30. Arbeitstagung

Bonn, 14. - 20. Juni 1991

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI/91-47

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3

und

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
5300 Bonn 1

Programm der Mathematischen Arbeitstagung 1991 (I)

Freitag, den 14.6.1991

16.00 - 17.00 Uhr	S.K. DONALDSON (Oxford) Glueing problems in Yang-Mills theory

Samstag, den 15.6.1991

10.15 - 11.15 Uhr	T. SHIODA (Rikkyo U., z.Zt. MPI) Mordell-Weil lattices: theory and applications
12.00 - 13.00 Uhr	D. ZAGIER (MPI Bonn und U. Utrecht) Polylogarithms
17.00 - 18.00 Uhr	B. OLIVER (Aarhus) Maps between classifying spaces of compact Lie groups

Sonntag, den 16.6.1991

10.15 - 11.15 Uhr	Yu.I. MANIN (Steklov Moskau, z.Zt. MPI) DeRham-complexes in non-commutative geometry
12.00 - 13.00 Uhr	G. WÜSTHOLZ (ETH Zürich) Faltings's proof of one of Lang's conjectures (rational points on subvarieties of Abelian varieties)
16.15 - 16.45 Uhr	H. HIRONAKA (Harvard) Miscellanea Mathematica
17.00 - 18.00 Uhr	J-P. SERRE (Collège de France Paris) Galois cohomology: recent results and open questions

Montag, den 17.6.1991

10.00 - 10.15 Uhr	Festlegung der restlichen Vorträge
10.15 - 11.15 Uhr	M. KONTSEVICH (Acad. Sci. Moskau, z.Zt. MPI) Intersection theory on the moduli space of curves and the matrix Airy function
13.00 Uhr	Schiffsfahrt nach Andernach. Abfahrt um 13.00 Uhr mit Motorschiff "Carmen Sylva", Ablegestelle Alter Zoll. Rückkehr ca. 20.00 Uhr

b.w.!
PTO!

Dienstag, den 18.6.1991

10.15 - 11.15 Uhr	M. WODZICKI (UC Berkeley) Excision in K-theory and proof of the Karoubi conjecture
12.00 - 13.00 Uhr	R. MACPHERSON (M.I.T.) Lefschetz numbers of Hecke correspondences
17.00 - 18.00 Uhr	S. LANG (Yale University, z.Zt. MPI) Degeneration of Riemann surfaces and Jorgenson's proof of a conjecture of Deligne

Die Vorträge finden alle im "*Großen Hörsaal*", Wegelerstraße 10, statt.

Erfrischungspausen mit Tee: Samstag, Sonntag und Dienstag 11.15 - 12.00 Uhr, Samstag und Dienstag 16.15-17.00 Uhr, Sonntag 15.30 - 16.15 Uhr, jeweils vor dem Großen Hörsaal.

Teilnehmerlisten und *Informationen* liegen vor dem Großen Hörsaal aus. Alle Teilnehmer mögen sich bitte in die Teilnehmerlisten eintragen.

Post liegt während der Teepausen aus.

Den *Tagungsbeitrag* bitte während der Teepausen vor dem Großen Hörsaal bezahlen.

Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zum *Empfang des Rektors* eingeladen. Zeit: Freitag, den 14.6., 20.00 Uhr. Ort: Festsaal der Universität, Hauptgebäude; Eingang von der Straße "Am Hof" durch das Tor gegenüber Buchhandlung Bouvier.

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3

und

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
5300 Bonn 1

Programm der Mathematischen Arbeitstagung 1991 (II)

Mittwoch, den 19.6.1991

10.15 - 11.15 Uhr	H. TSUJI (Tokyo Metropolitan U., z.Zt. MPI) Report on Mori Theory
12.00 - 13.00 Uhr	S. KLAINERMAN (Princeton U., z.Zt. SFB 256) On non-linear stability of Minkowski space
17.00 - 18.00 Uhr	C. BÄR (Bonn) On Killing spinors and exceptional holonomy groups

Donnerstag, den 20.6.1991

10.15 - 11.15 Uhr	R. PINK (Bonn) Deligne's conjecture on the Lefschetz trace formula in positive characteristic is now a theorem
12.00 - 13.00 Uhr	W. MÜLLER (Karl-Weierstraß-Institut, z.Zt. MPI) Analytic Torsion for non-unitary representations and Chern-Simons gauge theory
17.00 - 18.00 Uhr	F. OORT (Utrecht) Newton polygons and abelian varieties

Die Vorträge finden alle im "*Großen Hörsaal*", Wegelerstraße 10, statt.

Erfrischungspausen mit Tee: Mittwoch und Donnerstag, 11.15 - 12.00 Uhr und 16.15 - 17.00 Uhr vor dem Großen Hörsaal.

Post liegt während der Teepausen aus.

Informationen liegen vor dem Großen Hörsaal aus.

Den *Tagungsbeitrag* bitte während der Teepausen vor dem Großen Hörsaal bezahlen.

Teilnehmerliste

U. Abresch (Münster)
S. Adian (Steklov Moskau)
P. Arias (MPI)
H. Azad (z.Zt. MPI)
C. Bär (Bonn)
W. Ballmann (Bonn)
W. Barth (Erlangen)
G. Barthel (Konstanz)
S. Bauer (Göttingen)
H. Baum (Humboldt-U. Berlin)
H. Baumgärtel (KWI Berlin)
E. Bayer (Besançon)
A. Beauville (Paris-Sud)
H. Beckert (Leipzig)
K. Behnke (Hamburg)
M. Berger (I.H.E.S.)
T. Berger (MPI)
R. Berndt (Hamburg)
D. Blasius (UCLA)
S. Böcherer (Mannheim)
C.-F. Bödigheimer (Göttingen)
H.G. Bothe (KWI Berlin)
J.P. Bourguignon (École Polytechnique Palaiseau)
E. Brieskorn (Bonn)
C. Brinkmann (Bonn)
V. Brinzanescu (Bukarest)
W.D. Brownawell (Penn. State U.)
A. Brownstein (Rutgers U.)
U. Bunke (Greifswald)
D. Burde (Bonn)
D. Burghelea (Ohio State U.)
J. Buth (MPI)
C. Byrne (Springer-Verlag)
E. Calabi (U. of Pennsylvania)
R. Carlsson (Hamburg)
F. Catanese (Pisa)
J.-T. Chang (Oklahoma State U.)
J. Christophersen (Oslo)
H. Cohen (z.Zt. MPI)
B. Colbois (Bonn)
V. Cossart (Paris VI)
D. Dais (Bonn)
A. Deitmar (z.Zt. MPI)
A. Dessai (Bonn)
A. Dimca (z.Zt. MPI)
P. Dombrowski (Köln)
S.K. Donaldson (Oxford)
A. Durfee (Mt.Holyoke College)
W. Ebeling (Hannover)
B. Eckmann (ETH Zürich)
J. Eells (Warwick)
H. von Eitzen (Bonn)
D. Erle (Dortmund)
H. Esnault (Essen)
U. Everling (MPI)
C. Faber (z.Zt. MPI)
D. Feldhausen (Bonn)
J.-M. Feustel (KWI Berlin)
T. Fiedler (z.Zt. Göttingen)
B. Fischer (Humboldt-U. Berlin)
G. Fischer (Düsseldorf)
M. Flohr (Bonn)
R. Flume (Bonn)
H. Föllmer (Bonn)
O. Föllinger (Humboldt-U. Berlin)
M. Fontaine (MPI)
J. Franke (z.Zt. MPI)
E. Friedman (z.Zt. MPI)
R. Friedrich (Bonn)
T. Friedrich (Humboldt-U. Berlin)
M. Furushima (Okinawa)
H. Gangl (MPI)
L. Geatti (Sassari)
T. Geisser (Münster)
W. Gerdes (Brandeis U.)
L. Göttsche (MPI)
H. Götze (Springer-Verlag)
M. Goresky (Northeastern U.)
H. Grauert (Göttingen)
G.-M. Greuel (Kaiserslautern)
V. Gritsenko (LOMI Leningrad)
F. Grunewald (Bonn)
R. Gulliver (U. of Minnesota)
K. Haberland (Jena)
I. Hambleton (z.Zt. MPI)
U. Hamenstädt (Bonn)
H. Hamm (Münster)

G. Harder (Bonn)
Y. Hayashi (Bonn)
J. Heinze (Springer-Verlag)
H.-W. Henn (Heidelberg)
M. Hennes (MPI)
G. Henniart (Paris XI)
M. Herrmann (Köln)
C. Hertling (Bonn)
P. Hertling (Bonn)
S. Hildebrandt (Bonn)
T. Hintermann (Birkhäuser Verlag Basel)
F. Hirzebruch (MPI)
H. Hironaka (Harvard)
E. Hironaka (z.Zt. MPI)
T. Höfer (MPI)
G. Höhn (Bonn)
W. Hoffmann (KWI Berlin)
R.-P. Holzapfel (KWI Berlin)
Bizhong Hu (z.Zt. MPI)
G. Huber
B. Hunt (Kaiserslautern)
D. Husemoller (Haverford College)
D. Huybrechts (z.Zt. MPI)
E. Hyry (Helsinki)
H.-C. Im Hof (Basel)
J. Jahnel (Jena)
U. Jannsen (Köln)
A. Juhl (KWI Berlin)
B. Juhl-Jöricke (KWI Berlin)
R. Jung (MPI)
F. Kärsten (KWI Berlin)
B. Kahn (Paris VII)
U. Kaiser (Siegen)
T. Kakolewski (MPI)
H. Karcher (Bonn)
M. Karpinski (Bonn)
C. Kassel (Straßburg)
A. Katsuda (Grenoble)
R. Kellerhals (Bonn)
Hoil Kim (Bayreuth)
G. Kings (Münster)
S. Klainerman (Princeton U.)
J.R. Klein (Siegen)
W. Kleinert (Humboldt-U. Berlin)
M. Klingholz (Bonn)
A.A. Klyachko (z.Zt. MPI)
K. Knapp (Wuppertal)
H. Knörrer (ETH Zürich)
R. Kobayashi (z.Zt. MPI)
S. Kobayashi (UC Berkeley)
H. Koch (z.Zt. Bonn)
K. Köhler (Paris-Sud)
P. Koepke (Bonn)
W. Kohnen (MPI)
M. Kontsevich (z.Zt. MPI)
K. Kopfermann (Hannover)
J. Korbas (Bratislawa)
I. Kosarew (Zürich)
U. Koschorke (Siegen)
A. Kostrikin (U. Moskau)
D. Kotschick (Cambridge)
J. Kramer (ETH Zürich)
M. Kreck (z.Zt. MPI)
H. Kriete (Bochum)
R. Kühnau (Halle)
W. Kühnel (Duisburg)
W. Künzel
N.H. Kuiper (I.H.E.S.)
H. Kurke (Humboldt-U. Berlin)
K. Lamotke (Köln)
S. Lang (z.Zt. MPI)
Le Van Thanh (z.Zt. MPI)
R. Lee (Yale U.)
M. Lehn (Bonn)
J. Leiterer (Humboldt-U. Berlin)
L. Lemaire (Brüssel)
I. Lieb (Bonn)
F. Lorenz (Münster)
M. Lübke (Leiden)
M. Lüdde (Bonn)
R. MacPherson (M.I.T.)
H. Maennel (MPI)
Yu. Manin (z.Zt. MPI)
G. Masbaum (Nantes)
D. Masser (z.Zt. MPI)
S. Maurmann (MPI)
K.H. Mayer (Dortmund)
J. McKay (Montréal)
N. Mestrano (Toulouse)
W.T. Meyer (Münster)
R.J. Milgram (Stanford U.)
A. Miller (Bonn)

M. Min-Oo (McMaster U.)
B.Z. Moroz (z.Zt. MPI)
W. Müller (z.Zt. MPI)
A. Murase (z.Zt. MPI)
W. Nahm (Bonn)
H.-J. Nastold (Münster)
K.-H. Nech (Darmstadt)
K.-H. Neeb
J. Nekovár (Prag)
N. Netsvetaev (LOMI Leningrad)
W.D. Neumann (z.Zt. MPI)
K. Nomizu (z.Zt. Bonn)
S. Ochanine (z.Zt. MPI)
J. O'Hara (Tokyo Metropolitan)
C. Okonek (Bonn)
B. Oliver (Aarhus)
R. Olivier (Bonn)
F. Oort (Utrecht)
P. Orlik (U. of Wisconsin)
R. Palais (Brandeis)
S. Parmentier (z.Zt. MPI)
K. Peters (Bonn)
Vu Quoc Phong (z.Zt. MPI)
R. Piene (Oslo)
R. Pink (Bonn)
V. Platonov (z.Zt. MPI)
J. Podgor (Wroclaw)
K. Polthier (Bonn)
H. Popp (Mannheim)
V. Puppe (Konstanz)
M. Puschnigg (Heidelberg)
H.-B. Rademacher (Bonn)
A. Recknagel (MPI)
M. Reid (Warwick)
R. Remmert (Münster)
J.-E. Riedel (Bonn)
C. Riehm (McMaster U.)
Shi-shyr Roan (z.Zt. MPI)
A. Robertson (Oxford)
J. Rohlfs (Eichstätt)
Yu. Rozanov (Steklov Moskau)
K. Rozanova (z.Zt. Bonn)
Yongbin Ruan (UC Berkeley)
M. Saito (z.Zt. MPI)
J.H. Sampson (Johns Hopkins)
P. Schapira (Paris-Nord)
N. Schappacher (MPI)
P. Schenzel (Halle)
E. Schiavi (Trento)
G. Schmalz (KWI Berlin)
U. Schmickler-Hirzebruch (Springer-Verlag)
C. Schoen (z.Zt. MPI)
R. Schön (Heidelberg)
W. Schomburg (Bochum)
R. Schoof (Trento)
M. Schroeter (Münster)
B.-W. Schulze (KWI Berlin)
S. Seifarth (KWI Berlin)
W.K. Seiler (Mannheim)
U. Semmelmann (Humboldt-U. Berlin)
J-P. Serre (Collège de France)
S. Shatz (U. of Pennsylvania)
T. Shioda (z.Zt. MPI)
O. Sick (Bonn)
D. Siersma (Utrecht)
S. Sigg (Bonn)
C.T. Simpson (z.Zt.Toulouse)
P. Slodowy (Hamburg)
B. Smyth (U. of Notre Dame)
W. Soergel (z.Zt. MPI)
T.A. Springer (Utrecht)
K. Stein (München)
B. Steinert (Bonn)
L. Stoyanov (z.Zt. Darmstadt)
J. Strooker (Utrecht)
U. Stuhler (Wuppertal)
Wing-wah Sung (z.Zt. MPI)
M. Teicher (Bar-Ilan U.)
T. Terasoma (Mannheim)
Chuu-Lian Terng (Northeastern U.)
C. Thomas (Cambridge)
F. Thomas (Humboldt-U. Berlin)
G. Trautmann (Kaiserslautern)
J. Tschinkel (z.Zt. MPI)
H. Tsuji (z.Zt. MPI)
A. Van de Ven (Leiden)
R. Varnhagen (Bonn)
B. Venkov (z.Zt. MPI)
E. Viehweg (Essen)
W. Vogel (Halle)
H. Voskuil (z.Zt. MPI)

P. Vuillermot (z.Zt. MPI)
M. Waldschmidt (IHP Paris)
M. Weber (Bonn)
Lin Weng (z.Zt. MPI)
C. Wenzel (Wuppertal)
D. Wigner (Paris VI)
F. Wiid (z.Zt. MPI)
J. Wildeshaus (Münster)
M. Wodzicki (UC Berkeley)
M. Wohlgemuth (Bonn)
M. Wollenberg (KWI Berlin)
J. Wood (Leeds)
G. Wüstholz (ETH Zürich)
Gang Xiao (z.Zt. MPI)
M. Yoshida (Kyushu U.)
D. Zagier (MPI u. Utrecht)
M. Zaidenberg (z.Zt. MPI)
Yu. Zarhin (RCC Pushchino)
E. Zeidler (Leipzig)
E.W. Zink (KWI Berlin)
J. Zobel (Bonn)
P. Zograf (LOMI Leningrad)
Kang Zuo (Kaiserslautern)

Titel: Glueing problems in Yang-Mills Theory

Autor: S. Donaldson

Adresse: The Mathematical Institute,
24-29 St. Giles,
Oxford.

In order to calculate the invariants of 4-manifolds which are defined by Yang-Mills moduli spaces one would like information on their behaviour under "gluing" operations. Floer's theory of "instanton homology" provides a framework for many gluing problems but there are a number of points which are not covered by the theory at present.

Here we discuss a model problem in 2 dimensions which already contains many of the difficulties one encounters in 4-dimensions. Let Σ be a closed oriented surface ~~on~~ of genus g and $M = M(\Sigma)$ the moduli space of flat G-connections over Σ. The Kunneth components of characteristic classes of a universal bundle over $\Sigma \times M$ define cohomology classes over M. If Ω is a product of such classes, of top degree, we consider the pairing $\langle \Omega, [M] \rangle$, which is analogous to the 4-manifold invariants mentioned above. In particular if $G = SU(2)$ we are primarily interested in computing $I_{p,q} = \langle \alpha^p \beta^q, [M] \rangle$, where $\alpha \in H^2(M)$, $\beta \in H^4(M)$ are obtained from the Chern class in $H^4(M \times \Sigma)$, and $2p + 4q = 6g - 6 = \dim M$. (We can avoid technical difficulties by working with a "twisted" version of the problem, corresponding to an $SO(3)$ bundle with $w_2 \neq 0$). This problem has been solved by M. Thaddeus; his formula is:

$$I_{p,q} = (-1)^{g-1} 2^{2g} \frac{p!}{s!} (2^s - 2) B_s ,$$

where $s = q - g + 1$ and B_s is the Bernoulli number. (There should be similar formulae for other groups G.)

We discuss 3 different approaches to such calculations. Each hinges on a "gluing rule".

<u>Method 1</u> [1] This is based on complex geometry and conformal field theory. M can be given a complex structure and there is a positive line bundle $\mathcal{L} \to M$ with $c_1(\mathcal{L}) = \alpha$. The dimension of $V_k(\Sigma) = H^0(M(\Sigma), \mathcal{L}^k)$ can be expressed, via the Riemann-Roch theorem, in terms of the $I_{p,q}$. In conformal field theory there is a gluing rule for the $V_k(\Sigma)$ which allows ~~them~~ these dimensions to be calculated (by induction on g). The Riemann-Roch formula can be inverted to give the $I_{p,q}$.

<u>Method 2</u> This was introduced by Witten in [2]. He considers the measure on the moduli space defined by $\omega^n/n!$; where ω is a 2-form representing α. The measure can be obtained in a different way using

the torsion of a finite dimensional complex which computes the tangent space of M. There is a gluing rule for the volume of the moduli spaces, which enables the volumes to be calculated, and the pairings $I_{p,q}$ can be recovered from this.

Method 3 This uses ideas which are familiar in Floer homology. One deforms the moduli space by introducing perturbation into the flatness condition. The gluing rule takes the form:

$$(\dagger) \quad \langle \alpha^n, M(\Sigma) \rangle = \langle (\alpha + h)^n, \mathcal{H}(\Sigma') \rangle,$$

where $\mathcal{H}$ is a fibre bundle over $M(\Sigma')$, Σ' has genus $g-1$, and $h \in H^2(\mathcal{H})$. The cohomology ring of $\mathcal{H}$ is described by β, and this allows us to evaluate the right hand side of $(\dagger)$ in terms of the $I_{p,q}(\Sigma')$.

These three methods are related in a number of ways. One may hope that some of the ideas can be applied in 4 dimensions.

References

[1] M. Thaddeus. "Conformal Field Theory and the cohomology of the moduli space of stable bundles" To appear in Jour. Diff. Geom.

[2] E. Witten. "On Quantum gauge theories in two dimensions" Institute for Advanced Study Preprint.

Titel: Mordell-Weil Lattices : Theory and Applications

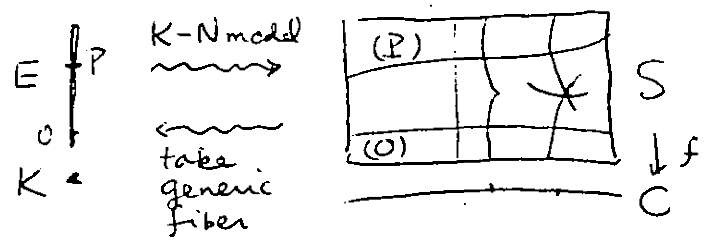
Autor: Tetsuji Shioda

Adresse: MPI, Bonn / Rikkyo Univ. (Tokyo, Japan)
(5-8/91)

Let E/K be an elliptic curve, $K = k(C)$ the function field of a curve C/k. Then the group of K-rational points of E, $E(K)$, is finitely generated under some mild assumption like (*) below, by the functional analogue of the Mordell-Weil theorem (cf. [L], [Se], [Si]).

The basic idea of Mordell-Weil lattices (abr. MWL) is 1) to view the <u>Mordell-Weil group</u> $E(K)$ as a <u>lattice</u> with respect to a natural pairing, which is defined in terms of the <u>Intersection Theory</u> on the associated elliptic surface and which turns out to coincide with the Néron-Tate canonical height. Another basic idea is 2) to view the MWL as a Galois lattice. This lead to <u>Gal. $\mathbb{Z}$-rep.</u> and <u>alg. equations</u> <u>using pts of ∞ order</u>.

I). First assume that k is alg. closed. For simplicity, we let $K = k(t)$ $(C = \mathbb{P}^1)$. Given E/K, one can uniquely associate an elliptic surface $f : S \to C$, called the Kodaira-Néron model of E/K.



S = smooth proj. surface
f : rel. minimal elliptic fibration

Almost all fibres of f are elliptic curves and there are finite number of singular fibres.
The K-rational points $P \in E(K)$ correspond bijectively to the sections $s : C \to S$. Thus we identify $s = P$.

Assume (*) f is not smooth, i.e. there is at least one singular fibre. Then we have a natural isom.

$$(\#)\qquad E(K) \simeq NS(S)/T$$

$$\begin{cases} P \longmapsto (P) \bmod T. \\ \mathrm{sum}(D|_E) \longleftarrow D \end{cases} \qquad \left(\begin{array}{l}(P)\subset S \text{ is the} \\ \text{curve defined by} \\ \text{the section } P: C \to S\end{array}\right)$$

Here $NS(S)$ is the Néron-Severi group of S, that is, $\{$divisors $D = \sum n_i \Gamma_i \mid n_i \in \mathbb{Z}, \Gamma_i$: irred. curves on $S\}$/alg. equiv., which is finitely generated and torsionfree under (*), and T is the subgroup spanned by $\{(O), F = $ fibre, $\forall$ irred. comp. of fibres$\}$. In particular, $r = \mathrm{rk}\, E(K) = \mathrm{rk}\, NS(S) - \mathrm{rk}\, T$
$= \rho - (2 + \sum_{v \in R} (m_v - 1))$ $\left(\begin{array}{l}\rho = \text{Picard number}, m_v = \#\text{irred comp. of } f^{-1}(v) \\ R = \{v \mid f^{-1}(v) \text{ reducible}\}\end{array}\right)$.

Now $NS(S)$ is an indefinite integral lattice w.r.t. the intersection pairing $(\cdot)$, with signature $(+, \rho-1)$. (Hodge index th.), and T is a sublattice having a direct sum decomp. $T = \langle (O), F \rangle \oplus \bigoplus_{v \in R} T_v$, where each T_v is a root lattice of type A_n, D_n or E_6, E_7 or E_8.

<u>Lemma 1</u> There is a unique map $\varphi: E(K) \to NS(S) \otimes \mathbb{Q}$
s.t. $\varphi(P) \equiv (P) \bmod T \otimes \mathbb{Q}$, $\mathrm{Im}(\varphi) \perp T$.
φ is a group homo. and $\mathrm{Ker}(\varphi) = E(K)_{tor}$.

<u>Lemma 2</u> $T^\perp$ is a negative-definite even integral lattice. (Use Hodge index th., adj. formula, can. bdle f.)

<u>Theorem</u> Define $\langle P, Q \rangle = -(\varphi(P) \cdot \varphi(Q))$ $(P, Q \in E(K))$.
Then $(E(K)/E(K)_{tor}, \langle\,,\,\rangle)$ is a positive-definite lattice, which we call $\boxed{\text{MWL}}$ of E/K (or $f: S \to C$).
The explicit formula of the "height pairing" $\langle\,,\,\rangle$ is:

(1) $\langle P, P\rangle = 2\chi + 2(PO) - \sum_{v\in R} \mathrm{contr}_v(P)$

where χ is ~~the~~ arithmetic genus of S (>0 under (*)), (PO) = the intersection number of the sections (P) and (O), and the "local contribution" term $\mathrm{contr}_v(P)$ is a non-neg. rational number determined by the type of the sing. fibre $f^{-1}(v)$ and the position of the comp. hit by (P).

Let $E(K)^0 = \{P \in E(K) \mid (P)$ passes through the identity comp. of every fibre $\}$

This is a subgroup of finite index in $E(K)$. Then

(2) $P \in E(K)^0 \Rightarrow \langle P, P\rangle = 2\chi + 2(PO)$ $(\in 2\mathbb{Z})$.

Hence $(E(K)^0, \langle\, ,\, \rangle)$ is a pos-def. even integral lattice, which we call the $\boxed{\text{narrow MWL}}$. Moreover

$$\min_{P\neq 0} \langle P, P\rangle \geq 2\chi \quad (\text{since } (PO)\geq 0 \text{ for } \forall P\neq O).$$

Also ~~the~~ formula (1) or (2) shows that our pairing $\langle\, ,\, \rangle$ coincides with Néron-Tate height.

As the first example, suppose that S is a <u>rational</u> ell. surface. Then $r = 8 - (\Sigma_v (m_v - 1)) \leq 8$, and $\chi = 1$.

<u>Structure Theorem</u>

	$r=8$	$r=7$	$r=6$	
$E(K) \cong$	E_8	E_7^*	E_6^*	D_6^*
$\cup$	$\parallel$	$\cup$ 2	$\cup$ 3	$\cup$ 4 (index)
$E(K)^0 \cong$	E_8	E_7	E_6	D_6
rem. conf	$R=\phi$	$R=\{v\}, m_v=2$	", $m_v=3$	$R=\{v,v'\}, m_v=m_{v'}=2$

Here E_r, D_r are the root lattices, and $*$ the dual lattices. As a consequence, we obtain effective generators of $E(K)$.

For the above, see [S1], [S2].

II) Next we consider the new situation:

$$\begin{cases} k_0 = \text{perfect field (e.g. } \mathbb{Q}, \mathbb{Q}(\lambda), \cdots) \\ k = \text{alg. closure of } k_0, \quad K = k(t) \\ E/k_0(t), \quad E(K) = E(k(t)) \text{ with } Gal(k/k_0)\text{-action.} \end{cases}$$

Then we get the <u>Galois representation</u> (cf. [S6])

$$\rho : Gal(k/k_0) \longrightarrow Aut(E(k), \langle , \rangle) = \text{a finite group.}$$

The ext. $\mathfrak{K}/k_0$ corresp. to $Ker(\rho)$ is called the splitting field; we have $Gal(\mathfrak{K}/k_0) = Im(\rho)$ and $\mathfrak{K}$ = the smallest ext. of k_0 s.t. $E(\mathfrak{K}(t)) = E(k(t))$.

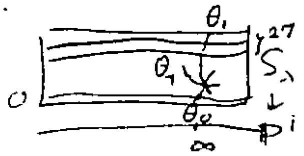
III) To see the nature of the Gal. rep., we pass to more concrete cases which lead to a unified approach to E_6, E_7, E_8. We treat the case E_6 here.

Consider the elliptic curve $E = E_\lambda$ defined by

$$(3) \qquad y^2 = x^3 + x\left(\sum_0^2 p_i t^i\right) + \left(\sum_0^2 q_j t^j + t^4\right),$$

$$\lambda = (p_0, p_1, p_2, q_0, q_1, q_2) \in \mathbb{A}^6, \quad k_0 = \mathbb{Q}(\lambda) = \mathbb{Q}(p_i, q_j).$$

[This equation defines the "univ. deformation" of E_6-sing. $y^2 = x^3 + t^4$ at $\lambda = 0$.] The assoc. ell. surface $f : S_\lambda \to \mathbb{P}^1$ is rational and has a reducible fibre of type IV ~~at $t=\infty$~~.



Assume (#) f has no other reducible fibres ~~than~~ $f^{-1}(\infty)$.

Then $E(k) \cong E_6^*$ by Sh. th. and hence we have

$$\rho_\lambda : Gal(k/k_0) \longrightarrow Aut(E_6) = W(E_6) \cdot \{\pm 1\} \subset GL_6(\mathbb{Z}),$$

where $W(E_6)$ is the Weyl group, $|W(E_6)| = 2^7 \cdot 3^4 \cdot 5$, which contains a simple group of index 2 ($\simeq U_4(2)$ in "Atlas"). In fact, we have $\rho_\lambda : Gal(k/k_0) \longrightarrow W(E_6)$.

Now E_6^* has 54 min. vectors of norm $\frac{4}{3}$. On the other hand,

$$\langle P, P \rangle = 2\chi + 2(PO) - \mathrm{contr}_\infty(P), \quad (= 0 \text{ or } \tfrac{2}{3})$$
$$\geq 4/3 \ ; \quad = \iff (PO) = 0 \ \& \ (P) \text{ meets } \Theta_1 \text{ or } \Theta_1.$$

It follows that there exist exactly 27 $P_i \in E(K)$ s.t.

(4) $\quad P_i = (\boxed{a_i} t + b_i, \ t^2 + d_i t + e_i) \quad (a_i, \cdots, e_i \in k)\ (1 \le i \le 27).$

Choose P_i s.t. $P_1, \cdots, P_6$ form a $\mathbb{Z}$-basis of $E(K) \cong E_6^*$.

<u>Theorem</u> Assume λ generic $/\mathbb{Q}$ (p_i, q_j alg. indep$/\mathbb{Q}$).

Then i) $\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}(\lambda)}/\mathbb{Q}(\lambda)) \longrightarrow W(E_6)$ is surjective.

ii) Let $\mathcal{K}_\lambda$ be the spl. field. Then

$$\left.\begin{array}{l} \mathcal{K}_\lambda = \mathbb{Q}(a_1, \cdots, a_6) \\ \cup \\ \mathbb{Q}(\lambda) = \mathbb{Q}(p_0, \cdots q_2) \end{array}\right\} \text{Gal. gr. } W(E_6)$$

iii) $\mathbb{Q}[a_1, \cdots, a_6]^{W(E_6)} = \mathbb{Q}[p_0, \cdots, q_2]$

iv) p_i or q_j = explicit fund. inv. of $W(E_6)$. (cf. [S5])

(<u>Idea of proof</u>. Substitute (4) into (3), and use explicit elimination. Then a_i will be the zeros of an alg. eq. $\Phi(X, \lambda) = 0$ of deg. 27, with $\Phi(X, \lambda) \in \mathbb{Z}[\lambda][X]$. Comparing the coeff. of X^d $(d = 2, 5, 6, 8, 9, 12)$, prove (iv), (iii), $\cdots$.)

IV) <u>Applications</u> ⓐ By specializing λ to $\lambda \in \mathbb{Q}^6$ and applying the Hilbert's irreducibility theorem, we have (cf. [S6])

<u>Th</u>. For most $\lambda \in \mathbb{Q}^6$, $\mathcal{K}_\lambda/\mathbb{Q}$ is a Gal ext. with Gal $= W(E_6)$. More precisely, those λ form the complement of a thin set in the sense of [Se]. Further every $W(E_6)$-ext. of $\mathbb{Q}$ arises this way from MWL of E_λ.

ⓑ By specializing to $(a_i) \in \mathbb{Q}^6$, we get ([S5])

<u>Th</u> For $\forall (a_1, \cdots a_6) \in \mathbb{Q}^6$ s.t. "discriminant" $\delta(a) \neq 0$, there is an ell. curve $E/\mathbb{Q}(t)$ of rank 6, having the generators $\{P_1, \cdots, P_6\}$ with $P_i = (\boxed{a_i} t + b_i, t^2 + d_i t + e_i)$.

For both ⓐ and ⓑ, we can give numerical examples. The same idea works for E_7 and E_8.

Appl. to 27 lines on a cubic surface.

Let V_λ denote the cubic surface in $\mathbb{P}^3$:

$$V_\lambda : Y^2W + 2YZ^2 = X^3 + X\left(\sum_0^2 p_i Z^i W^{2-i}\right) + \sum_0^2 q_j Z^j W^{3-j}.$$

This is a smooth surface iff (#) holds. Assume this. Then V_λ is obtained from the elliptic surface S_λ by blowing down (O), Θ_0, Θ_{-1} in this order, and the 27 minimal (P_i) (meeting Θ_1) are mapped to 27 lines l_i, whose equation is:

$$l_i \quad \begin{cases} X = a_i Z + b_i W \\ Y = d_i Z + e_i W \end{cases} \qquad (1 \le i \le 27).$$

The above results ⓐ, ⓑ can be translated into the results on cubic surfaces.

For a little more details, see [S6], and for deform. of E_r-Sing, [S4]

Application of MWL to Sphere packings. For this, see the excellent report of Elkies in the 29th Arbeitstagung.

REFERENCES (more ref. in [S]).

[E] Elkies, N.: On Mordell-Weil lattices, 1990 Arbeitstagung

[L] Lang, S.: Fundamentals of Diophantine Geometry, Springer ('83)

[Se] Serre, J-P.: Lectures on the Mordell-Weil theorem, Vieweg ('89)

[Si] Silverman, J.H: The arithmetic of ell. curves, Springer ('86).

[S] Shioda, T.: A Collection: Mordell-Weil Lattices, MPI '91; this contains 8 papers on MWL and related subjects, esp.
[S1] MWL & Gal Rep. [S2] On the MWL, [S5] Construction of ell. curves with high rank via the invariants of the Weyl groups.
[S6] Theory of MWL (ICM90). [S4] MWL of type E_8 and deformation of Sing.

Titel: Polylogarithms

Autor: Don Zagier

Adresse: Max-Planck-Institut für Mathematik, Bonn

The mth polylogarithm function Li_m, defined for $z \in \mathbf{C}$, $|z| < 1$ by the power series

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m},$$

is a classical non-elementary function which was studied by such mathematicians as Euler, Abel, Kummer, Lobachevsky and Ramanujan, and which has turned out in recent years to be connected with many interesting questions in topology, hyperbolic geometry, number theory, K-theory, and arithmetic algebraic geometry. For $m = 1$, of course, $\mathrm{Li}_1(z) = -\log(1-z)$ is simply the ordinary logarithm function developed in a Taylor series around 1, while the functions of higher order are obtained by successive integration: $\mathrm{Li}_m(z) = \int_0^z z^{-1}\,\mathrm{Li}_{m-1}(z)\,dz$. This latter definition gives the analytic continuation to all z in the cut plane $\mathbf{C} \smallsetminus [0,\infty]$ (or more invariantly, to the universal cover of $\mathbf{C} \smallsetminus \{0,1\} = \mathbf{P}^1(\mathbf{C}) \smallsetminus \{0,1,\infty\}$). For every natural number N, $\mathrm{Li}_m(z)$ satisfies the *distribution relation* $\mathrm{Li}_m(z^N) = N^{m-1}\sum \mathrm{Li}_m(\zeta z)$ (sum over all Nth roots of unity ζ), and there is also a functional equation expressing $\mathrm{Li}_m(z^{-1})$ in terms of $\mathrm{Li}_m(z)$. The latter equation can be used to compute $\mathrm{Li}_m(z)$ for $|z| \gg 1$, while for z near the unit circle (say $0.005 < |z| < 200$) a convenient method of calculation is afforded by the formula

$$\mathrm{Li}_m(e^x) = \sum_{\substack{n=0\\ n\neq m-1}}^{\infty} \zeta(m-n)\,\frac{x^n}{n!} - \Bigl(1 + \frac{1}{2} + \cdots + \frac{1}{m-1} - \log(-x)\Bigr)\frac{x^{m-1}}{(m-1)!}.$$

Besides the functional equations mentioned above, which hold for all orders m, the polylogarithms of low orders are known to satisfy certain functional equations with a far more interesting algebraic structure. It is this structure which makes the functions so interesting and in particular which is responsible for the connections to algebraic K-theory. In particular, the dilogarithm Li_2 satisfies a functional equation under $z \mapsto 1-z$ which together with the functional equation with respect to $z \mapsto 1/z$ already mentioned gives it a 6-fold symmetry, and also satisfies a much more interesting two-variable equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\Bigl(\frac{1-x}{1-xy}\Bigr) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\Bigl(\frac{1-y}{1-xy}\Bigr) = \cdots,$$

where "$\cdots$" on the right is a certain combination of products of logarithms. The functional equations become "clean" if we replace Li_2 by the *Bloch-Wigner function*

$D(z) = \Im\big(\operatorname{Li}_2(z) - |z| \operatorname{Li}_1(z)\big)$: $D(z)$ simply changes sign under $z \mapsto 1-z$ or $z \mapsto 1/z$, and in the 5-term equation just written one can replace "$\cdots$" by 0 if Li_2 is replaced everywhere by D. This 5-term equation, which replaces the fundamental functional equation $\operatorname{Li}_1(x) + \operatorname{Li}_1(y) = \operatorname{Li}_1(x+y-xy)$ of the unilogarithm, has a beautiful interpretation in terms of 3-dimensional hyperbolic geometry, as follows. Consider tetrahedra in the 3-dimensional hyperbolic (Lobachevsky) space H_3. Formulas of Lobachevsky imply that the volume of any such tetrahedron can be expressed in closed form in terms of values of the function D and in particular that the volume of an *ideal tetrahedron* Δ (one with all its vertices at infinity, i.e. in the boundary $\mathsf{P}^1(\mathsf{C})$ of H_3) equals $D(z(\Delta))$, where $z(\Delta)$ is the cross-ratio of the four vertices of Δ (this is invariant under the action of the isometry group $PGL(2,\mathsf{C})$ of H_3 on $\partial\mathsf{H}_3 = \mathsf{P}^1(\mathsf{C})$). The 6-fold symmetry of $D(z)$ now reflects the fact that four complex numbers have 6 different cross-ratios depending on the order in which they are taken, and the 5-term two-variable equation expresses the fact that the sum of the volumes of the 5 tetrahedra whose vertices are 4-subsets of a set of 5 points in $\mathsf{P}^1(\mathsf{C})$ vanishes (take the points to be 0, 1, ∞, x, and y).

Because any complete hyperbolic 3-manifold M, possibly after removing a set of measure 0, can be triangulated by ideal tetrahedra Δ_i, we can express the volume of M as a finite sum of values $D(z_i)$, $z_i = z(\Delta_i)$. This is of interest because the set of volumes of complete hyperbolic 3-manifolds is known by rigidity theorems to be a countable set and we would like to know this "volume spectrum," but not yet a very useful statement because the set of sums $\sum D(z_i)$ with complex arguments z_i is clearly equal to all of R. However, it follows from results of Dupont-Sah or of Neumann-Zagier that the arguments z_i of an arbitrary ideal triangulation of M satisfy the algebraic relation

$$\sum_i (z_i) \wedge (1 - z_i) = 0\,, \tag{*}$$

where the value $(z) \wedge (1-z)$ is to be interpreted as an element of the exterior square of the abelian group C^*, written additively as a Z-module. As an example, an explicit triangulation of the complement of the [redacted] link

given by Thurston implies that this complement, which has a hyperbolic structure, has

volume equal to $4D(a)+2D(b)$, where $a=\dfrac{1+\sqrt{-7}}{2}$ and $b=\dfrac{-1+\sqrt{-7}}{4}$, and the two identities $ab=-1$, $1-b=(1-a)^2b$ imply $4(a)\wedge(1-a)+2(b)\wedge(1-b)=0$. (Check this!) It turns out that the set of sums $\sum D(z_i)$ for collections of complex mumbers $\{z_i\}$ satisfying (*) is countable, overcoming the objection made above. Moreover, by restricting to the subset of arithmetic hyperbolic manifolds, the volumes of which are known to be expressible as simple multiples of the value at $s=2$ of the Dedekind zeta functions of algebraic number fields, one obtains the theorem that *the value of $\zeta_F(2)$ for any number field F can be expressed in terms of a finite number of values of the function $D(z)$ at algebraic arguments z.* For instance, one can show that the link complement above is a finite cover of the manifold $\mathrm{H}_3/SL(2,\mathcal{O})$, where $\mathcal{O}=\mathbb{Z}[a]$ is the ring of integers of $F=\mathbb{Q}(\sqrt{-7})$, and using this one shows that the value of $\zeta_F(2)$ for this particular number field is $2\pi^2/21\sqrt{7}$ times the volume $4D(a)+2D(b)$ of M.

The bulk of the talk was devoted to explaining the extensions of this last result to higher m. There is a modification $P_m(z)$ of the mth order polylogarithm $\mathrm{Li}_m(z)$ analogous to the modification $D(z)$ of $\mathrm{Li}_2(z)$, e.g., $P_3(z)=\Re\big(\mathrm{Li}_3(z)-\log|z|\,\mathrm{Li}_2(z)+\frac{1}{3}\log^2|z|\,\mathrm{Li}_1(z)\big)$, and one can formulate a rather precise form of the conjecture that *the value of $\zeta_F(m)$ for any number field F can be expressed as a finite combination of values of $P_m(z)$ at arguments z belonging to F* (in the various embeddings of F into $\mathbb{C}$). The key to the whole structure is the generalization to higher orders of the relation (*), which tells us which combinations of arguments one should look at. The details of the conjecture, as well as many examples, are given in my survey paper cited below, and will not be repeated here. This leads to a description in terms of algebraic K-theory (more precisely, in terms of the group $K_{2m-1}(F)$), the value of $\zeta_F(m)$ entering as the covolume of this K-group considered as a lattice in Euclidean space via the Borel regulator mapping. The conjecture is completely proved for $m=3$ by Goncharov (MPI preprint, 1990), while for higher m one at least knows that there is a map from an appropriate "polylogarithm group" to K-theory such that the Borel regulator map is expressed in terms of polylogarithms (Beilinson and Deligne, in preparation), so that the only thing still needed for the conjecture is the surjectivity of this map, which can be checked for any given F and m by a finite calculation. The main ingredient needed to extend Goncharov's proof to higher m would be a full theory of functional equations, but this is still missing. Kummer gave some functional equations for $m=3$, 4, and 5 in 1840, and H. Gangl (Bonn) has found functional equations for $m=6$ and 7 in the last 2 years, but for $m>3$ one does not have a complete set of functional equations and for

$m > 7$ no non-trivial functional equation is known at all.

The talk closed with a brief discussion of a generalization of the polylogarithm to elliptic curves, due to Bloch, Levin, and Beilinson; we do not describe this here.

L. Lewin, *Polylogarithms and Associated Functions*, North Holland 1981

D. Zagier, Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields, in *Arithmetic Algebraic Geometry* (eds. G. van der Geer, F. Oort, J. Steenbrink), Prog. in Math. **89**, Birkhäuser, Boston 1991, pp. 391–430.

Titel: Maps between classifying spaces of compact Lie groups

Autor: Bob Oliver

Adresse: Aarhus (z.Zt. Heidelberg)

This summary describes joint work with Stefan Jackowski and Jim McClure. It involves a program to study the homotopy theory of BG, for G a compact Lie group, by approximating BG as a homotopy direct limit of simpler spaces. In particular, we build on the program of Adams and Mahmud [AM], and develop procedures which in many cases can be used to classify homotopy classes of maps $f: BG \longrightarrow BG'$ when G and G' are compact Lie and G' is connected.

Here, I focus attention on the following result:

<u>Theorem 1</u> [JMO] Let G be any compact connected simple Lie group. Then the map

$$[BG, BG] \rightarrowtail End(H^*(BG;\mathbb{Q}))$$

is injective; i.e., homotopy classes of selfmaps of BG are detected by rational cohomology.

Note that Theorem 1 fails in general for semisimple G: ~~For~~ example, one can construct homomorphisms $\rho, \rho' : SU(3) \longrightarrow SO(8)$ such that

$B\rho \not\simeq B\rho'$ but $H^*(B\rho) = H^*(B\rho')$. Hence Theorem 1 fails when $G = SU(3) \times SO(8)$.

Theorem 1 is based on a new approximation of BG at the prime p, for any compact Lie group G and any prime p, described as follows. A group P is called <u>p-toral</u> if its identity component P_0 is a torus, and P/P_0 is a finite p-group. Consider the orbit category $\mathcal{O}(G)$, where

$$Ob(\mathcal{O}(G)) = \{G/H : H \leq G \text{ closed subgroup}\}$$

$$Mor(G/H, G/K) = \{\text{all } G\text{-equivariant maps}\}.$$

Let $\mathcal{R}_p(G) \subseteq \mathcal{O}(G)$ be the full subcategory on orbits G/P where

(1) P is p-toral

(2) $|N(P)/P| < \infty$

(3) $\nexists\, 1 \neq Q \triangleleft N(P)/P$, Q a p-subgroup.

For example, $\mathcal{R}_2(SO(3))$ contains just two (isomorphism classes of) objects:

$$SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2) \quad \text{and} \quad SO(3)/O(2).$$

In general, $\mathcal{R}_p(G)$ is a finite category, in that it contains finitely many isomorphism classes (orbit types), and finite morphism sets.

Our main approximation theorem is the following:

<u>Theorem 2</u> For any G, p, there is a $\mathbb{Z}_{(p)}$-homology equivalence

$$\underset{\substack{\longrightarrow \\ G/P\in\mathcal{R}_p(G)}}{\mathrm{hocolim}}(EG/P) \longrightarrow BG.$$

Note that $EG/P \simeq BP$; so Theorem 2 says that BG is approximated, at p, as the homotopy direct limit of BP's for p-toral $P \leq G$.

When P is p-toral, the mapping spaces $\mathrm{map}(BP, BG)$ have been studied by Dwyer + Zabrodsky and Notbohm. In particular, they showed the following:

<u>Theorem 3</u> [DZ][N] For any p-toral group P and any compact Lie group G, the map

$$\mathrm{Rep}(P,G) := \mathrm{Hom}(P,G)/\mathrm{Inn}(G) \xrightarrow{\cong} [BP, BG]$$

$(\rho \mapsto B\rho)$ is bijective.

The idea of the proof of Theorem 1 can now be sketched. Fix simple G, let T be a maximal torus, and let $N_p(T)/T \leq W = N(T)/T$ be a Sylow p-subgroup. Assume $f, f' : BG \to BG$ are such that $H^*(f;\mathbb{Q}) = H^*(f';\mathbb{Q})$. We first apply Theorem 3 to show that $f|BT \simeq f'|BT$; ~~and then~~

Now fix some prime p. We apply Theorem 3 again to show that ~~$f|BN_p(T) \simeq f'|$~~

$$f|BN_p(T) \simeq f'|BN_p(T)$$

(this is more complicated, and involves applying Theorem 3 to various p-toral subgroups $P \subseteq N_p(T)$.)

