-tH}\rho^{-1}g \;&=\; e^{-tH(\rho)}g\ .\end{aligned}$$

Proof. Define the (complex) Hilbert spaces $X_+\subset L^2\subset X_-$ setting $X_+ = \big(Q(H),\|u\|_+\big)$ with $\|u\|_+=\|(\lambda_0+H)^{1/2}u\|_2$, $X_-=(X_+)^*$, where, given $u,v\in X_+$, one has

$$\begin{aligned}
&0\;\leq\;\ell_3[u]\;\leq\;c_0\|u\|_+^2\ ,\\
&\big|\ell_1[u,v]\big|\;\leq\;\big\langle|u\rho^{-1}\nabla\rho|,|\nabla v|\big\rangle\;\leq\;\sqrt{\frac{c_0}{c_2}}\|u\|_+\|v\|_+\ ,\\
&\big|\ell_2[u,v]\big|\;=\;\big|\ell_1[v,u]\big|\;\leq\;\sqrt{\frac{c_0}{c_2}}\|u\|_+\|v\|_+\ ,\\
&{\rm Re}\,t[u]\;=\;\|u\|_+^2-\ell_3[u]-\lambda_0\|u\|_2^2\ ,\\
&{\rm Im}\,t[u]\;=\;\sqrt{-1}\,\big(\ell_1[u]-\ell_2[u]\big)\ ,\\
&(1-c_0)\|u\|_+^2\;\leq\;{\rm Re}\,t[u]+\lambda_0\|u\|_2^2\;\leq\;\|u\|_+^2\ .
\end{aligned}$$

The above shows that t is m-sectorial and therefore (see [K1, Ch. VI]) i) is proved.

It is also clear that

$$\begin{aligned}
&\langle H^{1/2}u,H^{1/2}v\rangle=\langle\widehat Hu,v\rangle \text{ where } \widehat H:\ X_+\mapsto X_-\,, \text{ and } D(H)\;=\;\{f\in X_+:\ \widehat Hf\in L^2\}\ ,\\
&\widehat H\mid D(H)=H,\quad \ell[u,v]=\langle\widehat Eu,v\rangle,\quad \ell_i[u,v]=\langle\widehat E_iu,v\rangle,\quad \widehat E,\widehat E_i:\,X_+\mapsto X_-\ ,\\
&H(\rho)=\widehat H-\widehat E\mid D\big(H(\rho)\big),\quad D\big(H(\rho)\big)=\{f\in X_+:\ \widehat Hf-\widehat Ef\in L^2\}\ .
\end{aligned}$$

Set $\widehat B:=\lambda+\widehat H-\widehat E_3$, $\lambda\geq\lambda_0$, and define $B:=\widehat B\mid\{f\in X_+:\ \widehat Bf\in L^2\}$ – the form sum of $\lambda+H$ and $-(\rho^{-1}\nabla\rho)^2$. Then $B^{-1/2}$: $X_-\mapsto L^2$, $B^{-1/2}$: $L^2\mapsto X_+$. Define $G:=-\sqrt{-1}\,B^{-1/2}(\widehat E_1+\widehat E_2)B^{-1/2}$ so that $G=G^*$: $L^2\mapsto L^2$ and

$$(**)\qquad \big(\lambda+H(\rho)\big)^{-1}\;=\;B^{-1/2}\big(1-\sqrt{-1}\,G\big)^{-1}B^{-1/2}\ .$$

Define ρ_n, $n\in N$, by

$$\rho_n(x)\ ;=\ \begin{cases}\rho(x) & \text{if } \frac1n\leq\rho(x)\leq n,\\ n & \text{if } \rho(x)\geq n,\\ 1/n & \text{if } \frac1n\geq\rho(x),\end{cases}$$

Then ρ_n and H are compatible (with the same constants c_i) and hence $(**)$ also holds for ρ_n. Let us show that

$$\big(\lambda + H(\rho_n)\big)^{-1} \;\to\; \big(\lambda + H(\rho)\big)^{-1} \text{ weakly in } L^2.$$

Indeed, since $B^{-1/2}(\rho_n) \to B^{-1/2}(\rho)$ strongly in L^2 by the monotone convergence theorem for a non-increasing sequence of symmetric quadratic forms [K1, Ch. VIII, sec. 3], we only have to prove that $\lim_n \big\langle \big(\widehat{E}_i(\rho) - \widehat{E}_i(\rho_n)\big) f, g\big\rangle = 0$ for $i = 1, 2$ and all $f, g \in X_+$. But, for instance, $\big|\big\langle \big(\widehat{E}_1(\rho) - \widehat{E}_1(\rho_n)\big) f, g\big\rangle\big| \le \|\nabla f\|_2 \big\langle (1 - 1_n)(\rho^{-1}\nabla\rho)^2 g, g\big\rangle^{1/2}$ where 1_n is the indicator of the support of ρ_n, and hence the required convergence follows e.g., from the Fatou lemma.

