Centralizer construction of the Yangian of the queer Lie superalgebra

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To Professor Anthony Joseph on the occasion of his 60th birthday

Summary. Consider the complex matrix Lie superalgebra $\mathfrak{gl}_{N|N}$ with the standard generators E_{ij} where $i,j=\pm 1,\ldots,\pm N$. Define an involutive automorphism η of $\mathfrak{gl}_{N|N}$ by $\eta(E_{ij})=E_{-i,-j}$. The queer Lie superalgebra $\mathfrak{q}_N$ is the fixed point subalgebra in $\mathfrak{gl}_{N|N}$ relative to η. Consider the twisted polynomial current Lie superalgebra

$$\mathfrak{g}=\{\,X(t)\in\mathfrak{gl}_{N|N}[t]\,:\,\eta(X(t))=X(-t)\,\}\,.$$

The enveloping algebra $\mathrm{U}(\mathfrak{g})$ of the Lie superalgebra $\mathfrak{g}$ has a deformation, called the Yangian of $\mathfrak{q}_N$. For each $M=1,2,\ldots$ denote by A_N^M the centralizer of $\mathfrak{q}_M\subset\mathfrak{q}_{N+M}$ in the associative superalgebra $\mathrm{U}(\mathfrak{q}_{N+M})$. We construct a sequence of surjective homomorphisms $\mathrm{U}(\mathfrak{q}_N)\leftarrow\mathrm{A}_N^1\leftarrow\mathrm{A}_N^2\leftarrow\ldots$. We describe the inverse limit of the sequence of centralizer algebras $\mathrm{A}_N^1,\mathrm{A}_N^2,\ldots$ in terms of the Yangian of $\mathfrak{q}_N$.

1. Main results

In this article we work with the queer Lie superalgebra $\mathfrak{q}_N$. This is perhaps the most interesting super-analogue of the general linear Lie algebra $\mathfrak{gl}_N$, see for instance [S2]. We will realize $\mathfrak{q}_N$ as a subalgebra in the general linear Lie superalgebra $\mathfrak{gl}_{N|N}$ over the complex field $\mathbb{C}$. Let the indices i,j run through $-N,\ldots,-1,1,\ldots,N$. Put $\bar\imath=0$ if $i>0$ and $\bar\imath=1$ if $i<0$. Take the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|N}$. Let $e_i\in\mathbb{C}^{N|N}$ be the standard basis vectors. The $\mathbb{Z}_2$-gradation on $\mathbb{C}^{N|N}$ is defined so that $\deg e_i=\bar\imath$. Let $E_{ij}\in\mathrm{End}(\mathbb{C}^{N|N})$ be the matrix units: $E_{ij}\,e_k=\delta_{jk}\,e_i$. The algebra $\mathrm{End}(\mathbb{C}^{N|N})$ is $\mathbb{Z}_2$-graded so that $\deg E_{ij}=\bar\imath+\bar\jmath$. We will also regard E_{ij} as basis elements of the Lie superalgebra $\mathfrak{gl}_{N|N}$. The *queer* Lie superalgebra $\mathfrak{q}_N$ is the fixed point subalgebra in $\mathfrak{gl}_{N|N}$ with respect to the involutive automorphism η defined by

$$\eta\,:\,E_{ij}\mapsto E_{-i,-j}\,. \tag{1.1}$$

Thus as a vector subspace, $\mathfrak{q}_N \subset \mathfrak{gl}_{N|N}$ is spanned by the elements

$$F_{ij} = E_{ij} + E_{-i,-j}\,.$$

Note that $F_{-i,-j} = F_{ij}$. The elements F_{ij} with $i > 0$ form a basis of $\mathfrak{q}_N$.