Any p-toral subgroup $P \leq G$ is conjugate to a subgroup of $N_p(T)$, so $f|BP \simeq f'|BP$ for all P. Since

$$[BG, BG_p^\wedge] \cong [\underrightarrow{hocolim}(\underset{\underset{BP}{\text{IS}}}{EG/P}), BG_p^\wedge]$$

by Theorem 2, the ~~computation~~ vanishing of certain inverse limits now shows that

$$f_p^\wedge \simeq f'^\wedge_p : BG \longrightarrow BG_p^\wedge .$$

This holds for each p, and Sullivan's arithmetic square for BG can now be used to show that $f \simeq f'$.

Upon combining Theorem 1 with earlier work of Sullivan, Wilkerson, Hubbuck, and Ishiguro, we get the following description of $[BG, BG]$:

<u>Theorem 4</u> [JMO] For any compact connected simple G, there is a bijection (monoid isomorphism)

$$[BG, BG] \cong \{0\} \amalg \left(Out(G) \times \{k > 0 : (k, |W|) = 1\}\right).$$

Here, W is the Weyl group, 0 corresponds to null homotopic maps, and (α, k) corresponds to $B\alpha \circ \psi^k$ where ψ^k is an "unstable Adams operation" of degree k.

As mentioned above, these methods can be used to describe $[BG, BG]$ in many other cases (see, fx, [JMO2]). As another example, $[B\Gamma, BSO(3)]$ can be explicitely described for any finite Γ.

References

{AM} J. F. Adams and Z. Mahmud, Maps between classifying spaces, Inventiones math. 35 (1976), 1-41

{DZ} W. Dwyer and A. Zabrodsky, Maps between classifying spaces, Algebraic topology, Barcelona, 1986, Lecture Notes in Math. 1298, Springer-Verlag (1987), 106-119

{JMO} S. Jackowski, J. McClure, and B. Oliver, Homotopy classification of self-maps of BG via G-actions, Annals of Math. (to appear)

{JMO2} S. Jackowski, J. McClure, and B. Oliver, Maps between classifying spaces revisited (in preparation)

{N} D. Notbohm, Abbildungen zwischen klassifizierenden Ra"umen, Dissertation, Go"ttingen (1988)

Titel: QUANTUM GROUPS AND NON-COMMUTATIVE DE RHAM COMPLEXES

Autor: MANIN Yu. I.

Adresse: STEKLOV INSTITUTE, MOSCOW
MAX-PLANCK-INSTITUTE FÜR MATH., BONN

INTRODUCTION. Let $\mathcal{M}$ be a category of "spaces" (e.g. finite sets, schemes / k, smooth manifolds ...) and F be a functor $\mathcal{M} \to \{\text{Linear spaces}\}$ such that $F(X \times Y) = F(X) \otimes F(Y)$, $F(pt) =$ ground field k. Then $F(X)$ is a commutative k-algebra (resp. co-commutative coalgebra) if F is contravariant (resp. covariant), w.r.t. $F(\Delta_X)$, where $\Delta_X : X \to X \times X$ is the diagonal. If G is a group in $\mathcal{M}$, $F(G)$ becomes a Hopf algebra, and if G acts upon Y, $F(Y)$ becomes a comodule (resp. module) over $F(G)$.

In the theory of quantum groups, one studies general Hopf algebras and their (co)actions upon algebras as if they were functions on "non-commutative", or "quantum", spaces. However, not all classical functors F were treated with equal attention.

Consider the following four functors:

1. Functions Φ.
2. Distribution with finite support U.
3. De Rham complex $\Omega^{\bullet}$.
4. Cohomology $H^{\bullet}$.

Quantum versions of U were constructed by V. Drinfeld [1] and M. Jimbo; functions Φ were considered by Faddeev et. al [2] and lecturer [3]; some variants of de Rham complexes were studied by S. L. Woronowicz [4].

In this lecture we shall concentrate upon $\Omega^\cdot$. We start with the Wess-Zumino de Rham complex of the quantum plane and discuss the following properties of quantum de Rham complexes:

- cohomology at roots of unity;
- operator algebras generated by vector fields;
- relations between Wess-Zumino and A. Connes' complexes.

1. WESS-ZUMINO AXIOMS. Let $\Omega^\cdot$ be a DGA over a field k. Assume that $\Omega^\cdot$ is generated by two subalgebras: $A = \Omega^0$ and $B = \underset{i \geq 0}{\oplus} B^i$. Consider the following properties:

WZ_i : $A \underset{k}{\otimes} B^i \xrightarrow{\text{mult.}} \Omega^i \xleftarrow{\text{mult}} B^i \underset{k}{\otimes} A$ are isomorphisms of linear spaces.

WZ : WZ_i are satisfied for all $i \geq 0$.

The main use of WZ_1 is that it allows to define vector fields. If $B^1 = \overset{n}{\underset{i=1}{\oplus}} k\xi_i$, define $X_i : A \to A$ by $df = \sum_{i=1}^{n} \xi_i X_i(f)$. From the Leibniz formula $d(fg) = df \cdot g + f\,dg$, $d : A \to \Omega^1$ one obtains the Leibniz formulas for X_i:

$$X_i(fg) = X_i(f)g + \sum_j \sigma_{ij}(f)\,X_j(g)\,, \qquad (1)$$

where $\sigma_{ij} : A \to A$ are defined by commutation rules implied by WZ_1: $f\xi_j = \sum_i \xi_i \sigma_{ij}(f)$. Since (1) essentially implies the comultiplication formulas

$$\Delta(X_i) = X_i \otimes 1 + \sum_j \sigma_{ij} \otimes X_j\,, \qquad (2)$$

it "explains" why in the $U_q(\mathfrak{g})$ Cartan generators appear in the exponentiated form. Namely, they correspond to certain σ_{ij} in a realization by vector fields, at least when $\mathfrak{g} = sl$.

2. A UNIVERSAL CONSTRUCTION. Let $(A, x_1, \dots, x_n)$ be a k-algebra together with a finite set of its generators. Consider all digrams $\varphi: A \to B \otimes A$ such that $\varphi(x_i) = \sum_j e_{ij} \otimes x_j$ for certain $e_{ij} \in B$. Among these diagrams there is a universal one $\delta: A \to E \otimes A$. Algebra E is automatically a bialgebra, and δ is a coaction. There exists a universal map $E \to H$ to a Hopf algebra H. If one imagines A as an algebra of functions on a quantum linear space, then E is an algebra of quantum linear endomorphisms of this space, and H is the function algebra of the quantum linear group of A.

This construction can be generalized in several directions:

a) Instead of one algebra (A, x_i) one can take a family $(A^{(\alpha)}, x_i^{(\alpha)})$; δ coacts in the same way upon all $(x_i^{(\alpha)})$.

b) Instead of k-algebras one can consider k-DGA's.

In other words, starting with functions on a quantum space, one can reconstruct functions on its quantum automorphism space; and starting with differential forms, one can reconstruct the differential forms.

3. WESS-ZUMINO COMPLEX OF QUANTUM PLANE. Apply the universal construction to the pair of algebras

$$A_q = k\langle x, y\rangle / (xy - q^{-1}yx); \quad B_p = k\langle \xi, \eta\rangle / (\xi^2, \eta^2, \xi\eta + p\eta\xi),$$

$p, q \in k^*; (pq)^2 \neq -1$. We shall obtain the Hopf algebra

$$\Phi[GL_{p,q}(2)] = k\langle a, b, c, d\rangle / (*) [\mathrm{DET}^{-1}]$$

of functions on the two-parametric quantum $GL(2)$, where $(*)$ denotes the following set of commutation relations:

$$\begin{aligned} ab &= p^{-1} ba; \quad ac = q^{-1} ca; \quad ad = da + (q^{-1} - p) cb; \\ bc &= pq^{-1} cb; \quad bd = q^{-1} db; \quad cd = p^{-1} dc, \end{aligned} \qquad (*)$$

and $\mathrm{DET} = ad - q^{-1} cb = da - pcb$.

Wess and Zumino interpreted (ξ,η) as differentials of (x,y). More precisely, they have shown that there exist exactly two structures of a DGA-algebra on $B_p \otimes A_q$, compatible with the action of $GL_{p,q}(2)$ and verifying $dx = \xi$, $dy = \eta$. One of the structures is given by the following cross-commutation relations:

$$x\xi = (pq)^{-1}\xi x \; ; \; y\eta = (pq)^{-1}\eta y \; ; \tag{4}$$
$$x\eta = (p^{-1}q^{-1}-1)\xi y + q^{-1}\eta x ; \; y\xi = p^{-1}\xi y .$$

The other one is obtained by $x \leftrightarrow y$, $\xi \leftrightarrow \eta$, $p \leftrightarrow p^{-1}$, $q \leftrightarrow q^{-1}$. A moral is: de Rham complex of A_q is not uniquely defined by the function ring. There are at least two one-parametric families of natural deRham complexes.

4. DE RHAM COMPLEXES OF 2×2 QUANTUM MATRICES.

Consider now the 4-dim quantum linear space (*). One can try to extend it to a DGA-algebra applying the universal construction of n°2 to one of the two WZ complexes $B_p \otimes A_q = \Omega^{\bullet}_{p,q}$. The resulting DGA-algebra is, however, too big and does not verify WZ. For each of the $\Omega^{\bullet}_{p,q}$'s, it has precisely three quotient DGA's verifying WZ_1 (and, as a consequence, WZ). This was checked by a long direct calculation by D.V. Zhdanovich and the lecturer. Hence (*) has six natural de Rham complexes.

5. DE RHAM COMPLEX OF A LINE. On the "coordinate axis" kx, (4) induces a non-trivial non-commutative differential geometry. Consider $\Omega = k\langle x^{\pm 1}, dx\rangle / (v\, dx.x - v^{-1} x\, dx)$. We have

$$d(x^n) = v^{n-1}[n]_v \, dx.x^{n-1}, \quad [n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}, \; v \in k^*.$$

Hence, if v is not a root of unity, $H^{\bullet}(\Omega^{\bullet}) \simeq k \oplus k\, dx.x^{-1}$, but if $v^{l} = 1$, l primitive, we have $H^{\bullet}(\Omega^{\bullet}) \simeq \bigoplus_m k\, x^{ml} \oplus k\, dx.x^{ml-1}$.

The residue functional $\int dx\,(\sum a_i x^i) = a_{-1}$

verifies only a twisted version of Connes' axiom:

$$\int [dx.x^n, x^m]_{v^m} = 0, \text{ where } [x,y]_w = wxy - w^{-1}xy.$$

Put $df = dx \cdot \partial_v f$, and $L_n = v x^{n+1}\partial_v$. The usual (zero charge Virasoro) commutation relations deform to

$$[L_n, L_m]_{v^{n-m}} = [m-n]_v L_{n+m}.$$

However, the Leibniz formula $L_n(fg) = L_n f \cdot g + \sigma(f) L_n g$, $\sigma(x^k) = v^{2k} x^k$, shows that closed comultiplication formula can be introduced only if one replaces L_0 by $K_0 = \sqrt{\sigma}$, which is essentially "exponentiated Cartan generator".

6. SOME STRUCTURAL RESULTS. In conclusion, we state in an imprecise form several structure theorems due to W.-Z., Yu.M, E.Mukhin.

a). Assume A and B given as in n°1. Then various structures of k-algebras upon $A \otimes B$ verifying WZ correspond to certain actions of $E(B)$ upon A, where $E(B)$ is the universally coacting on B bialgebra from n°2. (This is true if $\dim B < \infty$).

b). Let A, B be quadratic algebras, $A_1 = B_1$. Let $M(A,B)$ be the universally coacting on (A,B) bialgebra. Then there is a compatible structure of WZ_1-DGA on a quotient of $A * B$ iff $M(A,B)$ can be defined by a Yang-Baxter operator.

c). If A, B are quadratic algebras satisfying Diamond Lemma and conditions of b) are valid, then the WZ_1-quotient of $A * B$ satisfies the Diamond Lemma, so that WZ_1 implies WZ.

REFERENCES

[1] V.G. Drinfeld. Berkeley ICM talk.

[2]. N. Reshetikhin, L. Takhtajan, L. Faddeev. Algebra i Analiz, 1:1 (1989), 178-206.

[3]. Yu.I. Manin. Quantum groups and non-commutative geometry. Motre'al University, 1988.

[4]. S.L. Woronowicz. Comm. Math. Phys. 111 (1987), 613-665

Titel: On Faltings's recent proof of a conjecture of S. Lang
Autor: G. Wüstholz
Seite: 1
Adresse: Mathematikdepartement, ETH Zürich
Rämistr. 101
CH-8092 Zürich

0. Lang's conjectures

Let K be a field of finite type over $\mathbb{Q}$, A/K an abelian variety and $X \subseteq A$ a subvariety over K. It is then a very important diophantine problem to determine the set $X(K)$ of K-rational points on X. Clearly if $B \subseteq A$ is an abelian subvariety and $\xi + B \subseteq X$ then $(\xi+B)(K) \subseteq X(K)$ and so if $\xi \in X(K)$ and $B \neq 0$ we get a whole family of K-rational points in general, at least if B is also defined over K. Lang conjectured that these families are the only exceptions and that there are only finitely many of them. More precisely we have (see [L])

Lang's conjecture LCI: $X(K)$ lies in a finite number of translates $x+B$ of abelian subvarieties B with $x+B \subseteq X$.

Now let $\Gamma_0 \subseteq A(\mathbb{C})$ be a finitely generated subgroup and $\Gamma = \{\gamma \in A(\mathbb{C}),\ n\gamma \in \Gamma_0 \text{ for some } n>0, n \in \mathbb{Z}\}$ its division group.

Lang's conjecture LCII: $X(\mathbb{C}) \cap \Gamma$ lies in a finite number of translates $x+B$ of abelian subvarieties B with $x+B \subseteq X$.

1. Special case. We consider first the case that X does not contain a translate of a nontrivial abelian subvariety. Then the relation between LC I and LC II was clarified by Raynaud [R]. Namely he proved

Theorem (Raynaud). For every X, A and K as above LC I implies LC II.

The conjecture LC I in this case was solved by Faltings. Namely we have ([F1])

Theorem (Faltings, '89). $X(K)$ is finite.

Actually he proves it only for number fields but the extension to general K's is standard.

2. General case. We assume now X to be arbitrary. Then LC I was proved recently by Faltings ([F2]).

Theorem (Faltings, '90). For every X, A, K as above LC I holds.

One of the basic tools to deal with translates of abelian subvarieties is the following construction. Let A be an abelian variety, $X \subseteq A$ a subvariety. Then $S(X)$ is defined to be the connected component of the zero element of A of the stabilizer of X.

Then clearly $X + S(X) \subseteq X$ and so we obtain a fibration

$$\pi : X \longrightarrow X/S(X) =: Y .$$

Kawamata [K] proved now the following

Kawamata's structure Theorem ('80) If $S(X) = 0$ then there exists a finite set E of proper subvarieties Z of X with nonzero $S(Z)$ such that if $x + B$ is a translate of an abelian subvariety $B \neq 0$ with $x + B \subseteq X$ then

$$x + B \subseteq \bigcup_{Z \in E} Z =: Z(X).$$

3. Heights. Consider $\mathbb{P}^n_{\mathbb{Z}}$, the n-dimensional projective space over $\mathbb{Z}$, $L = \mathcal{O}(1)$ with the Fubini-Study metric and associated curvature form h. Then one defines the arithmetic Chowgroups $CH^p_{Ar}(\mathbb{P}^n)$ for $0 \leq p \leq n+1$ and obtains the arithmetic Chow ring $CH_{Ar}(\mathbb{P}^n) = \bigoplus CH^p_{Ar}(\mathbb{P}^n)$.

One has intersection pairings

$$CH^p_{Ar}(\mathbb{P}^n) \times CH^q_{Ar}(\mathbb{P}^n) \longrightarrow CH^{p+q}_{Ar}(\mathbb{P}^n)$$

and a degree map

$$\deg : CH^{n+1}_{Ar}(\mathbb{P}^n) \xrightarrow{\sim} \mathbb{R} .$$

Furthermore to every $X \in CH^p(\mathbb{P}^n)$ one attaches an element $\hat{X} \in CH^p_{Ar}(\mathbb{P}^n)$ and gets a map $CH(\mathbb{P}^n) \to CH_{Ar}(\mathbb{P}^n)$.

Then if $X \in CH^p(\mathbb{P}^n)$ we define its height as

$$h(X) = \deg(\hat{X} \cdot c_{1,Ar}(L, h')^{n+1-p})$$

where for any meromorphic section $0 \neq f$ we put

$$c_{1,Ar}(L, h') = \text{class of } (\operatorname{div}(f), \log h'(f,f)).$$

This has been further developed by W. Gubler in his thesis

4. The index. Let $P = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, $x \in P$, $L = \mathcal{O}(d_1, \ldots, d_m)$ and $f \in \Gamma(P, L)$. We trivialize L in a neighborhood of x so that we can regard f as a function on this neighborhood. In particular we get differential operators in a neighborhood of the projection of x onto the k-th factor. We denote them by D_k, $1 \leq k \leq m$. Then the index $i(x, f)$ is the maximum over all σ such that

$$D_1 \cdots D_m f(x) = 0$$

for all such differential operators D_k of order $\leq j_k$ for all integers $j_1, \ldots, j_m \geq 0$ with $j_1/d_1 + \cdots + j_m/d_m < \sigma$. With this we define closed subschemes $Z_\sigma(f)$ by

$$Z_\sigma(f) = \{x \in P;\ i(x, f) \geq \sigma\}.$$

They satisfy $Z_\sigma(f) \supseteq Z_\tau(f)$ if $\sigma \leq \tau$.

Product Theorem (Faltings). For all $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that if $d_j/d_{j+1} \geq r$ $(1 \leq j \leq m-1)$ and Z is an irreducible component of Z_σ and $Z_{\sigma+\varepsilon}$ we have

(i) $Z = Z_1 \times \cdots \times Z_m$, $Z_j \subseteq \mathbb{P}^{n_j}$

(ii) $\deg Z_j \leq c(\varepsilon)$, $(1 \leq i \leq m)$, $c(\varepsilon)$ a constant depending only on ε.

Complement: If f has integral coefficients bounded by C_v for $v|\infty$ then there are constants $c_1, c_2 > 0$ s.t.

(iii) $\sum d_i h(Z_i) \leq c_1 \sum_{v|\infty} C_v + c_2(d_1 + \cdots + d_m)$.

5. Faltings's line bundle. Let L be symmetric and ample and define for rational numbers ε and $s_1, \ldots, s_m > 0$ the line bundle on A^m

$$L(\varepsilon, s) = \sum (s_i x_i - s_{i-1} x_{i-1})^* L + \varepsilon \sum s_i^2 pr_i^* L$$

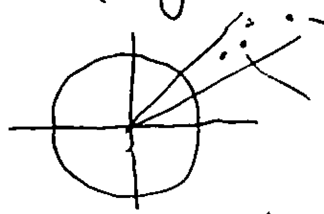
If $X \subseteq A$ define the morphism $\alpha_m : X^m \longrightarrow A^{m-1}$ by $(x_1, \ldots, x_m) \longmapsto (2x_1 - x_2, \ldots, 2x_{m-1} - x_m)$. Then it is possible to show that $\alpha_m : (X - Z(X))^m \longrightarrow A^{m-1}$ is quasi finite and a result of Burnol shows that on products $Y_1 \times \cdots \times Y_m$ with $Y_j \not\subseteq Z(X)$, $j = 1, \ldots, m$, the line bundle $L(-\varepsilon, s)|_{Y_1 \times \cdots \times Y_m}$ is represented by an effective divisor. Using this an application of Siegel's Lemma gives a section $0 \neq f \in \Gamma(Y, L(-\varepsilon, s)^{\otimes d})$, $d \gg 1$, $Y = Y_1 \times \cdots \times Y_m$, such that for the infinite places $v | \infty$ we have $\log \|f\|_v \ll d$.

6. Proof of the Theorem. One first shows that $(X - Z(X))(K)$ is finite if K is a number field. For this we assume that this is not the case and choose $x_1, \ldots, x_m \in (X - Z(X))(K)$ such that

$$h(x_1) \gg 1,$$
$$h(x_i)/h(x_{i-1}) \gg 1.$$

On $V = A(K) \otimes \mathbb{R}$ we have a bilinear form $\langle , \rangle$ coming from the Néron–Tate height on A. We then further assume that the x_i lie in a small segment of V:

$$\frac{\langle x_i, x_{i+1} \rangle}{\|x_i\| \, \|x_{i+1}\|} \geq 1 - \varepsilon/2$$



Put $x = (x_1, \dots, x_m)$ and let $\mathcal{O}_K$ = ring of integers of K. Then the degree of the projective module $x^*(L(-\varepsilon, S)^{\otimes d})$ over $\mathcal{O}_K$ satisfies

$$(*) \qquad \deg x^*(L(-\varepsilon, S)^{\otimes d}) \xrightarrow[d \to \infty]{} -\infty .$$

We now make induction over product varieties. First put $Y_j = X$ in section 5. Then because of $(*)$ the section f constructed there has large index $v := i(x, f)$. Choose $N > \dim X^m$. Then there exists a chain

$$x \in Z_N' \subset \cdots \subset Z_2' \subset Z_1' \subsetneq X^m$$

such that Z_j' is an irreducible component of $Z_{jv/N}$. By dimension reasons $Z_j' = Z_i'$ for some $i \neq j$. Put $Z = Z_i'$. Then the product theorem tells us that Z is a product and $x \in Z$. Now apply induction. This proves the claim. Therefore we only need to show that $Z(X)(K)$ lies in a finite number of translates. So let Z_j be a component. Then $S(Z_j) \neq 0$ and we finish by induction for $Z_j / S(Z_j)$.

7. References

[B] J.F. Burnol, Letter to G. Faltings, November 11, '90.

[F1] G. Faltings, Diophantine approximation on abelian varieties, Annals of Math. 129 (1991)

[F2] G. Faltings, The general case of S. Lang's conjecture, preprint 1991.

[K] S. Kawamata, On Bloch's Conjecture, Inv. math. 57 (1980), 97-100.

[L] S. Lang, Integral points on curves, Publ. Math. IHES 6 (1960), 27-43.

[R] M. Raynaud, Arround the Mordell conjecture for function fields and a conjecture of Serge Lang, in "Algebraic geometry", SLN 1016 (1983), 1-19.

Titel: Galois cohomology : recent results and open questions
Autor: J-P. Serre
Adresse: Collège de France, Paris

Galois cohomology (for semi-simple groups) is now reasonably well understood when the ground field is a number field : the Hasse local-global principle has been proved in full generality, thanks to the recent work of Chernousov and Premet.

But what about arbitrary fields? The aim of the talk will be to emphasize the many problems which are open, specially about the exceptional groups, such as F_4 or E_8.

J-P. Serre.

Référence : Chap. VI of "Algebraic Groups and Number Theory", by V. P. Platonov and A.S. Rapinchuk, Moskva, 1991.

Titel: Intersection theory on the moduli space of curves and the matrix Airy function

Autor: M. Kontsevich

Adresse: MPI (current), on leave from Acad. Sci. Moscow

This talk is devoted to the proof of a recent E. Witten's conjecture (see [1], [2]). The main object - "partition function in ∞ number of indeterminates t_i

$$F = F(t_0, t_1, \dots) = t_0^3/6 + t_1/24 + \dots \in \mathbb{Q}[[t_0, t_1, \dots]]$$

has six descriptions connected with

1. intersection theory on the moduli space of curves;
2. the matrix Airy function
$$A(Y) = \int \exp(\sqrt{-1}(\mathrm{Tr}\, X^3/3 - \mathrm{Tr}\, XY))\, dX \quad \text{, where}$$
X, Y are hermitean matrices of arbitrary size;
3. Asymptotic behavior of the number of triangulations by N triangles of a surface of genus g, $N \to +\infty$;
4. Asymptotic behavior of integrals $\int \exp(\mathrm{Tr}(P(X)))dX$ over the space of hermitean $N \times N$ matrices, $N \to +\infty$, P is a polynomial (depending on N).
5. Some special solution of KdV-hierarchy connected with the classical Airy function
$A(y) = \int \exp(\sqrt{-1}(x^3/3 - xy))dx$, $A''(y) + yA(y) = 0$;
6. highest weight vector for "one-half" of Virasoro algebra in bosonic representations.

The graph of connections between descriptions:

$$\begin{tikzcd}
1 \arrow[d] & 3 \arrow[d] & \text{surfaces, 2dim. gravity} \\
2 \arrow[d] & 4 \arrow[d] & \text{matrix integrals} \\
6 & 5 \arrow[l] & \text{integrable systems.}
\end{tikzcd}$$

Implications $3 \to 4 \to 5 \to 6$ are proven by

mathematical physicists (see [3], [4], [8]).
E. Witten conjectured that 1 is equivalent to ⟨3, 4, 5, 6⟩. This talk contains the proof of 1 → 2 → 6. At the moment we have no approach to the direct connections 1–3 or 2–4.

§1. Topological gravity in 2 dimensions.

Fix integers g, n ; $g \geq 0$, $n > 0$, $2-2g-n < 0$.
$M_{g,n}$ = the moduli space of smooth curves $C/\mathbb{C}$ of genus g with n distinct marked points x_i.
$\overline{M}_{g,n}$ = the Deligne-Mumford compactification of $M_{g,n}$ (see [5]).
$\mathcal{L}_i$, $i=1\ldots n$ - line bundles on $\overline{M}_{g,n}$; $\mathrm{fiber}(\mathcal{L}_i) = T^*_{x_i}C$.
Let $d_1, \ldots, d_n$ be integers, $\sum_{i=1}^{n} d_i = \dim_{\mathbb{C}} M_{g,n} = 3g-3+n$.
Denote by $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ the intersection index $\langle \prod_1^n c_1(\mathcal{L}_i)^{d_i}, [\overline{M}_{g,n}] \rangle \in \mathbb{Q}$ ($\overline{M}_{g,n}$ is an orbifold).
By definition

$$F(t_0, t_1, \ldots) = \sum_{k_0, k_1, \ldots} \langle \tau_0^{k_0} \tau_1^{k_1} \ldots \rangle \prod_{i=0}^{\infty} t_i^{k_i} / k_i! \; .$$

Witten's conjecture is equivalent to the fact that $U := \partial^2 F / \partial t_0^2$ satisfies the Korteweg-de Vries equation $\partial U / \partial t_1 = U \, \partial U / \partial t_0 + 1/12 \cdot (\partial / \partial t_0)^3 U$.

§2. Main theorem ("formula for F").

Let Λ be a positive definite hermitean $N \times N$ matrix, $t_i(\Lambda) := -(2i-1)!! \, \mathrm{Tr}\, \Lambda^{-(2i+1)}$, $n!! = n \cdot (n-2)(n-4)\ldots$
Then $F(t_\bullet(\Lambda))$ is an asymptotic expansion of

$$\log \left(\int \exp\left(\frac{\sqrt{-1}}{6} \mathrm{Tr}\, X^3 - \mathrm{Tr} \frac{X^2 \Lambda}{2} \right) dX \Big/ \int \exp\left(- \frac{\mathrm{Tr}\, X^2 \Lambda}{2} \right) dX \right),$$

where $\Lambda \to +\infty$.

This theorem gives arrow ①→②.

§3. A Combinatorial model for $M_{g,n}$

A ribbon graph (or a fatgraph) is a finite graph with cyclic order on the set of edges coming to each vertex. There is 1-1 correspondence between R.G. and cell decompositions of closed oriented surfaces. (RG associated with cell decomposition is 1-skeleton).
A metric on a RG is a map $l:\{\text{edges}\}\to\mathbb{R}_+$.
$M^{comb}_{g,n}$:= moduli space of RG with metric such that a) valency of each vertex ≥ 3, b) corresponding surface has genus g, c) the set of 2-cells is labeled $\{2\text{-cells}\}\xrightarrow{\sim}\{1,\dots,n\}$.

<u>Fact</u> $M^{comb}_{g,n}\approx M_{g,n}\times\mathbb{R}^n_+$. It follows from K. Strebel results on quadratic differentials, see [6], [7].
Projection $\pi: M^{comb}_{g,n}\to\mathbb{R}^n_+$ is $(p_1,\dots,p_n)$ where p_i is the perimeter of the boundary of i-th cell.

§4. A combinatorial model for $BU(1)$

$$BU(1)^{comb}=\bigcup_{k=1}^{\infty}\{(l_1,\dots,l_k)\mid l_i>0\}/\text{cyclic permutations}$$

is considered as an orbispace with some natural topology. There exists an universal S^1-bundle S_{un} on $BU(1)^{comb}$, fiber is a k-gon with edges of length $l_1,\dots,l_k$.
<u>Lemma</u> 2-form $\omega=\sum_{1\leq i<j\leq k-1} d\left(\frac{l_i}{p}\right)\wedge d\left(\frac{l_j}{p}\right)$, $p=l_1+\dots+l_k$ represents $c_1(S_{un})$.

The natural map $\varphi=(\varphi_1,\dots,\varphi_n): M_{g,n}^{comb}\to (BU(1)^{comb})^n$ which associates with RG the sequence of boundaries of 2-cell, can be prolonged to $\overline{M}_{g,n}\times\mathbb{R}^n_+$. $\varphi_i^*(S_{un})$ is isomorphic to the circle bundle arising from $\mathcal{L}_i$.

§ 5. Main computation

Fix real numbers $\lambda_i>0$, $i=1,\dots n$. Let $\Omega=\sum p_i^2\varphi_i^*(\omega)$

$$I_g(\lambda.):=\int_0^{+\infty}\!\!\cdots\!\int\prod_1^n dp_i\cdot\exp\left(-\sum\lambda_i p_i\right)\cdot\int_{\pi^{-1}(p.)}\Omega^d/d!\ ,\quad d=\dim_{\mathbb{C}} M_{g,n}$$

There are two ways to compute $I_g(\lambda.)$:

1° $$I_g(\lambda.)=\int_0^{+\infty}\!\!\cdots\!\int\prod dp_i\,\exp\left(-\sum\lambda_i p_i\right)\cdot\sum_{\substack{d_1\dots d_n\\ \sum d_i=d}}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\prod\frac{p_i^{2d_i}}{d_i!}=$$
$$=\sum_{d_1\dots d_n,\ \sum d_i=d}\ \prod_{i=1}^n\frac{(2d_i)!}{d_i!}\lambda_i^{-(2d_i+1)}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle$$

2° $$I_g(\lambda.)=\int_{M_{g,n}^{comb}}\exp\left(-\sum\lambda_i p_i\right)\cdot\rho\cdot\prod_{edges}d\,\ell(x)\ ,\quad\text{where}$$

$$\rho=\left(\prod_1^n d\ell_i\times\Omega^d/d!\right):\prod_{x-edges}d\ell(x)=2^{2n+5g-5}$$

$$I_g(\lambda.)=\rho\cdot\int_{M_{g,n}^{comb}}\exp\Big(-\sum_{x-edge\ i\|j}\ell(x)(\lambda_i+\lambda_j)\Big)\prod_{x-edge}d\ell(x)=$$
$$=\rho\cdot\sum_{\Gamma:\ 3\text{-valent RG}}\frac{1}{\#\mathrm{Aut}\,\Gamma}\cdot\prod_{edge\ i\|j}\frac{1}{\lambda_i+\lambda_j}.$$

Let $\Lambda=\mathrm{diag}(\Lambda_1,\dots,\Lambda_N)$, $\Lambda_\alpha>0$.

$$F(t.(\Lambda))=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\prod_{i=1}^n\left((2d_i-1)!!\sum_{\alpha_i=1}^N\Lambda_{\alpha_i}^{-(2d_i+1)}\right)=$$
$$=\sum_{g,n}(-1)^n\,2^{3-3g-n}\sum_{\alpha_1,\dots\alpha_n}I_g(\Lambda_{\alpha_1},\dots,\Lambda_{\alpha_n})=\sum_\Gamma\frac{1}{\#\mathrm{Aut}\,\Gamma}\cdot w(\Gamma).$$

Last summation is taken over all connected 3-valent RG with labeling $\{2\text{-cells}\}\to\{1,\dots,N\}$,

$$w(\Gamma)=\left(\frac{\sqrt{-1}}{2}\right)^{\#\text{vertices}}\prod_{\substack{edges\\ \alpha\|\beta}}\frac{2}{\Lambda_\alpha+\Lambda_\beta}.$$

By Feynman rules we obtain an asymptotic expansion for the integral from §2. The main thm is proven.

§6 Relation with Virasoro algebra.

The matrix Airy function $A(Y)=\int \exp(\sqrt{-1}(\mathrm{Tr}\frac{X^3}{3}-\mathrm{Tr}\,XY))dX$ obeyes equations $Y_{ji}A+\sum_k \frac{\partial^2 A}{\partial Y_{ik}\partial Y_{kj}}=0$. Airy function is conjugacy invariant, so it depends only on $\mathrm{Spec}\,Y=(Y_1 \ldots Y_N)$. Equations in new variables are

$$(*)\quad Y_i A+\frac{\partial^2 A}{\partial Y_i^2}+\sum_{j:\, j\neq i}\frac{\partial/\partial Y_j-\partial/\partial Y_i}{Y_j-Y_i}\cdot A=0.$$

If $Z=X-Y^{1/2}$ where $Y^{1/2}$ is some square root of Y then $\mathrm{Tr}(X^3/3-XY)=\mathrm{Tr}(Z^3/3+Z^2Y^{1/2})-2/3\,\mathrm{Tr}\,Y^{3/2}$. Applying the main thm one has an asymptotic expansion

$A(Y)\sim\sum_{Y^{1/2}} G\cdot R(Y^{1/2})$, $Y\to+\infty$, where $G=1+o(1)$ is a series in $T_1, T_3, T_5, \ldots, T_k=(\sqrt{-1})^k\,\mathrm{Tr}\,Y^{-k/2}/k$,

$R(Y^{1/2})=\exp\left(-\frac{2\sqrt{-1}}{3}\mathrm{Tr}\,Y^{3/2}\right)\cdot\int\exp(\sqrt{-1}\,\mathrm{Tr}\,Z^2Y^{1/2})dZ$ – an elementary function of Y.

Substitution in $(*)$ gives an equation $(a_i=\sqrt{-1}\,Y_i^{-1/2})$:

$$\left(\frac{1}{4}a_i^2T_1^2+\frac{1}{16}a_i^4\right)G+\sum_{n=1,3,5\ldots}\left(a_i^{n+1}+\sum_{l\leq n+2}\frac{lT_l}{2}a_i^{n+4-l}\right)\frac{\partial G}{\partial T_n}+\sum_{n,m}\frac{a_i^{n+m+4}}{4}\frac{\partial^2 G}{\partial T_n\partial T_m}=0$$

Coefficients in $a_i^2, a_i^4, a_i^6, \ldots$ give system of equations $(J_{2n+1}+L_{n-1})G=0$, $n=0,1,\ldots$, where $J_1=\frac{\partial}{\partial T_1}$, $J_3=\frac{\partial}{\partial T_3}, \ldots$, $J_{-1}=T_1$, $J_{-3}=3T_3, \ldots$ are bosons and

$L_n=\frac{1}{4}\sum_{k+l=2n}J_kJ_l$, $n\neq 0$; $L_0=\frac{1}{2}\sum_{k>0}J_{-k}J_k+\frac{1}{16}$ are generators of Virasoro algebra. This proves 2 $\rightarrow$ 6.

References:

[1]. M. Atiyah, talk on 29. Arbeitstagung. MPI /90-52
[2] E. Witten. Princeton preprint IASSNS-HEPP-90/45.
[3] E. Brezin, V. Kazakov. Phys. Lett. B236 (1990) 144
[4] D. Gross, A. Migdal, Phys. Rev. Lett. 64 (1990) 127
[5] D. Mumford, paper in "Arithmetic & Geometry", Birkhäuser 1983
[6]. K. Strebel, Quadratic differentials, Springer 1984
[7]. B. Zwiebach, Comm. Math. Phys. 136 (1991) 83.
[8] Dijkgraaf, E. & H. Verlinde. IAS preprint (1990).

Titel: Excision in algebraic K-theory and the proof of the Karoubi Conjecture
Autor: Mariusz Wodzicki
Seite: 1
Adresse: University of California, Berkeley

Let C be an arbitrary C^*-algebra and $\mathcal{K} = \mathcal{K}(H)$ denote the C^*-algebra of compact operators on the standard separable ∞-dimensional Hilbert space. There exists a unique up to equivalence C^*-algebra norm on $M(C) := \bigcup_n M_n(C)$ extending the norm on C. The induced completion $\overline{M(C)}$ is a C^*-algebra isomorphic to $C \tilde{\otimes} \mathcal{K}$ where $\tilde{\otimes}$ denotes the 'spatial' tensor product of C^*-algebras.

Theorem 1 ("Karoubi's Conjecture"). $K_*(C \tilde{\otimes} \mathcal{K}) \simeq K_*^{top}(C)$.

Every C^*-algebra C satisfies the condition $C = C^2$ which is suffient for the group $GL(C)$ to be quasi-perfect. In particular the space $BGL(C)^+$ exists.

Theorem 2. $BGL(C \tilde{\otimes} \mathcal{K})^+$ and $BGL^{top}(C)$ are homotopy equivalent.

The proof of the above theorems relies in an essential way on the excision property of the algebraic K-theory functor on the category of C^*-algebras and on some earlier work of G. Kasparov and N. Higson.

Let A be a 2-sided ideal in a unital ring R and $\overline{GL}(R/A) = \text{Image}\,(GL(R) \xrightarrow{p} GL(R/A))$. The group extension

$$1 \longrightarrow GL(A) \longrightarrow GL(R) \longrightarrow \overline{GL}(R/A) \longrightarrow 1$$

induces the map of (homotopy) fibrations:

$$\begin{tikzcd}
BGL(A) \arrow[r] \arrow[d, "\varphi^{R,A}"'] & BGL(R) \arrow[r, "Bp"] \arrow[d] & B\overline{GL}(R/A) \arrow[d] \\
F(R,A) \arrow[r] & BGL(R)^+ \arrow[r, "(Bp)^+"] & B\overline{GL}(R/A)^+
\end{tikzcd}$$

where $F(R,A)$ is the homotopy fibre of $(Bp)^+$.

<u>Def.</u> $K_i(R,A) := \pi_i F(R,A)$, $i \geq 1$;

$K_i(A) := K_i(\tilde{A};A)$, $i \geq 1$, where $\tilde{A} := \mathbb{Z} \ltimes A$.

The obvious morphism $(\tilde{A};A) \longrightarrow (R,A)$ induces the map $ex : F(\tilde{A},A) \longrightarrow F(R,A)$ (defined up to a homotopy) which will be called the <u>excision</u> map.

<u>Def.</u> A ring A <u>satisfies excision in algebraic K-theory</u> if the excision map ex defined above is a homotopy equivalence for any unital ring R containing A as a 2-sided ideal.

<u>Homological unitality</u>. Recall that a ring A is said to be H-<u>unital</u> (= homologically unital) if the Bar complex

$$B_*(A) = \{ A \xleftarrow{m} A \otimes_{\mathbb{Z}} A \xleftarrow{m \otimes 1_A - 1_A \otimes m} A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} A \longleftarrow \cdots$$

is <u>pure-acyclic</u>, i.e. $B_*(A) \otimes V$ remains acyclic for any abelian group V (m denotes the multiplication map), see [4].

<u>Theorem 3.</u> Let $A = A \otimes_{\mathbb{Z}} \mathbb{Q}$, then the following conditions are equivalent:

(a) A satisfies excision in algebraic K-theory;

(b) the map $\varphi^{R,A} : BGL(A) \longrightarrow F(R,A)$ is a homology equivalence for every $R \supset A$;

(c) the map $\varphi^{\tilde{A},A} : BGL(A) \longrightarrow F(\tilde{A},A)$ is a homology equivalence;

(d) $GL(R)$ acts trivially on $H_*(GL(A);\mathbb{Z})$ for every $R \supset A$;

(e) $GL(\mathbb{Z})$ acts trivially on $H_*(GL(A);\mathbb{Z})$;

(f) the natural inclusions

$$\begin{tikzcd} & GL(A) \arrow[dl] \arrow[dr] & \\ \widetilde{GL}(A) & & \widetilde{\widetilde{GL}}(A) \end{tikzcd}$$

where $\widetilde{GL}(A) := GL(A) \ltimes A^{\oplus\infty}$ and $\widetilde{\widetilde{GL}}(A) = A^{\oplus\infty} \rtimes GL(A)$, are homology equivalences;

(g) A satisfies excision in cyclic homology of $\mathbb{Z}$-algebras;

(h) A is H-unital.

More general results are proved in [3].

Universally flat rings. A ring A is said to be left universally flat if for every unital ring S containing A as a left ideal this ideal is flat as an S-module. Right universally flat rings are defined similarly.

Lemma. A ring A is H-unital if it is either left or, right, universally flat and its additive group $(A,+)$ is torsionless.

The following two theorems provide numerous examples of universally flat rings.

Theorem 4. All closed left ideals in an arbitrary C^*-algebra are left universally flat.

Theorem 5. 1) All 2-sided (not necessarily closed) ideals J in an arbitrary von Neumann algebra are flat both as left and as right ideals;

2)
$$\begin{array}{ccc} J \text{ is left universally flat} & \Longleftrightarrow & J \text{ is right universally flat} \\ \Updownarrow & & \Updownarrow \\ J \text{ is H-unital} & \Longleftrightarrow & J = J^2 \end{array}$$

(J is a 2-sided ideal in a von Neumann algebra).

Corollary. All closed (one-sided) ideals in any C^*-algebra satisfy excision in algebraic K-theory.

Higher index invariants

Let $\Psi_{X,E} = \bigcup_{m\in\mathbb{Z}} \Psi^m_{X,E}$ denote the ring of pseudodifferential operators acting in sections of a vector bundle E on a closed C^∞-manifold X and let $S_{X,E}$ be the corresponding quotient ring of (complete) symbols. The map $\sigma : \Psi_{X,E} \longrightarrow S_{X,E}$ associates to a pseudodifferential operator its complete symbol.

Theorem 6. There exist:

(a) a universal (i.e. not depending on X or E) graded abelian group $I_* = \bigoplus_{n=0}^{\infty} I_n$,

(b) functorial homomorphisms $j_* : I_* \longrightarrow K_*(\Psi_{X,E})$ and $\mathrm{index}_* : K_{*+1}(S_{X,E}) \longrightarrow I_*$

such that one has the following exact triangle:

$$\begin{tikzcd}
K_*(S_{X,E}) \arrow[rr, "\mathrm{index}", "\deg -1"'] & & I_* \arrow[dl, "j"] \\
& K_*(\Psi_{X,E}) \arrow[ul, "\mathrm{symbol}"] &
\end{tikzcd}$$

Remark. $I_* = K_*(\mathbb{L})$ where $\mathbb{L}$ is the ring of rapidly decaying complex-valued matrices (α_{ij}), $\sup_{ij} |\alpha_{ij}|(i+j)^N < \infty$ for all $N \in \mathbb{R}$; $I_0 = \mathbb{Z}$, $I_1 = \mathbb{C}^* \oplus (?)$, ...

The zeroth component $\mathrm{index}_0 : K_1(S_{X,E}) \longrightarrow \mathbb{Z}$ is the standard index of an (almost invertible) pseudodifferential operator.

Karoubi's construction of regulators gives numerical invariants $K_{2n-1}(S_{X,E}) \longrightarrow \mathbb{C}^*$, $n \in \mathbb{Z}_+$, which should be the subject of a generalization of the Atiyah-Singer Index Theorem to the higher algebraic K-theory of the ring of symbols.

The excision theorem 3 above is a joint work with Andrei Suslin.

REFERENCES

[1] A. SUSLIN, On the acyclicity of the sum of triangular complexes, Utrecht, April 1991 preprint

[2] ——— and M. WODZICKI, Excision in algebraic K-theory and Karoubi's Conjecture, Proc. Nat. Acad. Sci. U.S.A. 87 (1990), 9582-9584

[3] ——— and M. WODZICKI, Excision in algebraic K-theory, Ann. Math. (to appear)

[4] M. WODZICKI, Excision in cyclic homology and in rational algebraic K-theory, Ann. Math. 129 (1989), 591-639.

[5] ———, Homological properties of rings of functional-analytic type, Proc. Nat. Acad. Sci. U.S.A. 87 (1990), 4910-4911

[6] ———, On the algebraic K-theory of nonunital rings, K-theory (to appear)

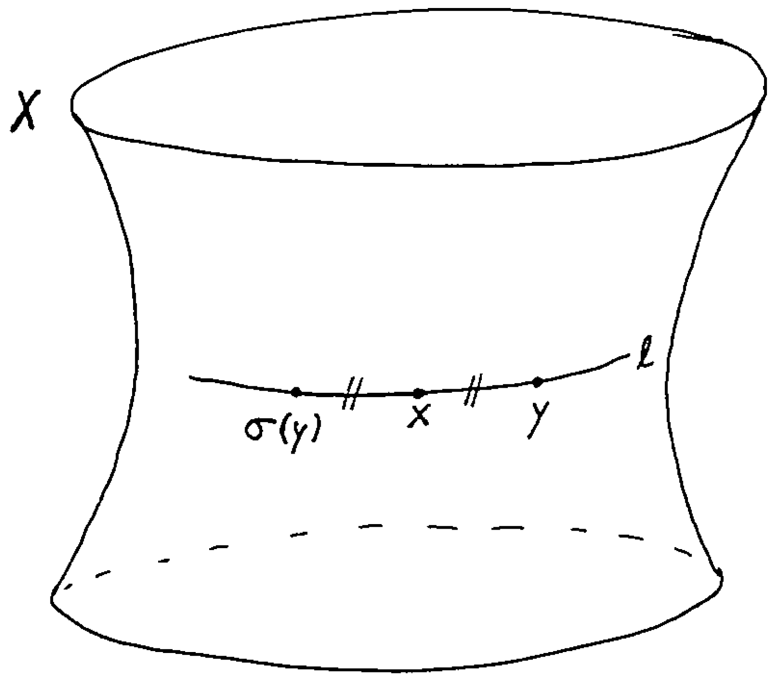
Titel: Lefschetz numbers of Hecke correspondences

Autor: Bob MacPherson

Adresse: Dept. of Mathematics, MIT, Cambridge MA, 02139 USA

All of the work described here is joint with Mark Goresky, and some of it is joint with Günter Harder and with Bob Kottwitz.

Recall the definition of a locally symmetric space. For every point x in a Riemannian manifold X there is a "reflection" map σ defined on points y close enough to x: if ℓ is the shortest geodesic from x to y, then $\sigma(y)$ is the point on ℓ the same distance from x in the opposite direction.



The Riemannian manifold X is a locally symmetric space if, for all $x \in X$, the reflection map is an isometry.