Next it is easily seen that $u \in Q(H)$ implies that $\rho_n^{\pm 1} u \in Q(H)$ and

$$(****) \qquad \langle H^{1/2}\rho_n^{-1} u, H^{1/2}\rho_n v\rangle \;=\; t(\rho_n)[u, v], \qquad u, v \in Q(H)\ .$$

Note that $T^t := \rho_n e^{-tH}\rho_n^{-1}$, $t \ge 0$, is a C_0-semigroup on L^2 with $\|T^t\|_{2\to 2} \le n^2$. Let $-A$ denote its generator. We claim that $A = H(\rho_n)$. Indeed, given $f \in L^2$ and $\lambda > 0$, define $u = (\lambda + A)^{-1} f$. Since $(\lambda + A)^{-1} = \theta(\lambda + H)^{-1}\theta^{-1}$, $\theta \equiv \rho_n$, we see that $\theta^{-1} u \in D(H)$ and $(\lambda + H)\theta^{-1} u = \theta^{-1} f$. Therefore, for all $v \in X_+$, $\big\langle (\lambda + H)\theta^{-1} u, \theta v\big\rangle = \langle f, v\rangle$ or $\langle f - \lambda u, v\rangle = \langle H^{1/2}\theta^{-1} u, H^{1/2}\theta v\rangle$. The latter means, due to $(****)$, that $u \in D\big(H(\rho_n)\big)$ and $\big\langle H(\rho_n) u, v\big\rangle = \langle f - \lambda u, v\rangle$ or $H(\rho_n) u = Au$. Thus $-A \subset -H(\rho_n)$, and since both of them are generators, the claim is proved. In particular, we have $\exp\big[-tH(\rho_n)\big] = \rho_n \exp[-tH]\rho_n^{-1}$. Assertion ii) of Proposition 5 now follows from $(****)$.

Now we construct a weight φ satisfying the conditions of Propositions 4 and 5 for $H^+ = -\Delta \dot{+} V$ where $V(x) \ge \lambda_1 + \gamma V_0(x)$ (see the Introduction). Set $\varphi(x) = |x|^{\sigma_1}$ if $|x| \le \frac{1}{2}$, $\varphi(x) = 1$ if $|x| \ge 1$ and $2^{-\sigma_1} \le \varphi \in C^2\big(R^d \backslash B_{1/2}(0)\big)$.

Proposition 6. *Let $\rho = \varphi$. Then i) ρ and H^+ are compatible with $c_0 \le \big(\sqrt{1+\gamma} - 1\big)^2 < \frac{1}{6}$; ii) $e^{-tH^+(\rho)}$ is L^1-quasi contraction; iii) $\|e^{-tH^+(\rho)}\|_{1\to q'} \le c e^{t\omega} t^{-d/(2q)}$.*

Proof. i) follows directly from the definitions of φ and H^+. Since $-\rho^{-1}(\Delta\rho) + V \ge -\omega$ for some finite constant ω and $\Delta \dot{-} 2\nabla \cdot (\rho^{-1}\nabla\rho)$ generates a L^1-contraction semigroup [LS1], we conclude, e.g., by the Trotter-Kato semigroup formula [K2], that $-H^+(\rho) = \Delta \dot{-} 2\nabla \cdot (\rho^{-1}\nabla\rho) \dot{-} \big(V - \rho^{-1}(\Delta\rho)\big)$ generates a quasi L^1-contraction semigroup, which proves ii). iii) follows from [LS1].

Proposition 6 shows that our weight φ does satisfy the conditions of Propositions 4 and 5 and therefore the upper bound on $e^{-tH^+}(x, y)$ is proved.

The lower bound on $e^{-tH^-}(x, y)$ follows now from the upper bounds on $e^{-tH^\pm}(x, y)$ via the inequality $\Lambda_t(x - y) \le \big(e^{-tH^+}(x, y)\big)^{1/2}\big(e^{-tH^-}(x, y)\big)^{1/2}$.

Concluding Remarks. 1. The proof of Theorem $2'$ can be obtained by a simple combination of Theorem 1 and the proof of Theorem 2. Slightly modifying the proof of Theorem 2 one can derive Theorem 3 either by applying non-linear inequality $(***)$ from section II or using the method of E.B. Davies [Da] as it was presented in [LS2]. We skip the details.

2. The ideas of this paper might be applied to the operator $-\Delta + b \cdot \nabla + V$ with a singular drift and potential. We hope to investigate it in a forthcoming paper.

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