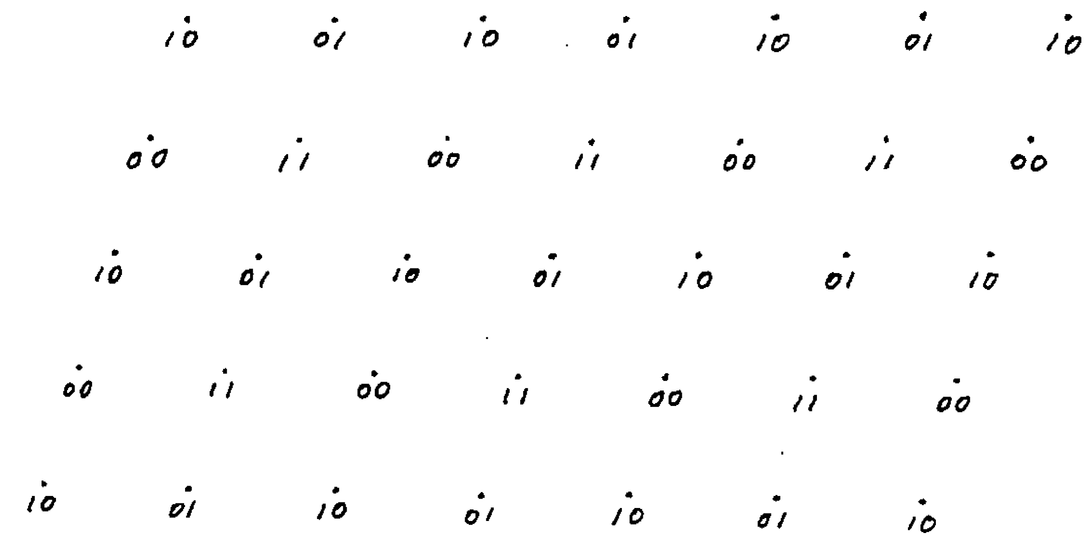
We will call the locally symmetric space X *arithmetic* if it is complete, has negative curvature, has finite volume, and has the following property: the group Γ of deck transformations $\tilde{X} \to \tilde{X}$ of the simply connected covering space $\tilde{X}$ of X is an arithmetic subgroup of the group $Aut(X)$ of Riemannian automorphisms of $\tilde{X}$. (By results of Margulis, this last requirement is automatic in most cases.)

A morphism of locally symmetric spaces is a local isometry. A *Hecke correspondence* on an arithmetic locally symmetric space X is another arithmetic locally symmetric space C equipped with two morphisms $s, t : C \rightrightarrows X$, called the source map and the target map. A Hecke correspondence acts on differential forms on X by the formula $C^*\omega = s_* t^*\omega$ (where the Gysin map s_* is defined since s will be a finite covering projection; it just adds the forms on each of the sheets). By this formula, C induces a self-map, also notated C^* and called a *Hecke operator*, on either the cohomology $H^*(X)$ of X or on the L_2 cohomology $H^*_{(2)}(X)$ of X. (Whenever we speak of L_2 cohomology, we assume that X is Hermitian, so that it is finite dimensional.)

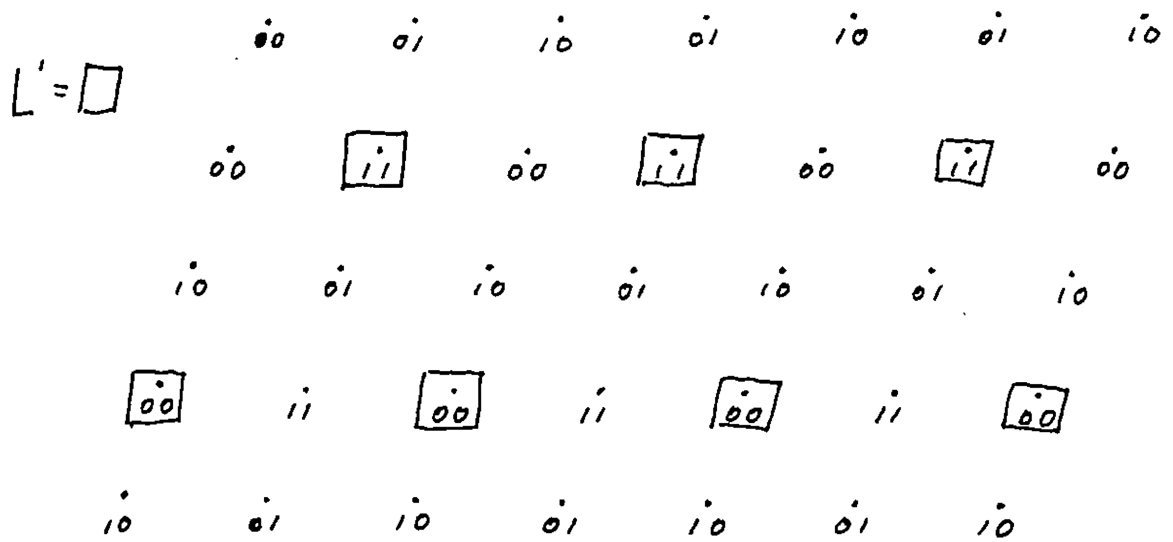
PROBLEM: Study the action of Hecke operators C^* on $H^*(X)$ or $H^*_{(2)}(X)$.

EXAMPLE

Let $X_l(n)$ be the space whose points are configurations consisting of a lattice L in $\mathbf{R}^n$ together with a surjection $L \to (\mathbf{Z}/l)^n$, called a marking. Two such configurations are considered equivalent if they differ by a rotation or a homothety (multiplication by a positive real number). For example, this is a configuration for $X_2(2)$:



If $l > 3$, $X_l(n)$ is nonsingular, and it is naturally an arithmetic locally symmetric space. For any prime p not dividing l, let T_p^i be the space whose points are configurations L and $L \to (\mathbf{Z}/l)^n$ as above, together with a sublattice $L' \subset L$ such that the abelian group L/L' is isomorphic to $(\mathbf{Z}/p)^i$ (again modulo rotations and homotheties). For example, this is a configuration for T_3^1:



Then T_p^i is a Hecke correspondence on $X_l(n)$ whose source map s forgets the sublattice L' completely, and whose target map t erases everything in L but L' and takes the restricted marking.

JUSTIFICATION

Why should we care about Hecke operators? The answer is that it is expected that interesting number theoretic information should be encoded in them. Consider the following example: The space $X_4(1)$ consists of two points α and β, represented by the following marked lattices:

0 1 2 3 0 1 2 3 0 1 2 3

0 3 2 1 0 3 2 1 0 3 2 1

Let J be the one dimensional subspace of $H^1(X_4(1))$ generated by 1 on α and -1 on β. For any odd prime p, the correspondence T_p^1 is actually a function from $X_4(1)$ to itself.

Exercise. Show that $(T_p^1)^*$ takes J into itself, and that it is the identity if and only if the prime p splits totally in the Gaussian integers $\mathbf{Z} \oplus \mathbf{Z}i$ (if and only if p is congruent to 1 mod 4).

Class field theory says that the splitting of a prime p in an extension of the rationals with an abelian Galois group is always governed by the action of $(T_p^1)^*$ on some subspace J of the cohomology of $X_l(1)$ for some l. Langlands philosophy suggests that for any extension of the rationals, the splitting of a prime p should be governed by the action of the set of operators $(T_p^i)^*$ on some subspace J of the cohomology, or L_2 cohomology, of $X_l(n)$ for some l and n.

LEFSCHETZ NUMBERS AND COMPACTIFICATIONS.

The actual calculation of Hecke operators C^* is probably hopeless; even the calculation of $H^*(X_l(3))$ on a supercomputer is beyond our abilities for reasonable l. However, as usual, alternating sums of cohomology groups are easier to deal with. Define the Lefschetz numbers $L(C) = \Sigma(-1)^i \, trace\ C^* : H^i(X) \to H^i(X)$ and $L_2(C) = \Sigma(-1)^i \, trace\ C^* : H^i_{(2)}(X) \to H^i_{(2)}(X)$. We modify our question to ask for the computation of these Lefschetz numbers. The fixed point set of a correspondence $s, t : C \rightrightarrows X$ is $\{c \in C | s(c) = t(c)\}$. Let K denote a connected component of the fixed point set. Then a Lefschetz Fixed Point Theorem says that there exist locally defined numbers $L(K)$ so that $L(C) = \sum_K L(K)$, and likewise for L_2 cohomology. However our spaces X are not compact, and no Lefschetz fixed point theorem is valid for noncompact spaces (as is shown by the map $\mathbf{R} \to \mathbf{R}$ sending x to $x + 1$, which has an empty fixed point set but a nonzero Lefschetz number).

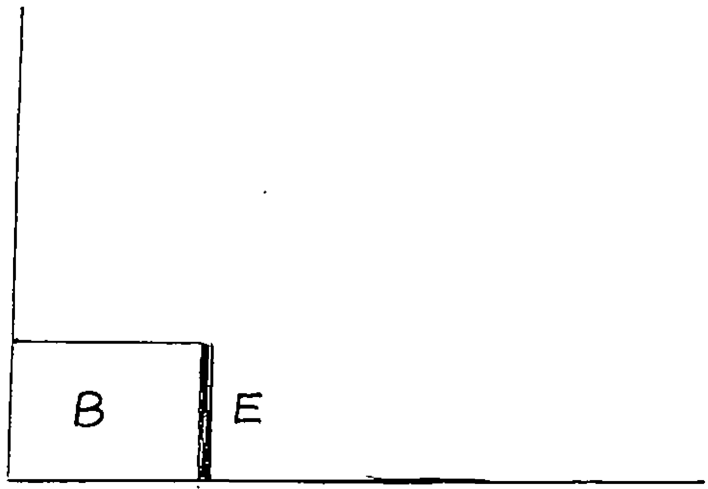
Compactifying X is a much studied subject, and several compactifications with different desirable properties have been defined. However, I know of no compactification that is both nonsingular and admits extensions of the Hecke correspondances. Therefore, we are forced to develop Lefschetz fixed point theory for singular varieties.

LEFSCHETZ FIXED POINT THEORY

There are Lefschetz fixed point theorems for singular spaces due to Dold (for ordinary cohomology) and Grothendieck-Illusie (for sheaf cohomology). However, what is wanted is a computable formula for the contributions $L(K)$.

Let s,t:$\bar{C} \rightrightarrows \bar{X}$ be a correspondence. A fixed point component K is called *weakly hyperbolic* if there is a map $(p_1, p_2) : U \to \mathbf{R}^{\geq 0} \times \mathbf{R}^{\geq 0}$, where U is a neighborhood of $s(K) = t(K)$, with the following two properties: 1. $(p_1, p_2)^{-1}(0,0) = K$ and 2. Near K, $p_1 s(c) \leq p_1 t(c)$ and $p_2 s(c) \geq p_2 t(c)$.

The intuition behind this definition is that the expanding directions must be mapped to the x-axis and the contracting directions must be mapped to the y-axis in $\mathbf{R}^{\geq 0} \times \mathbf{R}^{\geq 0}$. Take a small box B and and its edge E as in the following picture:



THEOREM [GM1] (Lefschetz fixed point formula)

$$L(K) = \sum(-1)^i \; trace \; \bar{C}^* : H^i((p_1,p_2)^{-1}B, (p_1,p_2)^{-1}E) \to (H^i((p_1,p_2)^{-1}B, (p_1,p_2)^{-1}E)$$

This same formula works for sheaf cohomology as well. Another (overlapping) fixed point formula has been proved by Kashiwara and Schipira.

THE REDUCTIVE BOREL-SERRE COMPACTIFICATION

Now, we return to our situation of an arithmetic locally symmetric space X. We work with the Reductive Borel-Serre compactification $\bar{X}$. This is distinguished by being the most natural compactification metrically, in the following sense: Suppose that two curves $\tau_i : [0,1) \to X$ converge to limit points $\tau_i(1)$ in $\bar{X}$. Then $\tau_1(1) = \tau_2(1)$ if and only if $\lim_{\delta \to 1} \; dist \; (\tau_1(\delta, 1), \tau_2(\delta, 1)) = 0$.

With this compactification, we are in a position to apply the Lefschetz fixed point formula for the following reasons:

THEOREM

1. The Hecke correspondance $s, t : C \rightrightarrows X$ extends canonically to a compactified Hecke correspondance $s, t : \bar{C} \rightrightarrows \bar{X}$.

2. The compactified Hecke correspondance $s, t : \bar{C} \rightrightarrows \bar{X}$ is weakly hyperbolic at each fixed point component.

3. There exist (derived) sheaves on $\bar{X}$ whose cohomology is $H^*(X)$ resp. $H^*_{(2)}(X)$.

Of these, the one that most deserves comment is 3. The fact that there is a sheaf on $\bar{X}$ whose cohomology is $H^{*}(X)$ is clear from Grothendieck's formalism of sheaf theory. (This would be true for any compactification.) The fact that there is a sheaf whose cohomology is $H^{*}_{(2)}(X)$ is deep. By the Zucker conjecture, proved by Looijenga and Saper-Stern, the cohomology of middle intersection homology sheaf $IC^{\bullet}$ on $\hat{X}$ computes $H^{*}_{(2)}(X)$, where $\hat{X}$ is the Baily-Borel compactification of X. There is a canonical projection $\pi : \bar{X} \to \hat{X}$.

THEOREM [GHM]
There is a "weighted cohomology sheaf" $WC^{\bullet}$ on $\bar{X}$ so that $R\pi_{*}WC^{\bullet} = IC^{\bullet}$.

With these results in hand, it remains to carry out the explicit computation of the Lefschetz fixed point formula. This calculation is made possible by the fact that the singularities of $\bar{X}$ are described by nilmanifolds, whose cohomology can be calculated by the Nomizu-Van Est theorem and Kostant's theorem. I will omit the calculation, which is in [GM2]. Analytic computations of the Lefschetz numbers of Hecke operators on L_2 cohomology have been carried out by Arthur, as part of his trace formula, and by Stern. The agreement of our formula with Arthur's formula is verified in [GKM]. It is interesting to note that Arthur's terms correspond to sums of terms $L_2(K)$. An analytic computation of the trace of Hecke operators on ordinary cohomology has been carried out by Franke. Finally, I would like to note that there is an intriguing similarity between our expression for the Lefschetz numbers of Hecke operators and Pink's formula for the Lefschetz numbers of Frobenius operators on the characteristic p reduction of the same space.

BIBLIOGRAPHY
[GM1] M. Goresky and R. MacPherson, *Local contribution to the Lefschetz Fixed Point Formula*, to appear in Inventiones.

[GM2] M. Goresky and R. MacPherson, *Lefschetz Numbers of Hecke Correspondences*, to appear in Montréal volume on $SU(2,1)$.

[GHM] M. Goresky, G. Harder, and R. MacPherson, *Weighted Cohomology*, to appear.

[GKM] M. Goresky, R. Kottwitz, and R. MacPherson, to appear.

Titel: Degeneration of Riemann surfaces and Jorgenson's proof of a conjecture of Deligne

Autor: Serge Lang

Adresse: Math Dept, Box 2155 Yale Station, New Haven CT 06520 USA
z Zt Max Planck Institut (Juni)

We start a purely algebraic version of a Deligne Riemann-Roch theorem. Since the Arbeitstagung started with Riemann-Roch theorems in 1957, it is appropriate that it should end with Riemann-Roch theorems.

Let X be a compact Riemann surface. Then we have its canonical sheaf $\mathcal{K}$. Let $\mathcal{L}$ be a line sheaf on X. Then we have a line (one dimensional vector space over the complex) defined by

$$\lambda(\mathcal{L}) = \det H^0(\mathcal{L}) \otimes \det H^1(\mathcal{L})^{-1}.$$

One defines a pairing between line sheaves by the formula

$$\langle \mathcal{L}, \mathcal{M} \rangle = \lambda(\mathcal{L}\otimes\mathcal{M}) \otimes \lambda(\mathcal{L})^{-1} \otimes \lambda(\mathcal{M})^{-1} \otimes \lambda(\mathcal{O}_X).$$

There ensues a canonical Deligne isomorphism

$$\lambda(\mathcal{L})^{12} \xrightarrow{\approx} \langle \mathcal{K}, \mathcal{K}\rangle \otimes \langle \mathcal{L}, \mathcal{L}\otimes\mathcal{K}^{-1}\rangle^{6}.$$

Following a philosophy started by Arakelov, there is an ongoing open ended program in algebraic geometry to put "natural metrics on all sheaves, so that the natural algebraic isomorphisms become isometries, possibly up to a constant factor. Essentially all of sheafy algebraic geometry is to be extended in this way. We must therefore now deal with metrics. Suppose given:

a positive (1,1)-form μ on X, which amounts to a metric on $\mathcal{K}$;
a metric ρ on $\mathcal{L}$.

These give rise to an L^2-hermitian product on $H^0(\mathcal{L})$, namely we have the hermitian product of two sections s, s' defined by

$$\langle s, s' \rangle_{\rho,\mu} = \int_X \langle s, s' \rangle_\rho \, \mu.$$

By Serre duality, for sections of $H^1(\mathcal{L})$ one also gets a hermitian product, because

$$H^1(\mathcal{L}) \approx H^0(\mathcal{L}^{-1}\otimes\mathcal{K})^{\vee}.$$

Thus we get what we call the L^2-metrics H_{L^2}, depending on ρ, μ.

We want more. Let $\Delta = \Delta_{\rho,\mu} : C^\infty(\mathcal{L}) \longrightarrow C^\infty(\mathcal{L})$ be the Laplacian, with the sign chosen that it is a positive operator. We let $0 < \lambda_1 \leqq \lambda_2 \leqq \cdots$ be the sequence of non-zero eigenvalues, and we define the (spectral) zeta function by the series

$$\zeta_\Delta(s) = \sum_{k=1}^{\infty} \lambda_k^{-1},$$

which converges for $\mathrm{Re}(s)$ large. A theorem of Seeley guarantees that ζ_Δ has a meromorphic continuation to $\mathbf{C}$, regular at 0. We can then define the determinant

$$\det{}^* \Delta = \exp(-\zeta'_\Delta(0)) \text{ "="} \prod_{k=1}^{\infty} \lambda_k.$$

The star in the superscript of det indicates that we are dealing with the non-zero eigenvalues. Associated with this determinant, we define an important constant

$$\boxed{C_{\Delta,\mu}(X) = \frac{\det^* \Delta_{\mathcal{O},\mu}}{\mathrm{Vol}_{\mu}(X)} \quad \text{and} \quad c_{\Delta,\mu} = \log C_{\Delta,\mu}.}$$

We shall eventually view this constant, with the hyperbolic metric on X, as a function on the moduli space.

We also define the Quillen metric

$$H_{Q,\rho,\mu} = H_{L^2}\cdot(\det^* \Delta_{\mathcal{O},\mu})^{-1}.$$

Deligne proved that the Riemann-Roch isomorphism stated above is an isometry up to a factor $\exp(a(g))$, where $a(g)$ is a constant which depends only on g.

Deligne conjecture. One has $a(g) = (1-g)a(0)$.

Jorgenson proved this conjecture by a fascinating method. The Deligne constant $a(g)$ has been expressed as a difference of log determinants, in what has been called the "spin 1/2 bosonization formula" in [A-B-M-N-V 87], published in a physics journal which makes everybody think all this has to do with physics. No matter what, suppose $g \geq 2$. Let

$\psi: X \longrightarrow J$ be a canonical map into the Jacobian, with $\psi(P) = 0$.

$W_{g-1} = \psi(X)+\ldots+\psi(X)$ (sum taken $g-1$ times).

θ = Rieman theta function, and Θ its divisor.

$\underset{\sim}{H}_X$ = the hermitian Riemann form associated with the polarization.

There is a unique divisor class D of degree $g-1$ such that

$$\Theta = W_{g-1} + \psi(D).$$

Let $\mathcal{J} = \mathcal{O}_X(D)$. Then $\mathcal{J}^2 \approx \mathcal{K}$, i.e. $\mathcal{J}$ is a square root of the canonical class. The expression of [A-B-M-N-V 87] is:

$$\boxed{\frac{1}{4}\,a(g) = \log\left(\det \Delta_{\mathcal{J},\rho^{1/2}} / \|\theta\|^2(0)\right) + \frac{1}{2}\,c_{\Delta,\rho}}$$

where

$$\|\theta\|^2(0) = (\det \underset{\sim}{H}_X)^{1/2}\,|\theta(0)|^2 .$$

This must be taken with a grain of salt. It may happen that $\theta(0) = 0$, but then the determinant will also be 0. One introduces a more complicated theta function, depending on a parameter $u \in J$, and one also introduces $\mathcal{L}_u$ $(u \in J)$ where $\mathcal{L}_u$ is a line sheaf of degree 0 with a flat metric. Then the quotient

$$\det \Delta_{\mathcal{J}\otimes\mathcal{L}_u,\rho^{1/2}} / \|\theta\|^2(0,u)$$

is well defined, positive, and independent of u because the determinant and the theta value vanish with the same order if they vanish at all.

For $g \geqq 2$, the idea is then to view the log determinant and other invariants of X as functions on the moduli space M_g, and to let X degenerate to a $\mathbb{P}^1$ with g nodes. Since $a(g)$ is constant, one finds its value from the limiting value of the log determinant term on this degenerate surface. This means that one has to keep track of the asymptotic behavior of several functions on the moduli space, of which $-\zeta'_\Delta(0)$ (hyperbolic metric), the theta value, the constant $c_{\Delta,\mu}$ are only the first examples.

Some of these functions tend to $\pm\infty$, but their differences may be continuous on the boundary of the moduli space. During the past few years, several people have systematically studied various such degeneracies, including Wolpert, Hejhal, Belavin-Kniznik, Taktajian, Zograf, Lundelius and Jorgenson, and others. Jorgenson proves appropriate limit formulas which allow him to determine Deligne's constant as conjectured.

It is a fairly vast enterprise to make a systematic tabulation of the behavior of all objects involved, namely in addition to the ones we have seen: small eigenvalues, small geodesics, and whatever. I shall select only some examples of theorems of Jorgenson giving the flavor of the observable phenomena.

I should also note that for the theory to be completely coherent, one must start from the beginning with non-compact Riemann surfaces having finite volume. All the objects such as Laplacians, zeta functions, etc. can be defined for such surfaces. In the limiting values with nodes, by deleting the nodes one obtains such surfaces. I started with compact surfaces only for simplicity, and to avoid taking certain precautions for the non-compact case.

<u>A limiting theorem</u>. I shall now describe one of Jorgenson's limit formulas.

By a <u>small eigenvalue</u> we mean an eigenvalue $< 1/4$.

By a <u>small geodesic</u>, we mean a geodesic of length $< \ell_0$, where ℓ_0 is the length of the smallest geodesic on $\mathbb{P}^1$ minus three points with the hyperbolic metric. These notions apply as well when X is not compact but has finite volume. It is known that the number of small eigenvalues is $\leqq 4g-3$ (Buser), and the number of small geodesics is $\leqq 3g-3$ (Bost). Define the products

$$\boxed{\prod_{sev}(X) = \prod_{small} \lambda_k \qquad \text{and} \qquad \prod_{sge}(X) = \prod_{small} \ell_j}$$

where ℓ_j ranges over the lengths of the small geodesics. We define a further constant

$$\boxed{\begin{aligned} C_\mu(X) &= \frac{\det \mathbb{H}_X}{\mathrm{Vol}_\mu(X)} \quad \text{if} \quad g \geqq 1 \\ &= 1/\mathrm{Vol}_\mu(X) \quad \text{if} \quad g = 0. \end{aligned}}$$

For degenerate surfaces, with several components and nodes, a similar definition can be given, multiplicative over the components. We omit it.

Finally, we define

$$\boxed{E(X) = c_{hyp}(X)\,\frac{\prod_{sev}(X)}{\prod_{sge}(X)}\ .}$$

One of Jorgenson's theorems is that:

<u>Theorem 1.</u> <u>For a degenerating family of Riemann surfaces</u> X, <u>degenerating to</u> X_o, <u>we have</u>

$$\lim_{X \to X_o} E(X) = E(X_o).$$

If $\{X_j\}$ are the irreducible components of X_o, then $E(X_o) := \prod E(X_j)$.

<u>The Selberg zeta function.</u> Essential to the study of the degeneration of $\zeta'_{hyp}(0)$ is the Selberg zeta function

$$Z(s) = \prod_{k=1}^{\infty} \prod_{\gamma} (1 - e^{-(s+k)\ell(\gamma)}),$$

where γ ranges over the primitive geodesics. One has a formula of D'Hoker and Phong [D'H-P 86]:

$$\boxed{\log \det{}^* \Delta_{hyp} = -\zeta'_{hyp}(0) = \log Z'(1) + c(g),}$$

where $c(g)$ is a constant which was determined to be

$$c(g) = (1-g)c(0)$$

and $c(0) = c_{\Delta,st}(\mathbf{P}^1) + \log 2 = -4\zeta'_{\mathbf{Q}}(-1) + \frac{1}{2} - \log 2\pi.$

This last explicit value is by a computation of Vardi. Here st denotes the standard metric on the projective line, namely

$$\mu_{st} = 4\pi\,\mu_{can} \qquad \text{and} \qquad \mu_{can} = \frac{\sqrt{-1}}{2\pi}\,\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}\ .$$

The constant $c(g)$ enters in the determination of $a(g)$, namely one has $a(g) + c(g) = d(g)$, which is still another constant but we won't go into that. One can define a twisted Selberg zeta function Z_1 by a character of order 2, and there is a similar formula of Sarnak [Sar 87]:

$$\boxed{\log \det \Delta_{\mathcal{S},hyp} = (1-g)(-4\zeta'_{\mathbf{Q}}(-1)) + \log Z_1(1/2),}$$

with a generalization to $\mathcal{S} \otimes \mathcal{L}_u$ in line with what we already remarked. Jorgenson studies the degenerations of these log determinants, eventually to get the constant values of their differences. We now turn to asymptotics.

Weil functions and potential functions. On any variety V let D be a Cartier divisor. By a (complex) Weil function associated to D we mean a function $g: V - \mathrm{supp}(D) \longrightarrow \mathbb{R}$, such that, if D is represented by a rational function $D = (\varphi)$ on a Zariski open set U, then there exists a continuous function α on U such that

$$g(P) = -\log|\varphi(P)|^2 + \alpha(P) \quad \text{for } P \notin \mathrm{supp}(D).$$

We use the letter g because natural choices of Weil functions lead to Green's functions (potential functions). If V is compact, then the difference of two Weil functions associated with the same divisor is continuous, and therefore bounded on V.

Theorem 2. The function $g_\Delta = 12 \log(C_{\Delta,\mathrm{hyp}} C_{\mathrm{hyp}})$ is a Weil function on the moduli space M_g, with respect to the boundary divisor. Furthermore, it is also a potential (Green) function for the Weil-Peterson metric, that is

$$dd^c g_\Delta = \mu_{WP} \quad \text{on } M_g,$$

possibly up to the factor $1/\pi^2$, depending how μ_{WP} is normalized.

I don't know to whom the first statement is due (about the Weil function). I learned it from Jorgenson. The second statement is due to Belavin-Kniznik [B-K 86] and Takhtajan-Zograf [T-Z 88], [TZ 91], who also prove the analogous formula for the non-compact case, involving Eisenstein series.

The asymptotics of another function $g_{\mathrm{geo}} = \sum (2\pi)^2/\ell_j$ are also very interesting, and have been studied, but I am running out of space.

Bibliography

[A-B-M-N-V 87] L. ALVAREZ-GAUMÉ, J.B. BOST, G. MOORE, P. NELSON, C. VAFA, Bosonization on higher genus Riemann surfaces, Comm. Math. Physics 112 (1987) pp. 503-552

[B-K 86] A. G. BELAVIN and V.G. KNIZNIK, Complex geometry and the theory of quantum strings, Sov. Phys. JETP 64 (1986) pp. 214-228

[D'H-P 86] E. D'HOKER and D.H. PHONG, On determinants of Laplacians on Riemann surfaces, Comm. Math. Physics 104 (1986) pp. 537-545

[Jo 91] J. JORGENSON, An evaluation of the constants in Deligne's Riemann-Roch theorem and small eigenvalues on compact hyperbolic Riemann surfaces, to appear

[J-L 91] J. JORGENSON and R. LUNDELIUS, Factorization theorems for determinants on finite volume Riemann surfaces, to appear

[Sa 87] P. SARNAK, Determinants of Laplacians, Comm. Math. Physics 110 (1987) pp. 113-120

[T-Z 91] L. TAKHTAJAN and P. ZOGRAF, A local index theorem for families of $\bar\partial$-operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces, Comm. Math. Phys.(1991)pp.399-426

[T-Z 88] L. TAKHTAJAN and P. ZOGRAF, The Selberg zeta function and a new Kähler metric on the moduli space of punctured Riemann surfaces, J. Geometry and Phasics Vol. 5 No. 4 (1988) pp. 553-570

[Wo 87] S. WOLPERT, Asymptotics of the spectrum and the Selberg zeta function for the spaces of Riemann surfaces, Commun. Math. Physics 112 (1987) pp. 283-315

Autor: H. Tsuji
Adresse: Max-Planck-Institut für Mathematik

This is a survey talk about Mori Theory on the classification of algebraic varieties.
The classification theory of algebraic varieties is an attempt to decompose algebraic varieties into three kinds of particles, i.e.

1. varieties with $-K_X > 0$.
2. varieties with $K_X \approx 0$, ($\approx$ num. equivalence)
3. varieties with $K_X > 0$.

This can be viewed as a higher dim. Riemann's uniformization.

I Cone theorem and Minimal Model Conjecture

To single out the particles of the 1st kind. Mori invented the following ~~theorem~~.

Theorem (cone theorem : Mori, Kawamata) ([M1. K])
X^n: proj var $/\mathbb{C}$ with only canonical sing.

$\overline{NE}(X)_{\mathbb{R}} = \{$ effective 1 cycle $\}/\approx$
with $\mathbb{R}$ coefficients

$\Rightarrow$ If we set

$$\overline{NE}^{+}(X)_{\mathbb{R}} = \{ [C] \in \overline{NE}(X)_{\mathbb{R}} \mid K_X \cdot C < 0 \},$$

then there exist rational curves $l_1, \ldots, l_i, \ldots$ with $0 > K_X \cdot l_i \geq -(n+1)$ s.t

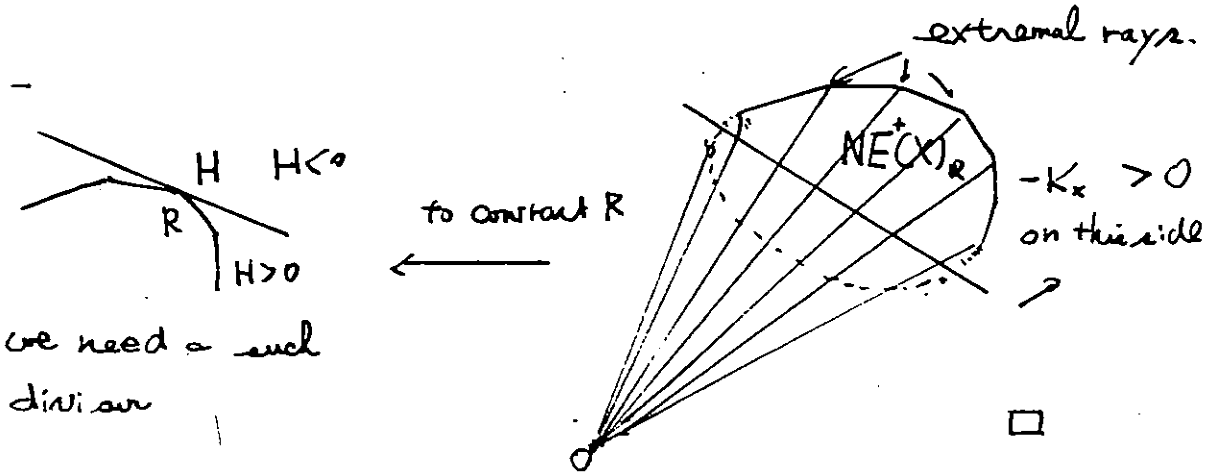
$$\overline{NE}^{+}(X)_{\mathbb{R}} = \sum_i \mathbb{R}_{\geq 0}[l_i] \quad \text{(cone of loc fin. edges)}$$

And for each edge $R = \mathbb{R}_{\geq 0}[l_i]$ (this is called an "extremal ray"), there is a surjective morphism of proj. varieties

$$\mathrm{cont}_R : X \longrightarrow \exists Y$$

s.t.

$$\mathrm{Ker}\{\mathrm{cont}_{R*} : H_2(X,\mathbb{R}) \longrightarrow H_2(Y,\mathbb{R})\} = \mathbb{R}[l_i].$$



□

In the above theorem canonical singularity means:

<u>Def</u>. X : normal alg. var. has only canonical singularity iff
(1) K_X (canonical Weil divisor) is $\mathbb{Q}$-Cartier.
(2) $\exists \mu : Y \longrightarrow X$ resolution s.t. excep. set is a divisor. $E = \sum E_i$, then

$$K_Y = \mu^* K_X + \sum a_i E_i \qquad (a_i \geq 0).$$ □

<u>Def</u>. X : normal proj var.

X : minimal $\Leftrightarrow$ (1) X has only can. sing
(2) K_X is num. semipositive.

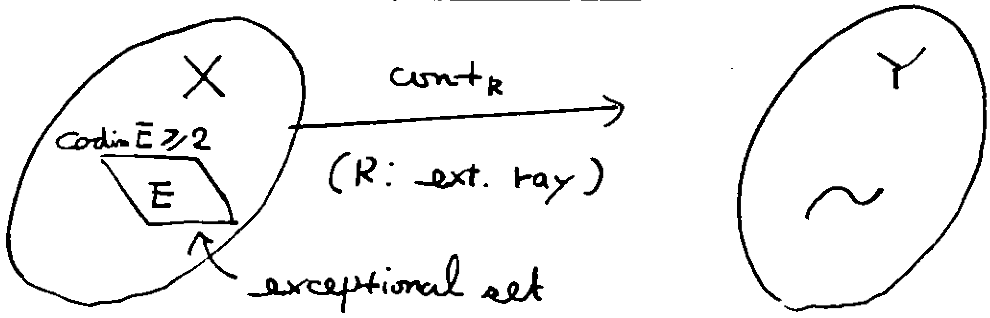
Minimal Model Conjecture

X : normal proj. var. If X is not uniruled (not covered by a family of rational curves), then there exists a minimal variety X_{min} birationally equivalent to X □

This conjecture was solved by Mori in the case of dim X = 3 (dim X ≤ 2 it is classically known) ([M2]).

Difficulty Let X be a non uniruled smooth proj. var. To construct X_{min} (minimal model) by "cone theorem" we encounter the following difficulty

bad contraction



If codim E ≥ 2 then K_Y is not $\mathbb{Q}$-Cartier.
We cannot continue the contraction!
To continue further, we need an additional operation so called "flip".

II Geometric Construction of Canonical Models.

X^n: smooth proj var. / $\mathbb{C}$ of general type

- construction of min. model of X as a Kähler-Einstein space.

ω_0: C^∞- Kähler form on X. We consider:

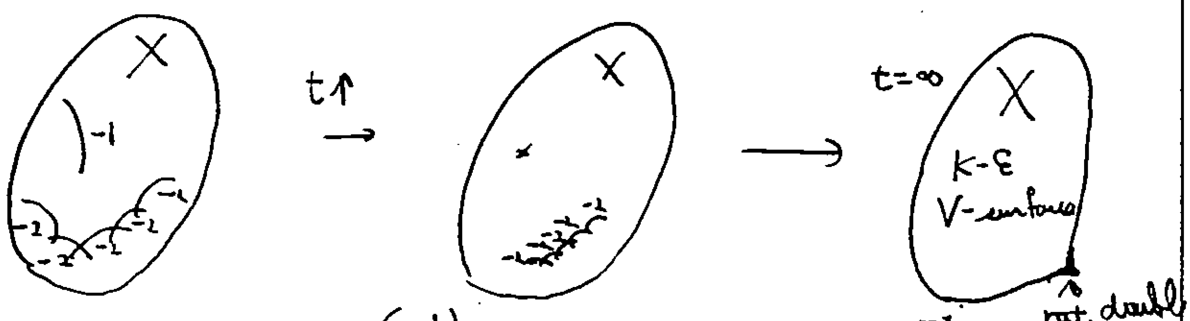
$$\begin{cases} \dfrac{\partial \omega}{\partial t} = -\mathrm{Ric}_\omega - \omega & \text{on } X \times [0, \infty). \\ \omega = \omega_0 & \text{on } X \times \{0\}. \end{cases}$$

where $\mathrm{Ric}_\omega = -\sqrt{-1}\,\partial\bar\partial \log \omega^n$.

Then we have

1. The solution ω exists as a d-closed positive (1.1)-current on $X \times [0, \infty)$.
2. $\omega_E := \lim\limits_{t\to\infty} \omega$ exists as a d-closed positive (1.1)-current and it is a C^∞ Kähler-Einstein form on a nonempty Zariski open subset of X.

<u>example</u> Case <u>dim $X = 2$</u>



⌒: rational curves

(-1) rat curves shrink in finite time

(-2) rat curves shrink in t=∞ to rational double pts.

This example indicates us that ω_E comes from the <u>canonical model</u> of X.

In fact we can prove:

<u>Theorem</u>. X as above: Then
$R(X, K_X) = \bigoplus_{\nu \geq 0} H^0(X, \mathcal{O}_X(\nu K_X))$ is finitely generated.
Hence the canonical (minimal) model
$X_{can} = \mathrm{Proj}\, R(X, K_X)$ exists. $\square$

In particular ω_E comes from the K-E metric on X_{can}.

The relation between Minimal Model Conjecture and this result will be discussed in the talk.

<u>References</u>

[M-1] 3-folds whose canonical bdls are not numerically effective Ann Math. (1982)

[M-2] Flip Conjecture and the existence of minimal models for 3-folds, J. AMS 1 (1988)

[K] The cone of curves of algebraic varieties, Ann. of Math. 120 (1984)

Titel: NONLINEAR STABILITY OF MINKOWSKI SPACE
Autor: S. KLAINERMAN
Adresse: DEPT. OF MATH., PRINCETON UNIV.
N.J. 08544 U.S.A.

AT THE HEART OF THE MODERN THEORY OF SPACE, TIME AND GRAVITY LIES THE GREAT UNIFICATION MADE BY EINSTEIN ACCORDING TO WHICH, IN THE ABSCENCE OF MATTER, the PHYSICAL "SPACE-TIME" CONSISTS OF A PAIR $(\mathcal{M}, g)$ WHERE $\mathcal{M}$ IS A 3+1 DIM. MANIFOLD AND g A LORENTZ METRIC WITH VANISHING RICCI CURVATURE,

$$(1) \qquad R_{\alpha\beta} = 0 \qquad \alpha, \beta = 0,1,2,3$$

RECALL THAT IF $R_{\alpha\beta\gamma\delta}$ DENOTES THE RIEMANN CURVATURE TENSOR OF g, $R_{\alpha\beta} = g^{\mu\nu} R_{\alpha\mu\beta\nu}$.

THE MINKOWSKI SPACE-TIME $\mathbb{M}^{1+3}$, WHICH PROVIDES THE GEOMETRIC FRAMEWORK OF SPECIAL RELATIVITY, IS A TRIVIAL SOLUTION OF (1). IN FACT $\mathbb{M}^{1+3}$ IS FLAT I.E. $R_{\alpha\beta\gamma\delta} \equiv 0$. A SPACE-TIME WHICH LOOKS MINKOWSKIAN OUTSIDE SOME COMPACT REGION WILL BE CALLED, IN WHAT FOLLOWS, GLOBALLY ASYMPTOTICALLY FLAT OR G.A.F. G.A.F SPACE-TIMES CORRESPOND TO ISOLATED PHYSICAL SYSTEMS AND PLAY A FUNDAMENTAL ROLE IN GENERAL RELATIVITY. THE QUESTION OF EXISTENCE AND DESCRIPTION OF SUCH SPACE-TIMES IS INTIMATELY CONNECTED WITH THAT OF THE

Titel:

Autor:

Adresse:

STABILITY OF MINKOWSKI SPACE-TIME.
THIS QUESTION CAN BE FRASED AS FOLLOWS:

ASSUME THAT $(\mathcal{M}, g)$ IS A SOLUTION TO (1), i.e. RICCI FLAT, AND THAT $\mathcal{H}$ IS A SPACE-LIKE HYPERSURFACE EMBEDDED IN $(\mathcal{M}, g)$ with $\underline{g}, \underline{k}$ THE FIRST AND SECOND FUNDAMENTAL FORM. THE FACT THAT $\mathcal{H}$ IS SPACE-LIKE MEANS PRECISELY THAT THE INDUCED METRIC $\underline{g}$ IS RIEMANNIAN. IN VIEW OF THE EQUATION (1) $\underline{g}, \underline{k}$ SATISFY THE GAUSS-CODAZZI EQUATIONS ON $\mathcal{H}$,

$$(2)\quad \begin{cases} \bar{\nabla}^i k_{ij} - \bar{\nabla}_j \operatorname{tr}_g k = 0 \\ \underline{R} - |k|^2 + (\operatorname{tr}_g k)^2 = 0 \end{cases}$$

WHERE $\bar{\nabla}$ DENOTES THE INDUCED COVARIANT DERIVATIVE ON $\mathcal{H}$ AND $\underline{R}$ THE SCALAR CURVATURE OF $\underline{g}$.

THE TRIPLET $(\mathcal{H}, g, k)$, VERIFYING THE "CONSTRAINT EQTS" (2), CAN BE INTERPRETED AS AN INITIAL DATA SET. THE SPACE-TIME $(\mathcal{M}, g)$

Titel:

Autor: Seite: 3

Adresse:

CAN THEN BE VIEWED AS ITS
CAUCHY DEVELOPMENT. IF
$(\mathcal{M}, g)$ IS THE MINKOWSKI
SPACE-TIME AND $\mathcal{H}$ A
MAXIMAL HYPERSURFACE, I.E
$tr_{\underline{g}}\, \underline{k} = 0$, THEN $(\mathcal{H}, \underline{g}, \underline{k})$
IS FLAT I.E. $\underline{g}$ IS THE
EUCLIDEAN METRIC e AND $\underline{k} \equiv 0$.

THE QUESTION OF GLOBAL NONLINEAR
STABILITY OF THE MINKOWSKI
SPACE-TIME IS THAT OF STUDY
THE RELATION BETWEEN SMALL
PERTURBATION OF THE FLAT
INITIAL DATA SET $(\mathcal{H}, e, 0)$
AND THE BEHAVIOUR OF THEIR
CAUCHY DEVELOPMENTS RELATIVE
TO THE MINKOWSKI SPACE. IN
OTHER WORDS IF $(\mathcal{H}, \underline{g}, \underline{k})$ IS
A SMALL PERTURBATION OF
$(\mathcal{H}, e, 0)$ AND $(\mathcal{M}, g)$ ITS
DEVELOPMENT HOW CLOSE IS
$(\mathcal{M}, g)$ to $\mathbb{M}^{3+1}$. IN PARTICULAR
IS IT COMPLETE, IS IT
GLOBALLY ASYMPT. FLAT ?

Titel:

Autor: Seite: 4

Adresse:

WHEN ONE CONSIDERS THE QUESTION
OF THE STABILITY OF THE MINKOWSKI
SPACE-TIME IT IS NATURAL TO RESTRICT
OURSELVES TO INITIAL DATA SETS WHICH
LOOK FLAT OUTSIDE A SUFFICIENTLY
LARGE SET IN $\mathcal{H}$. SUCH DATA
SETS ARE CALLED ASYMPTOTICALLY
FLAT, OR A.F. THE ONLY
KNOWN, EXPLICIT DEVELOPMENTS OF
A.F. INITIAL DATA SETS, ~~BESID~~ INCLUDING
THE MINKOWSKI SPACE-TIME AS WELL
AS THE SCHWARSCHILD SPACE-TIME,
IS GIVEN BY A 2-PARAMETER
FAMILY OF SOLUTIONS CALLED THE
KERR SPACE-TIMES. WITH THE
OBVIOUS EXCEPTION OF THE M^{1+3}
ALL OTHER KERR SPACE-TIMES
ARE INCOMPLETE. IT WAS
THUS AN OPEN QUESTION WHETHER
ANY GLOBAL, COMPLETE RICCI FLAT
SOLUTIONS EXIST AT ALL. MOREOVER
ALL SOLUTIONS OF THE KERR FAMILY
ARE STATIONARY THEREFORE HAVE
NO INTERESTING TIME EVOLUTION.

THE QUESTION OF STABILITY
OF THE MINKOWSKI SPACE-TIME WAS
RECENTLY SOLVED BY D. CHRISTODOULOU
AND MYSELF. IT PROVIDES A LARGE

CLASS OF SOLUTIONS WHICH ARE SMOOTH, COMPLETE AND DYNAMICALLY INTERESTING.

THEOREM (D. CHRISTODOULOU - S. K.)

ANY A.F., MAXIMAL, (I.E. $\mathrm{tr}_g k$) INITIAL DATA SET, SUFFICIENTLY CLOSE TO THE FLAT ONE, HAS A SMOOTH, GLOBALY ASYMPT. FLAT DEVELOPMENT VERIFYING THE EINSTEIN FIELD EQUATIONS (1)

REF. D. CHRISTODOULOU - S. K.

THE GLOBAL NONL. STABILITY OF MINKOWSK SPACE - PREPRINT

Titel: On Killing spinors and exceptional holonomy groups
Autor: Christian Bär
Seite: 1
Adresse: Mathematisches Institut der Univ. Bonn
Heckenheimer Allee 160
5300 Bonn 1, Germany

Let M be an n-dimensional complete Riemannian spin manifold. A spinor field ψ is called Killing spinor with Killing constant α if for all tangent vectors X the equation $\nabla_X \psi = \alpha \cdot X \cdot \psi$ holds. Here $X \cdot \psi$ means Clifford multiplication. Killing spinors are of physical interest, see [DNP], but they also occur in purely mathematical context. For example, Friedrich has proved that if M is compact and the scalar curvature satisfies $S \geqslant S_0 > 0$, $S_0 \in \mathbb{R}$, then for all eigenvalues λ of the Dirac operator the estimate $\lambda^2 \geqslant \frac{1}{4} \frac{n}{n-1} S_0$ holds [F1]. If we have equality in this estimate, then the corresponding eigenspinor is a Killing spinor.

If M carries a Killing spinor, then M is an Einstein manifold with Ricci curvature $Ric = 4(n-1)\alpha^2$. In particular, we have three distinct cases; α can be purely imaginary, then M is noncompact and we call ψ an imaginary Killing spinor, α can be 0, in this case ψ is a parallel spinor field, and finally α can be real, then M is compact and ψ is called a real Killing spinor.

Hitchin showed that manifolds with parallel spinor fields can be characterized by their holonomy group [Hit, thm. 12 and footnote p. 54], see also [F3] and [W].

Manifolds with imaginary Killing spinors have been classified by Baum in [B1] - [B3], shortly later the classification has been extended by Rademacher to generalized imaginary

Killing spinors where we allow the Killing "constant" α to be an imaginary function [R].

Most results on real Killing spinors known so far are statements for particular low dimensions. For example, Friedrich showed in [F2] that a complete 4-dimensional manifold with real Killing spinor is isometric to the standard sphere. The analogous result in dimension 8 is due to Hijazi [Hi]. We show

<u>Theorem 1.</u> Let M be a complete Riemannian spin manifold of dimension n carrying a Killing spinor with Killing constant $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$ (This can be achieved by rescaling the metric). If n is even, $n \neq 6$, then M is isometric to the standard sphere. □

We say that M is of type (p,q) if M carries exactly p linearly independant Killing spinors for $\alpha = \frac{1}{2}$ and exactly q linearly independant Killing spinors for $\alpha = -\frac{1}{2}$ or vice versa.

In dimension 6 we recover a theorem of Grunewald [G]

<u>Theorem 2.</u> Let M be a 6-dimensional complete 1-connected Riemannian spin manifold with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. Then there are two possibilities:

(i) $M = S^6$

(ii) M is of type $(1,1)$ and M is nearly Kähler, non-Kähler

Conversely, if a complete 1-connected Riem. mfd. $M \neq S^6$ is nearly Kähler, non-Kähler, then M is of type $(1,1)$. □

Furthermore, we get (M is always a complete 1-connected Riem. spin manifold of dimension n with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$)

Theorem 3. If $n = 2m-1$, $m \geq 3$ odd, then there are two possibilities:

(i) $M = S^n$

(ii) M is of type $(1,1)$ and M is an Einstein-Sasaki manifold. $\square$

In dimension 5 this theorem can also be found in [FK1].

Theorem 4. If $n = 4m-1$, $m \geq 3$, then there are three possibilities:

(i) $M = S^n$

(ii) M is of type $(2,0)$ and M is an Einstein-Sasaki manifold, but does not carry a Sasaki-3-structure.

(iii) M is of type $(m+1, 0)$ and M carries a Sasaki-3-structure. $\square$

Theorem 5. If $n = 7$, then there are four possibilities:

(i) $M = S^7$

(ii) M is of type $(1,0)$ and M carries a 3-form φ with $\nabla\varphi = *\varphi$ which can be induced by the multiplication of imaginary Cayley numbers, but M does not carry a Sasaki structure.

(iii) M is of type (2,0) and carries a Sasaki structure, but not a Sasaki-3-structure

(iv) M is of type (3,0) and carries a Sasaki-3-structure

In theorems 3-5 the converses are also true. For example, if $n = 2m-1$, $M \neq S^n$ is a complete 1-connected Einstein-Sasaki manifold, then M is of type (1,1).

The method of proof is as follows. First we modify the spinor connection because we want to interprete Killing spinors as parallel sections. To do this we have to enlarge the structure group $Spin(n)$ of the spinor bundle to $Spin(n+1)$. Then we show that this connection is related to the Levi-Civita connection of the cone over the original manifold. Since Killing spinors now correspond to fixpoints of the holonomy group of the cone we can use the Berger-Simons classification of possible holonomy groups to see how the cone can possibly look like. Finally, this information is retranslated into conditions on the original manifold itself.

Exceptional holonomy groups.

The study of the exceptional dimension 6 provides us with a construction principle of Riemannian manifolds with exceptional holonomy group G_2. The recipe is as follows. Take any compact 1-connected nearly Kähler, non-Kähler manifold of dimension 6, normalize the metric such that $Ric = 5$, then the cone over this manifold has holonomy group G_2. Using this method we recover Bryant's first example which was the cone over the complex flag manifold $SU(3)/T^2$ [Br]. Further examples are obtained by taking the cones over $S^3 \times S^3$ and $\mathbb{C}P^3$ with certain non-standard metrics.

Similarly, the cones over certain 7-manifolds have exceptional holonomy group Spin(7). Explicit new examples are the cones over the squashed 7-sphere and over the Wallach manifolds.

References

[B] C. Bär, Real Killing spinors and holonomy, preprint Bonn 1991

[B1] H. Baum, Variétés riem. admettant des spineurs de Killing imaginaires, CR Acad Sci Paris 309 (1989) 47-49

[B2] H. Baum, Odd-dim. Riem. mfds with im. Killing spinors, Ann. Glob. Anal. Geom. 7 (1989) 141-153

[B3] H. Baum, Complete Riem. mfds with im. Killing spinors, Ann. Glob. Anal. Geom. 7 (1989) 205-226

[SB] Baum, Friedrich, Grunewald, Kath, Twistor and Killing spinors on Riem. mfds, Seminarbericht 108, Humboldt-Univ. Berlin 1990

[Br] R.L. Bryant, Metrics with exceptional holonomy, Ann. Math. 126 (1987) 525-576

[F1] T. Friedrich, Der erste Eigenwert..., Math. Nach. 97 (1980) 117-146

[F2] T. Friedrich, A remark ..., Math. Nach. 102 (1981) 53-56

[F3] T. Friedrich, Zur Existenz..., Colloq. Math. 44 (1981) 277-290

[DNP] Duff, Nilsson, Pope, Kaluza-Klein supergravity, Phys. Rep. 130 (1986) 1-142

[G] R. Grunewald, Six-dimensional ..., Ann. Glob. Anal. Geom. 8 (1990) 43-59

[Hi] O. Hijazi, Caractérisation de la sphère..., CR Acad Sci Paris 303 (1986) 417-419

[Hit] N. Hitchin, Harmonic spinors, Adv. in Math. 14 (1974) 1-55

[R] H-B. Rademacher, Generalized Killing spinors..., to appear in Proc. conf. Glob. Anal. Glob. Diff. Geom., Berlin 1990, Springer LN.

[W] M. Wang, Parallel spinors and parallel forms, Ann. Glob. Anal. Geom. 7 (1989) 59-68

Titel: Deligne's Conjecture on the Lefschetz Trace Formula in positive Characteristic

Autor: Richard Pink

Seite: 1

Adresse: Mathematisches Institut der Universität
Beringstr. 4
5300 Bonn 1, GERMANY (WEST)

Grothendieck's LTF (= Lefschetz Trace Formula) for the Frobenius morphism plays a central role in the study of the Galois representations on the étale cohomology of algebraic varieties (see SGA $4\frac{1}{2}$ Rapport 3.2). To obtain the same information for direct factors of the total cohomology that are "cut out" by correspondences one needs a LTF for correspondences twisted by Frobenius. There exists a very general such LTF (see SGA 5, III, Cor. 4.7) but its local terms are difficult to calculate.

Deligne's conjecture asserts that, under certain conditions sufficient for the above-mentioned purpose, there is a LTF of a very simple form.

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of a finite field $\mathbb{F}_q$ with q elements. A <u>correspondence</u> is a diagram

$$X \xleftarrow{b_1} B \xrightarrow{b_2} X$$

of morphisms and compactifiable separated schemes of finite type over $\overline{\mathbb{F}}_q$. Assume that X comes from a scheme over $\mathbb{F}_q$, i.e. that it comes with a Frobenius morphism $\Phi_q : X \longrightarrow X$. The <u>twisted correspondence</u> $b^{(n)} := (b_1^{(n)}, b_2^{(n)})$ is defined by the diagram

$$\begin{tikzcd} X & X \arrow[l, "\Phi_q^n"'] & B \arrow[l, "b_1"'] \arrow[ll, bend left, "b_1^{(n)}"] \arrow[r, "b_2"] \arrow[r, bend right, "b_2^{(n)}"'] & X \end{tikzcd} .$$

Assume: (1) b_1 is proper.

(2) b_2 is quasifinite of degree $< q^n$.

It is easy to see that

$$\mathrm{Fix}(b^{(n)}) := \{ \beta \in B \mid b_1^{(n)}(\beta) = b_2^{(n)}(\beta) \}$$

is finite ("isolated fixed points").

Let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on X, $\ell \nmid q$, and consider a homomorphism

$$u : b_{2!} b_1^* \mathcal{F} \longrightarrow \mathcal{F} .$$

Assume that we are given an isomorphism $\Phi_q^* \mathcal{F} \cong \mathcal{F}$, ~~then~~ we can twist u:

$$\begin{tikzcd} b^{(n)}_{2\,!}\, b_1^{(n)*} \mathcal{F} \arrow[r, "u^{(n)}"] \arrow[d, equal] & \mathcal{F} \\ b_{2!}\, b_1^* \Phi_q^{n*} \mathcal{F} \arrow[r, phantom, "\cong"] & b_{2!}\, b_1^* \mathcal{F} \arrow[u, "u"'] \end{tikzcd} .$$

Now the "<u>cohomological correspondence</u>" $(b^{(n)}, u^{(n)})$ has a <u>global term</u>, the trace of the endomorphism $u_!^{(n)}$:

$$\begin{tikzcd} H_c^i(X, \mathcal{F}) \arrow[r, "u_!^{(n)}"] \arrow[d] & H_c^i(X, \mathcal{F}) \\ H_c^i(B, b_1^{(n)*}\mathcal{F}) \arrow[r, phantom, "\cong"] & H_c^i(X, b_{2!}^{(n)} b_1^{(n)*} \mathcal{F}) \arrow[u, "u^{(n)}"'] \end{tikzcd}$$

and <u>local terms</u> for every fixed point $\beta \in \mathrm{Fix}(b^{(n)})$:

Put $x := b_1^{(n)}(\beta) = b_2^{(n)}(\beta)$, then we get an endomorphism of the stalk

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow{u_\beta^{(n)}} & \mathcal{F}_x \\ \| & & \uparrow u^{(n)} \\ (b_1^{(n)*}\mathcal{F})_\beta & \underset{\text{canonical}}{\hookrightarrow} & (b_{2}^{(n)}{}_! \, b_1^{(n)*}\mathcal{F})_x \end{array}.$$

<u>Conjecture</u> (Deligne, 1970es): For n sufficiently large

$$\sum_i (-1)^i \operatorname{tr}\left(u_!^{(n)} \middle| H_c^i(X,\mathcal{F})\right) = \sum_{\beta \in \operatorname{Fix} b^{(n)}} \operatorname{tr}\left(u_\beta^{(n)}\right).$$

Special cases of this were known (Grothendieck, Deligne-Lusztig, Illusie (dim = 1), Zink (dim = 2), ...)

<u>Theorem</u> (Pink 1990): Deligne's conjecture follows from resolution of singularities over $\mathbb{F}_q$.
(A somewhat weaker result has been proved by Shpiz, see ref.)

The precise resolution assumptions are as follows:

(1) <u>RESOLUTION</u>: X of finite type over k,
$Z \subset X$ closed such that $X \setminus Z$ is smooth
$\Longrightarrow \exists\, \pi: \tilde{X} \longrightarrow X$ proper modification with:

π is an isomorphism over $X \setminus Z$,
$\tilde{X}$ is smooth, and $\pi^{-1}(Z)$ is a divisor with normal crossings.

(2) SEPARATION: X of finite type over k, $Z_1, Z_2 \subset X$ closed
$\Longrightarrow \exists\, \pi : \tilde{X} \to X$ proper modification with:
π is an isomorphism outside $Z_1 \cap Z_2$, and the proper transforms of Z_1, Z_2 are disjoint.

These two hypotheses are known in characteristic zero (Hironaka), but not at present over $k = \mathbb{F}_q$.

Nevertheless, suppose now that we are given a cohomological correspondence with the above properties, but defined over a number field K. For all but finitely many primes of K we obtain a reduction at $\mathfrak{p}$. The resolutions of singularities necessary in the above theorem can be done once and for all over K, and can be used at almost all primes.

$\Longrightarrow$ Theorem: Given a cohomological correspondence over a number field, Deligne's conjecture is true for its reduction at all but finitely many primes.

An important case is where X is (a reduction of) a Shimura variety or a(n algebraic) compactification thereof, where (B, b) is a Hecke-correspondence, and $\mathcal{F}$ is an automorphic locally constant sheaf respectively a suitable extension to the compactification (extension by zero, intersection complex, ...).

References:

SGA $4\frac{1}{2}$: Séminaire de géometrie algebrique $4\frac{1}{2}$, Cohomologie étale, Springer LN 569 (1977)

SGA 5 : Cohomologie ℓ-adique et fonctions L Springer LN 589 (1977)

Pink, On the calculation of local terms in the Lefschetz-Verdier trace formula and its application to a conjecture of Deligne, to appear in: Annals of Mathematics

E. Shpiz, Ph.D. thesis, Harvard (1990).

Titel: Analytic torsion for non-unitary representations and Chern-Simons gauge theory

Autor: Werner Müller

Seite: 1

Adresse: MPI

The purpose of this talk is to give a report on some new developements related to analytic torsion.

Introduction The concept of torsion was introduced in 1935 by Reidemeister, Franz and de Rham. Let K be a finite simplicial complex and $\rho : \pi_1(M) \to O(N)$ an orthogonal representation with associated flat bundle E_ρ. Assume that ρ is acyclic, that is, $H^*(K; E_\rho) = 0$. Then the *Reidemeister-Franz torsion* (or *R-torsion*) $\tau_M(\rho) \in \mathbf{R}^+$ is defined. The torsion $\tau_M(\rho)$ is a kind of determinant which describes how the simplices of $\tilde{K}$ are fitted together with respect to the action of $\pi_1(K)$. It is known to be a combinatorial invariant in the sense that it is invariant under subdivision [Mi].

In particular, if K is a smooth triangulation of a closed C^∞-manifold M, then the R-torsion depends only on the smooth structure of M and we denote the torsion by $\tau_M(\rho)$.

In [RS], Ray and Singer introduced the analytic torsion $T_M(\rho)$ as analytic counterpart to R-torsion. To define $T_M(\rho)$ one has to choose a Riemannian metric g on M. Together with the canonical metric on E_ρ which is compatible with the flat connection we get an inner product on the twisted de Rham complex $\Lambda^*(M; E_\rho)$ of E_ρ-valued differential q-forms on M. Let Δ_q be the Laplacian on $\Lambda^q(M; E_\rho)$ and $\zeta_q(s;\rho) = \sum_{\lambda_j} \lambda_j^{-s}$, $\mathrm{Re}(s) > n/2$, $n = \dim M$, the zeta function of Δ_q. Then $T_M(\rho)$ is defined as

$$T_M(\rho) = \exp\left(\frac{1}{2}\sum_{q=0}^{n}(-1)^q q\,\frac{d}{ds}\zeta_q(s;\rho)\big|_{s=0}\right). \tag{1}$$

For acyclic ρ, $T_M(\rho)$ is independent of the choice of the Riemmanian metric on M.It was conjectured by Ray and Singer that $T_M(\rho) = \tau_M(\rho)$ for all acyclic orthogonal representations ρ. This conjecture was proved independently by Cheeger [C] and the author [Mü1].

Recently, torsion has found interesting applications in low dimensional topology and topological quantum field theory. D.Johnson has shown that R-torsion is closely related to Casson's invariant [J]. In [W4], Witten has used the relation of the weak couplind limit of three dimensional Chern-Simons gauge theory to analytic torsion to study two dimensional quantum Yang-Mills theory. This leads to formulas for the volumes of moduli spaces of representations of fundamental groups of compact surfaces.

1. Torsion for unimodular representations. The restriction to orthogonal (or unitary) representations is certainly a limitation of the applicability of the equality of the two torsions if the fundamantal group is infinite. We remove this limitation essentially. Namely, let $\rho : \pi_1(M) \to GL(E)$ be a representation on a finite dimensional real or complex vector space E. ρ is called *unimodular* if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(M)$. Then the definition of R-torsion also makes sense for any unimodular representation. To define the analytic torsion we follow an idea of Schwarz [S]. We choose a metric h on E_ρ and with respect to this metric we define the torsion $T_M(\rho; h)$ by a formula similar to (1). If $\dim M$ is odd and ρ is acyclic, it turns out that $T_M(\rho; h)$ is independent of h and also on the Riemannian metric on M. We call the common value $T_M(\rho)$.

Theorem 1. *Let $\dim M$ be odd. For all acyclic unimodular representations $\rho : \pi_1(M) \to \mathrm{GL}(E)$ we have*

$$T_M(\rho) = \tau_M(\rho).$$

For the proof see [Mü2]. We remark that both analytic torsion and R-torsion can be defined for all unimodular representations. If the representation is not acyclic, then both invariants will depent on the choice of the metrics on M and E_ρ. Nevertheless, the equality of Theorem 1 remains valid.

Next we discuss some applications of this result.

2.Locally symmetric manifolds. Let G be a connected real semi-simple Lie group with finite center. Assume that G has no compact factors and let $K \subset G$ be a maximal compact subgroup. Then $X = G/K$ is a symmetric space. Let $\Gamma \subset G$ be a discrete, torsion free, co-compact subgroup. Then $M = \Gamma\backslash X$ is a compact locally symmetric manifold. Examples are hyperbolic 3-manifolds. Given a finite dimensional representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ we get by restriction, a representation ρ_Γ of Γ and a flat bundle E_ρ over $\Gamma\backslash X$. Note that ρ_Γ is unimodular. The flat bundle has a natural locally homogeneous metric [MM]. Let θ be the Cartan involution of $(\mathfrak{g}, \mathfrak{k})$ and assume that (E, ρ) is irreducible with heighest weight $\Lambda - \underline{\rho}$ ($\underline{\rho} = \frac{1}{2}$sum of positive roots). If $\theta\Lambda \neq \Lambda$, it follows from Theorem 6.7 of section VII in [BW] that $H^*(\Gamma\backslash X; E) = 0$. If $\dim G/K$ is odd, then $\mathrm{rk}\, G > \mathrm{rk}\, K$ and a generic irreducible representation has vanishing cohomology. We also note that for $\mathrm{rk}\, G > 1$, superrigidity implies that all representations of Γ arise from representations of G [Ma]. Thus we can define the analytic torsion for all these representations and we expect them to be interesting invariants of the locally symmetric manifold $\Gamma\backslash X$.

3.Lefschetz numbers for flows. As an application of Theorem 1 we obtain the extension of a result of Moscovici and Stanton [MS]. Consider the geodesic flow Φ on the unit sphere bundle to $\Gamma\backslash X$. The connected components of the periodic set are parametrized by the non-trivial conjugacy classes $\{\gamma\}$ in Γ. Each connected component X_γ is itself a compact locally symmetric manifold of non-positive sectional curvature and Φ restricts to a periodic flow on X_γ. The quotient $\hat{X}_\gamma = X_\gamma/\Phi$ is a smooth orbifold. Let l_γ be the common length of the orbits in X_γ and let μ_γ be the multiplicity of a generic orbit of $\Phi|X_\gamma$. Set

$$Z_\rho(s) = \exp - \sum_{\{\gamma\}\neq 1} \mathrm{Tr}\rho(\gamma)\,\chi(\hat{X}_\gamma)\,\frac{e^{-sl_\gamma}}{\mu_\gamma}.$$

Theorem 2. *$Z_\rho(s)$ is analytic for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s = 0$. If $\dim G/K$ is odd, then*

$$Z_\rho(0) = \tau_M(\rho)^2 \quad \text{where} \quad M = \Gamma\backslash X.$$

For orthogonal (or unitary) representations ρ, this result is due to Moscovici and Stanton [MS].

4. Chern-Simons gauge theory. Chern-Simons theory is a three dimensional gauge field theory with pure Chern-Simons action. It was used by Witten [W1] to introduce new 3-manifold invariants. The basic setting for Chern-Simons theory is a compact oriented three dimensional manifold M without boundary and a Lie group G. We start with the case where G is compact and for simplicity, we take G to be $\mathrm{SU}(N)$. Consider the space $\mathcal{A}$ of all G-connections on the trivial G-bundle over M. In fact, every principal G-bundle over M is trivial. The space $\mathcal{A}$ may be identified with the space $\Lambda^1(M, \mathfrak{g})$ of differential 1-forms on M with values in the Lie algebra $\mathfrak{g}$ of G. For a given connection $A \in \mathcal{A}$, the Chern-Simons action is defined to be

$$I(A) = \frac{1}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \tag{2}$$

where Tr is the trace of $\mathfrak{su}(N)$ in the standard representation. This is a real valued non-linear functional on $\mathcal{A}$. The gauge group $\mathcal{G} = \mathrm{Map}(M, G)$ acts on $\mathcal{A}$ by the usual prescription $A^g = g^{-1}Ag + g^{-1}dg$, $g \in G$, $A \in \mathcal{A}$. Let $k \in \mathbf{N}$. Then $e^{ikI(A)}$ is a $\mathcal{G}$-invariant function on $\mathcal{A}$ and Witten's invariant of M is defined as the path integral

$$Z_M(k) = \int e^{ikI(A)}\, \mathcal{D}A \tag{3}$$

where the integration is over all gauge equivalence classes of connections. This, however, has to be considered as a formal expression, because no measure $\mathcal{D}A$ has been constructed up to now. Part of this theory can be made rigorous and Witten gave an explicit recipe for computing $Z_M(k)$.

A standard way to study functional integrals like (3) is to use the method of stationary phase approximation which predicts the behaviour of $Z_M(k)$ for large k. In the present context this method is again not based on solid ground, but it gives very interesting results. By the method of stationary phase, the leading order contribution to $Z_M(k)$ comes from the critical points of the action (2). The critical points of (2) are precisely the connections with vanishing curvature, that is, the flat connections on the bundle $P = M \times G$. Assume that the topology of M is such that there exists only a finite number of gauge equivalence classes of flat connections on P, say $A_1, \ldots, A_m$ and that $A_1, \ldots, A_m$ are all irreducible. Then Witten's formula for the stationary phase approximation of the path inytegral (3) is

$$Z_M(k) \sim \frac{1}{\# Z(G)} \sum_{j=1}^{m} e^{i\eta(\rho_{\alpha_j})} \sqrt{T_M(\rho_{\alpha_j})} \tag{4}$$

where $Z(G)$ is the center of G, $\alpha_j : \pi_1(M) \to G$ is the representation determined by A_j, $\eta(\alpha_j)$ is a certain phase factor described in [W1] and $T_M(\rho_{\alpha_j})$ is the analytic torsion of $\rho_{\alpha_j} = \mathrm{Ad} \circ \alpha_j : \pi_1(M) \to \mathrm{GL}(\mathfrak{g})$. Since each ρ_{α_j} is acyclic, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric g on M and, by [C], [Mül], it coincides with the R-torsion $\tau_M(\rho_{\alpha_j})$. As we know, the R-torsion $\tau_M(\rho_{\alpha_j})$ can be computed from a triangulation K of M in a pure combinatorial way. This suggests that one may be able to develop a rigorous treatment of the path integral (3) on the combinatorial level and derive the asymptotic behaviour (4) in this way.

So far we considered the case of a compact gauge group. Witten has also started to investigate Chern-Simons theory with non-compact gauge group [W3]. There exist several motivations to develop such a theory. For example, $2+1$ dimensional gravity is related to Chern-Simons gauge theory with gauge group $\mathrm{SL}(2,\mathbf{C})$, $ISO(2,1)$ or $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ depending on wether the cosmological constant is positive, zero, or negative [W2]. For a general non-compact Lie group G, the quantization of Chern-Simons gauge theory with gauge grouyp G is not yet understood. Nevertheless, one can study the perturbative expansion of the corresponding path integral [BNW].

The perturbative treatment of Chern-Simons gauge theory with non-compact gauge group requires again gauge fixing. Since the Killing form is indefinite there exists no obvious gauge fixing as in the compact case and different approaches are possible [BNW].For a semi-simple Lie group G, the most natural gauge fixing seems to be the unitary gauge fixing described in section 4 of [BNW]. Let A be a flat connection on the trivial G-bundle over M with holonomy representation $\alpha : \pi_1(M) \to G$. As above, let $\mathfrak{g}_\alpha$ be the flat bundle defined by $\rho_\alpha = \mathrm{Ad} \circ \alpha$. Then the unitary gauge fixing amounts to the choice of a riemannian metric g on M and a hermitian metric h on $\mathfrak{g}_\alpha$. We observe that $\rho_\alpha : \pi_1(M) \to \mathrm{GL}(\mathfrak{g})$ is unimodular. In fact, since $\mathfrak{g}$ is semi-simple, the Killing form is non-degenerate. Hence, for each $g \in G$, $\mathrm{Ad}(g)$ preserves a non-degenerate symmetric bilinear form on $\mathfrak{g}$ which implies that $|\det \mathrm{Ad}(g)| = 1$. This is precisely the setting of section 1.

Under the same assumption as above, one gets a formula for the one loop approximation of the path integral which is similar to (4). The analytic torsion $T_M(\rho_{\alpha_j})$ is now defined as discribed in section 1. For the discussion of the phase factor see section 4 of [BNW]. By assumption, each representation ρ_{α_j} is acyclic and therefore, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric on M and $\mathfrak{g}_\alpha$. Moreover, by Theorem 1, $T_M(\rho_{\alpha_j})$ equals the R-torsion $\tau_M(\rho_{\alpha_j})$ which has again a pure combinatorial describtion. This suggests that Chern-Simons gauge theory with a non-compact, but semi-simple gauge group should also be accessible to a combinatorial treatment.

References

[BNW] Bar-Natan,D., Witten,E.: Perturbative expansion of Chern-Simons theory with non-compact gauge group. Preprint, IASSNS-HEP-91/4, Princeton, 1991.

[BW] Borel,A., Wallach,N.: Continuous cohomology, discrete subgroups, and representations of reductive groups. Annals of Math. Studies **94**, Princeton Univ. Press, princeton, 1980.

[C] Cheeger,J.: Analytic torsion and the heat equation. Annals of Math. **109**, 259 - 322 (1979).

[J] Johnson,D.: A geometric form of Casson's invariant, and its connection to Reidemeister torsion. unpublished lecture notes.

[Ma] Margulis,G.A.: Discrete Subgroups of Semisimple Lie Groups. Springer-Verlag, Berlin Heidelberg New York, 1991.

[MM] Matsushima,Y., Murakami,S.: On vector bundle valued harmonic forms and automorphic forms on symmetric spaces. Annals of Math. **78**, 365 - 416 (1963).

[Mi] Milnor,J.: Whitehead torsion. Bull. Amer. Math Soc. **72**, 358 -426 (1966).

[MS] Moscovici,H., Stanton,R.J.: R-torsion and zeta functions for locally symmetric manifolds. Preprint, Ohio State Univ., 1990.

[Mü1] Müller,W.: Analytic torsion and R-torsion of Riemannian manifolds. Advances in Math. **28**, 233 - 305 (1978).

[Mü2] Müller,W.: Analytic torsion and R-torsion for unimodular representations. MPI-Preprint, 1991.

[RS] Ray,D.B., Singer,I.M.: R-torsion and the Laplacian on Riemannian manifolds. Advances in Math. **7**, 145 - 210 (1971).

[S] Schwarz,A.: The partition function of degenerate quadratic functional and Ray-Singer invariants. Lett. Math. Phys. **2**, 247 - 252 (1978).

[W1] Witten,E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys. **121**, 351 - 399 (1988).

[W2] Witten,E.: $2+1$ dimensional gravity as an exactly soluble system. Nuclear Phys. B **311**, 46 - 78 (1988/89).

[W3] Witten,E.: Quantization of Chern-Simons gauge theory with complex gauge group. Commun. Math. Phys. **137**, 29 - 66 (1991).

[W4] Witten,E.: On quantum gauge theories in two dimensions. Preprint IASSNS-HEP-91/3, Princeton 1991.

Titel: Newton polygons and abelian varieties.

Autor: Frans OORT

Adresse: Mathematical Institute
Budapestlaan 6
NL - 3508 TA UTRECHT
The Netherlands

1 Introduction. Geometric points of a moduli space correspond to isomorphism classes of certain objects (which on wants to study, wants to classify). If one requires those objects to have some extra properties (some additional structure) one obtains a subset of that moduli space (which because it is nature-given is interesting, and can have some nice properties). Many proof in algebraic geometry are given using properties of such substes of a moduli space.

Today we discuss closed subsets of the moduli space of principally polarized abelian varieties in positive characteristic given by Newton polygons. This stratification refines the one by the p-rank.

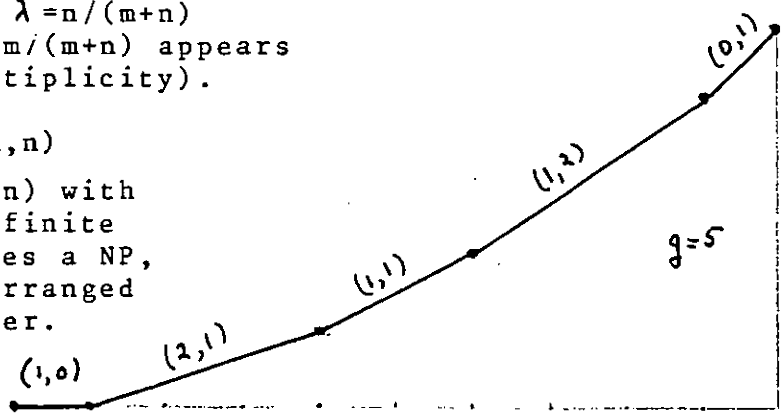
From the precise information we obtain for the strata defined by the various Newton polygons we derive a proof of a conjecture by Manin (1963: every symmetric Newton poly-on can be realized by an abelian varietiy), and we show a strengthened form of a conjecture by Koblitz (1975: an ordered pair of Newton polygons can be realized by a specialization of abelian varieties). Our inspiration came from these conjectures by Manin and Koblitz, from results by Mumford, Grothendieck, Katz (and many others), and from cooperation with Tadao Oda, T.Katsura, P.Norman, T.Ekedahl and K.-Z.Li.

Some notation: $g=\dim(AV)$, p is a prime number, n is a positive integer prime to p, we write NP for Newton polygon, $\mathcal{N}(X)$ is the NP of the abelian variety X, we write $f=f(X)$ for the p-rank of X, we write $a(X)= \dim \operatorname{Hom}(\alpha_p,X)$, $G_{m,n}$ is a p-divisible group of dimension m, whose (Serre-)dual has dimension n.

2 Newton polygons.

NP: lower convex polygon in $\mathbb{Q}\times\mathbb{Q}$, breakpoints in $\mathbb{Z}\times\mathbb{Z}$, starts at (0,0), ends at (2g,g), and is symmetric (if a slope $\lambda=n/(m+n)$ appears then $1-\lambda=m/(m+n)$ appears with the same multiplicity).

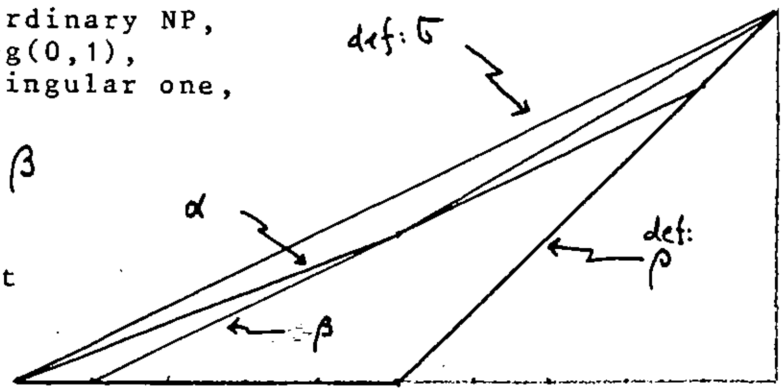
$G_{m,n}$, or the pair (m,n) gives the slope $n/(m+n)$ with multiplicity $m+n$. A finite set of such pairs gives a NP, when the slopes are arranged in non-decreasing order.



We write ρ for the ordinary NP,
i.e. given by g(1,0)+g(0,1),
and σ for the supersingular one,
i,e, given by g(1,1),
and we write

$$\alpha \prec \beta$$

(and we say that β is below α) if no point of α is strictly below β.
!! note the reverse in this order!!



Smallest: σ (supersingular), largest: ρ (ordinary).
The NPs form a directed graph: $\forall \alpha: \sigma \prec \alpha \prec \rho$.

3 <u>The NP of an abelian variety</u>.
If k is a field, char(k)=p > 0, and X is an abelian variety over k, then X determines a NP. If k happens to be a finite field this can be given by the NP of the char.pol. of the geometric Frobenius of X (suitably normalized if k has more than p elements). In general on takes the p-divisible group G of X over an algebraic closure of k. By Dieudonné-Manin theory we can write $G \sim f.(G_{1,0}+G_{0,1}) + \sum (G_{m_i,n_i}+G_{n_i,m_i}) + s.G_{1,1}$
and the pairs (m,n) thus obtained give the NP $\mathcal{N}(X)$. Note that X has a polarization, hence is isogenous to its dual, and this gives the symmetry of $\mathcal{N}(X)$. Note that $\mathcal{N}(X)$ depends only on the isogeny class of X over some field containing k.

4 <u>Conjecture, Manin, 1963</u>: Suppose given g, and a prime number p, and a NP (symmetric), then there exists an AV having this NP in characteristic p (cf. (8), page 76).
Remark: this was proved in 1967 by Honda, and by Serre, via reducing a well-choosen CM abelian variety from char. zero to char. p; below we indicate another proof .

5 <u>Theorem, Grothendieck</u>: If an abelian variety X_ξ specializes to an abelian variety X_0, then the NP goes up: $(X_\xi \rightsquigarrow X_0) \Rightarrow \mathcal{N}(X_\xi) \succ \mathcal{N}(X_0)$ cf. (1), page 91.

6 <u>Conjecture, Koblitz, 1975</u>: The converse of this theorem should be true (i.e. every ordered pair of NPs can be realized by a specialization of AVs),
cf. (6), page 211 .

7 <u>Notation</u>: $\mathcal{A} := \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$, i.e. for every algebraically closed field k of char(k)=p, the set $\mathcal{A}(k)$ of k-rational points is the set of isomorphism classes of triples (X, λ, γ), where X is an AV of dimension g over k, where λ is a <u>principal</u> polarization (i.e. deg(λ) =1), and where γ is a symplectic level-n-structure on X.

For any NP α we write W_α for the set

$$W_\alpha = \{(X,\lambda,\gamma) \mid \mathcal{N}(X) \prec \alpha\}/\cong \;\subset \mathcal{A}.$$

By Grothendieck-Katz we know that this is a closed subset of $\mathcal{A}$. NB!! there is no a-priori reason why any point of W_α should correspond with an AV having $\mathcal{N}(X) = \alpha$. (closed: cf. (5), page 143, Th. 2.3.1 & Coroll. 2.3.2).

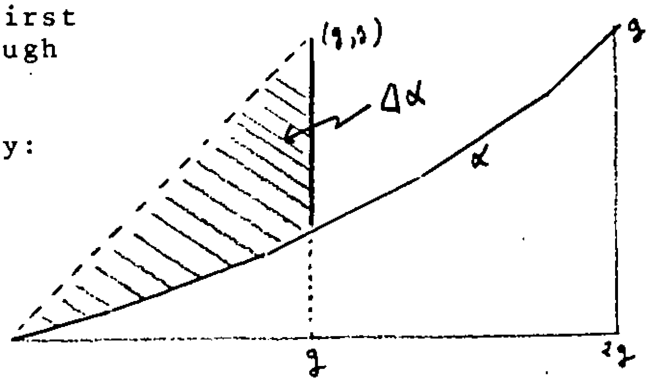
We write $\Delta\alpha$ for the region of the plane:

$\Delta(\alpha) := \{(x,y) \mid y < x,\ 0 \le x \le g$, and this point is on or above $\alpha\}$

(note that we only use the first half of the NP, which is enough for our purposes, because of symmetry).

We define the number $d(\alpha)$ by:

$$d(\alpha) := \#\big((\mathbb{Z}\times\mathbb{Z}) \cap \Delta\alpha\big).$$



Note: $d(\sigma) = \left[\frac{g^2}{4}\right]$,

$d(\rho) = \frac{1}{2}g(g+1) = \dim \mathcal{A}$,

note!!: $d(\rho) - d(\alpha)$ = length of longest path from α to ρ in the NP-graph (and below: this will equal $\mathrm{codim}(W_\alpha \subset \mathcal{A})$, !!).

8 Theorem (cf. (12)): Fix g, p, and n as above, consider the moduli space as above of principally polarized abelian varieties in characteristic p, and let k be an algebraically closed field, char(k)=p.

a) Let W be an irreducible component of $W_\alpha \otimes k$, let $\eta \in W_\alpha$ be its generic point, then $\mathcal{N}(X_\eta) = \alpha$, $a(X_\eta) \le 1$, $\boxed{\dim W = d(\alpha)}$.

b) Let $\alpha \prec \beta$ be an ordered pair of NPs, and consider geometrically irreducible components:
for every $W \subset W_\alpha \otimes k$ there is a unique $W' \subset W_\beta \otimes k$ with $W \subset W'$,
for every $W' \subset W_\beta \otimes k$ there is a $W \subset W_\alpha \otimes k$ with $W \subset W'$,
in particular the set of geometrically irreducible components of W_α maps surjectively onto the same of W_β.
(We see: for any geometrically irreducible component $W \subset W_\alpha \otimes k$ the set of all components of all W_β is the same as the part of the NP-graph of all NPs below α).

9 Corollary (proof of conjecture by Manin, cf. 4): For any NP α there is a point in W_α (even defined over an algebraic closure of $\mathbb{F}_p$) having $\mathcal{N}(X) = \alpha$.

10 Corollary (A strengthened form of a conjecture by Koblitz, cf. 6): Let (X_0, λ_0) be a principally polarized abelian variety in positive characteristic, and let

$$\mathcal{N}(X_0) = \alpha \prec \beta.$$

Then there exist a specialization $(X_\xi, \lambda_\xi) \rightsquigarrow (X_0, \lambda_0)$ such that $\mathcal{N}(X_\xi) = \beta$.

11 Remarks: In the theorem, and in the last corollary it is essential that we work with polarizations with degree prime to p. Counterexamples to more general cases (already for g=3) can be found in (4). - The fact that the supersingular locus W has dimension equal to $[g^2/4]$ was conjectured in (10), and a proof will appear in (7), cf. theorem 12 below. The fact that $\frac{1}{2}g(g+1) - [g^2/4]$ exactly equals the length of the longest path in the NP-graph gave the clue to theorem 8. For those closed sets of the NP-stratification given by the p-rank the dimension formula in theorem 8 was proved in (6), also see (9).

12 Theorem (T.Ekedahl &FO), cf. (2): For any $g \geq 2$, and any NP α the set W_α is connected.
Corollary (Chai-Faltings): For any g the moduli space $\mathcal{A}$ is irreducible.

13 Theorem (K.-Z.Li & FO), cf. (7): For any g the supersingular locus $\mathcal{S} = W_\sigma$ has dimension equal to $[g^2/4]$, and the number of components of $\mathcal{S} \otimes k$ (where k is an algebraically closed field of char. p) is given by a class number as conjectured in (4).

14 Remarks: We see that the number of geometric components of $W_\alpha \otimes k$ is less or equal to the class number

$H_g(p,1)$ (g is odd), respectively $H_g(1,p)$ (g is even), which is the number of geometric components of the supersingular locus for that g. We have no complete information on the number of components for every W_α.

Note that the case g=1, the computation of the number of supersingular j-invariants is classical (Deuring-Eichler, and Igusa). The case g=2 was settled in (3), and for g=3 we find the answer in (4).

REFERENCES:

(1) M.Demazure - Lectures on p-divisible groups. Lect. N. Math. 302, Springer-Verlag, 1972.

(2) T.Ekedahl & F.Oort - Connected substes of moduli spaces of abelian varieties. (to appear)

(3) T.Katsura & F.Oort - Families of supersingular abelian surfaces. Compos. Math. 62 (1987), 107-167.

(4) T.Katsura & F.Oort - Supersingular abelian varieties of dimension two and three and class numbers. Adv. St. Pure Math. 10, 1987 (Algebraic Geom., Sendai, 1985; Ed. T.Oda), Kinokuniya Cy, Tokyo Japan, and North-Holl. Cy, Amsterdam, 1987; pp. 253-281.

(5) N.M.Katz - Slope filtrations of F-crystals. Journ. Géom. Alg. Rennes, Vol. I, Astérisque 63, Soc. Math. France, 1979; pp. 113-164.

(6) N.Koblitz - p-adic variation of the zeta-function over families of varieties defined over finite fields. Compos. Math. 31 (1975), 119-218.

(7) K.-Z.Li & F.Oort - Moduli of supersingular abelian varieties. (to appear)

(8) Yu.I.Manin - The theory of commutative formal groups over fields of finite characteristic. Usp. Math. 18 (1963), 3-90; Russ. Math. Surveys 18 (1963), 1-80.

(9) P.Norman & F.Oort - Moduli of abelian varieties. Ann. Math. 112 (1980), 413-439.

(10) T.Oda & F.Oort - Supersingular abelian varieties. Intl. Symp. on Algebraic Geom., Kyoto 1977 (Ed. M.Nagata), Kinokuniya Book-store, 1978; pp. 595-621.

(11) F.Oort - Subvarieties of moduli spaces. Invent. Math. 24 (1974), 95-119.

(12) F.Oort - Moduli of abelian varieties and Newton polygons. C. R. Acad. Sci. Paris, 312 (1991), 385-389.

(13) F.Oort - Moduli of abelian varieties in positive characteristic. (to appear: Barsotti memorial sympos. on algebr. geom., Padova, 1991)

Frans OORT
Mathematical Institute
Budapestlaan 6
NL - 3508 TA UTRECHT,
The Netherlands
oort@math.ruu.nl

Titel:

Autor:

Adresse:

Seite: 1

Reihenfolge:

von Frau Suter *

- Titelblatt — 1 Seite
- Programm — 3 Seiten
- Teilnehmerliste — 4
- Donaldson — 5
- Shioda — 6
- Zagier — 4 (bitte ausschneiden & aufkleben)
- Oliver — 5
- Manin — 5
- Wüstholz — 7
- Serre — 1
- Kontsevich — 5
- Wodzicki — 6
- MacPherson — 5
- Lang — 5
- Tsuji — 5
- Klainerman — 5
- Bär — 6
- Pink — 5
- Müller — 5
- Oort — 5

jeweils die erste Seite rechts

Bitte 3 Exemplare heften, davon eins mit Rückenband.

Alle anderen Exemplare bleiben bis Donnerstag, 18 Uhr, in den Kartons.

* Falls Frau Suter um 10 Uhr nicht da ist, bitte anrufen!

30. Arbeitstagung

Bonn, 14. - 20. Juni 1991

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI/91-47

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3

und

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
5300 Bonn 1

Programm der Mathematischen Arbeitstagung 1991 (I)

Freitag, den 14.6.1991

16.00 - 17.00 Uhr	S.K. DONALDSON (Oxford) Glueing problems in Yang-Mills theory

Samstag, den 15.6.1991

10.15 - 11.15 Uhr	T. SHIODA (Rikkyo U., z.Zt. MPI) Mordell-Weil lattices: theory and applications
12.00 - 13.00 Uhr	D. ZAGIER (MPI Bonn und U. Utrecht) Polylogarithms
17.00 - 18.00 Uhr	B. OLIVER (Aarhus) Maps between classifying spaces of compact Lie groups

Sonntag, den 16.6.1991

10.15 - 11.15 Uhr	Yu.I. MANIN (Steklov Moskau, z.Zt. MPI) DeRham-complexes in non-commutative geometry
12.00 - 13.00 Uhr	G. WÜSTHOLZ (ETH Zürich) Faltings's proof of one of Lang's conjectures (rational points on subvarieties of Abelian varieties)
16.15 - 16.45 Uhr	H. HIRONAKA (Harvard) Miscellanea Mathematica
17.00 - 18.00 Uhr	J-P. SERRE (Collège de France Paris) Galois cohomology: recent results and open questions

Montag, den 17.6.1991

10.00 - 10.15 Uhr	Festlegung der restlichen Vorträge
10.15 - 11.15 Uhr	M. KONTSEVICH (Acad. Sci. Moskau, z.Zt. MPI) Intersection theory on the moduli space of curves and the matrix Airy function
13.00 Uhr	Schiffsfahrt nach Andernach. Abfahrt um 13.00 Uhr mit Motorschiff "Carmen Sylva", Ablegestelle Alter Zoll. Rückkehr ca. 20.00 Uhr

b.w.!
PTO!

Dienstag, den 18.6.1991

10.15 - 11.15 Uhr	M. WODZICKI (UC Berkeley) Excision in K-theory and proof of the Karoubi conjecture
12.00 - 13.00 Uhr	R. MACPHERSON (M.I.T.) Lefschetz numbers of Hecke correspondences
17.00 - 18.00 Uhr	S. LANG (Yale University, z.Zt. MPI) Degeneration of Riemann surfaces and Jorgenson's proof of a conjecture of Deligne

Die Vorträge finden alle im "*Großen Hörsaal*", Wegelerstraße 10, statt.

Erfrischungspausen mit Tee: Samstag, Sonntag und Dienstag 11.15 - 12.00 Uhr, Samstag und Dienstag 16.15-17.00 Uhr, Sonntag 15.30 - 16.15 Uhr, jeweils vor dem Großen Hörsaal.

Teilnehmerlisten und *Informationen* liegen vor dem Großen Hörsaal aus. Alle Teilnehmer mögen sich bitte in die Teilnehmerlisten eintragen.

Post liegt während der Teepausen aus.

Den *Tagungsbeitrag* bitte während der Teepausen vor dem Großen Hörsaal bezahlen.

Alle Tagungsteilnehmer mit ihren Damen oder Herren sind herzlich zum *Empfang des Rektors* eingeladen. Zeit: Freitag, den 14.6., 20.00 Uhr. Ort: Festsaal der Universität, Hauptgebäude; Eingang von der Straße "Am Hof" durch das Tor gegenüber Buchhandlung Bouvier.

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3

und

Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
5300 Bonn 1

Programm der Mathematischen Arbeitstagung 1991 (II)

Mittwoch, den 19.6.1991

10.15 - 11.15 Uhr	H. TSUJI (Tokyo Metropolitan U., z.Zt. MPI) Report on Mori Theory
12.00 - 13.00 Uhr	S. KLAINERMAN (Princeton U., z.Zt. SFB 256) On non-linear stability of Minkowski space
17.00 - 18.00 Uhr	C. BÄR (Bonn) On Killing spinors and exceptional holonomy groups

Donnerstag, den 20.6.1991

10.15 - 11.15 Uhr	R. PINK (Bonn) Deligne's conjecture on the Lefschetz trace formula in positive characteristic is now a theorem
12.00 - 13.00 Uhr	W. MÜLLER (Karl-Weierstraß-Institut, z.Zt. MPI) Analytic Torsion for non-unitary representations and Chern-Simons gauge theory
17.00 - 18.00 Uhr	F. OORT (Utrecht) Newton polygons and abelian varieties

Die Vorträge finden alle im "*Großen Hörsaal*", Wegelerstraße 10, statt.

Erfrischungspausen mit Tee: Mittwoch und Donnerstag, 11.15 - 12.00 Uhr und 16.15 - 17.00 Uhr vor dem Großen Hörsaal.

Post liegt während der Teepausen aus.

Informationen liegen vor dem Großen Hörsaal aus.

Den *Tagungsbeitrag* bitte während der Teepausen vor dem Großen Hörsaal bezahlen.

Teilnehmerliste

U. Abresch (Münster)
S. Adian (Steklov Moskau)
P. Arias (MPI)
H. Azad (z.Zt. MPI)
C. Bär (Bonn)
W. Ballmann (Bonn)
W. Barth (Erlangen)
G. Barthel (Konstanz)
S. Bauer (Göttingen)
H. Baum (Humboldt-U. Berlin)
H. Baumgärtel (KWI Berlin)
E. Bayer (Besançon)
A. Beauville (Paris-Sud)
H. Beckert (Leipzig)
K. Behnke (Hamburg)
M. Berger (I.H.E.S.)
T. Berger (MPI)
R. Berndt (Hamburg)
D. Blasius (UCLA)
S. Böcherer (Mannheim)
C.-F. Bödigheimer (Göttingen)
H.G. Bothe (KWI Berlin)
J.P. Bourguignon (École Polytechnique Palaiseau)
E. Brieskorn (Bonn)
C. Brinkmann (Bonn)
V. Brinzanescu (Bukarest)
W.D. Brownawell (Penn. State U.)
A. Brownstein (Rutgers U.)
U. Bunke (Greifswald)
D. Burde (Bonn)
D. Burghelea (Ohio State U.)
J. Buth (MPI)
C. Byrne (Springer-Verlag)
E. Calabi (U. of Pennsylvania)
R. Carlsson (Hamburg)
F. Catanese (Pisa)
J.-T. Chang (Oklahoma State U.)
J. Christophersen (Oslo)
H. Cohen (z.Zt. MPI)
B. Colbois (Bonn)
V. Cossart (Paris VI)
D. Dais (Bonn)
A. Deitmar (z.Zt. MPI)
A. Dessai (Bonn)
A. Dimca (z.Zt. MPI)
P. Dombrowski (Köln)
S.K. Donaldson (Oxford)
A. Durfee (Mt.Holyoke College)
W. Ebeling (Hannover)
B. Eckmann (ETH Zürich)
J. Eells (Warwick)
H. von Eitzen (Bonn)
D. Erle (Dortmund)
H. Esnault (Essen)
U. Everling (MPI)
C. Faber (z.Zt. MPI)
D. Feldhausen (Bonn)
J.-M. Feustel (KWI Berlin)
T. Fiedler (z.Zt. Göttingen)
B. Fischer (Humboldt-U. Berlin)
G. Fischer (Düsseldorf)
M. Flohr (Bonn)
R. Flume (Bonn)
H. Föllmer (Bonn)
O. Föllinger (Humboldt-U. Berlin)
M. Fontaine (MPI)
J. Franke (z.Zt. MPI)
E. Friedman (z.Zt. MPI)
R. Friedrich (Bonn)
T. Friedrich (Humboldt-U. Berlin)
M. Furushima (Okinawa)
H. Gangl (MPI)
L. Geatti (Sassari)
T. Geisser (Münster)
W. Gerdes (Brandeis U.)
L. Göttsche (MPI)
H. Götze (Springer-Verlag)
M. Goresky (Northeastern U.)
H. Grauert (Göttingen)
G.-M. Greuel (Kaiserslautern)
V. Gritsenko (LOMI Leningrad)
F. Grunewald (Bonn)
R. Gulliver (U. of Minnesota)
K. Haberland (Jena)
I. Hambleton (z.Zt. MPI)
U. Hamenstädt (Bonn)
H. Hamm (Münster)

G. Harder (Bonn)
Y. Hayashi (Bonn)
J. Heinze (Springer-Verlag)
H.-W. Henn (Heidelberg)
M. Hennes (MPI)
G. Henniart (Paris XI)
M. Herrmann (Köln)
C. Hertling (Bonn)
P. Hertling (Bonn)
S. Hildebrandt (Bonn)
T. Hintermann (Birkhäuser Verlag Basel)
F. Hirzebruch (MPI)
H. Hironaka (Harvard)
E. Hironaka (z.Zt. MPI)
T. Höfer (MPI)
G. Höhn (Bonn)
W. Hoffmann (KWI Berlin)
R.-P. Holzapfel (KWI Berlin)
Bizhong Hu (z.Zt. MPI)
G. Huber
B. Hunt (Kaiserslautern)
D. Husemoller (Haverford College)
D. Huybrechts (z.Zt. MPI)
E. Hyry (Helsinki)
H.-C. Im Hof (Basel)
J. Jahnel (Jena)
U. Jannsen (Köln)
A. Juhl (KWI Berlin)
B. Juhl-Jöricke (KWI Berlin)
R. Jung (MPI)
F. Kärsten (KWI Berlin)
B. Kahn (Paris VII)
U. Kaiser (Siegen)
T. Kakolewski (MPI)
H. Karcher (Bonn)
M. Karpinski (Bonn)
C. Kassel (Straßburg)
A. Katsuda (Grenoble)
R. Kellerhals (Bonn)
Hoil Kim (Bayreuth)
G. Kings (Münster)
S. Klainerman (Princeton U.)
J.R. Klein (Siegen)
W. Kleinert (Humboldt-U. Berlin)
M. Klingholz (Bonn)
A.A. Klyachko (z.Zt. MPI)
K. Knapp (Wuppertal)
H. Knörrer (ETH Zürich)
R. Kobayashi (z.Zt. MPI)
S. Kobayashi (UC Berkeley)
H. Koch (z.Zt. Bonn)
K. Köhler (Paris-Sud)
P. Koepke (Bonn)
W. Kohnen (MPI)
M. Kontsevich (z.Zt. MPI)
K. Kopfermann (Hannover)
J. Korbas (Bratislawa)
I. Kosarew (Zürich)
U. Koschorke (Siegen)
A. Kostrikin (U. Moskau)
D. Kotschick (Cambridge)
J. Kramer (ETH Zürich)
M. Kreck (z.Zt. MPI)
H. Kriete (Bochum)
R. Kühnau (Halle)
W. Kühnel (Duisburg)
W. Künzel
N.H. Kuiper (I.H.E.S.)
H. Kurke (Humboldt-U. Berlin)
K. Lamotke (Köln)
S. Lang (z.Zt. MPI)
Le Van Thanh (z.Zt. MPI)
R. Lee (Yale U.)
M. Lehn (Bonn)
J. Leiterer (Humboldt-U. Berlin)
L. Lemaire (Brüssel)
I. Lieb (Bonn)
F. Lorenz (Münster)
M. Lübke (Leiden)
M. Lüdde (Bonn)
R. MacPherson (M.I.T.)
H. Maennel (MPI)
Yu. Manin (z.Zt. MPI)
G. Masbaum (Nantes)
D. Masser (z.Zt. MPI)
S. Maurmann (MPI)
K.H. Mayer (Dortmund)
J. McKay (Montréal)
N. Mestrano (Toulouse)
W.T. Meyer (Münster)
R.J. Milgram (Stanford U.)
A. Miller (Bonn)

M. Min-Oo (McMaster U.)
B.Z. Moroz (z.Zt. MPI)
W. Müller (z.Zt. MPI)
A. Murase (z.Zt. MPI)
W. Nahm (Bonn)
H.-J. Nastold (Münster)
K.-H. Nech (Darmstadt)
K.-H. Neeb
J. Nekovář (Prag)
N. Netsvetaev (LOMI Leningrad)
W.D. Neumann (z.Zt. MPI)
K. Nomizu (z.Zt. Bonn)
S. Ochanine (z.Zt. MPI)
J. O'Hara (Tokyo Metropolitan)
C. Okonek (Bonn)
B. Oliver (Aarhus)
R. Olivier (Bonn)
F. Oort (Utrecht)
P. Orlik (U. of Wisconsin)
R. Palais (Brandeis)
S. Parmentier (z.Zt. MPI)
K. Peters (Bonn)
Vu Quoc Phong (z.Zt. MPI)
R. Piene (Oslo)
R. Pink (Bonn)
V. Platonov (z.Zt. MPI)
J. Podgor (Wroclaw)
K. Polthier (Bonn)
H. Popp (Mannheim)
V. Puppe (Konstanz)
M. Puschnigg (Heidelberg)
H.-B. Rademacher (Bonn)
A. Recknagel (MPI)
M. Reid (Warwick)
R. Remmert (Münster)
J.-E. Riedel (Bonn)
C. Riehm (McMaster U.)
Shi-shyr Roan (z.Zt. MPI)
A. Robertson (Oxford)
J. Rohlfs (Eichstätt)
Yu. Rozanov (Steklov Moskau)
K. Rozanova (z.Zt. Bonn)
Yongbin Ruan (UC Berkeley)
M. Saito (z.Zt. MPI)
J.H. Sampson (Johns Hopkins)
P. Schapira (Paris-Nord)
N. Schappacher (MPI)
P. Schenzel (Halle)
E. Schiavi (Trento)
G. Schmalz (KWI Berlin)
U. Schmickler-Hirzebruch
(Springer-Verlag)
C. Schoen (z.Zt. MPI)
R. Schön (Heidelberg)
W. Schomburg (Bochum)
R. Schoof (Trento)
M. Schroeter (Münster)
B.-W. Schulze (KWI Berlin)
S. Seifarth (KWI Berlin)
W.K. Seiler (Mannheim)
U. Semmelmann (Humboldt-U. Berlin)
J-P. Serre (Collège de France)
S. Shatz (U. of Pennsylvania)
T. Shioda (z.Zt. MPI)
O. Sick (Bonn)
D. Siersma (Utrecht)
S. Sigg (Bonn)
C.T. Simpson (z.Zt.Toulouse)
P. Slodowy (Hamburg)
B. Smyth (U. of Notre Dame)
W. Soergel (z.Zt. MPI)
T.A. Springer (Utrecht)
K. Stein (München)
B. Steinert (Bonn)
L. Stoyanov (z.Zt. Darmstadt)
J. Strooker (Utrecht)
U. Stuhler (Wuppertal)
Wing-wah Sung (z.Zt. MPI)
M. Teicher (Bar-Ilan U.)
T. Terasoma (Mannheim)
Chuu-Lian Terng (Northeastern U.)
C. Thomas (Cambridge)
F. Thomas (Humboldt-U. Berlin)
G. Trautmann (Kaiserslautern)
J. Tschinkel (z.Zt. MPI)
H. Tsuji (z.Zt. MPI)
A. Van de Ven (Leiden)
R. Varnhagen (Bonn)
B. Venkov (z.Zt. MPI)
E. Viehweg (Essen)
W. Vogel (Halle)
H. Voskuil (z.Zt. MPI)

P. Vuillermot (z.Zt. MPI)
M. Waldschmidt (IHP Paris)
M. Weber (Bonn)
Lin Weng (z.Zt. MPI)
C. Wenzel (Wuppertal)
D. Wigner (Paris VI)
F. Wiid (z.Zt. MPI)
J. Wildeshaus (Münster)
M. Wodzicki (UC Berkeley)
M. Wohlgemuth (Bonn)
M. Wollenberg (KWI Berlin)
J. Wood (Leeds)
G. Wüstholz (ETH Zürich)
Gang Xiao (z.Zt. MPI)
M. Yoshida (Kyushu U.)
D. Zagier (MPI u. Utrecht)
M. Zaidenberg (z.Zt. MPI)
Yu. Zarhin (RCC Pushchino)
E. Zeidler (Leipzig)
E.W. Zink (KWI Berlin)
J. Zobel (Bonn)
P. Zograf (LOMI Leningrad)
Kang Zuo (Kaiserslautern)

Titel: Glueing problems in Yang-Mills Theory
Autor: S. Donaldson

Adresse: The Mathematical Institute,
24-29 St. Giles,
Oxford.

In order to calculate the invariants of 4-manifolds which are defined by Yang-Mills moduli spaces one would like information on their behaviour under "gluing" operations. Floer's theory of "instanton homology" provides a framework for many gluing problems but there are a number of points which are not covered by the theory at present.

Here we discuss a model problem in 2 dimensions which already contains many of the difficulties one encounters in 4-dimensions. Let Σ be a closed oriented surface ~~on~~ of genus g and $M = M(\Sigma)$ the moduli space of flat G-connections over Σ. The Kunneth components of characteristic classes of a universal bundle over $\Sigma \times M$ define cohomology classes over M. If Ω is a product of such classes, of top degree, we consider the pairing $\langle \Omega, [M] \rangle$, which is analogous to the 4-manifold invariants mentioned above. In particular if $G = SU(2)$ we are primarily interested in computing $I_{p,q} = \langle \alpha^p \beta^q, [M] \rangle$, where $\alpha \in H^2(M)$, $\beta \in H^4(M)$ are obtained from the Chern class in $H^4(M \times \Sigma)$, and $2p + 4q = 6g - 6 = \dim M$. (We can avoid technical difficulties by working with a "twisted" version of the problem, corresponding to an $SO(3)$ bundle with $w_2 \neq 0$). This problem has been solved by M. Thaddeus; his formula is:

$$I_{p,q} = (-1)^{g-1} 2^{2g} \frac{p!}{s!} (2^s - 2) B_s ,$$

where $s = q - g + 1$ and B_s is the Bernoulli number. (There should be similar formulae for other groups G.)

We discuss 3 different approaches to such calculations. Each hinges on a "gluing rule".

Method 1 [1] This is based on complex geometry and conformal field theory. M can be given a complex structure and there is a positive line bundle $\mathcal{L} \to M$ with $c_1(\mathcal{L}) = \alpha$. The dimension of $V_k(\Sigma) = H^0(M(\Sigma), \mathcal{L}^k)$ can be expressed, via the Riemann-Roch theorem, in terms of the $I_{p,q}$. In conformal field theory there is a gluing rule for the $V_k(\Sigma)$ which allows ~~them~~ these dimensions to be calculated (by induction on g). The Riemann-Roch formula can be inverted to give the $I_{p,q}$.

Method 2 This was introduced by Witten in [2]. He considers the measure on the moduli space defined by $\omega^n/n!$; where ω is a 2-form representing α. The measure can be obtained in a different way using the torsion of a finite dimensional complex which computes the tangent space of M. There is a gluing rule for the <u>volume</u> of the moduli spaces, which enables the volumes to be calculated, and the pairings $I_{p,q}$ can be recovered from this.

<u>Method 3</u> This uses ideas which are familiar in Floer homology. One deforms the moduli space by introducing perturbation into the flatness condition. The gluing rule takes the form:

$$(\dagger) \quad \langle \alpha^n, M(\Sigma) \rangle = \langle (\alpha + h)^n, \mathcal{X}(\Sigma') \rangle ,$$

where $\mathcal{X}$ is a fibre bundle over $M(\Sigma')$, Σ' has genus $g-1$, and $h \in H^2(\mathcal{X})$. The cohomology ring of $\mathcal{X}$ is described by β, and this allows us to evaluate the right hand side of $(\dagger)$ in terms of the $I_{p,q}(\Sigma')$.

These three methods are related in a number of ways. One may hope that some of the ideas can be applied in 4 dimensions.

References

[1] M. Thaddeus. "Conformal Field Theory and the cohomology of the moduli space of stable bundles" To appear in Jour. Diff. Geom.

[2] E. Witten. "On Quantum gauge theories in two dimensions" Institute for Advanced Study Preprint.

Titel: Mordell-Weil Lattices: Theory and Applications

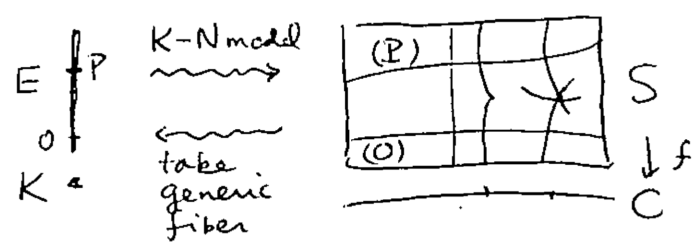
Autor: Tetsuji Shioda

Adresse: MPI, Bonn / Rikkyo Univ. (Tokyo, Japan)
(5–8/91)

Let E/K be an elliptic curve, $K = k(C)$ the function field of a curve C/k. Then the group of K-rational points of E, $E(K)$, is finitely generated under some mild assumption like (*) below, by the functional analogue of the Mordell-Weil theorem (cf. [L], [Se], [Si]).

The basic idea of Mordell-Weil lattices (abr. MWL) is 1) to view the Mordell–Weil group $E(K)$ as a lattice with respect to a natural pairing, which is defined in terms of the Intersection Theory on the associated elliptic surface and which turns out to coincide with the Néron-Tate canonical height. Another basic idea is 2) to view the MWL as a Galois lattice. This lead to Gal. $\mathbb{Z}$-rep. and alg. equations using pts of ∞ order.

I). First assume that k is alg. closed. For simplicity, we let $K = k(t)$ $(C = \mathbb{P}^1)$. Given E/K, one can uniquely associate an elliptic surface $f: S \to C$, called the Kodaira-Néron model of E/K.



(S = smooth proj. surface
f: rel. minimal elliptic fibration)

Almost all fibres of f are elliptic curves, and there are finite number of singular fibres.
The K-rational points $P \in E(K)$ correspond bijectively to the sections $s: C \to S$. Thus we identify $s = P$.

Assume (*) f is not smooth, i.e. there is at least one singular fibre. Then we have a natural isom.

$$(\#)\qquad E(K) \simeq NS(S)/T$$

$$\begin{cases} P \longmapsto (P) \bmod T, \\ \mathrm{sum}(D|_E) \longleftarrow D \end{cases} \qquad \left(\begin{array}{l}(P)\subset S \text{ is the} \\ \text{curve defined by} \\ \text{the section } P: C\to S\end{array}\right)$$

Here $NS(S)$ is the Néron-Severi group of S, that is, $\{$divisors $D=\sum n_i\Gamma_i \mid n_i\in\mathbb{Z},\ \Gamma_i$: irred. curves on $S\}$/alg. equiv., which is finitely generated and torsionfree under (*), and T is the subgroup spanned by $\{(O),\ F=$ fibre, $\forall$ irred. comp. of fibres$\}$. In particular, $r = \mathrm{rk}\, E(K) = \mathrm{rk}\, NS(S) - \mathrm{rk}\, T = \rho - (2 + \sum_{v\in R}(m_v - 1))$ $\left(\begin{array}{l}\rho = \text{Picard number},\ m_v = \#\text{irred. comp. of } f^{-1}(v) \\ R = \{v \mid f^{-1}(v) \text{ reducible}\}\end{array}\right)$.

Now $NS(S)$ is an indefinite integral lattice w.r.t. the intersection pairing $(\cdot)$, with signature $(1, \rho-1)$, (Hodge index th.), and T is a sublattice having a direct sum decomp. $T = \langle (O), F\rangle \oplus \bigoplus_{v\in R} T_v$, where each T_v is a root lattice of type A_n, D_n or E_6, E_7 or E_8.

<u>Lemma 1</u> There is a unique map $\varphi: E(K) \to NS(S)\otimes\mathbb{Q}$ s.t. $\varphi(P) \equiv (P) \bmod T\otimes\mathbb{Q}$, $\mathrm{Im}(\varphi) \perp T$.

φ is a group homo. and $\mathrm{Ker}(\varphi) = E(K)_{tor}$.

<u>Lemma 2</u> $T^{\perp}$ is a negative-definite even integral lattice. (Use Hodge index th., adj. formula, can. bdle f.)

<u>Theorem</u> Define $\langle P, Q\rangle = -(\varphi(P)\cdot\varphi(Q))$ $(P, Q\in E(K))$. Then $(E(K)/E(K)_{tor}, \langle\,,\,\rangle)$ is a positive-definite lattice, which we call $\boxed{\text{MWL}}$ of E/K (or $f: S\to C$).

The explicit formula of the "height pairing" $\langle\,,\,\rangle$ is:

(1) $$\langle P, P\rangle = 2\chi + 2(PO) - \sum_{v\in R} \mathrm{contr}_v(P)$$

where χ is ~~the~~ arithmetic genus of S (>0 under (*)), (PO) = the intersection number of the sections (P) and (O), and the "local contribution" term $\mathrm{contr}_v(P)$ is a non-neg. rational number determined by the type of the sing. fibre $f^{-1}(v)$ and the position of the comp. hit by (P).

Let $E(K)^0 = \{P \in E(K) \mid (P)$ passes through the identity comp. of every fibre$\}$

This is a subgroup of finite index in $E(K)$. Then

(2) $P \in E(K)^0 \Rightarrow \langle P, P\rangle = 2\chi + 2(PO)$ $(\in 2\mathbb{Z})$.

Hence $(E(K)^0, \langle\,,\,\rangle)$ is a pos-def. even integral lattice, which we call the $\boxed{\text{narrow MWL}}$. Moreover

$$\min_{P\neq 0} \langle P, P\rangle \geq 2\chi \quad (\text{since } (PO)\geq 0 \text{ for } {}^\forall P\neq 0).$$

Also ~~the~~ formula (1) or (2) shows that our pairing $\langle\,,\,\rangle$ coincides with Néron-Tate height.

As the first example, suppose that S is a rational ell. surface. Then $r = 8 - (\Sigma_v(m_v-1)) \leq 8$, and $\chi = 1$.

Structure Theorem.

	$r=8$	$r=7$	$r=6$	
$E(K) \cong$	E_8	E_7^*	E_6^*	D_6^*
$\cup$	$\parallel$	$\cup$ 2	$\cup$ 3	$\cup$ 4 (index)
$E(K)^0 \cong$	E_8	E_7	E_6	D_6
rem. conf	$R=\phi$	$R=\{v\}, m_v=2$	", $m_v=3$	$R=\{v,v'\}, m_v=m_{v'}=2$

Here E_r, D_r are the root lattices, and $*$ the dual lattices. As a consequence, we obtain effective generators of $E(K)$.

For the above, see [S1], [S2].

II) Next we consider the new situation:

$$\begin{cases} k_0 = \text{perfect field } (\text{e.g. } \mathbb{Q}, \mathbb{Q}(\lambda), \cdots) \\ k = \text{alg. closure of } k_0, \quad K = k(t) \\ E/k_0(t), \quad E(K) = E(k(t)) \text{ with } Gal(k/k_0)\text{-action.} \end{cases}$$

Then we get the <u>Galois representation</u> (cf. [S6])

$$\rho : Gal(k/k_0) \longrightarrow Aut(E(K), \langle , \rangle) = \text{a finite group.}$$

The ext. $\tilde{k}/k_0$ corresp. to $Ker(\rho)$ is called the splitting field; we have $Gal(\tilde{k}/k_0) = Im(\rho)$ and $\tilde{k}$ = the smallest ext. of k_0 s.t. $E(\tilde{k}(t)) = E(k(t))$.

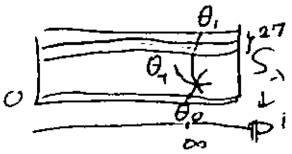
III) To see the nature of the Gal. rep., we pass to more concrete cases which lead to a unified approach to E_6, E_7, E_8. We treat the case E_6 here.

Consider the elliptic curve $E = E_\lambda$ defined by

$$(3) \qquad y^2 = x^3 + x\Big(\sum_0^2 p_i t^i\Big) + \Big(\sum_0^2 q_j t^j + t^4\Big),$$

$$\lambda = (p_0, p_1, p_2, q_0, q_1, q_2) \in \mathbb{A}^6, \quad k_0 = \mathbb{Q}(\lambda) = \mathbb{Q}(p_i, q_j).$$

[This equation defines the "univ. deformation" of E_6-sing. $y^2 = x^3 + t^4$ at $\lambda = 0$.] The assoc. ell. surface $f : S_\lambda \to \mathbb{P}^1$ is rational and has a reducible fibre of type IV at ~~(t=)~~ ∞.



Assume (#) f has no other reducible fibres than $f^{-1}(\infty)$.

Then $E(K) \cong E_6^*$ by Sh. Th. and hence we have

$$\rho_\lambda : Gal(k/k_0) \longrightarrow Aut(E_6) = W(E_6)\cdot\{\pm 1\} \subset GL_6(\mathbb{Z}),$$

where $W(E_6)$ is the Weyl group, $|W(E_6)| = 2^7.3^4.5$, which contains a simple group of index 2 ($\simeq U_4(2)$ in "Atlas"). In fact, we have $\rho_\lambda : Gal(k/k_0) \longrightarrow W(E_6)$.

Now E_6^* has 54 min. vectors of norm $\frac{4}{3}$. On the other hand,

$$\langle P,P\rangle = 2\chi + 2(PO) - \mathrm{contr}_\infty(P) \;,\; (=0 \text{ or } \tfrac{2}{3})$$
$$\geqq 4/3 \;;\; = \Leftrightarrow (PO)=0 \;\&\; (P) \text{ meets } \Theta_1 \text{ or } \Theta_1.$$

It follows that there exist exactly 27 $P_i \in E(K)$ s.t.

(4) $P_i = (\boxed{a_i}t+b_i,\ t^2+d_it+e_i)$ $(a_i,\cdots,e_i\in k)$ $(1\le i\le 27)$.

Choose P_i s.t. $P_1,\cdots,P_6$ form a $\mathbb{Z}$-basis of $E(K)\cong E_6^*$.

Theorem Assume λ generic $/\mathbb{Q}$ (p_i, q_j alg. indep$/\mathbb{Q}$).

Then i) $\rho_\lambda : Gal(\overline{\mathbb{Q}(\lambda)}/\mathbb{Q}(\lambda)) \longrightarrow W(E_6)$ is surjective.

ii) let $\mathcal{K}_\lambda$ be the spl. field. Then

$$\left.\begin{array}{l}\mathcal{K}_\lambda = \mathbb{Q}(a_1,\cdots,a_6)\\ \quad\cup\\ \mathbb{Q}(\lambda)=\mathbb{Q}(p_0,\cdots q_2)\end{array}\right\}\ Gal.\,gr.\,W(E_6)$$

iii) $\mathbb{Q}[a_1,\cdots,a_6]^{W(E_6)} = \mathbb{Q}[p_0,\cdots,q_2]$

iv) p_i or q_j = explicit fund. inv. of $W(E_6)$. (cf. [S5])

(Idea of proof. Substitute (4) into (3), and use explicit elimination. Then a_i will be the zeros of an alg. eq. $\Phi(X,\lambda)=0$ of deg. 27, with $\Phi(X,\lambda)\in\mathbb{Z}[\lambda][X]$. Comparing the coeff. of X^d $(d=2,5,6,8,9,12)$, prove (iv), (iii), $\cdots$.)

IV) Applications ⓐ By specializing λ to $\lambda\in\mathbb{Q}^6$ and applying the Hilbert's irreducibility theorem, we have (cf. [S6])

Th. For most $\lambda\in\mathbb{Q}^6$, $\mathcal{K}_\lambda/\mathbb{Q}$ is a Gal. ext. with $Gal = W(E_6)$. More precisely, those λ form the complement of a thin set in the sense of [Se]. Further every $W(E_6)$-ext. of $\mathbb{Q}$ arises this way from MWL of E_λ.

ⓑ By specializing to $(a_i)\in\mathbb{Q}^6$, we get ([S5])

Th For $\forall(a_1,\cdots a_6)\in\mathbb{Q}^6$ s.t. "discriminant" $\delta(a)\neq 0$, there is an ell. curve $E/\mathbb{Q}(t)$ of rank 6, having the generators $\{P_1,\cdots,P_6\}$ with $P_i=(\boxed{a_i}t+b_i,\ t^2+d_it+e_i)$.

For both ⓐ and ⓑ, we can give numerical examples. The same idea works for E_7 and E_8.

Appl. to 27 lines on a cubic surface.

Let V_λ denote the cubic surface in $\mathbb{P}^3$:

$$V_\lambda : Y^2W + 2YZ^2 = X^3 + X\Big(\sum_0^2 p_i Z^i W^{2-i}\Big) + \sum_0^2 q_j Z^j W^{3-j}.$$

This is a smooth surface iff (#) holds. Assume this. Then V_λ is obtained from the elliptic surface S_λ by blowing down (O), Θ_0, Θ_{-1} in this order, and the 27 minimal (P_i) (meeting Θ_1) are mapped to 27 lines l_i, whose equation is:

$$l_i \quad \begin{cases} X = a_i Z + b_i W \\ Y = d_i Z + e_i W \end{cases} \qquad (1 \le i \le 27).$$

The above results ⓐ, ⓑ can be translated into the results on cubic surfaces.

For a little more details, see [S6], and for deform. of E_r-sing, [S4].

Application of MWL to Sphere packings. For this, see the excellent report of Elkies in the 29th Arbeitstagung.

REFERENCES (more ref. in [S:]).

[E] Elkies, N.: On Mordell-Weil lattices, 1990 Arbeitstagung

[L] Lang, S.: Fundamentals of Diophantine Geometry, Springer ('83)

[Se] Serre, J-P.: Lectures on the Mordell-Weil theorem, Vieweg ('89)

[Si] Silverman, J.H: The arithmetic of ell. curves, Springer ('86)

[S] Shioda, T.: A Collection: Mordell-Weil lattices, MPI '91; this contains 8 papers on MWL and related subjects, esp. [S1] MWL & Gal Rep. [S2] On the MWL, [S5] Construction of ell. curves with high rank via the invariants of the Weyl groups. [S6] Theory of MWL. (ICM90). [S4] MWL of type E_8 and deformation of sing.

The mth polylogarithm function Li_m, defined for $z \in \mathbf{C}$, $|z| < 1$ by the power series

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m},$$

is a classical non-elementary function which was studied by such mathematicians as Euler, Abel, Kummer, Lobachevsky and Ramanujan, and which has turned out in recent years to be connected with many interesting questions in topology, hyperbolic geometry, number theory, K-theory, and arithmetic algebraic geometry. For $m = 1$, of course, $\mathrm{Li}_1(z) = -\log(1-z)$ is simply the ordinary logarithm function developed in a Taylor series around 1, while the functions of higher order are obtained by successive integration: $\mathrm{Li}_m(z) = \int_0^z z^{-1}\,\mathrm{Li}_{m-1}(z)\,dz$. This latter definition gives the analytic continuation to all z in the cut plane $\mathbf{C} \smallsetminus [0,\infty]$ (or more invariantly, to the universal cover of $\mathbf{C} \smallsetminus \{0, 1\} = \mathbf{P}^1(\mathbf{C}) \smallsetminus \{0, 1, \infty\}$). For every natural number N, $\mathrm{Li}_m(z)$ satisfies the ***distribution relation*** $\mathrm{Li}_m(z^N) = N^{m-1} \sum \mathrm{Li}_m(\zeta z)$ (sum over all Nth roots of unity ζ), and there is also a functional equation expressing $\mathrm{Li}_m(z^{-1})$ in terms of $\mathrm{Li}_m(z)$. The latter equation can be used to compute $\mathrm{Li}_m(z)$ for $|z| \gg 1$, while for z near the unit circle (say $0.005 < |z| < 200$) a convenient method of calculation is afforded by the formula

$$\mathrm{Li}_m(e^x) = \sum_{\substack{n=0\\ n\neq m-1}}^{\infty} \zeta(m-n)\,\frac{x^n}{n!} - \Bigl(1 + \frac{1}{2} + \cdots + \frac{1}{m-1} - \log(-x)\Bigr)\frac{x^{m-1}}{(m-1)!}.$$

Besides the functional equations mentioned above, which hold for all orders m, the polylogarithms of low orders are known to satisfy certain functional equations with a far more interesting algebraic structure. It is this structure which makes the functions so interesting and in particular which is responsible for the connections to algebraic K-theory. In particular, the dilogarithm Li_2 satisfies a functional equation under $z \mapsto 1-z$ which together with the functional equation with respect to $z \mapsto 1/z$ already mentioned gives it a 6-fold symmetry, and also satisfies a much more interesting two-variable equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\Bigl(\frac{1-x}{1-xy}\Bigr) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\Bigl(\frac{1-y}{1-xy}\Bigr) = \cdots,$$

where "$\cdots$" on the right is a certain combination of products of logarithms. The functional equations become "clean" if we replace Li_2 by the *Bloch-Wigner function*

$D(z) = \Im\big(\mathrm{Li}_2(z) - |z|\,\mathrm{Li}_1(z)\big)$: $D(z)$ simply changes sign under $z \mapsto 1-z$ or $z \mapsto 1/z$, and in the 5-term equation just written one can replace "$\cdots$" by 0 if Li_2 is replaced everywhere by D. This 5-term equation, which replaces the fundamental functional equation $\mathrm{Li}_1(x) + \mathrm{Li}_1(y) = \mathrm{Li}_1(x+y-xy)$ of the unilogarithm, has a beautiful interpretation in terms of 3-dimensional hyperbolic geometry, as follows. Consider tetrahedra in the 3-dimensional hyperbolic (Lobachevsky) space H_3. Formulas of Lobachevsky imply that the volume of any such tetrahedron can be expressed in closed form in terms of values of the function D and in particular that the volume of an *ideal tetrahedron* Δ (one with all its vertices at infinity, i.e. in the boundary $\mathsf{P}^1(\mathsf{C})$ of H_3) equals $D(z(\Delta))$, where $z(\Delta)$ is the cross-ratio of the four vertices of Δ (this is invariant under the action of the isometry group $PGL(2,\mathsf{C})$ of H_3 on $\partial\mathsf{H}_3 = \mathsf{P}^1(\mathsf{C})$). The 6-fold symmetry of $D(z)$ now reflects the fact that four complex numbers have 6 different cross-ratios depending on the order in which they are taken, and the 5-term two-variable equation expresses the fact that the sum of the volumes of the 5 tetrahedra whose vertices are 4-subsets of a set of 5 points in $\mathsf{P}^1(\mathsf{C})$ vanishes (take the points to be 0, 1, ∞, x, and y).

Because any complete hyperbolic 3-manifold M, possibly after removing a set of measure 0, can be triangulated by ideal tetrahedra Δ_i, we can express the volume of M as a finite sum of values $D(z_i)$, $z_i = z(\Delta_i)$. This is of interest because the set of volumes of complete hyperbolic 3-manifolds is known by rigidity theorems to be a countable set and we would like to know this "volume spectrum," but not yet a very useful statement because the set of sums $\sum D(z_i)$ with complex arguments z_i is clearly equal to all of R. However, it follows from results of Dupont-Sah or of Neumann-Zagier that the arguments z_i of an arbitrary ideal triangulation of M satisfy the algebraic relation

$$\sum_i (z_i) \wedge (1 - z_i) = 0\,, \tag{*}$$

where the value $(z) \wedge (1-z)$ is to be interpreted as an element of the exterior square of the abelian group C^*, written additively as a Z-module. As an example, an explicit triangulation of the complement of the [redacted] link

given by Thurston implies that this complement, which has a hyperbolic structure, has

volume equal to $4D(a)+2D(b)$, where $a=\dfrac{1+\sqrt{-7}}{2}$ and $b=\dfrac{-1+\sqrt{-7}}{4}$, and the two identities $ab=-1$, $1-b=(1-a)^2b$ imply $4(a)\wedge(1-a)+2(b)\wedge(1-b)=0$. (Check this!) It turns out that the set of sums $\sum D(z_i)$ for collections of complex mumbers $\{z_i\}$ satisfying (*) is countable, overcoming the objection made above. Moreover, by restricting to the subset of arithmetic hyperbolic manifolds, the volumes of which are known to be expressible as simple multiples of the value at $s=2$ of the Dedekind zeta functions of algebraic number fields, one obtains the theorem that *the value of $\zeta_F(2)$ for any number field F can be expressed in terms of a finite number of values of the function $D(z)$ at algebraic arguments z.* For instance, one can show that the link complement above is a finite cover of the manifold $\mathbf{H}_3/SL(2,\mathcal{O})$, where $\mathcal{O}=\mathbf{Z}[a]$ is the ring of integers of $F=\mathbf{Q}(\sqrt{-7})$, and using this one shows that the value of $\zeta_F(2)$ for this particular number field is $2\pi^2/21\sqrt{7}$ times the volume $4D(a)+2D(b)$ of M.

The bulk of the talk was devoted to explaining the extensions of this last result to higher m. There is a modification $P_m(z)$ of the mth order polylogarithm $\mathrm{Li}_m(z)$ analogous to the modification $D(z)$ of $\mathrm{Li}_2(z)$, e.g., $P_3(z)=\Re\big(\mathrm{Li}_3(z)-\log|z|\,\mathrm{Li}_2(z)+\frac{1}{3}\log^2|z|\,\mathrm{Li}_1(z)\big)$, and one can formulate a rather precise form of the conjecture that *the value of $\zeta_F(m)$ for any number field F can be expressed as a finite combination of values of $P_m(z)$ at arguments z belonging to F* (in the various embeddings of F into $\mathbf{C}$). The key to the whole structure is the generalization to higher orders of the relation (*), which tells us which combinations of arguments one should look at. The details of the conjecture, as well as many examples, are given in my survey paper cited below, and will not be repeated here. This leads to a description in terms of algebraic K-theory (more precisely, in terms of the group $K_{2m-1}(F)$), the value of $\zeta_F(m)$ entering as the covolume of this K-group considered as a lattice in Euclidean space via the Borel regulator mapping. The conjecture is completely proved for $m=3$ by Goncharov (MPI preprint, 1990), while for higher m one at least knows that there is a map from an appropriate "polylogarithm group" to K-theory such that the Borel regulator map is expressed in terms of polylogarithms (Beilinson and Deligne, in preparation), so that the only thing still needed for the conjecture is the surjectivity of this map, which can be checked for any given F and m by a finite calculation. The main ingredient needed to extend Goncharov's proof to higher m would be a full theory of functional equations, but this is still missing. Kummer gave some functional equations for $m=3$, 4, and 5 in 1840, and H. Gangl (Bonn) has found functional equations for $m=6$ and 7 in the last 2 years, but for $m>3$ one does not have a complete set of functional equations and for

$m > 7$ no non-trivial functional equation is known at all.

The talk closed with a brief discussion of a generalization of the polylogarithm to elliptic curves, due to Bloch, Levin, and Beilinson; we do not describe this here.

L. Lewin, *Polylogarithms and Associated Functions*, North Holland 1981

D. Zagier, Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields, in *Arithmetic Algebraic Geometry* (eds. G. van der Geer, F. Oort, J. Steenbrink), Prog. in Math. **89**, Birkhäuser, Boston 1991, pp. 391–430.

↑ Open here ↑

Please read the enclosed yellow page first.

ecturer's name: D. Zagier

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D – 5300 Bonn 3

4 Rahmenseiten,
4 Textseiten
(4 verkleinerte Textseiten

The mth polylogarithm function Li_m, defined for $z \in \mathbf{C}$, $|z| < 1$ by the power series

$$\mathrm{Li}_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m},$$

is a classical non-elementary function which was studied by such mathematicians as Euler, Abel, Kummer, Lobachevsky and Ramanujan, and which has turned out in recent years to be connected with many interesting questions in topology, hyperbolic geometry, number theory, K-theory, and arithmetic algebraic geometry. For $m = 1$, of course, $\mathrm{Li}_1(z) = -\log(1-z)$ is simply the ordinary logarithm function developed in a Taylor series around 1, while the functions of higher order are obtained by successive integration: $\mathrm{Li}_m(z) = \int_0^z z^{-1}\,\mathrm{Li}_{m-1}(z)\,dz$. This latter definition gives the analytic continuation to all z in the cut plane $\mathbf{C} \setminus [0,\infty]$ (or more invariantly, to the universal cover of $\mathbf{C} \setminus \{0,\, 1\} = \mathbf{P}^1(\mathbf{C}) \setminus \{0,\, 1,\, \infty\}$). For every natural number N, $\mathrm{Li}_m(z)$ satisfies the *distribution relation* $\mathrm{Li}_m(z^N) = N^{m-1} \sum \mathrm{Li}_m(\zeta z)$ (sum over all Nth roots of unity ζ), and there is also a functional equation expressing $\mathrm{Li}_m(z^{-1})$ in terms of $\mathrm{Li}_m(z)$. The latter equation can be used to compute $\mathrm{Li}_m(z)$ for $|z| \gg 1$, while for z near the unit circle (say $0.005 < |z| < 200$) a convenient method of calculation is afforded by the formula

$$\mathrm{Li}_m(e^x) = \sum_{\substack{n=0 \\ n \neq m-1}}^{\infty} \zeta(m-n)\,\frac{x^n}{n!} - \Big(1 + \frac{1}{2} + \cdots + \frac{1}{m-1} - \log(-x)\Big)\frac{x^{m-1}}{(m-1)!}.$$

Besides the functional equations mentioned above, which hold for all orders m, the polylogarithms of low orders are known to satisfy certain functional equations with a far more interesting algebraic structure. It is this structure which makes the functions so interesting and in particular which is responsible for the connections to algebraic K-theory. In particular, the dilogarithm Li_2 satisfies a functional equation under $z \mapsto 1 - z$ which together with the functional equation with respect to $z \mapsto 1/z$ already mentioned gives it a 6-fold symmetry, and also satisfies a much more interesting two-variable equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2\Big(\frac{1-x}{1-xy}\Big) + \mathrm{Li}_2(1-xy) + \mathrm{Li}_2\Big(\frac{1-y}{1-xy}\Big) = \cdots,$$

where "$\cdots$" on the right is a certain combination of products of logarithms. The functional equations become "clean" if we replace Li_2 by the *Bloch-Wigner function*

$D(z) = \Im\big(\mathrm{Li}_2(z) - |z|\,\mathrm{Li}_1(z)\big)$: $D(z)$ simply changes sign under $z \mapsto 1-z$ or $z \mapsto 1/z$, and in the 5-term equation just written one can replace "$\cdots$" by 0 if Li_2 is replaced everywhere by D. This 5-term equation, which replaces the fundamental functional equation $\mathrm{Li}_1(x) + \mathrm{Li}_1(y) = \mathrm{Li}_1(x+y-xy)$ of the unilogarithm, has a beautiful interpretation in terms of 3-dimensional hyperbolic geometry, as follows. Consider tetrahedra in the 3-dimensional hyperbolic (Lobachevsky) space $\mathbf{H}_3$. Formulas of Lobachevsky imply that the volume of any such tetrahedron can be expressed in closed form in terms of values of the function D and in particular that the volume of an *ideal tetrahedron* Δ (one with all its vertices at infinity, i.e. in the boundary $\mathbf{P}^1(\mathbf{C})$ of $\mathbf{H}_3$) equals $D(z(\Delta))$, where $z(\Delta)$ is the cross-ratio of the four vertices of Δ (this is invariant under the action of the isometry group $PGL(2,\mathbf{C})$ of $\mathbf{H}_3$ on $\partial\mathbf{H}_3 = \mathbf{P}^1(\mathbf{C})$). The 6-fold symmetry of $D(z)$ now reflects the fact that four complex numbers have 6 different cross-ratios depending on the order in which they are taken, and the 5-term two-variable equation expresses the fact that the sum of the volumes of the 5 tetrahedra whose vertices are 4-subsets of a set of 5 points in $\mathbf{P}^1(\mathbf{C})$ vanishes (take the points to be 0, 1, ∞, x, and y).

Because any complete hyperbolic 3-manifold M, possibly after removing a set of measure 0, can be triangulated by ideal tetrahedra Δ_i, we can express the volume of M as a finite sum of values $D(z_i)$, $z_i = z(\Delta_i)$. This is of interest because the set of volumes of complete hyperbolic 3-manifolds is known by rigidity theorems to be a countable set and we would like to know this "volume spectrum," but not yet a very useful statement because the set of sums $\sum D(z_i)$ with complex arguments z_i is clearly equal to all of $\mathbf{R}$. However, it follows from results of Dupont-Sah or of Neumann-Zagier that the arguments z_i of an arbitrary ideal triangulation of M satisfy the algebraic relation

$$\sum_i (z_i) \wedge (1 - z_i) = 0\,, \qquad (*)$$

where the value $(z) \wedge (1-z)$ is to be interpreted as an element of the exterior square of the abelian group $\mathbf{C}^*$, written additively as a $\mathbf{Z}$-module. As an example, an explicit triangulation of the complement of the ~~knot~~ link

given by Thurston implies that this complement, which has a hyperbolic structure, has

volume equal to $4D(a)+2D(b)$, where $a=\dfrac{1+\sqrt{-7}}{2}$ and $b=\dfrac{-1+\sqrt{-7}}{4}$, and the two identities $ab=-1$, $1-b=(1-a)^2b$ imply $4(a)\wedge(1-a)+2(b)\wedge(1-b)=0$. (Check this!) It turns out that the set of sums $\sum D(z_i)$ for collections of complex mumbers $\{z_i\}$ satisfying (*) is countable, overcoming the objection made above. Moreover, by restricting to the subset of arithmetic hyperbolic manifolds, the volumes of which are known to be expressible as simple multiples of the value at $s=2$ of the Dedekind zeta functions of algebraic number fields, one obtains the theorem that *the value of $\zeta_F(2)$ for any number field F can be expressed in terms of a finite number of values of the function $D(z)$ at algebraic arguments z.* For instance, one can show that the link complement above is a finite cover of the manifold $\mathsf{H}_3/SL(2,\mathcal{O})$, where $\mathcal{O}=\mathbf{Z}[a]$ is the ring of integers of $F=\mathbf{Q}(\sqrt{-7})$, and using this one shows that the value of $\zeta_F(2)$ for this particular number field is $2\pi^2/21\sqrt{7}$ times the volume $4D(a)+2D(b)$ of M.

The bulk of the talk was devoted to explaining the extensions of this last result to higher m. There is a modification $P_m(z)$ of the mth order polylogarithm $\mathrm{Li}_m(z)$ analogous to the modification $D(z)$ of $\mathrm{Li}_2(z)$, e.g., $P_3(z)=\Re\big(\mathrm{Li}_3(z)-\log|z|\,\mathrm{Li}_2(z)+\frac{1}{3}\log^2|z|\,\mathrm{Li}_1(z)\big)$, and one can formulate a rather precise form of the conjecture that *the value of $\zeta_F(m)$ for any number field F can be expressed as a finite combination of values of $P_m(z)$ at arguments z belonging to F* (in the various embeddings of F into $\mathbf{C}$). The key to the whole structure is the generalization to higher orders of the relation (*), which tells us which combinations of arguments one should look at. The details of the conjecture, as well as many examples, are given in my survey paper cited below, and will not be repeated here. This leads to a description in terms of algebraic K-theory (more precisely, in terms of the group $K_{2m-1}(F)$), the value of $\zeta_F(m)$ entering as the covolume of this K-group considered as a lattice in Euclidean space via the Borel regulator mapping. The conjecture is completely proved for $m=3$ by Goncharov (MPI preprint, 1990), while for higher m one at least knows that there is a map from an appropriate "polylogarithm group" to K-theory such that the Borel regulator map is expressed in terms of polylogarithms (Beilinson and Deligne, in preparation), so that the only thing still needed for the conjecture is the surjectivity of this map, which can be checked for any given F and m by a finite calculation. The main ingredient needed to extend Goncharov's proof to higher m would be a full theory of functional equations, but this is still missing. Kummer gave some functional equations for $m=3$, 4, and 5 in 1840, and H. Gangl (Bonn) has found functional equations for $m=6$ and 7 in the last 2 years, but for $m>3$ one does not have a complete set of functional equations and for

$m > 7$ no non-trivial functional equation is known at all.

The talk closed with a brief discussion of a generalization of the polylogarithm to elliptic curves, due to Bloch, Levin, and Beilinson; we do not describe this here.

L. Lewin, *Polylogarithms and Associated Functions*, North Holland 1981

D. Zagier, Polylogarithms, Dedekind zeta functions, and the algebraic K-theory of fields, in *Arithmetic Algebraic Geometry* (eds. G. van der Geer, F. Oort, J. Steenbrink), Prog. in Math. **89**, Birkhäuser, Boston 1991, pp. 391–430.

Titel: Maps between classifying spaces of compact Lie groups
Autor: Bob Oliver
Adresse: Aarhus (z.Zt. Heidelberg)

This summary describes joint work with Stefan Jackowski and Jim McClure. It involves a program to study the homotopy theory of BG, for G a compact Lie group, by approximating BG as a homotopy direct limit of simpler spaces. In particular, we build on the program of Adams and Mahmud [AM], and develop procedures which in many cases can be used to classify homotopy classes of maps $f: BG \longrightarrow BG'$ when G and G' are compact Lie and G' is connected.

Here, I focus attention on the following result:

<u>Theorem 1</u> [JMO] Let G be any compact connected simple Lie group. Then the map

$$[BG, BG] \longrightarrow \mathrm{End}(H^*(BG;\mathbb{Q}))$$

is injective; i.e., homotopy classes of selfmaps of BG are detected by rational cohomology.

Note that Theorem 1 fails in general for semisimple G: ~~For~~ For example, one can construct homomorphisms $\rho, \rho': SU(3) \longrightarrow SO(8)$ such that

$B\rho \not\simeq B\rho'$ but $H^*(B\rho) = H^*(B\rho')$. Hence Theorem 1 fails when $G = SU(3) \times SO(8)$.

Theorem 1 is based on a new approximation of BG at the prime p, for any compact Lie group G and any prime p, described as follows. A group P is called <u>p-toral</u> if its identity component P_0 is a torus, and P/P_0 is a finite p-group. Consider the orbit category $\mathcal{O}(G)$, where

$$Ob(\mathcal{O}(G)) = \{G/H : H \leq G \text{ closed subgroup}\}$$

$$Mor(G/H, G/K) = \{\text{all } G\text{-equivariant maps}\}.$$

Let $\mathcal{R}_p(G) \subseteq \mathcal{O}(G)$ be the full subcategory on orbits G/P where

(1) P is p-toral

(2) $|N(P)/P| < \infty$

(3) $\nexists\ 1 \neq Q \trianglelefteq N(P)/P$, Q a p-subgroup.

For example, $\mathcal{R}_2(SO(3))$ contains just two (isomorphism classes of) objects:

$$SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2) \quad \text{and} \quad SO(3)/O(2).$$

In general, $\mathcal{R}_p(G)$ is a finite category, in that it contains finitely many isomorphism classes (orbit types), and finite morphism sets.

Our main approximation theorem is the following:

Theorem 2 For any G, p, there is a $\mathbb{Z}_{(p)}$-homology equivalence

$$\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} (EG/P) \longrightarrow BG .$$

Note that $EG/P \simeq BP$; so Theorem 2 says that BG is approximated, at p, as the homotopy direct limit of BP's for p-toral $P \subseteq G$.

When P is p-toral, the mapping spaces $\mathrm{map}(BP, BG)$ have been studied by Dwyer & Zabrodsky and Notbohm. In particular, they showed the following:

Theorem 3 [DZ][N] For any p-toral group P and any compact Lie group G, the map

$$\mathrm{Rep}(P,G) := \mathrm{Hom}(P,G)/\mathrm{Inn}(G) \xrightarrow{\ \cong\ } [BP, BG]$$

$(\rho \mapsto B\rho)$ is bijective.

The idea of the proof of Theorem 1 can now be sketched. Fix simple G, let T be a maximal torus, and let $N_p(T)/T \subseteq W = N(T)/T$ be a ~~p~~Sylow p-subgroup. Assume $f, f' : BG \longrightarrow BG$ are such that $H^*(f;\mathbb{Q}) = H^*(f';\mathbb{Q})$. We first apply Theorem 3 to show that $f|BT \simeq f'|BT$. ~~and then~~

Now fix some prime p. We apply Theorem 3 again to show that $\cancel{f|N_p(T) \simeq f'|}$

$$f|BN_p(T) \simeq f'|BN_p(T)$$

(this is more complicated, and involves applying Theorem 3 to various p-toral subgroups $P \subseteq N_p(T)$.)
Any p-toral subgroup $P \le G$ is conjugate to a subgroup of $N_p(T)$, so $f|BP \simeq f'|BP$ for all P. Since

$$[BG, BG^\wedge_p] \cong [\underrightarrow{\mathrm{hocolim}}\,(\underset{\overset{\simeq}{BP}}{EG/P}), BG^\wedge_p]$$

by Theorem 2, the ~~computation~~ vanishing of certain inverse limits now shows that

$$f^\wedge_p \simeq f'^\wedge_p : BG \longrightarrow BG^\wedge_p .$$

This holds for each p, and Sullivan's arithmetic square for BG can now be used to show that $f \simeq f'$.

Upon combining Theorem 1 with earlier work of Sullivan, Wilkerson, Hubbuck, and Ishiguro, we get the following description of $[BG, BG]$:

<u>Theorem 4</u> [JMO] For any compact connected simple G, there is a bijection (monoid isomorphism)

$$[BG,BG] \cong \{0\} \amalg \left(\mathrm{Out}(G) \times \{k>0 : (k,|W|)=1\}\right).$$

Here, W is the Weyl group, 0 corresponds to null homotopic maps, and (α,k) corresponds to $B\alpha \circ \psi^k$ where ψ^k is an "unstable Adams operation" of degree k.

As mentioned above, these methods can be used to describe $[BG, BG]$ in many other

Autor: Bob Oliver Seite: 5

cases (see, fx, [JMO2]). As another example, $[B\Gamma, BSO(3)]$ can be explicitely described for any finite Γ.

References

{AM} J. F. Adams and Z. Mahmud, Maps between classifying spaces, Inventiones math. 35 (1976), 1-41

{DZ} W. Dwyer and A. Zabrodsky, Maps between classifying spaces, Algebraic topology, Barcelona, 1986, Lecture Notes in Math. 1298, Springer-Verlag (1987), 106-119

{JMO} S. Jackowski, J. McClure, and B. Oliver, Homotopy classification of self-maps of BG via G-actions, Annals of Math. (to appear)

{JMO2} S. Jackowski, J. McClure, and B. Oliver, Maps between classifying spaces revisited (in preparation)

{N} D. Notbohm, Abbildungen zwischen klassifizierenden Ra"umen, Dissertation, Go"ttingen (1988)

Titel: QUANTUM GROUPS AND NON-COMMUTATIVE DE RHAM COMPLEXES

Autor: MANIN Yu. I.

Adresse: STEKLOV INSTITUTE, MOSCOW

MAX-PLANCK-INSTITUTE FÜR MATH., BONN

INTRODUCTION. Let $\mathcal{M}$ be a category of "spaces" (e.g. finite sets, schemes/k, smooth manifolds ...) and F be a functor $\mathcal{M} \to \{\text{Linear spaces}\}$ such that $F(X \times Y) = F(X) \otimes F(Y)$, $F(pt) =$ ground field k. Then $F(X)$ is a commutative k-algebra (resp. co-commutative coalgebra) if F is contravariant (resp. covariant), w.r.t. $F(\Delta_X)$, where $\Delta_X : X \to X \times X$ is the diagonal. If G is a group in $\mathcal{M}$, $F(G)$ becomes a Hopf algebra, and if G acts upon Y, $F(Y)$ becomes a comodule (resp. module) over $F(G)$.

In the theory of quantum groups, one studies general Hopf algebras and their (co)actions upon algebras as if they were functions on "non-commutative", or "quantum", spaces. However, not all classical functors F were treated with equal attention.

Consider the following four functors:

1. Functions Φ.
2. Distribution with finite support U.
3. De Rham complex $\Omega^\bullet$.
4. Cohomology $H^\bullet$.

Quantum versions of U were constructed by V. Drinfeld [1] and M. Jimbo; functions Φ were considered by Faddeev et. al [2] and lecturer [3]; some variants of de Rham complexes were studied by S. L. Woronowicz [4].

In this lecture we shall concentrate upon $\Omega^\cdot$. We start with the Wess-Zumino de Rham complex of the quantum plane and discuss the following properties of quantum de Rham complexes:

- cohomology at roots of unity;
- operator algebras generated by vector fields;
- relations between Wess-Zumino and A. Connes' complexes.

1. WESS-ZUMINO AXIOMS. Let $\Omega^\cdot$ be a DGA over a field k. Assume that $\Omega^\cdot$ is generated by two subalgebras: $A = \Omega^0$ and $B = \underset{i\geq 0}{\oplus} B^i$. Consider the following properties:

$$WZ_i: \qquad A \underset{k}{\otimes} B^i \xrightarrow{mult} \Omega^i \xleftarrow{mult} B^i \underset{k}{\otimes} A$$

are isomorphisms of linear spaces.

$$WZ: \quad WZ_i \text{ are satisfied for all } i \geq 0.$$

The main use of WZ_1 is that it allows to define vector fields. If $B^1 = \overset{n}{\underset{i=1}{\oplus}} k\xi_i$, define $X_i: A \to A$ by $df = \sum_{i=1}^{n} \xi_i X_i(f)$. From the Leibniz formula $d(fg) = df \cdot g + f dg$, $d: A \to \Omega^1$ one obtains the Leibniz formulas for X_i:

$$X_i(fg) = X_i(f)g + \sum_j \sigma_{ij}(f) X_j(g), \qquad (1)$$

where $\sigma_{ij}: A \to A$ are defined by commutation rules implied by WZ_1: $f\xi_j = \sum_i \xi_i \sigma_{ij}(f)$. Since (1) essentially implies the comultiplication formulas

$$\Delta(X_i) = X_i \otimes 1 + \sum_j \sigma_{ij} \otimes X_j, \qquad (2)$$

it "explains" why in the $U_q(\mathfrak{g})$ Cartan generators appear in the exponentiated form. Namely, they correspond to certain σ_{ij} in a realization by vector fields, at least when $\mathfrak{g} = sl$.

2. A UNIVERSAL CONSTRUCTION. Let $(A, x_1, \dots, x_n)$ be a k-algebra together with a finite set of its generators. Consider all digrams $\varphi: A \to B \otimes A$ such that $\varphi(x_i) = \sum_j e_{ij} \otimes x_j$ for certain $e_{ij} \in B$. Among these diagrams there is a universal one $\delta: A \to E \otimes A$. Algebra E is automatically a bialgebra, and δ is a coaction. There exists a universal map $E \to H$ to a Hopf algebra H. If one imagines A as an algebra of functions on a quantum linear space, then E is an algebra of quantum linear endomorphisms of this space, and H is the function algebra of the quantum linear group of A.

This construction can be generalized in several directions:

a) Instead of one algebra (A, x_i) one can take a family $(A^{(\alpha)}, x_i^{(\alpha)})$; δ coacts in the same way upon all $(x_i^{(\alpha)})$.

b) Instead of k-algebras one can consider k-DGA's.

In other words, starting with functions on a quantum space, one can reconstruct functions on its quantum automorphism space; and starting with differential forms, one can reconstruct the differential forms.

3. WESS-ZUMINO COMPLEX OF QUANTUM PLANE. Apply the universal construction to the pair of algebras

$$A_q = k\langle x,y\rangle/(xy - q^{-1}yx); \quad B_p = k\langle \xi,\eta\rangle/(\xi^2, \eta^2, \xi\eta + p\eta\xi),$$

$p, q \in k^*$; $(pq)^2 \neq -1$. We shall obtain the Hopf algebra

$$\Phi[GL_{p,q}(2)] = k\langle a,b,c,d\rangle/(*)[\mathrm{DET}^{-1}]$$

of functions on the two-parametric quantum $GL(2)$, where $(*)$ denotes the following set of commutation relations:

$$\begin{aligned} ab &= p^{-1}ba; \; ac = q^{-1}ca; \; ad = da + (q^{-1} - p)cb; \\ bc &= pq^{-1}cb; \; bd = q^{-1}db; \; cd = p^{-1}dc, \end{aligned} \qquad (*)$$

and $\mathrm{DET} = ad - q^{-1}cb = da - pcb$.

Wess and Zumino interpreted (ξ,η) as differentials of (x,y). More precisely, they have shown that there exist exactly two structures of a DGA-algebra on $B_p \otimes A_q$, compatible with the action of $GL_{p,q}(2)$ and verifying $dx = \xi$, $dy = \eta$. One of the structures is given by the following cross-commutation relations:

$$x\xi = (pq)^{-1}\xi x \; ; \; y\eta = (pq)^{-1}\eta y \; ; \qquad (4)$$
$$x\eta = (p^{-1}q^{-1}-1)\xi y + q^{-1}\eta x ; \; y\xi = p^{-1}\xi y .$$

The other one is obtained by $x \leftrightarrow y$, $\xi \leftrightarrow \eta$, $p \leftrightarrow p^{-1}$, $q \leftrightarrow q^{-1}$. A moral is: de Rham complex of A_q is <u>not</u> uniquely defined by the function ring. There are at least two one-parametric families of natural deRham complexes.

4. <u>DE RHAM COMPLEXES OF 2×2 QUANTUM MATRICES</u>.

Consider now the 4-dim quantum linear space (*). One can try to extend it to a DGA-algebra applying the universal construction of n°2 to one of the two WZ complexes $B_p \otimes A_q = \Omega^{\bullet}_{p,q}$. The resulting DGA-algebra is, however, too big and does not verify WZ. For each of the $\Omega^{\bullet}_{p,q}$'s, it has precisely <u>three</u> quotient DGA's verifying WZ_1 (and, as a consequence, WZ). This was checked by a long direct calculation by D.V. Zhdanovich and the lecturer. Hence (*) has six natural de Rham complexes.

5. <u>DE RHAM COMPLEX OF A LINE</u>. On the "coordinate axis" kx, (4) induces a non-trivial non-commutative differential geometry. Consider $\Omega = k\langle x^{\pm 1}, dx\rangle / (v\, dx.x - v^{-1} x\, dx)$. We have

$$d(x^n) = v^{n-1}[n]_v \, dx.x^{n-1}, \quad [n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}, \; v \in k^*.$$

Hence, if v is not a root of unity, $H^{\bullet}(\Omega^{\bullet}) \simeq k \oplus k\, dx.x^{-1}$, but if $v^{\ell} = 1$, ℓ primitive, we have $H^{\bullet}(\Omega^{\bullet}) \simeq \bigoplus_m k\, x^{m\ell} \oplus k\, dx.x^{m\ell-1}$.

The residue functional $\int dx\,(\Sigma a_i x^i) = a_{-1}$

verifies only a twisted version of Connes' axiom:

$$\int [dx.x^n, x^m]_{v^m} = 0, \text{ where } [x,y]_w = wxy - w^{-1}xy.$$

Put $df = dx \cdot \partial_v f$, and $L_n = v x^{n+1}\partial_v$. The usual (zero charge Virasoro) commutation relations deform to

$$[L_n, L_m]_{v^{n-m}} = [m-n]_v L_{n+m}.$$

However, the Leibniz formula $L_n(fg) = L_n f \cdot g + \sigma(f) L_n g$, $\sigma(x^k) = v^{2k} x^k$, shows that closed comultiplication formula can be introduced only if one replaces L_0 by $K_0 = \sqrt{\sigma}$, which is essentially "exponentiated Cartan generator".

6. SOME STRUCTURAL RESULTS. In conclusion, we state in an imprecise form several structure theorems due to W.-Z., Yu.M, E. Mukhin.

a). Assume A and B given as in n°1. Then various structures of k-algebras upon $A \otimes B$ verifying WZ correspond to certain actions of $E(B)$ upon A, where $E(B)$ is the universally coacting on B bialgebra from n°2. (This is true if $\dim B < \infty$).

b). Let A, B be quadratic algebras, $A_1 = B_1$. Let $M(A,B)$ be the universally coacting on (A,B) bialgebra. Then there is a compatible structure of WZ_1-DGA on a quotient of $A * B$ iff $M(A,B)$ can be defined by a Yang-Baxter operator.

c). If A, B are quadratic algebras satisfying Diamond Lemma and conditions of b) are valid, then the WZ_1-quotient of $A*B$ satisfies the Diamond Lemma, so that WZ_1 implies WZ.

REFERENCES

[1] V. G. Drinfeld. Berkeley ICM talk.

[2]. N. Reshetikhin, L. Takhtajan, L. Faddeev. Algebra i Analiz, 1:1 (1989), 178-206.

[3]. Yu. I. Manin. Quantum groups and non-commutative geometry. Montréal University, 1988.

[4]. S. L. Woronowicz. Comm. Math. Phys. 111 (1987), 613-665

Titel: On Faltings's recent proof of a conjecture of S. Lang
Autor: G. Wüstholz
Adresse: Mathematikdepartement, ETH Zürich
Rämistr. 101
CH-8092 Zürich

0. Lang's conjectures

Let K be a field of finite type over $\mathbb{Q}$, A/K an abelian variety and $X \subseteq A$ a subvariety over K. It is then a very important diophantine problem to determine the set $X(K)$ of K-rational points on X. Clearly if $B \subseteq A$ is an abelian subvariety and $\xi + B \subseteq X$ then $(\xi + B)(K) \subseteq X(K)$ and so if $\xi \in X(K)$ and $B \neq 0$ we get a whole family of K-rational points in general, at least if B is also defined over K. Lang conjectured that these families are the only exceptions and that there are only finitely many of them. More precisely we have (see [L])

Lang's conjecture LC I: $X(K)$ lies in a finite number of translates $x+B$ of abelian subvarieties B with $x+B \subseteq X$.

Now let $\Gamma_0 \subseteq A(\mathbb{C})$ be a finitely generated subgroup and $\Gamma = \{\gamma \in A(\mathbb{C}),\ n\gamma \in \Gamma_0 \text{ for some } n>0, n \in \mathbb{Z}\}$ its division group.

Lang's conjecture LC II: $X(\mathbb{C}) \cap \Gamma$ lies in a finite number of translates $x+B$ of abelian subvarieties B with $x+B \subseteq X$.

1. <u>Special case</u>. We consider first the case that X does not contain a translate of a nontrivial abelian subvariety. Then the relation between LC I and LC II was clarified by Raynaud [R]. Namely he proved

<u>Theorem</u> (Raynaud). For every X, A and K as above LC I implies LC II.

The conjecture LC I in this case was solved by Faltings. Namely we have ([F1])

<u>Theorem</u> (Faltings, '89). $X(K)$ is finite.

Actually he proves it only for number fields but the extension to general K's is standard.

2. <u>General case</u>. We assume now X to be arbitrary. Then LC I was proved recently by Faltings ([F2]).

<u>Theorem</u> (Faltings, '90). For every X, A, K as above LC I holds.

One of the basic tools to deal with translates of abelian subvarieties is the following construction. Let A be an abelian variety, $X \subseteq A$ a subvariety. Then $S(X)$ is defined to be the connected component of the zero element of A of the stabilizer of X.

Then clearly $X+S(X)\subseteq X$ and so we obtain a fibration

$$\pi: X \longrightarrow X/S(X) =: Y.$$

Kawamata [K] proved now the following

<u>Kawamata's structure Theorem</u> ('80) If $S(X)=0$ then there exists a finite set E of proper subvarieties Z of X with nonzero $S(Z)$ such that if $x+B$ is a translate of an abelian subvariety $B\neq 0$ with $x+B\subseteq X$ then

$$x+B \subseteq \bigcup_{Z\in E} Z =: Z(X).$$

<u>3. Heights</u>. Consider $\mathbb{P}^n_{\mathbb{Z}}$, the n-dimensional projective space over $\mathbb{Z}$, $L=\mathcal{O}(1)$ with the Fubini-Study metric and associated curvature form h. Then one defines the arithmetic Chowgroups $CH^p_{Ar}(\mathbb{P}^n)$ for $0\leq p\leq n+1$ and obtains the arithmetic Chow ring $CH_{Ar}(\mathbb{P}^n)=\bigoplus CH^p_{Ar}(\mathbb{P}^n)$. One has intersection pairings

$$CH^p_{Ar}(\mathbb{P}^n)\times CH^q_{Ar}(\mathbb{P}^n)\longrightarrow CH^{p+q}_{Ar}(\mathbb{P}^n)$$

and a degree map

$$\deg: CH^{n+1}_{Ar}(\mathbb{P}^n)\xrightarrow{\sim}\mathbb{R}.$$

Furthermore to every $X\in CH^p(\mathbb{P}^n)$ one attaches an element $\hat{X}\in CH^p_{Ar}(\mathbb{P}^n)$ and gets a map $CH(\mathbb{P}^n)\to CH_{Ar}(\mathbb{P}^n)$. Then if $X\in CH^p(\mathbb{P}^n)$ we define its height as

$$h(X)=\deg(\hat{X}\cdot c_{1,Ar}(L,h')^{n+1-p})$$

where for any meromorphic section $0\neq f$ we put

$$c_{1,Ar}(L,h') = \text{class of } (\mathrm{div}(f), \log h'(f,f)).$$

This has been further developed by W. Gubler in his thesis

4. The index. Let $P = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$, $x \in P$, $L = \mathcal{O}(d_1, \ldots, d_m)$ and $f \in \Gamma(P, L)$. We trivialize L in a neighborhood of x so that we can regard f as a function on this neighborhood. In particular we get differential operators in a neighborhood of the projection of x onto the k-th factor. We denote them by D_k, $1 \leq k \leq m$. Then the index $i(x, f)$ is the maximum over all σ such that

$$D_1 \cdots D_m f(x) = 0$$

for all such differential operators D_k of order $\leq j_k$ for all integers $j_1, \ldots, j_m \geqq 0$ with $j_1/d_1 + \cdots + j_m/d_m < \sigma$. With this we define closed subschemes $Z_\sigma(f)$ by

$$Z_\sigma(f) = \{x \in P;\ i(x, f) \geqq \sigma\}.$$

They satisfy $Z_\sigma(f) \supseteq Z_\tau(f)$ if $\sigma \leq \tau$.

Product Theorem (Faltings). For all $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that if $d_j/d_{j+1} \geq r$ $(1 \leq j \leq m-1)$ and Z is an irreducible component of Z_σ and $Z_{\sigma+\varepsilon}$ we have

(i) $Z = Z_1 \times \cdots \times Z_m$, $Z_j \subseteq \mathbb{P}^{n_j}$

(ii) $\deg Z_j \leq c(\varepsilon)$, $(1 \leq i \leq m)$, $c(\varepsilon)$ a constant depending only on ε.

Complement: If f has integral coefficients bounded by C_v for $v | \infty$ then there are constants $c_1, c_2 > 0$ s.t.

(iii) $\sum d_i h(Z_i) \leq c_1 \sum_{v|\infty} C_v + c_2(d_1 + \cdots + d_m)$.

5. Faltings's line bundle. Let L be symmetric and ample and define for rational numbers ε and $s_1, \dots, s_m > 0$ the line bundle on A^m

$$L(\varepsilon, s) = \sum (s_i x_i - s_{i-1} x_{i-1})^* L + \varepsilon \sum s_i^2 pr_i^* L$$

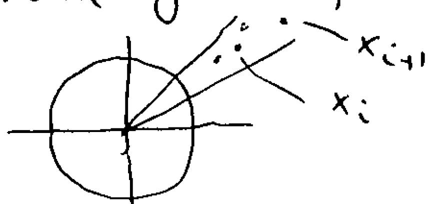
If $X \subseteq A$ define the morphism $\alpha_m : X^m \to A^{m-1}$ by $(x_1, \dots, x_m) \mapsto (2x_1 - x_2, \dots, 2x_{m-1} - x_m)$. Then it is possible to show that $\alpha_m : (X - Z(X))^m \to A^{m-1}$ is quasi finite and a result of Burnol [B] shows that on products $Y_1 \times \dots \times Y_m$ with $Y_j \not\subseteq Z(X)$, $j = 1, \dots, m$, the line bundle $L(-\varepsilon, s)|_{Y_1 \times \dots \times Y_m}$ is represented by an effective divisor. Using this an application of Siegel's Lemma gives a section $0 \neq f \in \Gamma(Y, L(-\varepsilon, s)^{\otimes d})$, $d \gg 1$, $Y = Y_1 \times \dots \times Y_m$, such that for the infinite places $v | \infty$ we have $\log \|f\|_v \ll d$.

6. Proof of the Theorem. One first shows that $(X - Z(X))(K)$ is finite if K is a number field. For this we assume that this is not the case and choose $x_1, \dots, x_m \in (X - Z(X))(K)$ such that

$$h(x_1) \gg 1,$$
$$h(x_i)/h(x_{i-1}) \gg 1.$$

On $V = A(K)\otimes\mathbb{R}$ we have a bilinear form $\langle\ ,\ \rangle$ coming from the Néron-Tate height on A. We then further assume that the x_i lie in a small segment of V:

$$\frac{\langle x_i, x_{i+1}\rangle}{\|x_i\|\,\|x_{i+1}\|} \geq 1-\varepsilon/2$$



Put $x=(x_1,\dots,x_m)$ and let O_K = ring of integers of K. Then the degree of the projective module $x^*(L(-\varepsilon,s)^{\otimes d})$ over O_K satisfies

$$(*)\qquad \deg x^*(L(-\varepsilon,s)^{\otimes d}) \xrightarrow[d\to\infty]{} -\infty .$$

We now make induction over product varieties.

First put $Y_j = X$ in section 5. Then because of $(*)$ the section f constructed there has large index $v = i(x,f)$. Choose $N > \dim X^m$. Then there exist a chain

$$x \in Z'_N \subset \cdots \subset Z'_2 \subset Z'_1 \subsetneq X^m$$

such that Z'_j is an irreducible component of $Z_{jv/N}$.
By dimension reasons $Z'_j = Z'_i$ for some $i \neq j$.
Put $Z = Z'_i$. Then the product theorem tells us that Z is a product and $x \in Z$. Now apply induction.
This proves the claim. Therefore we only need to show that $Z(X)(K)$ lies in a finite number of translates. So let Z_j be a component. Then $S(Z_j) \neq 0$ and we finish by induction for $Z_j/S(Z_j)$.

7 References

[B] J.F. Burnol, Letter to G. Faltings, November 11, '90.

[F1] G. Faltings, Diophantine approximation on abelian varieties, Annals of Math. 129 (1991)

[F2] G. Faltings, The general case of S. Lang's conjecture, preprint 1991.

[K] S. Kawamata, On Bloch's Conjecture, Inv. math. 57 (1980), 97-100.

[L] S. Lang, Integral points on curves, Publ. Math. IHES 6 (1960), 27-43.

[R] M. Raynaud, Arround the Mordell conjecture for function fields and a conjecture of Serge Lang, in "Algebraic geometry", SLN 1016 (1983), 1-19.

Titel: Galois cohomology : recent results and open questions
Autor: J-P. Serre
Seite: 1

Adresse: Collège de France, Paris

Galois cohomology (for semi-simple groups) is now reasonably well understood when the ground field is a number field : the Hasse local-global principle has been proved in full generality, thanks to the recent work of Chernousov and Premet.

But what about arbitrary fields? The aim of the talk will be to emphasize the many problems which are open, especially about the exceptional groups, such as F_4 or E_8.

J-P. Serre.

Référence : Chap. VI of "Algebraic Groups and Number Theory", by V.P. Platonov and A.S. Rapinchuk, Moskva, 1991.

Titel: Intersection theory on the moduli space of curves and the matrix Airy function

Autor: M. Kontsevich

Seite: 1

Adresse: MPI (current), on leave from Acad. Sci. Moscow

This talk is devoted to the proof of a recent E. Witten's conjecture (see [1], [2]).
The main object – "partition function in ∞ number of indeterminates t_i

$$F = F(t_0, t_1, \dots) = t_0^3/6 + t_1/24 + \dots \in \mathbb{Q}[[t_0, t_1, \dots]]$$

has six descriptions connected with

① intersection theory on the moduli space of curves;

② the matrix Airy function

$$A(Y) = \int \exp\left(\sqrt{-1}\left(\mathrm{Tr}\, X^3/3 - \mathrm{Tr}\, XY\right)\right) dX \ , \text{where}$$

X, Y are hermitean matrices of arbitrary size;

③ Asymptotic behavior of the number of triangulations by N triangles of a surface of genus g, $N \to +\infty$;

④ Asymptotic behavior of integrals $\int \exp(\mathrm{Tr}(P(X)))dX$ over the space of hermitean $N \times N$ matrices, $N \to +\infty$, P is a polynomial (depending on N).

⑤ Some special solution of KdV-hierarchy connected with the classical Airy function

$$A(y) = \int \exp\left(\sqrt{-1}\left(x^3/3 - xy\right)\right) dx \ , \quad A''(y) + yA(y) = 0;$$

⑥ highest weight vector for "one-half" of Virasoro algebra in bosonic representations.

The graph of connections between descriptions:

$$\begin{tikzcd} ① \arrow[d] & ③ \arrow[d] & \text{surfaces, 2-dim. gravity} \\ ② \arrow[d] & ④ \arrow[d] & \text{matrix integrals} \\ ⑥ & ⑤ \arrow[l] & \text{integrable systems.} \end{tikzcd}$$

Implications ③ $\to$ ④ $\to$ ⑤ $\to$ ⑥ are proven by

mathematical physicists (see [3], [4], [8]).
E. Witten conjectured that 1 is equivalent to 3, 4, 5, 6. This talk contains the proof of 1 → 2 → 6. At the moment we have no approach to the direct connections 1–3 or 2–4.

§1. Topological gravity in 2 dimensions.

Fix integers g, n; $g \geq 0$, $n > 0$, $2-2g-n < 0$.

$M_{g,n}$ = the moduli space of smooth curves $C/\mathbb{C}$ of genus g with n distinct marked points x_i.

$\overline{M}_{g,n}$ = the Deligne-Mumford compactification of $M_{g,n}$ (see [5]).

$\mathcal{L}_i$, $i=1\dots n$ - line bundles on $\overline{M}_{g,n}$; $\mathrm{fiber}(\mathcal{L}_i) = T^*_{x_i}C$.

Let $d_1, \dots, d_n$ be integers, $\sum_{i=1}^{n} d_i = \dim_{\mathbb{C}} M_{g,n} = 3g-3+n$.

Denote by $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ the intersection index $\left\langle \prod_1^n c_1(\mathcal{L}_i)^{d_i}, [\overline{M}_{g,n}] \right\rangle \in \mathbb{Q}$ ($\overline{M}_{g,n}$ is an orbifold).

By definition

$$F(t_0, t_1, \dots) = \sum_{k_0, k_1, \dots} \langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \prod_{i=0}^{\infty} t_i^{k_i} / k_i! \,.$$

Witten's conjecture is equivalent to the fact that $U := \partial^2 F / \partial t_0^2$ satisfies the Korteweg-de Vries equation $\partial U / \partial t_1 = U \, \partial U / \partial t_0 + 1/12 \cdot (\partial / \partial t_0)^3 U$.

§2. Main theorem ("formula for F").

Let Λ be a positive definite hermitean $N \times N$ matrix, $t_i(\Lambda) := -(2i-1)!! \operatorname{Tr} \Lambda^{-(2i+1)}$, $n!! = n \cdot (n-2)(n-4) \dots$

Then $F(t_\cdot(\Lambda))$ is an asymptotic expansion of

$$\log \left(\int \exp\left(\frac{\sqrt{-1}}{6} \operatorname{Tr} X^3 - \operatorname{Tr} \frac{X^2 \Lambda}{2} \right) dX \Big/ \int \exp\left(- \frac{\operatorname{Tr} X^2 \Lambda}{2} \right) dX \right),$$ where $\Lambda \to +\infty$.

This theorem gives arrow ① → ②.

§3. A. Combinatorial model for $M_{g,n}$

A ribbon graph (or a fatgraph) is a finite graph with cyclic order on the set of edges coming to each vertex. There is 1-1 correspondence between R.G. and cell decompositions of closed oriented surfaces. (RG associated with cell decomposition is 1-skeleton). A metric on a RG is a map $l: \{\text{edges}\} \to \mathbb{R}_+$.

$M^{comb}_{g,n}$:= moduli space of RG with metric such that a) valency of each vertex ≥ 3,
b) corresponding surface has genus g,
c) the set of 2-cells is labeled $\{\text{2-cells}\} \xrightarrow{\sim} \{1, \dots, n\}$.

Fact $M^{comb}_{g,n} \approx M_{g,n} \times \mathbb{R}^n_+$. It follows from K. Strebel results on quadratic differentials, see [6], [7].

Projection $\pi : M^{comb}_{g,n} \to \mathbb{R}^n_+$ is $(p_1, \dots, p_n)$ where p_i is the perimeter of the boundary of i-th cell.

§4. A combinatorial model for $BU(1)$.

$$BU(1)^{comb} = \bigcup_{k=1}^{\infty} \{(l_1, \dots, l_k) \mid l_i > 0\} / \text{cyclic permutations}$$

is considered as an orbispace with some natural topology. There exists an universal S^1 bundle S_{un} on $BU(1)^{comb}$, fiber is a k-gon with edges of length $l_1, \dots, l_k$.

Lemma 2-form $\omega = \sum_{1 \leq i < j \leq k-1} d\left(\frac{l_i}{p}\right) \wedge d\left(\frac{l_j}{p}\right)$, $p = l_1 + \dots + l_k$ represents $c_1(S_{un})$.

The natural map $\varphi = (\varphi_1, \dots, \varphi_n) : M_{g,n}^{comb} \to (BU(1)^{comb})^n$ which associates with RG the sequence of boundaries of 2-cell, can be prolonged to $\overline{M}_{g,n} \times \mathbb{R}^n_+$. $\varphi_i^*(S_{un})$ is isomorphic to the circle bundle arising from $\mathcal{L}_i$.

§5. Main computation

Fix real numbers $\lambda_i > 0$, $i = 1, \dots n$. Let $\Omega = \sum p_i^2 \varphi_i^*(\omega)$.

$$I_g(\lambda.) := \int_0^{+\infty}\!\!\cdots\!\int \prod_1^n dp_i \cdot \exp\left(-\sum \lambda_i p_i\right) \cdot \int_{\pi^{-1}(p.)} \Omega^d/d!\,, \quad d = \dim_{\mathbb{C}} M_{g,n}$$

There are two ways to compute $I_g(\lambda.)$:

1° $$I_g(\lambda.) = \int_0^{+\infty}\!\!\cdots\!\int \prod dp_i \exp\left(-\sum \lambda_i p_i\right) \cdot \sum_{\substack{d_1 \dots d_n \\ \sum d_i = d}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod \frac{p_i^{2d_i}}{d_i!} =$$

$$= \sum_{d_1 \dots d_n, \sum d_i = d} \prod_{i=1}^n \frac{(2d_i)!}{d_i!} \lambda_i^{-(2d_i+1)} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle$$

2° $$I_g(\lambda.) = \int_{M_{g,n}^{comb}} \exp\left(-\sum \lambda_i p_i\right) \cdot \rho \cdot \prod_{edges} d\ell(x)\,,$$ where

$$\rho = \left(\prod_1^n d\ell_i \times \Omega^d/d!\right) : \prod_{x-edges} d\ell(x) = 2^{2n+5g-5}.$$

$$I_g(\lambda.) = \rho \cdot \int_{M_{g,n}^{comb}} \exp\left(-\sum_{x\text{-edge } i \| j} \ell(x)(\lambda_i + \lambda_j)\right) \prod_{x\text{-edge}} d\ell(x) =$$

$$= \rho \cdot \sum_{\Gamma:\, 3\text{-valent RG}} \frac{1}{\#\mathrm{Aut}\,\Gamma} \cdot \prod_{edge\ i \| j} \frac{1}{\lambda_i + \lambda_j}.$$

Let $\Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_N)$, $\Lambda_\alpha > 0$.

$$F(t.(\Lambda)) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^n \left((2d_i - 1)!! \sum_{\alpha_i = 1}^N \Lambda_{\alpha_i}^{-(2d_i+1)} \right) =$$

$$= \sum_{g,n} (-1)^n 2^{3-3g-n} \sum_{\alpha_1 \dots \alpha_n} I_g(\Lambda_{\alpha_1}, \dots, \Lambda_{\alpha_n}) = \sum_{\Gamma} \frac{1}{\#\mathrm{Aut}\,\Gamma} \cdot w(\Gamma).$$

Last summation is taken over all connected 3-valent RG with labeling $\{2\text{-cells}\} \to \{1, \dots, N\}$,

$$w(\Gamma) = \left(\frac{\sqrt{-1}}{2}\right)^{\#\text{vertices}} \prod_{\substack{edges \\ \alpha \| \beta}} \frac{2}{\Lambda_\alpha + \Lambda_\beta}.$$

By Feynmann rules we obtain an asymptotic expansion for the integral from §2. The main thm is proven.

§6 Relation with Virasoro algebra.

The matrix Airy function $A(Y)=\int \exp(\sqrt{-1}(\mathrm{Tr}\frac{X^3}{3}-\mathrm{Tr}\,XY))dX$ obeyes equations $Y_{ji}A+\sum_k \frac{\partial^2 A}{\partial Y_{ik}\partial Y_{kj}}=0$. Airy function is conjugacy invariant, so it depends only on $\mathrm{Spec}\,Y=(Y_1 \ldots Y_N)$. Equations in new variables are

$$(*)\quad Y_i A+\frac{\partial^2 A}{\partial Y_i^2}+\sum_{j:\,j\neq i}\frac{\partial/\partial Y_j-\partial/\partial Y_i}{Y_j-Y_i}.A=0.$$

If $Z=X-Y^{1/2}$ where $Y^{1/2}$ is some square root of Y then $\mathrm{Tr}(X^3/3-XY)=\mathrm{Tr}(Z^3/3+Z^2Y^{1/2})-2/3\,\mathrm{Tr}\,Y^{3/2}$. Applying the main thm one has an asymptotic expansion.

$A(Y)\sim\sum_{Y^{1/2}} G\cdot R(Y^{1/2})$, $Y\to+\infty$, where $G=1+o(1)$ is a series in $T_1,T_3,T_5,\ldots,T_k=(\sqrt{-1})^k\,\mathrm{Tr}\,Y^{-k/2}/k$,

$R(Y^{1/2})=\exp\left(-\frac{2\sqrt{-1}}{3}\mathrm{Tr}\,Y^{3/2}\right).\int\exp(\sqrt{-1}\,\mathrm{Tr}\,Z^2Y^{1/2})dZ$ – an elementary function of Y.

Substitution in $(*)$ gives an equation $(a_i=\sqrt{-1}\,Y_i^{-1/2})$:

$$\left(\frac{1}{4}a_i^2T_1^2+\frac{1}{16}a_i^4\right)G+\sum_{n=1,3,5..}\left(a_i^{n+1}+\sum_{l\leq n+2}\frac{lT_l}{2}a_i^{n+4-l}\right)\frac{\partial G}{\partial T_n}+\sum_{n,m}\frac{a_i^{n+m+4}}{4}\frac{\partial^2 G}{\partial T_n\partial T_m}=0$$

Coefficients in $a_i^2, a_i^4, a_i^6,\ldots$ give system of equations $(J_{2n+1}+L_{n-1})G=0$, $n=0,1,\ldots$, where $J_1=\frac{\partial}{\partial T_1}$, $J_3=\frac{\partial}{\partial T_3},\ldots$, $J_{-1}=T_1$, $J_{-3}=3T_3,\ldots$ are bosons and

$$L_n=\frac{1}{4}\sum_{k+l=2n}J_kJ_l,\ n\neq 0;\quad L_0=\frac{1}{2}\sum_{k>0}J_{-k}J_k+\frac{1}{16}$$

are generators of Virasoro algebra. This proves 2 $\longrightarrow$ 6.

References:

[1]. M. Atiyah, talk on 29. Arbeitstagung. MPI/90-52
[2] E. Witten. Princeton preprint IASSNS-HEPP-90/45.
[3] E. Brezin, V. Kazakov, Phys. Lett. B236 (1990) 144
[4] D. Gross, A. Migdal, Phys. Rev. Lett. 64 (1990) 127
[5] D. Mumford, paper in "Arithmetic & Geometry", Birkhäuser 1983
[6]. K. Strebel, Quadratic differentials, Springer 1984
[7]. B. Zwiebach, Comm. Math. Phys. 136 (1991) 83.
[8] Dijkgraaf, E. & H. Verlinde, IAS preprint (1990).

Titel: Excision in algebraic K-theory and the proof of the Karoubi Conjecture

Autor: Mariusz Wodzicki
Adresse: University of California, Berkeley

Let C be an arbitrary C^*-algebra and $\mathcal{K} = \mathcal{K}(H)$ denote the C^*-algebra of compact operators on the standard separable ∞-dimensional Hilbert space. There exists a unique up to equivalence C^*-algebra norm on $M(C) := \bigcup_n M_n(C)$ extending the norm on C. The induced completion $\overline{M(C)}$ is a C^*-algebra isomorphic to $C \tilde{\otimes} \mathcal{K}$ where $\tilde{\otimes}$ denotes the 'spatial' tensor product of C^*-algebras.

Theorem 1. ("Karoubi's Conjecture"). $K_*(C \tilde{\otimes} \mathcal{K}) \simeq K_*^{top}(C)$.

Every C^*-algebra C satisfies the condition $C = C^2$ which is suffient for the group $GL(C)$ to be quasi-perfect. In particular the space $BGL(C)^+$ exists.

Theorem 2. $BGL(C \tilde{\otimes} \mathcal{K})^+$ and $BGL^{top}(C)$ are homotopy equivalent.

The proof of the above theorems relies in an essential way on the excision property of the algebraic K-theory functor on the category of C^*-algebras and on some earlier work of G. Kasparov and N. Higson.

Let A be a 2-sided ideal in a unital ring R and $\overline{GL}(R/A) = \mathrm{Image}\,(GL(R) \xrightarrow{p} GL(R/A))$. The group extension

$$1 \longrightarrow GL(A) \longrightarrow GL(R) \longrightarrow \overline{GL}(R/A) \longrightarrow 1$$

induces the map of (homotopy) fibrations:

$$\begin{tikzcd} BGL(A) \arrow[r] \arrow[d, "\varphi^{R,A}"'] & BGL(R) \arrow[r, "Bp"] \arrow[d] & B\overline{GL}(R/A) \arrow[d] \\ F(R,A) \arrow[r] & BGL(R)^+ \arrow[r, "(Bp)^+"] & B\overline{GL}(R/A)^+ \end{tikzcd}$$

where $F(R,A)$ is the homotopy fibre of $(Bp)^+$.

<u>Def.</u> $K_i(R,A) := \pi_i F(R,A)$, $i \geq 1$;

$K_i(A) := K_i(\tilde{A}, A)$, $i \geq 1$, where $\tilde{A} := \mathbb{Z} \ltimes A$.

The obvious morphism $(\tilde{A}, A) \longrightarrow (R,A)$ induces the map $ex : F(\tilde{A}, A) \longrightarrow F(R,A)$ (defined up to a homotopy) which will be called the <u>excision</u> map.

<u>Def.</u> A ring A <u>satisfies</u> <u>excision</u> <u>in</u> <u>algebraic</u> K-<u>theory</u> if the excision map ex defined above is a homotopy equivalence for any unital ring R containing A as a 2-sided ideal.

<u>Homological unitality</u>. Recall that a ring A is said to be H-<u>unital</u> (= homologically unital) if the Bar complex

$$B_*(A) = \{ A \xleftarrow{m} A \otimes_{\mathbb{Z}} A \xleftarrow{m \otimes 1_A - 1_A \otimes m} A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} A \longleftarrow \cdots$$

is <u>pure</u>-<u>acyclic</u>, i.e. $B_*(A) \otimes V$ remains acyclic for any abelian group V (m denotes the multiplication map), see [4].

<u>Theorem 3</u>. Let $A = A \otimes_{\mathbb{Z}} \mathbb{Q}$, then the following conditions are equivalent:

(a) A satisfies excision in algebraic K-theory;

(b) the map $\varphi^{R,A} : BGL(A) \longrightarrow F(R,A)$ is a homology equivalence for every $R \supset A$;

(c) the map $\varphi^{\tilde{A},A} : BGL(A) \longrightarrow F(\tilde{A},A)$ is a homology equivalence;

(d) $GL(R)$ acts trivially on $H_*(GL(A);\mathbb{Z})$ for every $R \supset A$;

(e) $GL(\mathbb{Z})$ acts trivially on $H_*(GL(A);\mathbb{Z})$;

(f) the natural inclusions

$$\begin{tikzcd} & GL(A) \arrow[dl] \arrow[dr] & \\ \widetilde{GL}(A) & & \widetilde{\widetilde{GL}}(A) \end{tikzcd}$$

where $\widetilde{GL}(A) := GL(A) \ltimes A^{\oplus\infty}$ and $\widetilde{\widetilde{GL}}(A) = A^{\oplus\infty} \rtimes GL(A)$, are homology equivalences;

(g) A satisfies excision in cyclic homology of $\mathbb{Z}$-algebras;

(h) A is H-unital.

More general results are proved in [3].

Universally flat rings. A ring A is said to be left universally flat if for every unital ring S containing A as a left ideal this ideal is flat as an S-module. Right universally flat rings are defined similarly.

Lemma. A ring A is H-unital if it is either left or, right, universally flat and its additive group $(A,+)$ is torsionless.

The following two theorems provide numerous examples of universally flat rings.

Theorem 4. All closed left ideals in an arbitrary C^*-algebra are left universally flat.

Theorem 5. 1) All 2-sided (not necessarily closed) ideals J in an arbitrary von Neumann algebra are flat both as left and as right ideals;

2)
$$\begin{array}{ccc} J \text{ is left universally flat} & \Leftrightarrow & J \text{ is right universally flat} \\ \Updownarrow & & \Updownarrow \\ J \text{ is H-unital} & \Leftrightarrow & J = J^2 . \end{array}$$

(J is a 2-sided ideal in a von Neumann algebra).

Corollary. All closed (one-sided) ideals in any C^*-algebra satisfy excision in algebraic K-theory.

Higher index invariants

Let $\Psi_{X,E} = \bigcup_{m\in\mathbb{Z}} \Psi^m_{X,E}$ denote the ring of pseudodifferential operators acting in sections of a vector bundle E on a closed C^∞-manifold X and let $S_{X,E}$ be the corresponding quotient ring of (complete) symbols. The map $\sigma : \Psi_{X,E} \longrightarrow S_{X,E}$ associates to a pseudodifferential operator its complete symbol.

Theorem 6. There exist:

(a) a universal (i.e. not depending on X or E) graded abelian group $I_* = \overset{\infty}{\underset{n=0}{\oplus}} I_n$,

(b) functorial homomorphisms $j_* : I_* \longrightarrow K_*(\Psi_{X,E})$ and $\mathrm{index}_* : K_{*+1}(S_{X,E}) \longrightarrow I_*$

such that one has the following exact triangle:

$$\begin{tikzcd} K_*(S_{X,E}) \arrow[rr, "\mathrm{index}", "\deg -1"'] & & I_* \arrow[dl, "j"] \\ & K_*(\Psi_{X,E}) \arrow[ul, "\mathrm{symbol}"] & \end{tikzcd}$$

Remark. $I_* = K_*(\mathbb{L}_0)$ where $\mathbb{L}_0$ is the ring of rapidly decaying complex-valued matrices (α_{ij}), $\sup_{ij} |\alpha_{ij}|(i+j)^N < \infty$ for all $N \in \mathbb{R}$; $I_0 = \mathbb{Z}$, $I_1 = \mathbb{C}^* \oplus (?)$, ...

The zeroth component $\mathrm{index}_0 : K_1(S_{X,E}) \longrightarrow \mathbb{Z}$ is the standard index of an (almost invertible) pseudodifferential operator.

Karoubi's construction of regulators gives numerical invariants $K_{2n-1}(S_{X,E}) \longrightarrow \mathbb{C}^*$, $n \in \mathbb{Z}_+$, which should be the subject of a generalization of the Atiyah-Singer Index Theorem to the higher algebraic K-theory of the ring of symbols.

The excision theorem 3 above is a joint work with Andrei Suslin.

REFERENCES

[1] A. SUSLIN, On the acyclicity of the sum of triangular complexes, Utrecht, April 1991 preprint

[2] ——— and M. WODZICKI, Excision in algebraic K-theory and Karoubi's Conjecture, Proc. Nat. Acad. Sci. U.S.A. 87 (1990), 9582-9584

[3] ——— and M. WODZICKI, Excision in algebraic K-theory, Ann. Math. (to appear)

[4] M. WODZICKI, Excision in cyclic homology and in rational algebraic K-theory, Ann. Math. 129 (1989), 591-639.

[5] ———, Homological properties of rings of functional-analytic type, Proc. Nat. Acad. Sci. U.S.A. 87 (1990), 4910-4911

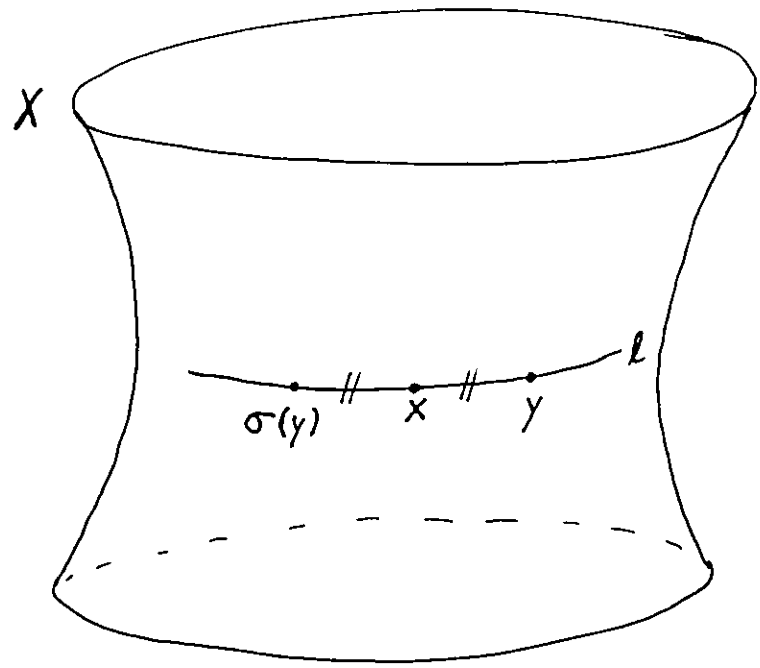
[6] ———, On the algebraic K-theory of nonunital rings, K-theory (to appear)

Titel: Lefschetz numbers of Hecke correspondences
Autor: Bob MacPherson

Adresse: Dept. of Mathematics, MIT, Cambridge MA, 02139
USA

All of the work described here is joint with Mark Goresky, and some of it is joint with Günter Harder and with Bob Kottwitz.

Recall the definition of a locally symmetric space. For every point x in a Riemannian manifold X there is a "reflection" map σ defined on points y close enough to x: if ℓ is the shortest geodesic from x to y, then $\sigma(y)$ is the point on ℓ the same distance from x in the opposite direction.



The Riemannian manifold X is a locally symmetric space if, for all $x \in X$, the reflection map is an isometry.

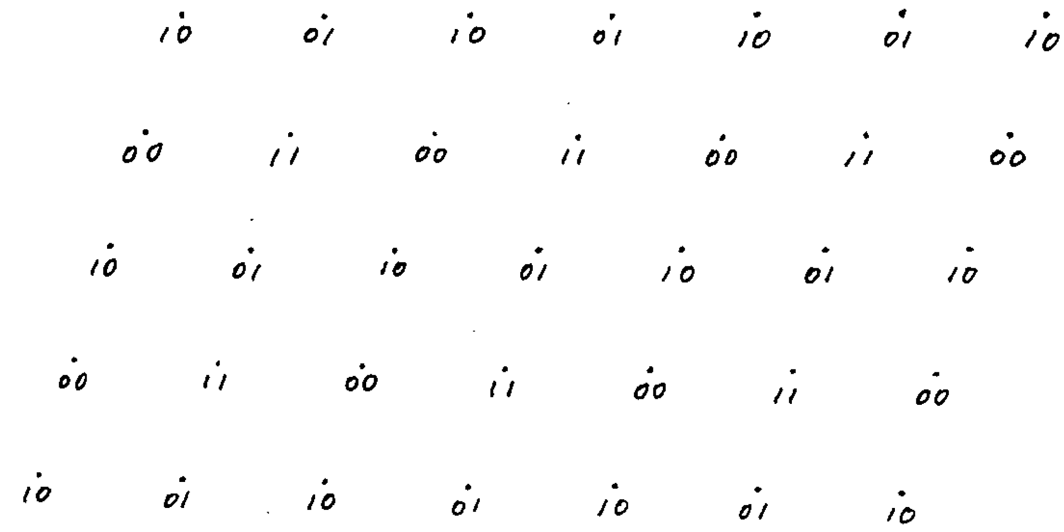
We will call the locally symmetric space X *arithmetic* if it is complete, has negative curvature, has finite volume, and has the following property: the group Γ of deck transformations $\tilde{X} \to \tilde{X}$ of the simply connected covering space $\tilde{X}$ of X is an arithmetic subgroup of the group $Aut(\tilde{X})$ of Riemannian automorphisms of $\tilde{X}$. (By results of Margulis, this last requirement is automatic in most cases.)

A morphism of locally symmetric spaces is a local isometry. A *Hecke correspondence* on an arithmetic locally symmetric space X is another arithmetic locally symmetric space C equipped with two morphisms $s, t : C \rightrightarrows X$, called the source map and the target map. A Hecke correspondence acts on differential forms on X by the formula $C^*\omega = s_* t^* \omega$ (where the Gysin map s_* is defined since s will be a finite covering projection; it just adds the forms on each of the sheets). By this formula, C induces a self-map, also notated C^* and called a *Hecke operator*, on either the cohomology $H^*(X)$ of X or on the L_2 cohomology $H^*_{(2)}(X)$ of X. (Whenever we speak of L_2 cohomology, we assume that X is Hermitian, so that it is finite dimensional.)

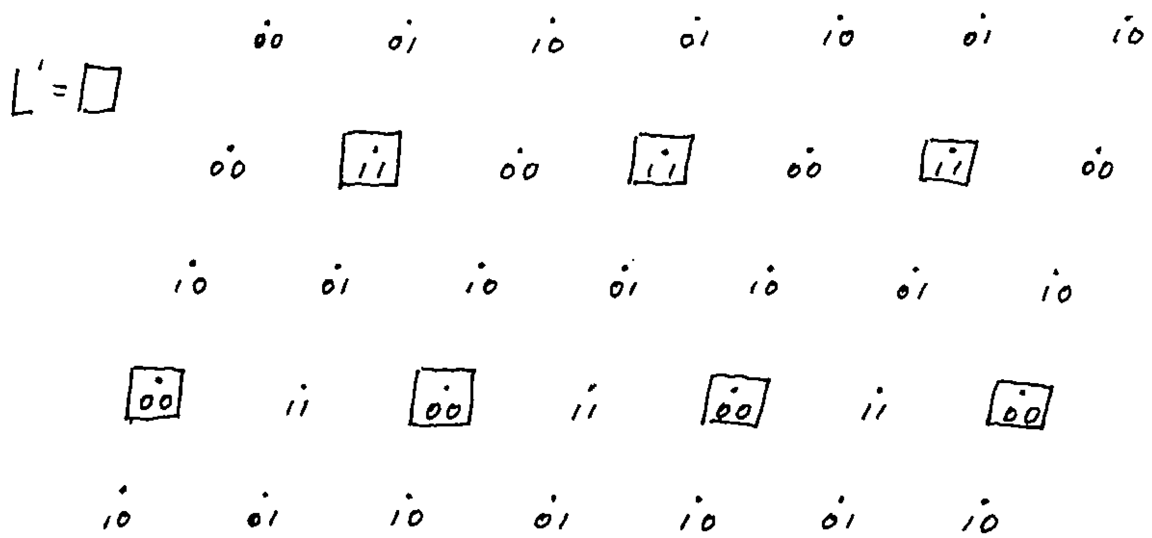
PROBLEM: Study the action of Hecke operators C^* on $H^*(X)$ or $H^*_{(2)}(X)$.

EXAMPLE

Let $X_l(n)$ be the space whose points are configurations consisting of a lattice L in $\mathbf{R}^n$ together with a surjection $L \to (\mathbf{Z}/l)^n$, called a marking. Two such configurations are considered equivalent if they differ by a rotation or a homothety (multiplication by a positive real number). For example, this is a configuration for $X_2(2)$:



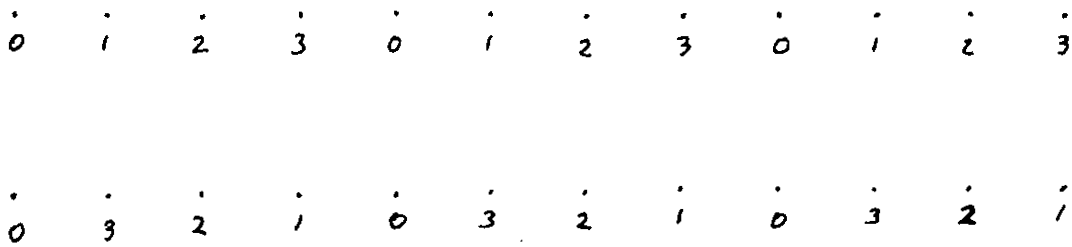
If $l > 3$, $X_l(n)$ is nonsingular, and it is naturally an arithmetic locally symmetric space. For any prime p not dividing l, let T_p^i be the space whose points are configurations L and $L \to (\mathbf{Z}/l)^n$ as above, together with a sublattice $L' \subset L$ such that the abelian group L/L' is isomorphic to $(\mathbf{Z}/p)^i$ (again modulo rotations and homotheties). For example, this is a configuration for T_3^1:



Then T_p^i is a Hecke correspondence on $X_l(n)$ whose source map s forgets the sublattice L' completely, and whose target map t erases everything in L but L' and takes the restricted marking.

JUSTIFICATION

Why should we care about Hecke operators? The answer is that it is expected that interesting number theoretic information should be encoded in them. Consider the following example: The space $X_4(1)$ consists of two points α and β, represented by the following marked lattices:



Let J be the one dimensional subspace of $H^1(X_4(1))$ generated by 1 on α and -1 on β. For any odd prime p, the correspondence T_p^1 is actually a function from $X_4(1)$ to itself.

Exercise. Show that $(T_p^1)^*$ takes J into itself, and that it is the identity if and only if the prime p splits totally in the Gaussian integers $\mathbf{Z} \oplus \mathbf{Z}i$ (if and only if p is congruent to 1 mod 4).

Class field theory says that the splitting of a prime p in an extension of the rationals with an abelian Galois group is always governed by the action of $(T_p^1)^*$ on some subspace J of the cohomology of $X_l(1)$ for some l. Langlands philosophy suggests that for any extension of the rationals, the splitting of a prime p should be governed by the action of the set of operators $(T_p^i)^*$ on some subspace J of the cohomology, or L_2 cohomology, of $X_l(n)$ for some l and n.

LEFSCHETZ NUMBERS AND COMPACTIFICATIONS.

The actual calculation of Hecke operators C^* is probably hopeless; even the calculation of $H^*(X_l(3))$ on a supercomputer is beyond our abilities for reasonable l. However, as usual, alternating sums of cohomology groups are easier to deal with. Define the Lefschetz numbers $L(C) = \Sigma(-1)^i\, trace\ C^* : H^i(X) \to H^i(X)$ and $L_2(C) = \Sigma(-1)^i\, trace\ C^* : H^i_{(2)}(X) \to H^i_{(2)}(X)$. We modify our question to ask for the computation of these Lefschetz numbers. The fixed point set of a correspondence $s, t : C \rightrightarrows X$ is $\{c \in C | s(c) = t(c)\}$. Let K denote a connected component of the fixed point set. Then a Lefschetz Fixed Point Theorem says that there exist locally defined numbers $L(K)$ so that $L(C) = \sum_K L(K)$, and likewise for L_2 cohomology. However our spaces X are not compact, and no Lefschetz fixed point theorem is valid for noncompact spaces (as is shown by the map $\mathbf{R} \to \mathbf{R}$ sending x to $x+1$, which has an empty fixed point set but a nonzero Lefschetz number).

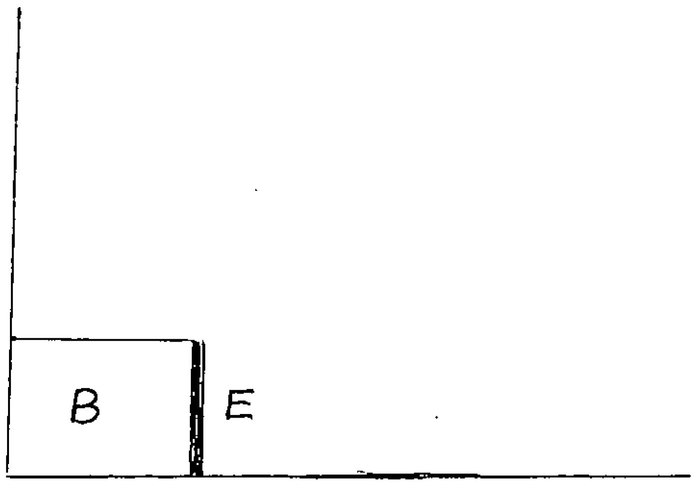
Compactifying X is a much studied subject, and several compactifications with different desirable properties have been defined. However, I know of no compactification that is both nonsingular and admits extensions of the Hecke correspondances. Therefore, we are forced to develop Lefschetz fixed point theory for singular varieties.

LEFSCHETZ FIXED POINT THEORY

There are Lefschetz fixed point theorems for singular spaces due to Dold (for ordinary cohomology) and Grothendieck-Illusie (for sheaf cohomology). However, what is wanted is a computable formula for the contributions $L(K)$.

Let s,t:$\bar{C} \rightrightarrows \bar{X}$ be a correspondence. A fixed point component K is called ***weakly hyperbolic*** if there is a map $(p_1, p_2) : U \to \mathbf{R}^{\geq 0} \times \mathbf{R}^{\geq 0}$, where U is a neighborhood of $s(K) = t(K)$, with the following two properties: 1. $(p_1, p_2)^{-1}(0,0) = K$ and 2. Near K, $p_1 s(c) \leq p_1 t(c)$ and $p_2 s(c) \geq p_2 t(c)$.

The intuition behind this definition is that the expanding directions must be mapped to the x-axis and the contracting directions must be mapped to the y-axis in $\mathbf{R}^{\geq 0} \times \mathbf{R}^{\geq 0}$. Take a small box B and and its edge E as in the following picture:



THEOREM [GM1] (Lefschetz fixed point formula)

$$L(K) = \sum (-1)^i \; trace \; \bar{C}^* : H^i((p_1,p_2)^{-1}B, (p_1,p_2)^{-1}E) \to (H^i((p_1,p_2)^{-1}B, (p_1,p_2)^{-1}E)$$

This same formula works for sheaf cohomology as well. Another (overlapping) fixed point formula has been proved by Kashiwara and Schipira.

THE REDUCTIVE BOREL-SERRE COMPACTIFICATION

Now, we return to our situation of an arithmetic locally symmetric space X. We work with the Reductive Borel-Serre compactification $\bar{X}$. This is distinguished by being the most natural compactification metrically, in the following sense: Suppose that two curves $\tau_i : [0,1) \to X$ converge to limit points $\tau_i(1)$ in $\bar{X}$. Then $\tau_1(1) = \tau_2(1)$ if and only if $\lim_{\delta \to 1} \; dist \; (\tau_1(\delta, 1), \tau_2(\delta, 1)) = 0$.

With this compactification, we are in a position to apply the Lefschetz fixed point formula for the following reasons:

THEOREM

1. The Hecke correspondance $s, t : C \rightrightarrows X$ extends canonically to a compactified Hecke correspondance $s, t : \bar{C} \rightrightarrows \bar{X}$.

2. The compactified Hecke correspondance $s, t : \bar{C} \rightrightarrows \bar{X}$ is weakly hyperbolic at each fixed point component.

3. There exist (derived) sheaves on $\bar{X}$ whose cohomology is $H^*(X)$ resp. $H^*_{(2)}(X)$.

Of these, the one that most deserves comment is 3. The fact that there is a sheaf on $\bar{X}$ whose cohomology is $H^*(X)$ is clear from Grothendieck's formalism of sheaf theory. (This would be true for any compactification.) The fact that there is a sheaf whose cohomology is $H^*_{(2)}(X)$ is deep. By the Zucker conjecture, proved by Looijenga and Saper-Stern, the cohomology of middle intersection homology sheaf $IC^\bullet$ on $\hat{X}$ computes $H^*_{(2)}(X)$, where $\hat{X}$ is the Baily-Borel compactification of X. There is a canonical projection $\pi : \bar{X} \to \hat{X}$.

THEOREM [GHM]
There is a "weighted cohomology sheaf" $WC^\bullet$ on $\bar{X}$ so that $R\pi_* WC^\bullet = IC^\bullet$.

With these results in hand, it remains to carry out the explicit computation of the Lefschetz fixed point formula. This calculation is made possible by the fact that the singularities of $\bar{X}$ are described by nilmanifolds, whose cohomology can be calculated by the Nomizu-Van Est theorem and Kostant's theorem. I will omit the calculation, which is in [GM2]. Analytic computations of the Lefschetz numbers of Hecke operators on L_2 cohomology have been carried out by Arthur, as part of his trace formula, and by Stern. The agreement of our formula with Arthur's formula is verified in [GKM]. It is interesting to note that Arthur's terms correspond to sums of terms $L_2(K)$. An analytic computation of the trace of Hecke operators on ordinary cohomology has been carried out by Franke. Finally, I would like to note that there is an intriguing similarity between our expression for the Lefschetz numbers of Hecke operators and Pink's formula for the Lefschetz numbers of Frobenius operators on the characteristic p reduction of the same space.

BIBLIOGRAPHY

[GM1] M. Goresky and R. MacPherson, *Local contribution to the Lefschetz Fixed Point Formula*, to appear in Inventiones.

[GM2] M. Goresky and R. MacPherson, *Lefschetz Numbers of Hecke Correspondences*, to appear in Montréal volume on $SU(2,1)$.

[GHM] M. Goresky, G. Harder, and R. MacPherson, *Weighted Cohomology*, to appear.

[GKM] M. Goresky, R. Kottwitz, and R. MacPherson, to appear.

Titel: Degeneration of Riemann surfaces and Jorgenson's proof of a conjecture of Deligne

Autor: Serge Lang

Adresse: Math Dept, Box 2155 Yale Station, New Haven CT 06520 USA
z Zt Max Planck Institut (Juni)

We start a purely algebraic version of a Deligne Riemann-Roch theorem. Since the Arbeitstagung started with Riemann-Roch theorems in 1957, it is appropriate that it should end with Riemann-Roch theorems.

Let X be a compact Riemann surface. Then we have its canonical sheaf $\mathcal{K}$. Let $\mathcal{L}$ be a line sheaf on X. Then we have a line (one dimensional vector space over the complex) defined by

$$\lambda(\mathcal{L}) = \det H^0(\mathcal{L}) \otimes \det H^1(\mathcal{L})^{-1}.$$

One defines a pairing between line sheaves by the formula

$$\langle \mathcal{L}, \mathcal{M} \rangle = \lambda(\mathcal{L}\otimes\mathcal{M}) \otimes \lambda(\mathcal{L})^{-1} \otimes \lambda(\mathcal{M})^{-1} \otimes \lambda(\mathcal{O}_X).$$

There ensues a canonical Deligne isomorphism

$$\lambda(\mathcal{L})^{12} \xrightarrow{\approx} \langle \mathcal{K}, \mathcal{K} \rangle \otimes \langle \mathcal{L}, \mathcal{L}\otimes\mathcal{K}^{-1} \rangle^{6}.$$

Following a philosophy started by Arakelov, there is an ongoing open ended program in algebraic geometry to put "natural metrics on all sheaves, so that the natural algebraic isomorphisms become isometries, possibly up to a constant factor. Essentially all of sheafy algebraic geometry is to be extended in this way. We must therefore now deal with metrics. Suppose given:

a positive (1,1)-form μ on X, which amounts to a metric on $\mathcal{K}$;
a metric ρ on $\mathcal{L}$.

These give rise to an L^2-hermitian product on $H^0(\mathcal{L})$, namely we have the hermitian product of two sections s, s' defined by

$$\langle s, s' \rangle_{\rho,\mu} = \int_X \langle s, s' \rangle_\rho \, \mu.$$

By Serre duality, for sections of $H^1(\mathcal{L})$ one also gets a hermitian product, because

$$H^1(\mathcal{L}) \approx H^0(\mathcal{L}^{-1}\otimes\mathcal{K})^\vee.$$

Thus we get what we call the L^2-metrics H_{L^2}, depending on ρ, μ.

We want more. Let $\Delta = \Delta_{\rho,\mu} : C^\infty(\mathcal{L}) \longrightarrow C^\infty(\mathcal{L})$ be the Laplacian, with the sign chosen that it is a positive operator. We let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the sequence of non-zero eigenvalues, and we define the (spectral) zeta function by the series

$$\zeta_\Delta(s) = \sum_{k=1}^{\infty} \lambda_k^{-1},$$

which converges for $\mathrm{Re}(s)$ large. A theorem of Seeley guarantees that ζ_Δ has a meromorphic continuation to $\mathbb{C}$, regular at 0. We can then define the determinant

$$\det{}^* \Delta = \exp(-\zeta_\Delta'(0)) \text{ "=" } \prod_{k=1}^{\infty} \lambda_k.$$

The star in the superscript of det indicates that we are dealing with the non-zero eigenvalues. Associated with this determinant, we define an important constant

$$\boxed{C_{\Delta,\mu}(X) = \frac{\det^* \Delta_{\mathcal{O},\mu}}{\mathrm{Vol}_{\mu}(X)} \quad \text{and} \quad c_{\Delta,\mu} = \log C_{\Delta,\mu}.}$$

We shall eventually view this constant, with the hyperbolic metric on X, as a function on the moduli space.

We also define the Quillen metric

$$H_{Q,\rho,\mu} = H_{L^2}\cdot(\det^* \Delta_{\mathcal{O},\mu})^{-1}.$$

Deligne proved that the Riemann-Roch isomorphism stated above is an isometry up to a factor $\exp(a(g))$, where $a(g)$ is a constant which depends only on g.

Deligne conjecture. One has $a(g) = (1-g)a(0)$.

Jorgenson proved this conjecture by a fascinating method. The Deligne constant $a(g)$ has been expressed as a difference of log determinants, in what has been called the "spin 1/2 bosonization formula" in [A-B-M-N-V 87], published in a physics journal which makes everybody think all this has to do with physics. No matter what, suppose $g \geq 2$. Let

$\psi: X \longrightarrow J$ be a canonical map into the Jacobian, with $\psi(P) = 0$.

$W_{g-1} = \psi(X)+\ldots+\psi(X)$ (sum taken $g-1$ times).

θ = Rieman theta function, and Θ its divisor.

H_X = the hermitian Riemann form associated with the polarization.

There is a unique divisor class D of degree $g-1$ such that

$$\Theta = W_{g-1} + \psi(D).$$

Let $\mathcal{J} = \mathcal{O}_X(D)$. Then $\mathcal{J}^2 \approx \mathcal{K}$, i.e. $\mathcal{J}$ is a square root of the canonical class. The expression of [A-B-M-N-V 87] is:

$$\boxed{\frac{1}{4}\, a(g) = \log\left(\det \Delta_{\mathcal{J},\rho^{1/2}} / \|\theta\|^2(0)\right) + \frac{1}{2}\, c_{\Delta,\rho}}$$

where

$$\|\theta\|^2(0) = (\det H_X)^{1/2}\, |\theta(0)|^2 .$$

This must be taken with a grain of salt. It may happen that $\theta(0) = 0$, but then the determinant will also be 0. One introduces a more complicated theta function, depending on a parameter $u \in J$, and one also introduces $\mathcal{L}_u$ $(u \in J)$ where $\mathcal{L}_u$ is a line sheaf of degree 0 with a flat metric. Then the quotient

$$\det \Delta_{\mathcal{J}\otimes\mathcal{L}_u,\rho^{1/2}} / \|\theta\|^2(0,u)$$

is well defined, positive, and independent of u because the determinant and the theta value vanish with the same order if they vanish at all.

For $g \geqq 2$, the idea is then to view the log determinant and other invariants of X as functions on the moduli space M_g, and to let X degenerate to a $\mathbb{P}^1$ with g nodes. Since $a(g)$ is constant, one finds its value from the limiting value of the log determinant term on this degenerate surface. This means that one has to keep track of the asymptotic behavior of several functions on the moduli space, of which $-\zeta'_\Delta(0)$ (hyperbolic metric), the theta value, the constant $c_{\Delta,\mu}$ are only the first examples.

Some of these functions tend to $\pm\infty$, but their differences may be continuous on the boundary of the moduli space. During the past few years, several people have systematically studied various such degeneracies, including Wolpert, Hejhal, Belavin-Kniznik, Taktajian, Zograf, Lundelius and Jorgenson, and others. Jorgenson proves appropriate limit formulas which allow him to determine Deligne's constant as conjectured.

It is a fairly vast enterprise to make a systematic tabulation of the behavior of all objects involved, namely in addition to the ones we have seen: small eigenvalues, small geodesics, and whatever. I shall select only some examples of theorems of Jorgenson giving the flavor of the observable phenomena.

I should also note that for the theory to be completely coherent, one must start from the beginning with non-compact Riemann surfaces having finite volume. All the objects such as Laplacians, zeta functions, etc. can be defined for such surfaces. In the limiting values with nodes, by deleting the nodes one obtains such surfaces. I started with compact surfaces only for simplicity, and to avoid taking certain precautions for the non-compact case.

<u>A limiting theorem</u>. I shall now describe one of Jorgenson's limit formulas.

By a <u>small eigenvalue</u> we mean an eigenvalue $< 1/4$.

By a <u>small geodesic</u>, we mean a geodesic of length $< \ell_0$, where ℓ_0 is the length of the smallest geodesic on $\mathbb{P}^1$ minus three points with the hyperbolic metric. These notions apply as well when X is not compact but has finite volume. It is known that the number of small eigenvalues is $\leqq 4g-3$ (Buser), and the number of small geodesics is $\leqq 3g-3$ (Bost). Define the products

$$\boxed{\prod_{\mathrm{sev}}(X) = \prod_{\mathrm{small}} \lambda_k \qquad \text{and} \qquad \prod_{\mathrm{sge}}(X) = \prod_{\mathrm{small}} \ell_j}$$

where ℓ_j ranges over the lengths of the small geodesics. We define a further constant

$$\boxed{\begin{aligned} C_\mu(X) &= \frac{\det \mathbb{H}_X}{\mathrm{Vol}_\mu(X)} \quad \text{if } g \geqq 1 \\ &= 1/\mathrm{Vol}_\mu(X) \quad \text{if } g = 0. \end{aligned}}$$

For degenerate surfaces, with several components and nodes, a similar definition can be given, multiplicative over the components. We omit it.

Finally, we define

$$\boxed{E(X) = c_{hyp}(X)\,\frac{\prod_{s\ddot{e}v}(X)}{\prod_{sge}(X)}\,.}$$

One of Jorgenson's theorems is that:

<u>Theorem 1.</u> <u>For a degenerating family of Riemann surfaces</u> X, <u>degenerating to</u> X_o, <u>we have</u>

$$\lim_{X\to X_o} E(X) = E(X_o).$$

If $\{X_j\}$ are the irreducible components of X_o, then $E(X_o) := \prod E(X_j)$.

<u>The Selberg zeta function.</u> Essential to the study of the degeneration of $\zeta'_{hyp}(0)$ is the Selberg zeta function

$$Z(s) = \prod_{k=1}^{\infty}\prod_{\gamma}\left(1 - e^{-(s+k)\ell(\gamma)}\right),$$

where γ ranges over the primitive geodesics. One has a formula of D'Hoker and Phong [D'H-P 86]:

$$\boxed{\log \det{}^* \Delta_{hyp} = -\zeta'_{hyp}(0) = \log Z'(1) + c(g),}$$

where $c(g)$ is a constant which was determined to be

$$c(g) = (1-g)c(0)$$

and $c(0) = c_{\Delta,st}(\underset{\sim}{P}^1) + \log 2 = -4\zeta'_{\underset{\sim}{Q}}(-1) + \frac{1}{2} - \log 2\pi.$

This last explicit value is by a computation of Vardi. Here st denotes the standard metric on the projective line, namely

$$\mu_{st} = 4\pi\,\mu_{can} \qquad \text{and} \qquad \mu_{can} = \frac{\sqrt{-1}}{2\pi}\,\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}\,.$$

The constant $c(g)$ enters in the determination of $a(g)$, namely one has $a(g) + c(g) = d(g)$, which is still another constant but we won't go into that. One can define a twisted Selberg zeta function Z_1 by a character of order 2, and there is a similar formula of Sarnak [Sar 87]:

$$\boxed{\log \det \Delta_{\mathcal{S},hyp} = (1-g)(-4\zeta'_{\underset{\sim}{Q}}(-1)) + \log Z_1(1/2),}$$

with a generalization to $\mathcal{S}\otimes\mathcal{L}_u$ in line with what we already remarked. Jorgenson studies the degenerations of these log determinants, eventually to get the constant values of their differences. We now turn to asymptotics.

Weil functions and potential functions. On any variety V let D be a Cartier divisor. By a (complex) Weil function associated to D we mean a function $g\colon V - \operatorname{supp}(D) \longrightarrow \mathbb{R}$, such that, if D is represented by a rational function $D = (\varphi)$ on a Zariski open set U, then there exists a continuous function α on U such that

$$g(P) = -\log|\varphi(P)|^2 + \alpha(P) \quad \text{for } P \notin \operatorname{supp}(D).$$

We use the letter g because natural choices of Weil functions lead to Green's functions (potential functions). If V is compact, then the difference of two Weil functions associated with the same divisor is continuous, and therefore bounded on V.

Theorem 2. The function $g_{\Delta} = 12\log(C_{\Delta,\mathrm{hyp}} \cdot C_{\mathrm{hyp}})$ is a Weil function on the moduli space M_g, with respect to the boundary divisor. Furthermore, it is also a potential (Green) function for the Weil-Peterson metric, that is

$$dd^c g_{\Delta} = \mu_{WP} \quad \text{on } M_g,$$

possibly up to the factor $1/\pi^2$, depending how μ_{WP} is normalized.

I don't know to whom the first statement is due (about the Weil function). I learned it from Jorgenson. The second statement is due to Belavin-Kniznik [B-K 86] and Takhtajan-Zograf [T-Z 88], [TZ 91], who also prove the analogous formula for the non-compact case, involving Eisenstein series.

The asymptotics of another function $g_{\mathrm{geo}} = \sum (2\pi)^2/\ell$, are also very interesting, and have been studied, but I am running out of space.

Bibliography

[A-B-M-N-V 87] L. ALVAREZ-GAUMÉ, J.B. BOST, G. MOORE, P. NELSON, C. VAFA, Bosonization on higher genus Riemann surfaces, Comm. Math. Physics 112 (1987) pp. 503-552

[B-K 86] A. G. BELAVIN and V.G. KNIZNIK, Complex geometry and the theory of quantum strings, Sov. Phys. JETP 64 (1986) pp. 214-228

[D'H-P 86] E. D'HOKER and D.H. PHONG, On determinants of Laplacians on Riemann surfaces, Comm. Math. Physics 104 (1986) pp. 537-545

[Jo 91] J. JORGENSON, An evaluation of the constants in Deligne's Riemann-Roch theorem and small eigenvalues on compact hyperbolic Riemann surfaces, to appear

[J-L 91] J. JORGENSON and R. LUNDELIUS, Factorization theorems for determinants on finite volume Riemann surfaces, to appear

[Sa 87] P. SARNAK, Determinants of Laplacians, Comm. Math. Physics 110 (1987) pp. 113-120

[T-Z 91] L. TAKHTAJAN and P. ZOGRAF, A local index theorem for families of $\bar\partial$-operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces, Comm. Math. Phys.(1991)pp.399-426

[T-Z 88] L. TAKHTAJAN and P. ZOGRAF, The Selberg zeta function and a new Kähler metric on the moduli space of punctured Riemann surfaces, J. Geometry and Phasics Vol. 5 No. 4 (1988) pp. 553-570

[Wo 87] S. WOLPERT, Asymptotics of the spectrum and the Selberg zeta function for the spaces of Riemann surfaces, Commun. Math. Physics 112 (1987) pp. 283-315

Titel: Report on Mori Theory
Autor: H. Tsuji
Adresse: Max-Planck-Institut für Mathematik

This is a survey talk about Mori Theory on the classification of algebraic varieties.
The classification theory of algebraic varieties is an attempt to decompose algebraic varieties into three kinds of particles, i.e.

1. varieties with $-K_X > 0$,
2. varieties with $K_X \approx 0$, ($\approx$ num. equivalence)
3. varieties with $K_X > 0$.

This can be viewed as a higher dim. Riemann's uniformization.

I Cone theorem and Minimal Model Conjecture

To single out the particles of the 1st kind. Mori invented the following theorem.

Theorem (cone theorem : Mori, Kawamata) ([M1. K])
X^n: proj var / $\mathbb{C}$ with only canonical sing.
$NE(X)_{\mathbb{R}} = \{$ effective 1 cycle with $\mathbb{R}$ coefficients $\}/\approx$
$\Rightarrow$ If we set

$$NE^+(X)_{\mathbb{R}} = \{ [C] \in NE(X)_{\mathbb{R}} \mid K_X \cdot C < 0 \},$$

then there exist rational curves $l_1, \dots, l_i, \dots$
with $0 > K_X \cdot l_i \geqq -(n+1)$ s.t

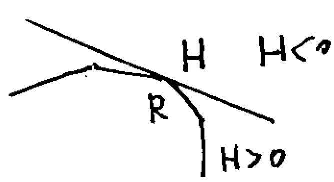
$$NE^+(X)_{\mathbb{R}} = \sum_i \mathbb{R}_{\geq 0}[l_i] \quad \text{(cone of loc fin. edges)}$$

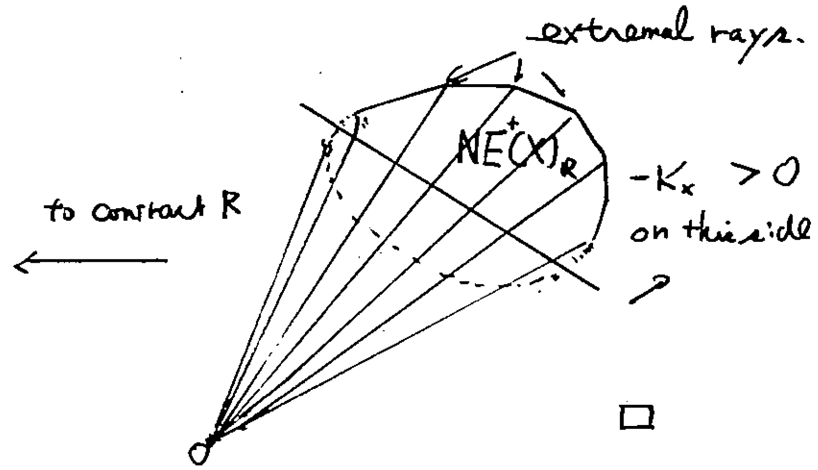
And for each edge $R = \mathbb{R}_{\geq 0}[l_i]$ (this is called an "extremal ray"), there is a surjective morphism of proj. varieties

$$\mathrm{cont}_R : X \longrightarrow {}^{\exists}Y$$

s.t.

$$\mathrm{Ker}\{\mathrm{cont}_{R*} : H_2(X,\mathbb{R}) \longrightarrow H_2(Y,\mathbb{R})\} = \mathbb{R}[l_i].$$





we need a such divisor

□

In the above theorem canonical singularity means:

<u>Def</u>. X : $\overset{\text{normal}}{\text{alg. var.}}$ has only canonical singularity iff
(1) K_X (canonical Weil divisor) is $\mathbb{Q}$-Cartier.
(2) $\exists \mu : Y \longrightarrow X$ resolution s.t. excep. set is a divisor. $E = \sum E_i$, then

$$K_Y = \mu^* K_X + \sum a_i E_i \qquad (a_i \geq 0).$$ □

<u>Def</u>. X : normal proj var.
X : minimal $\Longleftrightarrow$ ~~def~~ (1) X has only can. sing
(2) K_X is num. semipositive.

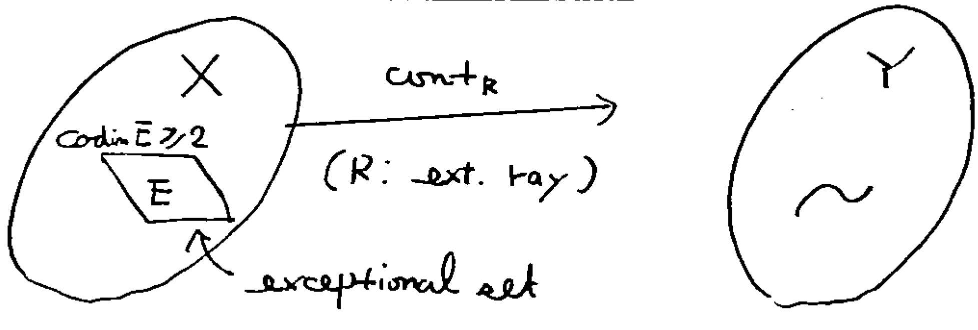
Minimal Model Conjecture

X: normal proj. var. If X is not uniruled (not covered by a family of rational curves), then there exists a minimal variety X_{min} birationally equivalent to X □

This conjecture was solved by Mori in the case of $\dim X = 3$ ($\dim X \leq 2$ it is classically known)([M2])

Difficulty Let X be a non uniruled smooth proj. var. To construct X_{min} (minimal model) by "cone theorem" we encounter the following difficulty.

bad contraction



If codim $E \geq 2$ then K_Y is not $\mathbb{Q}$-Cartier.

We cannot continue the contraction!

To continue further, we need an additional operation so called "flip".

II Geometric Construction of Canonical Models.

X^n: smooth proj var. $/\mathbb{C}$ of general type

- construction of min. model of X as a Kähler-Einstein space.

ω_0: C^∞- Kähler form on X. We consider:

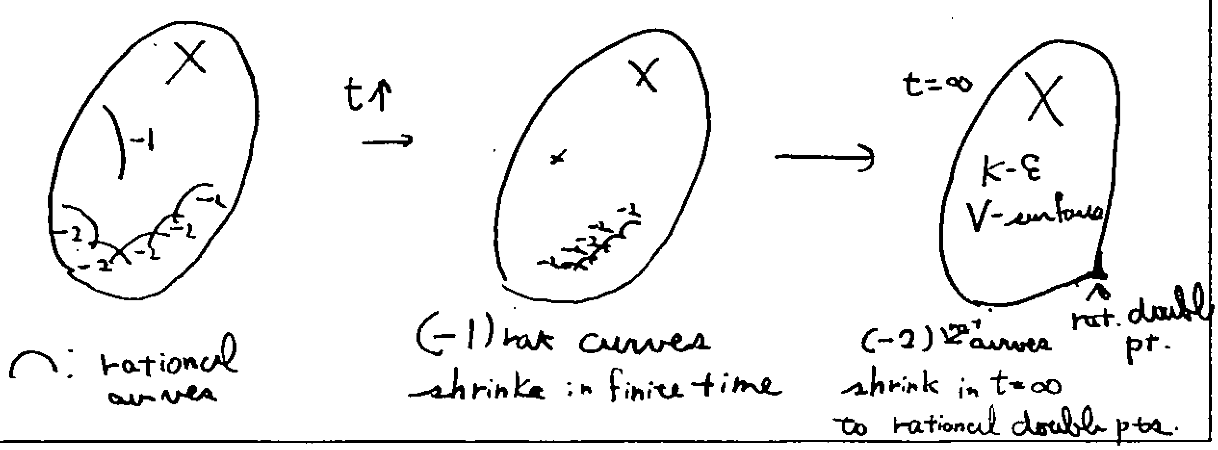
$$\begin{cases} \dfrac{\partial \omega}{\partial t} = -\mathrm{Ric}_\omega - \omega & \text{on } X \times [0, \infty). \\ \omega = \omega_0 & \text{on } X \times \{0\}. \end{cases}$$

where $\mathrm{Ric}_\omega = -\sqrt{-1}\,\partial\bar\partial \log \omega^n$.

Then we have

1. The solution ω exists as a d-closed positive (1.1)-current on $X \times [0, \infty)$.
2. $\omega_E := \lim\limits_{t\to\infty} \omega$ exists as a d-closed positive (1.1)-current and it is a C^∞ Kähler-Einstein form on a nonempty Zariski open subset of X.

example Case $\dim X = 2$



This example indicates us that ω_E comes from the <u>canonical model</u> of X.

In fact we can prove:

<u>Theorem</u>. X as above: Then
$R(X, K_X) = \bigoplus_{\nu \geq 0} H^0(X, \mathcal{O}_X(\nu K_X))$ is finitely generated.

Hence the canonical (minimal) model
$X_{can} = \mathrm{Proj}\, R(X, K_X)$ exists. $\square$

In particular ω_E comes from the K-E metric on X_{can}.

The relation between Minimal Model Conjecture and this result will be discussed in the talk.

<u>References</u>

[M-1] 3-folds whose canonical bdls are not numerically effective Ann Math. (1982)

[M-2] Flip Conjecture and the existence of minimal models for 3-folds, J. AMS 1 (1988)

[K] The cone of curves of algebraic varieties, Ann. of Math. 120 (1984)

Titel: NONLINEAR STABILITY OF MINKOWSKI SPACE

Autor: S. KLAINERMAN

Seite: 1

Adresse: DEPT. OF MATH., PRINCETON UNIV.
N.J. 08544 U.S.A.

AT THE HEART OF THE MODERN THEORY OF SPACE, TIME AND GRAVITY LIES THE GREAT UNIFICATION MADE BY EINSTEIN ACCORDING TO WHICH, IN THE ABSCENCE OF MATTER, the PHYSICAL "SPACE-TIME" CONSISTS OF A PAIR $(\mathcal{M}, g)$ WHERE $\mathcal{M}$ IS A 3+1 DIM. MANIFOLD AND g A LORENTZ METRIC WITH VANISHING RICCI CURVATURE,

$$(1) \qquad R_{\alpha\beta} = 0 \qquad \alpha, \beta = 0,1,2,3$$

RECALL THAT IF $R_{\alpha\beta\gamma\delta}$ DENOTES THE RIEMANN CURVATURE TENSOR OF g, $R_{\alpha\beta} = g^{\mu\nu} R_{\alpha\mu\beta\nu}$.

THE MINKOWSKI SPACE-TIME $\mathbb{M}^{1+3}$, WHICH PROVIDES THE GEOMETRIC FRAMEWORK OF SPECIAL RELATIVITY, IS A TRIVIAL SOLUTION OF (1). IN FACT $\mathbb{M}^{1+3}$ IS FLAT I.E. $R_{\alpha\beta\gamma\delta} \equiv 0$. A SPACE-TIME WHICH LOOKS MINKOWSKIAN OUTSIDE SOME COMPACT REGION WILL BE CALLED, IN WHAT FOLLOWS, GLOBALLY ASYMPTOTICALLY FLAT OR G.A.F. G.A.F SPACE-TIMES CORRESPOND TO ISOLATED PHYSICAL SYSTEMS AND PLAY A FUNDAMENTAL ROLE IN GENERAL RELATIVITY. THE QUESTION OF EXISTENCE AND DESCRIPTION OF SUCH SPACE-TIMES IS INTIMATELY CONNECTED WITH THAT OF THE

Titel:

Autor:

Adresse:

STABILITY OF MINKOWSKI SPACE-TIME.
THIS QUESTION CAN BE FRASED
AS FOLLOWS:

ASSUME THAT $(\mathcal{M}, g)$ IS A SOLUTION TO (1), I.E. RICCI FLAT, AND THAT $\mathcal{H}$ IS A SPACE-LIKE HYPERSURFACE EMBEDDED IN $(\mathcal{M}, g)$ with $\underline{g}, \underline{k}$ THE FIRST AND SECOND FUNDAMENTAL FORM. THE FACT THAT $\mathcal{H}$ IS SPACE-LIKE MEANS PRECISELY THAT THE INDUCED METRIC $\underline{g}$ IS RIEMANNIAN. IN VIEW OF THE EQUATION (1) $\underline{g}, \underline{k}$ SATISFY THE GAUSS-CODAZZI EQUATIONS ON $\mathcal{H}$,

$$(2) \quad \begin{cases} \bar{\nabla}^i k_{ij} - \bar{\nabla}_j \, tr_g k = 0 \\ \underline{R} - |k|^2 + (tr_g k)^2 = 0 \end{cases}$$

WHERE $\bar{\nabla}$ DENOTES THE INDUCED COVARIANT DERIVATIVE ON $\mathcal{H}$ AND $\underline{R}$ THE SCALAR CURVATURE OF $\underline{g}$.

THE TRIPLET $(\mathcal{H}, g, k)$, VERIFYING THE "CONSTRAINT EQTS" (2), CAN BE INTERPRETED AS AN INITIAL DATA SET. THE SPACE-TIME $(\mathcal{M}, g)$

Titel:

Autor: Seite: 3

Adresse:

CAN THEN BE VIEWED AS ITS
CAUCHY DEVELOPMENT. IF
$(\mathcal{M}, g)$ IS THE MINKOWSKI
SPACE-TIME AND $\mathcal{H}$ A
MAXIMAL HYPERSURFACE, I.E
$tr_{\underline{g}}\, \underline{k} = 0$, THEN $(\mathcal{H}, \underline{g}, \underline{k})$
IS FLAT I.E. $\underline{g}$ IS THE
EUCLIDEAN METRIC e AND $\underline{k} \equiv 0$.

THE QUESTION OF GLOBAL NONLINEAR
STABILITY OF THE MINKOWSKI
SPACE-TIME IS THAT OF STUDY
THE RELATION BETWEEN SMALL
PERTURBATION OF THE FLAT
INITIAL DATA SET $(\mathcal{H}, e, 0)$
AND THE BEHAVIOUR OF THEIR
CAUCHY DEVELOPMENTS RELATIVE
TO THE MINKOWSKI SPACE. IN
OTHER WORDS IF $(\mathcal{H}, \underline{g}, \underline{k})$ IS
A SMALL PERTURBATION OF
$(\mathcal{H}, e, 0)$ AND $(\mathcal{M}, g)$ ITS
DEVELOPMENT HOW CLOSE IS
$(\mathcal{M}, g)$ to $\mathbb{M}^{3+1}$. IN PARTICULAR
IS IT COMPLETE, IS IT
GLOBALLY ASYMPT. FLAT ?

Titel:

Autor: Seite: 4

Adresse:

WHEN ONE CONSIDERS THE QUESTION OF THE STABILITY OF THE MINKOWSKI SPACE-TIME IT IS NATURAL TO RESTRICT OURSELVES TO INITIAL DATA SETS WHICH LOOK FLAT OUTSIDE A SUFFICIENTLY LARGE SET IN $\mathcal{H}$. SUCH DATA SETS ARE CALLED ASYMPTOTICALLY FLAT, OR A.F. THE ONLY KNOWN, EXPLICIT DEVELOPMENTS OF A.F. INITIAL DATA SETS, ~~BESID~~ INCLUDING THE MINKOWSKI SPACE-TIME AS WELL AS THE SCHWARSCHILD SPACE-TIME, IS GIVEN BY A 2-PARAMETER FAMILY OF SOLUTIONS CALLED THE KERR SPACE-TIMES. WITH THE OBVIOUS EXCEPTION OF THE M^{1+3} ALL OTHER KERR SPACE-TIMES ARE INCOMPLETE. IT WAS THUS AN OPEN QUESTION WHETHER ANY GLOBAL, COMPLETE RICCI FLAT SOLUTIONS EXIST AT ALL. MOREOVER ALL SOLUTIONS OF THE KERR FAMILY ARE STATIONARY THEREFORE HAVE NO INTERESTING TIME EVOLUTION.

THE QUESTION OF STABILITY OF THE MINKOWSKI SPACE-TIME WAS RECENTLY SOLVED BY D. CHRISTODOULOU AND MYSELF. IT PROVIDES A LARGE

CLASS OF SOLUTIONS WHICH ARE SMOOTH, COMPLETE AND DYNAMICALLY INTERESTING.

<u>THEOREM</u> (D. CHRISTODOULOU - S. K.)

ANY A.F., MAXIMAL, (I.B. r_g^b) INITIAL DATA SET, SUFFICIENTLY CLOSE TO THE FLAT ONE, HAS A SMOOTH, GLOBALY ASYMPT. FLAT DEVELOPMENT VERIFYING THE EINSTEIN FIELD EQUATIONS (1)

<u>REF.</u> D. CHRISTODOULOU - S. K.

THE GLOBAL NONL. STABILITY OF MINKOWSK SPACE - PREPRINT

Titel: On Killing spinors and exceptional holonomy groups

Autor: Christian Bär

Adresse: Mathematisches Institut der Univ. Bonn
Heckenheimer Allee 160
5300 Bonn 1, Germany

Let M be an n-dimensional complete Riemannian spin manifold. A spinor field ψ is called Killing spinor with Killing constant α if for all tangent vectors X the equation $\nabla_X\psi = \alpha \cdot X \cdot \psi$ holds. Here $X\cdot\psi$ means Clifford multiplication. Killing spinors are of physical interest, see [DNP], but they also occur in purely mathematical context. For example, Friedrich has proved that if M is compact and the scalar curvature satisfies $S \geq S_0 > 0$, $S_0 \in \mathbb{R}$, then for all eigenvalues λ of the Dirac operator the estimate $\lambda^2 \geq \frac{1}{4}\frac{n}{n-1} S_0$ holds [F1]. If we have equality in this estimate, then the corresponding eigenspinor is a Killing spinor.

If M carries a Killing spinor, then M is an Einstein manifold with Ricci curvature $Ric = 4(n-1)\alpha^2$. In particular, we have three distinct cases; α can be purely imaginary, then M is noncompact and we call ψ an imaginary Killing spinor, α can be 0, in this case ψ is a parallel spinor field, and finally α can be real, then M is compact and ψ is called a real Killing spinor.

Hitchin showed that manifolds with parallel spinor fields can be characterized by their holonomy group [Hit, thm. 1.2 and footnote p. 54], see also [F3] and [W].

Manifolds with imaginary Killing spinors have been classified by Baum in [B1] – [B3], shortly later the classification has been extended by Rademacher to generalized imaginary

Killing spinors where we allow the Killing "constant" α to be an imaginary function [R].

Most results on real Killing spinors known so far are statements for particular low dimensions. For example, Friedrich showed in [F2] that a complete 4-dimensional manifold with real Killing spinor is isometric to the standard sphere. The analogous result in dimension 8 is due to Hijazi [Hi]. We show

Theorem 1. Let M be a complete Riemannian spin manifold of dimension n carrying a Killing spinor with Killing constant $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$ (This can be achieved by rescaling the metric). If n is even, $n \neq 6$, then M is isometric to the standard sphere. □

We say that M is of type (p,q) if M carries exactly p linearly independant Killing spinors for $\alpha = \frac{1}{2}$ and exactly q linearly independant Killing spinors for $\alpha = -\frac{1}{2}$ or vice versa.

In dimension 6 we recover a theorem of Grunewald [G]

Theorem 2. Let M be a 6-dimensional complete 1-connected Riemannian spin manifold with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. Then there are two possibilities:

(i) $M = S^6$

(ii) M is of type $(1,1)$ and M is nearly Kähler, non-Kähler.

Conversely, if a complete 1-connected Riem. mfd. $M \neq S^6$ is nearly Kähler, non-Kähler, then M is of type $(1,1)$. □

Furthermore, we get (M is always a complete 1-connected Riem. spin manifold of dimension n with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$)

<u>Theorem 3.</u> If $n = 2m-1$, $m \geq 3$ odd, then there are two possibilities:

(i) $M = S^n$

(ii) M is of type $(1,1)$ and M is an Einstein-Sasaki manifold. □

In dimension 5 this theorem can also be found in [FK1].

<u>Theorem 4.</u> If $n = 4m-1$, $m \geq 3$, then there are three possibilities:

(i) $M = S^n$

(ii) M is of type $(2,0)$ and M is an Einstein-Sasaki manifold, but does not carry a Sasaki-3-structure.

(iii) M is of type $(m+1, 0)$ and M carries a Sasaki-3-structure. □

<u>Theorem 5.</u> If $n = 7$, then there are four possibilities:

(i) $M = S^7$

(ii) M is of type $(1,0)$ and M carries a 3-form φ with $\nabla\varphi = *\varphi$ which can be induced by the multiplication of imaginary Cayley numbers, but M does not carry a Sasaki structure.

(iii) M is of type $(2,0)$ and carries a Sasaki structure, but not a Sasaki-3-structure

(iv) M is of type $(3,0)$ and carries a Sasaki-3-structure

In theorems 3–5 the converses are also true. For example, if $n = 2m-1$, $M \neq S^n$ is a complete 1-connected Einstein-Sasaki manifold, then M is of type $(1,1)$.

The method of proof is as follows. First we modify the spinor connection because we want to interprete Killing spinors as parallel sections. To do this we have to enlarge the structure group $Spin(n)$ of the spinor bundle to $Spin(n+1)$. Then we show that this connection is related to the Levi-Civita connection of the cone over the original manifold. Since Killing spinors now correspond to fixpoints of the holonomy group of the cone we can use the Berger-Simons classification of possible holonomy groups to see how the cone can possibly look like. Finally, this information is retranslated into conditions on the original manifold itself.

Exceptional holonomy groups.

The study of the exceptional dimension 6 provides us with a construction principle of Riemannian manifolds with exceptional holonomy group G_2. The recipe is as follows. Take any compact 1-connected nearly Kähler, non-Kähler manifold of dimension 6, normalize the metric such that $Ric = 5$, then the cone over this manifold has holonomy group G_2. Using this method we recover Bryant's first example which was the cone over the complex flag manifold $SU(3)/T^2$ [Br]. Further examples are obtained by taking the cones over $S^3 \times S^3$ and $\mathbb{C}P^3$ with certain non-standard metrics.

Similarly, the cones over certain 7-manifolds have exceptional holonomy group Spin(7). Explicit new examples are the cones over the squashed 7-sphere and over the Wallach manifolds.

References

[B] C. Bär, Real Killing spinors and holonomy, preprint Bonn 1991

[B1] H. Baum, Variétés riem. admettant des spineurs de Killing imaginaires, CR Acad Sci Paris 309 (1989) 47-49

[B2] H. Baum, Odd-dim. Riem. mfds with im. Killing spinors, Ann. Glob. Anal. Geom. 7 (1989) 141-153

[B3] H. Baum, Complete Riem. mfds with im. Killing spinors, Ann. Glob. Anal. Geom. 7 (1989) 205-226

[SB] Baum, Friedrich, Grunewald, Kath, Twistor and Killing spinors on Riem. mfds, Seminarbericht 108, Humboldt-Univ. Berlin 1990

[Br] R.L. Bryant, Metrics with exceptional holonomy, Ann. Math. 126 (1987) 525-576

[F1] T. Friedrich, Der erste Eigenwert ..., Math. Nach. 97 (1980) 117-146

[F2] T. Friedrich, A remark ..., Math. Nach. 102 (1981) 53-56

[F3] T. Friedrich, Zur Existenz ..., Colloq. Math. 44 (1981) 277-290

[DNP] Duff, Nilsson, Pope, Kaluza-Klein supergravity, Phys. Rep. 130 (1986) 1-142

[G] R. Grunewald, Six-dimensional ..., Ann. Glob. Anal. Geom. 8 (1990) 43-59

Autor: Christian Bär Seite: 6

[Hi] O. Hijazi, Caractérisation de la sphère..., CR Acad Sci Paris 303 (1986) 417-419

[Hit] N. Hitchin, Harmonic spinors, Adv. in Math. 14 (1974) 1-55

[R] H-B. Rademacher, Generalized Killing spinors..., to appear in Proc. conf. Glob. Anal. Glob. Diff. Geom., Berlin 1990, Springer LN.

[W] M. Wang, Parallel spinors and parallel forms, Ann. Glob. Anal. Geom. 7 (1989) 59-68

Titel: Deligne's Conjecture on the Lefschetz Trace Formula in positive Characteristic

Autor: Richard Pink

Adresse: Mathematisches Institut der Universität
Beringstr. 4
5300 Bonn 1, GERMANY (WEST)

Grothendieck's LTF (= Lefschetz Trace Formula) for the Frobenius morphism plays a central role in the study of the Galois representations on the étale cohomology of algebraic varieties (see SGA $4\frac{1}{2}$ Rapport 3.2). To obtain the same information for direct factors of the total cohomology that are "cut out" by correspondences one needs a LTF for correspondences twisted by Frobenius. There exists a very general such LTF (see SGA 5, III, Cor. 4.7) but its local terms are difficult to calculate.
Deligne's conjecture asserts that, under certain conditions sufficient for the above-mentioned purpose, there is a LTF of a very simple form.

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of a finite field $\mathbb{F}_q$ with q elements. A <u>correspondence</u> is a diagram

$$X \xleftarrow{b_1} B \xrightarrow{b_2} X$$

of morphisms and compactifiable separated schemes of finite type over $\overline{\mathbb{F}}_q$. Assume that X comes from a scheme over $\mathbb{F}_q$, i.e. that it comes with a Frobenius morphism $\Phi_q : X \longrightarrow X$. The <u>twisted correspondence</u> $b^{(n)} := (b_1^{(n)}, b_2^{(n)})$ is defined by the diagram

$$\begin{tikzcd} X & X \arrow[l, "\Phi_q^n"'] & B \arrow[l, "b_1"'] \arrow[ll, bend left, "b_1^{(n)}"] \arrow[r, "b_2"] \arrow[r, bend right, "b_2^{(n)}"'] & X \end{tikzcd}.$$

Assume: (1) b_1 is proper.

(2) b_2 is quasifinite of degree $< q^n$.

It is easy to see that

$$\mathrm{Fix}(b^{(n)}) := \{\beta \in B \mid b_1^{(n)}(\beta) = b_2^{(n)}(\beta)\}$$

is finite ("isolated fixed points").

Let $\mathcal{F}$ be a constructible $\mathbb{Q}_\ell$-sheaf on X, $\ell \nmid q$, and consider a homomorphism

$$u : b_{2!}\, b_1^* \mathcal{F} \longrightarrow \mathcal{F}.$$

Assume that we are given an isomorphism $\Phi_q^* \mathcal{F} \cong \mathcal{F}$, ~~then~~ we can twist u:

$$\begin{tikzcd} b_{2!}^{(n)}\, b_1^{(n)*} \mathcal{F} \arrow[r, "u^{(n)}"] \arrow[d, equal] & \mathcal{F} \\ b_{2!}\, b_1^* \Phi_q^{n*} \mathcal{F} \arrow[r, phantom, "\cong"] & b_{2!}\, b_1^* \mathcal{F} \arrow[u, "u"'] \end{tikzcd}.$$

Now the "<u>cohomological correspondence</u>" $(b^{(n)}, u^{(n)})$ has a <u>global term</u>, the trace of the endomorphism $u_!^{(n)}$:

$$\begin{tikzcd} H_c^i(X, \mathcal{F}) \arrow[r, "u_!^{(n)}"] \arrow[d] & H_c^i(X, \mathcal{F}) \\ H_c^i(B, b_1^{(n)*}\mathcal{F}) \arrow[r, phantom, "\cong"] & H_c^i(X, b_{2!}^{(n)}\, b_1^{(n)*}\mathcal{F}) \arrow[u, "u^{(n)}"'] \end{tikzcd}$$

and <u>local terms</u> for every fixed point $\beta \in \mathrm{Fix}(b^{(n)})$:

Put $x := b_1^{(n)}(\beta) = b_2^{(n)}(\beta)$, then we get an endomorphism of the stalk

$$\begin{tikzcd} \mathcal{F}_x \arrow[r, "u_\beta^{(n)}"] \arrow[d, equal] & \mathcal{F}_x \\ (b_1^{(n)*}\mathcal{F})_\beta \arrow[r, hook, "\text{canonical}"'] & (b_{2!}^{(n)} b_1^{(n)*}\mathcal{F})_x \arrow[u, "u^{(n)}"'] \end{tikzcd}$$

<u>Conjecture</u> (Deligne, 1970es): For n sufficiently large

$$\sum_i (-1)^i \operatorname{tr}\left(u_!^{(n)} \middle| H_c^i(X,\mathcal{F})\right) = \sum_{\beta \in \operatorname{Fix} b^{(n)}} \operatorname{tr}\left(u_\beta^{(n)}\right).$$

Special cases of this were known (Grothendieck, Deligne-Lusztig, Illusie (dim = 1), Zink (dim = 2), ...)

<u>Theorem</u> (Pink 1990): Deligne's conjecture follows from resolution of singularities over $\mathbb{F}_q$.

(A somewhat weaker result has been proved by Shpiz, see ref.)

The precise resolution assumptions are as follows:

(1) <u>RESOLUTION</u>: X of finite type over k, $Z \subset X$ closed such that $X \setminus Z$ is smooth
$\Longrightarrow \exists\, \pi : \tilde{X} \longrightarrow X$ proper modification with:

π is an isomorphism over $X \setminus Z$,
$\tilde{X}$ is smooth, and $\pi^{-1}(Z)$ is a divisor with normal crossings.

(2) <u>SEPARATION</u>: X of finite type over k, $Z_1, Z_2 \subset X$ closed
$\Longrightarrow \exists\, \pi : \tilde{X} \to X$ proper modification with:
π is an isomorphism outside $Z_1 \cap Z_2$, and the proper transforms of Z_1, Z_2 are disjoint.

These two hypotheses are known in characteristic zero (Hironaka), but not at present over $k = \mathbb{F}_q$.

Nevertheless, suppose now that we are given a cohomological correspondence with the above properties, but defined over a number field K. For all but finitely many primes of K we obtain a reduction at $\mathfrak{p}$. The resolutions of singularities necessary in the above theorem can be done once and for all over K, and can be used at almost all primes.

$\Longrightarrow$ <u>Theorem</u>: Given a cohomological correspondence over a number field, Deligne's conjecture is true for its reduction at all but finitely many primes.

An important case is where X is (a reduction of) a Shimura variety or a(n algebraic) compactification thereof, where (B, b) is a Hecke-correspondence, and $\mathcal{F}$ is an automorphic locally constant sheaf respectively a suitable extension to the compactification (extension by zero, intersection complex, ...).

References:

SGA $4\frac{1}{2}$: Séminaire de géometrie algebrique $4\frac{1}{2}$, Cohomologie étale, Springer LN 569 (1977)

SGA 5 : Cohomologie l-adique et fonctions L Springer LN 589 (1977)

Pink, On the calculation of local terms in the Lefschetz-Verdier trace formula and its application to a conjecture of Deligne, to appear in: Annals of Mathematics

E. Shpiz, Ph.D. thesis, Harvard (1990).

Titel: Analytic torsion for non-unitary representations and Chern-Simons gauge theory
Autor: Werner Müller
Adresse: MPI

The purpose of this talk is to give a report on some new developements related to analytic torsion.

Introduction The concept of torsion was introduced in 1935 by Reidemeister, Franz and de Rham. Let K be a finite simplicial complex and $\rho : \pi_1(M) \to O(N)$ an orthogonal representation with associated flat bundle E_ρ. Assume that ρ is acyclic, that is, $H^*(K; E_\rho) = 0$. Then the *Reidemeister-Franz torsion* (or *R-torsion*) $\tau_M(\rho) \in \mathbf{R}^+$ is defined. The torsion $\tau_M(\rho)$ is a kind of determinant which describes how the simplices of $\hat{K}$ are fitted together with respect to the action of $\pi_1(K)$. It is known to be a combinatorial invariant in the sense that it is invariant under subdivision [Mi].

In particular, if K is a smooth triangulation of a closed C^∞-manifold M, then the R-torsion depends only on the smooth structure of M and we denote the torsion by $\tau_M(\rho)$.

In [RS], Ray and Singer introduced the analytic torsion $T_M(\rho)$ as analytic counterpart to R-torsion. To define $T_M(\rho)$ one has to choose a Riemannian metric g on M. Together with the canonical metric on E_ρ which is compatible with the flat connection we get an inner product on the twisted de Rham complex $\Lambda^*(M; E_\rho)$ of E_ρ-valued differential q-forms on M. Let Δ_q be the Laplacian on $\Lambda^q(M; E_\rho)$ and $\zeta_q(s;\rho) = \sum_{\lambda_j} \lambda_j^{-s}$, $\mathrm{Re}(s) > n/2$, $n = \dim M$, the zeta function of Δ_q. Then $T_M(\rho)$ is defined as

$$T_M(\rho) = \exp\Big(\tfrac{1}{2}\sum_{q=0}^{n}(-1)^q q \,\frac{d}{ds}\zeta_q(s;\rho)\big|_{s=0}\Big). \tag{1}$$

For acyclic ρ, $T_M(\rho)$ is independent of the choice of the Riemmanian metric on M.It was conjectured by Ray and Singer that $T_M(\rho) = \tau_M(\rho)$ for all acyclic orthogonal representations ρ. This conjecture was proved independently by Cheeger [C] and the author [Mü1].

Recently, torsion has found interesting applications in low dimensional topology and topological quantum field theory. D.Johnson has shown that R-torsion is closely related to Casson's invariant [J]. In [W4], Witten has used the relation of the weak couplind limit of three dimensional Chern-Simons gauge theory to analytic torsion to study two dimensional quantum Yang-Mills theory. This leads to formulas for the volumes of moduli spaces of representations of fundamental groups of compact surfaces.

1. Torsion for unimodular representations. The restriction to orthogonal (or unitary) representations is certainly a limitation of the applicability of the equality of the two torsions if the fundamantal group is infinite. We remove this limitation essentially. Namely, let $\rho : \pi_1(M) \to \mathrm{GL}(E)$ be a representation on a finite dimensional real or complex vector space E. ρ is called *unimodular* if $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(M)$. Then the definition of R-torsion also makes sense for any unimodular representation. To define the analytic torsion we follow an idea of Schwarz [S]. We choose a metric h on E_ρ and with respect to this metric we define the torsion $T_M(\rho; h)$ by a formula similar to (1). If $\dim M$ is odd and ρ is acyclic, it turns out that $T_M(\rho; h)$ is independent of h and also on the Riemannian metric on M. We call the common value $T_M(\rho)$.

Theorem 1. *Let $\dim M$ be odd. For all acyclic unimodular representations $\rho : \pi_1(M) \to \mathrm{GL}(E)$ we have*

$$T_M(\rho) = \tau_M(\rho).$$

For the proof see [Mü2]. We remark that both analytic torsion and R-torsion can be defined for all unimodular representations. If the representation is not acyclic, then both invariants will depent on the choice of the metrics on M and E_ρ. Nevertheless, the equality of Theorem 1 remains valid.

Next we discuss some applications of this result.

2.Locally symmetric manifolds. Let G be a connected real semi-simple Lie group with finite center. Assume that G has no compact factors and let $K \subset G$ be a maximal compact subgroup. Then $X = G/K$ is a symmetric space. Let $\Gamma \subset G$ be a discrete, torsion free, co-compact subgroup. Then $M = \Gamma\backslash X$ is a compact locally symmetric manifold. Examples are hyperbolic 3-manifolds. Given a finite dimensional representation $\rho : \pi_1(M) \to \mathrm{GL}(E)$ we get by restriction, a representation ρ_Γ of Γ and a flat bundle E_ρ over $\Gamma\backslash X$. Note that ρ_Γ is unimodular. The flat bundle has a natural locally homogeneous metric [MM]. Let θ be the Cartan involution of $(\mathfrak{g}, \mathfrak{k})$ and assume that (E, ρ) is irreducible with heighest weight $\Lambda - \underline{\rho}$ ($\underline{\rho} = \frac{1}{2}$sum of positive roots). If $\theta\Lambda \neq \Lambda$, it follows from Theorem 6.7 of section VII in [BW] that $H^*(\Gamma\backslash X; E) = 0$. If $\dim G/K$ is odd, then $\operatorname{rk} G > \operatorname{rk} K$ and a generic irreducible representation has vanishing cohomology. We also note that for $\operatorname{rk} G > 1$, superrigidity implies that all representations of Γ arise from representations of G [Ma]. Thus we can define the analytic torsion for all these representations and we expect them to be interesting invariants of the locally symmetric manifold $\Gamma\backslash X$.

3.Lefschetz numbers for flows. As an application of Theorem 1 we obtain the extension of a result of Moscovici and Stanton [MS]. Consider the geodesic flow Φ on the unit sphere bundle to $\Gamma\backslash X$. The connected components of the periodic set are parametrized by the non-trivial conjugacy classes $\{\gamma\}$ in Γ. Each connected component X_γ is itself a compact locally symmetric manifold of non-positive sectional curvature and Φ restricts to a periodic flow on X_γ. The quotient $\hat{X}_\gamma = X_\gamma/\Phi$ is a smooth orbifold. Let l_γ be the common length of the orbits in X_γ and let μ_γ be the multiplicity of a generic orbit of $\Phi|X_\gamma$. Set

$$Z_\rho(s) = \exp - \sum_{\{\gamma\}\neq 1} \operatorname{Tr}\rho(\gamma)\, \chi(\hat{X}_\gamma)\, \frac{e^{-sl_\gamma}}{\mu_\gamma}.$$

Theorem 2. *$Z_\rho(s)$ is analytic for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s = 0$. If $\dim G/K$ is odd, then*

$$Z_\rho(0) = \tau_M(\rho)^2 \quad \text{where} \quad M = \Gamma\backslash X.$$

For orthogonal (or unitary) representations ρ, this result is due to Moscovici and Stanton [MS].

4.Chern-Simons gauge theory. Chern-Simons theory is a three dimensional gauge field theory with pure Chern-Simons action. It was used by Witten [W1] to introduce new 3-manifold invariants. The basic setting for Chern-Simons theory is a compact oriented three dimensional manifold M without boundary and a Lie group G. We start with the case where G is compact and for simplicity, we take G to be $\mathrm{SU}(N)$. Consider the space $\mathcal{A}$ of all G-connections on the trivial G-bundle over M. In fact, every principal G-bundle over M is trivial. The space $\mathcal{A}$ may be identified with the space $\Lambda^1(M, \mathfrak{g})$ of differential 1-forms on M with values in the Lie algebra $\mathfrak{g}$ of G. For a given connection $A \in \mathcal{A}$, the Chern-Simons action is defined to be

$$I(A) = \frac{1}{4\pi} \int_M \mathrm{Tr}(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A) \tag{2}$$

where Tr is the trace of $\mathfrak{su}(N)$ in the standard representation. This is a real valued non-linear functional on $\mathcal{A}$. The gauge group $\mathcal{G} = \mathrm{Map}(M, G)$ acts on $\mathcal{A}$ by the usual prescription $A^g = g^{-1}Ag + g^{-1}dg$, $g \in G$, $A \in \mathcal{A}$. Let $k \in \mathbf{N}$. Then $e^{ikI(A)}$ is a $\mathcal{G}$-invariant function on $\mathcal{A}$ and Witten's invariant of M is defined as the path integral

$$Z_M(k) = \int e^{ikI(A)}\, \mathcal{D}A \tag{3}$$

where the integration is over all gauge equivalence classes of connections. This, however, has to be considered as a formal expression, because no measure $\mathcal{D}A$ has been constructed up to now. Part of this theory can be made rigorous and Witten gave an explicit recipe for computing $Z_M(k)$.

A standard way to study functional integrals like (3) is to use the method of stationary phase approximation which predicts the behaviour of $Z_M(k)$ for large k. In the present context this method is again not based on solid ground, but it gives very interesting results. By the method of stationary phase, the leading order contribution to $Z_M(k)$ comes from the critical points of the action (2). The critical points of (2) are precisely the connections with vanishing curvature, that is, the flat connections on the bundle $P = M \times G$. Assume that the topology of M is such that there exists only a finite number of gauge equivalence classes of flat connections on P, say $A_1, \dots, A_m$ and that $A_1, \dots, A_m$ are all irreducible. Then Witten's formula for the stationary phase approximation of the path inytegral (3) is

$$Z_M(k) \sim \frac{1}{\#Z(G)} \sum_{j=1}^{m} e^{i\eta(\rho_{\alpha_j})} \sqrt{T_M(\rho_{\alpha_j})} \tag{4}$$

where $Z(G)$ is the center of G, $\alpha_j : \pi_1(M) \to G$ is the representation determined by A_j, $\eta(\alpha_j)$ is a certain phase factor described in [W1] and $T_M(\rho_{\alpha_j})$ is the analytic torsion of $\rho_{\alpha_j} = \mathrm{Ad} \circ \alpha_j : \pi_1(M) \to \mathrm{GL}(\mathfrak{g})$. Since each ρ_{α_j} is acyclic, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric g on M and, by [C], [Mül], it coincides with the R-torsion $\tau_M(\rho_{\alpha_j})$. As we know, the R-torsion $\tau_M(\rho_{\alpha_j})$ can be computed from a triangulation K of M in a pure combinatorial way. This suggests that one may be able to develop a rigorous treatment of the path integral (3) on the combinatorial level and derive the asymptotic behaviour (4) in this way.

So far we considered the case of a compact gauge group. Witten has also started to investigate Chern-Simons theory with non-compact gauge group [W3]. There exist several motivations to develop such a theory. For example, $2+1$ dimensional gravity is related to Chern-Simons gauge theory with gauge group $\mathrm{SL}(2,\mathbf{C})$, $ISO(2,1)$ or $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ depending on wether the cosmological constant is positive, zero, or negative [W2]. For a general non-compact Lie group G, the quantization of Chern-Simons gauge theory with gauge grouyp G is not yet understood. Nevertheless, one can study the perturbative expansion of the corresponding path integral [BNW].

The perturbative treatment of Chern-Simons gauge theory with non-compact gauge group requires again gauge fixing. Since the Killing form is indefinite there exists no obvious gauge fixing as in the compact case and different approaches are possible [BNW].For a semi-simple Lie group G, the most natural gauge fixing seems to be the unitary gauge fixing described in section 4 of [BNW]. Let A be a flat connection on the trivial G-bundle over M with holonomy representation $\alpha : \pi_1(M) \to G$. As above, let $\mathfrak{g}_\alpha$ be the flat bundle defined by $\rho_\alpha = \mathrm{Ad} \circ \alpha$. Then the unitary gauge fixing amounts to the choice of a riemannian metric g on M and a hermitian metric h on $\mathfrak{g}_\alpha$. We observe that $\rho_\alpha : \pi_1(M) \to \mathrm{GL}(\mathfrak{g})$ is unimodular. In fact, since $\mathfrak{g}$ is semi-simple, the Killing form is non-degenerate. Hence, for each $g \in G$, $\mathrm{Ad}(g)$ preserves a non-degenerate symmetric bilinear form on $\mathfrak{g}$ which implies that $|\det \mathrm{Ad}(g)| = 1$. This is precisely the setting of section 1.

Under the same assumption as above, one gets a formula for the one loop approximation of the path integral which is similar to (4). The analytic torsion $T_M(\rho_{\alpha_j})$ is now defined as discribed in section 1. For the discussion of the phase factor see section 4 of [BNW]. By assumption, each representation ρ_{α_j} is acyclic and therefore, $T_M(\rho_{\alpha_j})$ is independent of the choice of the metric on M and $\mathfrak{g}_\alpha$. Moreover, by Theorem 1, $T_M(\rho_{\alpha_j})$ equals the R-torsion $\tau_M(\rho_{\alpha_j})$ which has again a pure combinatorial describtion. This suggests that Chern-Simons gauge theory with a non-compact, but semi-simple gauge group should also be accessible to a combinatorial treatment.

References

[BNW] Bar-Natan,D., Witten,E.: Perturbative expansion of Chern-Simons theory with non-compact gauge group. Preprint, IASSNS-HEP-91/4, Princeton, 1991.

[BW] Borel,A., Wallach,N.: Continuous cohomology, discrete subgroups, and representations of reductive groups. Annals of Math. Studies **94**, Princeton Univ. Press, princeton, 1980.

[C] Cheeger,J.: Analytic torsion and the heat equation. Annals of Math. **109**, 259 - 322 (1979).

[J] Johnson,D.: A geometric form of Casson's invariant, and its connection to Reidemeister torsion. unpublished lecture notes.

[Ma] Margulis,G.A.: Discrete Subgroups of Semisimple Lie Groups. Springer-Verlag, Berlin Heidelberg New York, 1991.

[MM] Matsushima,Y., Murakami,S.: On vector bundle valued harmonic forms and automorphic forms on symmetric spaces. Annals of Math. **78**, 365 - 416 (1963).

[Mi] Milnor,J.: Whitehead torsion. Bull. Amer. Math Soc. **72**, 358 -426 (1966).

[MS] Moscovici,H., Stanton,R.J.: R-torsion and zeta functions for locally symmetric manifolds. Preprint, Ohio State Univ., 1990.

[Mü1] Müller,W.: Analytic torsion and R-torsion of Riemannian manifolds. Advances in Math. **28**, 233 - 305 (1978).

[Mü2] Müller,W.: Analytic torsion and R-torsion for unimodular representations. MPI-Preprint, 1991.

[RS] Ray,D.B., Singer,I.M.: R-torsion and the Laplacian on Riemannian manifolds. Advances in Math. **7**, 145 - 210 (1971).

[S] Schwarz,A.: The partition function of degenerate quadratic functional and Ray-Singer invariants. Lett. Math. Phys. **2**, 247 - 252 (1978).

[W1] Witten,E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys. **121**, 351 - 399 (1988).

[W2] Witten,E.: $2+1$ dimensional gravity as an exactly soluble system. Nuclear Phys. **B 311**, 46 - 78 (1988/89).

[W3] Witten,E.: Quantization of Chern-Simons gauge theory with complex gauge group. Commun. Math. Phys. **137**, 29 - 66 (1991).

[W4] Witten,E.: On quantum gauge theories in two dimensions. Preprint IASSNS-HEP-91/3, Princeton 1991.

Titel: Newton polygons and abelian varieties.

Autor: Frans OORT

Adresse: Mathematical Institute
Budapestlaan 6
NL - 3508 TA UTRECHT
The Netherlands

1 Introduction. Geometric points of a moduli space correspond to isomorphism classes of certain objects (which on wants to study, wants to classify). If one requires those objects to have some extra properties (some additional structure) one obtains a subset of that moduli space (which because it is nature-given is interesting, and can have some nice properties). Many proof in algebraic geometry are given using properties of such substes of a moduli space.

Today we discuss closed subsets of the moduli space of principally polarized abelian varieties in positive characteristic given by Newton polygons. This stratification refines the one by the p-rank.

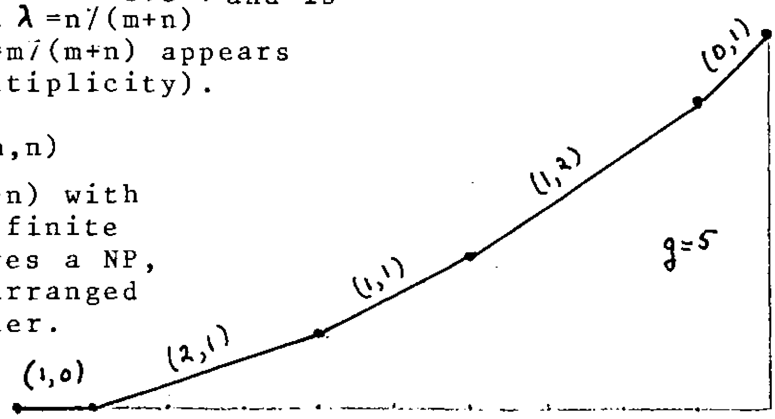
From the precise information we obtain for the strata defined by the various Newton polygons we derive a proof of a conjecture by Manin (1963: every symmetric Newton poly-on can be realized by an abelian varietiy), and we show a strengthened form of a conjecture by Koblitz (1975: an ordered pair of Newton polygons can be realized by a specialization of abelian varieties). Our inspiration came from these conjectures by Manin and Koblitz, from results by Mumford, Grothendieck, Katz (and many others), and from cooperation with Tadao Oda, T.Katsura, P.Norman, T.Ekedahl and K.-Z.Li.

Some notation: $g=\dim(AV)$, p is a prime number, n is a positive integer prime to p, we write NP for Newton polygon, $\mathcal{N}(X)$ is the NP of the abelian variety X, we write $f=f(X)$ for the p-rank of X, we write $a(X)= \dim \mathrm{Hom}(\alpha_p,X)$, $G_{m,n}$ is a p-divisible group of dimension m, whose (Serre-)dual has dimension n.

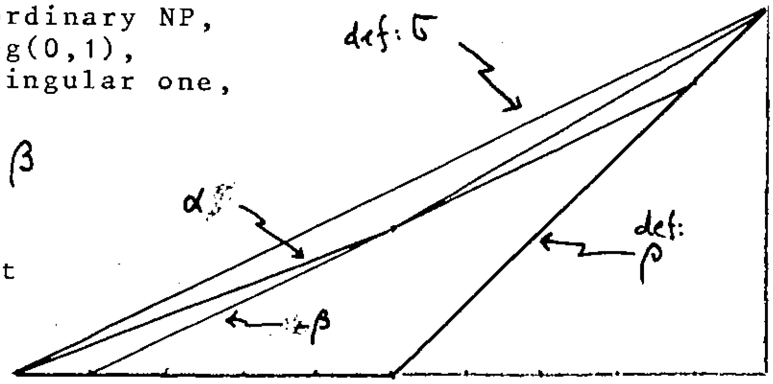
2 Newton polygons.

NP: lower convex polygon in $\mathbb{Q}\times\mathbb{Q}$, breakpoints in $\mathbb{Z}\times\mathbb{Z}$, starts at (0,0), ends at (2g,g), and is symmetric (if a slope $\lambda=n/(m+n)$ appears then $1-\lambda=m/(m+n)$ appears with the same multiplicity).

$G_{m,n}$, or the pair (m,n) gives the slope $n/(m+n)$ with multiplicity m+n. A finite set of such pairs gives a NP, when the slopes are arranged in non-decreasing order.



We write ρ for the ordinary NP, i.e. given by g(1,0)+g(0,1), and σ for the supersingular one, i,e, given by g(1,1), and we write $\alpha \prec \beta$



(and we say that β is below α) if no point of α is strictly below β.
!! note the reverse in this order!!
Smallest: σ (supersingular), largest: ρ (ordinary).
The NPs form a directed graph: $\forall \alpha : \sigma \prec \alpha \prec \rho$.

3 <u>The NP of an abelian variety</u>.
If k is a field, char(k)=p > 0, and X is an abelian variety over k, then X determines a NP. If k happens to be a finite field this can be given by the NP of the char.pol. of the geometric Frobenius of X (suitably normalized if k has more than p elements). In general on takes the p-divisible group G of X over an algebraic closure of k. By Dieudonné-Manin theory we can write $G \sim f.(G_{1,0}+G_{0,1}) + \sum (G_{m_i,n_i}+G_{n_i,m_i}) + s.G_{1,1}$ and the pairs (m,n) thus obtained give the NP $\mathcal{N}(X)$. Note that X has a polarization, hence is isogenous to its dual, and this gives the symmetry of $\mathcal{N}(X)$. Note that $\mathcal{N}(X)$ depends only on the isogeny class of X over some field containing k.

4 <u>Conjecture, Manin, 1963</u>: Suppose given g, and a prime number p, and a NP (symmetric), then there exists an AV having this NP in characteristic p (cf. (8), page 76).
Remark: this was proved in 1967 by Honda, and by Serre, via reducing a well-choosen CM abelian variety from char. zero to char. p; below we indicate another proof .

5 <u>Theorem, Grothendieck</u>: If an abelian variety X_ξ specializes to an abelian variety X_o , then the NP goes up: $(X_\xi \rightsquigarrow X_o) \Rightarrow \mathcal{N}(X_\xi) \succ \mathcal{N}(X_o)$ cf. (1), page 91.

6 <u>Conjecture, Koblitz, 1975</u>: The converse of this theorem should be true (i.e. every ordered pair of NPs can be realized by a specialization of AVs), cf. (6), page 211 .

7 <u>Notation</u>: $\mathcal{A} := \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$, i.e. for every algebraically closed field k of char(k)=p, the set $\mathcal{A}$(k) of k-rational points is the set of isomorphism classes of triples (X, λ, γ), where X is an AV of dimension g over k, where λ is a <u>principal</u> polarization (i.e. deg(λ) =1), and where γ is a symplectic level-n-structure on X.

For any NP α we write W_α for the set

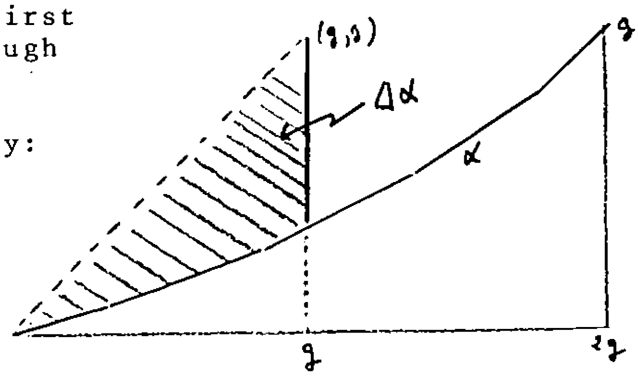
$$W_\alpha = \{(X,\lambda,\gamma) \mid \mathcal{N}(X) \prec \alpha\}/\cong\ \subset \mathcal{A}.$$

By Grothendieck-Katz we know that this is a closed subset of $\mathcal{A}$. NB!! there is no a-priori reason why any point of W_α should correspond with an AV having $\mathcal{N}(X) = \alpha$. (closed: cf. (5), page 143, Th. 2.3.1 & Coroll. 2.3.2).

We write $\Delta\alpha$ for the region of the plane:

$$\Delta(\alpha) := \{(x,y) \mid y < x,\ 0 \leq x \leq g, \text{ and this point is on or above } \alpha\}$$

(note that we only use the first half of the NP, which is enough for our purposes, because of symmetry).



We define the number $d(\alpha)$ by:

$$d(\alpha) := \#\left((\mathbb{Z}\times\mathbb{Z}) \cap \Delta\alpha\right).$$

Note: $d(\sigma) = \left\lfloor \frac{g^2}{4} \right\rfloor$,

$d(\rho) = \frac{1}{2}g(g+1) = \dim \mathcal{A}$,

note!!: $d(\rho) - d(\alpha)$ = length of longest path from α to ρ in the NP-graph (and below: this will equal $\operatorname{codim}(W_\alpha \subset \mathcal{A})$, !!).

8 Theorem (cf. (12)): Fix g, p, and n as above, consider the moduli space as above of principally polarized abelian varieties in characteristic p, and let k be an algebraically closed field, char(k)=p.

a) Let W be an irreducible component of $W_\alpha \otimes k$, let $\eta \in W_\alpha$ be its generic point, then $\mathcal{N}(X_\eta) = \alpha$, $a(X_\eta) \leq 1$, $\boxed{\dim W = d(\alpha)}$.

b) Let $\alpha \prec \beta$ be an ordered pair of NPs, and consider geometrically irreducible components:

for every $W \subset W_\alpha \otimes k$ there is a unique $W' \subset W_\beta \otimes k$ with $W \subset W'$,

for every $W' \subset W_\beta \otimes k$ there is a $W \subset W_\alpha \otimes k$ with $W \subset W'$,

in particular the set of geometrically irreducible components of W_α maps surjectively onto the same of W_β.

(We see: for any geometrically irreducible component $W \subset W_\alpha \otimes k$ the set of all components of all W_β is the same as the part of the NP-graph of all NPs below α).

9 Corollary (proof of conjecture by Manin, cf. 4): For any NP α there is a point in W_α (even defined over an algebraic closure of $\mathbb{F}_p$) having $\mathcal{N}(X) = \alpha$.

10 Corollary (A strengthened form of a conjecture by Koblitz, cf. 6): Let (X_0, λ_0) be a principally polarized abelian variety in positive characteristic, and let

$$\mathcal{N}(X_0) = \alpha \prec \beta .$$

Then there exist a specialisation $(X_\xi, \lambda_\xi) \rightsquigarrow (X_0, \lambda_0)$ such that $\mathcal{N}(X_\xi) = \beta$.

11 Remarks: In the theorem, and in the last corollary it is essential that we work with polarizations with degree prime to p. Counterexamples to more general cases (already for g=3) can be found in (4). - The fact that the supersingular locus W has dimension equal to $[g^2/4]$ was conjectured in (10), and a proof will appear in (7), cf. theorem 12 below. The fact that $\frac{1}{2}g(g+1) - [g^2/4]$ exactly equals the length of the longest path in the NP-graph gave the clue to theorem 8. For those closed sets of the NP-stratification given by the p-rank the dimension formula in theorem 8 was proved in (6), also see (9).

12 Theorem (T.Ekedahl &FO), cf. (2): For any $g \geq 2$, and any NP α the set W_α is connected.
Corollary (Chai-Faltings): For any g the moduli space $\mathcal{A}$ is irreducible.

13 Theorem (K.-Z.Li & FO), cf. (7): For any g the supersingular locus $\mathcal{S} = W_\sigma$ has dimension equal to $[g^2/4]$, and the number of components of $\mathcal{S} \otimes k$ (where k is an algebraically closed field of char. p) is given by a class number as conjectured in (4).

14 Remarks: We see that the number of geometric components of $W_\alpha \otimes k$ is less or equal to the class number

$H_g(p,1)$ (g is odd), respectively $H_g(1,p)$ (g is even), which is the number of geometric components of the supersingular locus for that g. We have no complete information on the number of components for every W_α .

Note that the case g=1, the computation of the number of supersingular j-invariants is classical (Deuring-Eichler, and Igusa). The case g=2 was settled in (3), and for g=3 we find the answer in (4).

REFERENCES:

(1) M.Demazure - Lectures on p-divisible groups. Lect. N. Math. 302, Springer-Verlag, 1972.

(2) T.Ekedahl & F.Oort - Connected substes of moduli spaces of abelian varieties. (to appear)

(3) T.Katsura & F.Oort - Families of supersingular abelian surfaces. Compos. Math. 62 (1987), 107-167.

(4) T.Katsura & F.Oort - Supersingular abelian varieties of dimension two and three and class numbers. Adv. St. Pure Math. 10, 1987 (Algebraic Geom., Sendai, 1985; Ed. T.Oda), Kinokuniya Cy, Tokyo Japan, and North-Holl. Cy, Amsterdam, 1987; pp. 253-281.

(5) N.M.Katz - Slope filtrations of F-crystals. Journ. Géom. Alg. Rennes, Vol. I, Astérisque 63, Soc. Math. France, 1979; pp. 113-164.

(6) N.Koblitz - p-adic variation of the zeta-function over families of varieties defined over finite fields. Compos. Math. 31 (1975), 119-218.

(7) K.-Z.Li & F.Oort - Moduli of supersingular abelian varieties. (to appear)

(8) Yu.I.Manin - The theory of commutative formal groups over fields of finite characteristic. Usp. Math. 18 (1963), 3-90; Russ. Math. Surveys 18 (1963), 1-80.

(9) P.Norman & F.Oort - Moduli of abelian varieties. Ann. Math. 112 (1980), 413-439.

(10) T.Oda & F.Oort - Supersingular abelian varieties. Intl. Symp. on Algebraic Geom., Kyoto 1977 (Ed. M.Nagata), Kinokuniya Book-store, 1978; pp. 595-621.

(11) F.Oort - Subvarieties of moduli spaces. Invent. Math. 24 (1974), 95-119.

(12) F.Oort - Moduli of abelian varieties and Newton polygons. C. R. Acad. Sci. Paris, 312 (1991), 385-389.

(13) F.Oort - Moduli of abelian varieties in positive characteristic. (to appear: Barsotti memorial sympos. on algebr. geom., Padova, 1991)

Frans OORT
Mathematical Institute
Budapestlaan 6
NL - 3508 TA UTRECHT,
The Netherlands oort@math.ruu.nl