The vector subspace of $\mathrm{End}\,(\mathbb{C}^{N|N})$ spanned by the elements F_{ij} is closed with respect to the usual matrix multiplication. Hence we can also regard it as an associative algebra. Denote this associative algebra by Q_N, to distinguish its structure from that of the Lie superalgebra $\mathfrak{q}_N$. Both $\mathrm{End}\,(\mathbb{C}^{N|N})$ and Q_N are simple as associative $\mathbb{Z}_2$-graded algebras, see [J, Theorem 2.6].

The enveloping algebra $\mathrm{U}(\mathfrak{q}_N)$ of the Lie superalgebra $\mathfrak{q}_N$ is a $\mathbb{Z}_2$-graded associative unital algebra. In this article we will always keep to the following convention. Let A and B be any two associative $\mathbb{Z}_2$-graded algebras. Their tensor product $\mathrm{A}\otimes\mathrm{B}$ is a $\mathbb{Z}_2$-graded algebra such that for any homogeneous elements $X, X' \in \mathrm{A}$ and $Y, Y' \in \mathrm{B}$

$$(X\otimes Y)(X'\otimes Y') = (-1)^{\deg X' \deg Y}\, X X' \otimes Y Y', \tag{1.2}$$

$$\deg\,(X\otimes Y) = \deg X + \deg Y\,. \tag{1.3}$$

By definition, an anti-homomorphism $\omega : \mathrm{A} \to \mathrm{B}$ is any linear map which preserves the $\mathbb{Z}_2$-gradation and satisfies any homogeneous $X, X' \in \mathrm{A}$

$$\omega(X X') = (-1)^{\deg X \deg X'}\, \omega(X')\, \omega(X)\,. \tag{1.4}$$

For any Lie superalgebra $\mathfrak{a}$, the *principal* anti-automorphism of the enveloping $\mathbb{Z}_2$-graded algebra $\mathrm{U}(\mathfrak{a})$ is determined by the assignment $X \mapsto -X$ for $X \in \mathfrak{a}$.

The *supercommutator* of any two homogeneous elements $X, Y \in \mathrm{A}$ is by definition

$$[\,X\,,Y\,] \,=\, XY - (-1)^{\deg X \deg Y}\, YX. \tag{1.5}$$

This definition extends to arbitrary elements $X, Y \in \mathrm{A}$ by linearity. It is the bracket (1.5) that defines the Lie superalgebra structure on the vector space $\mathrm{A} = \mathrm{End}\,(\mathbb{C}^{N|N})$. Thus for any indices $i\,,j\,,k\,,l = \pm1\,,\,\ldots\,,\,\pm N$ we have

$$\begin{aligned}
[\,E_{ij}\,,E_{kl}\,] \,&=\, \delta_{kj}\,E_{il} - (-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)}\,\delta_{il}\,E_{kj}\,;\\
[\,F_{ij}\,,F_{kl}\,] \,&=\, \delta_{kj}\,F_{il} - (-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)}\,\delta_{il}\,F_{kj}\,+\\
&\quad \delta_{-k,j}\,F_{-i,l} - (-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)}\,\delta_{i,-l}\,F_{k,-j}\,.
\end{aligned} \tag{1.6}$$

For any A and any subset $\mathrm{C} \subset \mathrm{A}$, by the *centralizer* of C in A we mean the collection of all elements $X \in \mathrm{A}$ such that $[\,X\,,Y\,] = 0$ for any $Y \in \mathrm{C}$. To remind the reader about this convention, we shall then refer to the $\mathbb{Z}_2$-graded algebra A as a superalgebra. The centre $\mathrm{Z}(\mathfrak{q}_N)$ of the enveloping algebra $\mathrm{U}(\mathfrak{q}_N)$ will be always taken in the superalgebra sense. A set of generators of the algebra $\mathrm{Z}(\mathfrak{q}_N)$ was given in [S1]. In particular, all central elements of $\mathrm{U}(\mathfrak{q}_N)$ were shown to have $\mathbb{Z}_2$-degree 0. A distinguished basis of the vector space $\mathrm{Z}(\mathfrak{q}_N)$ was constructed in [N2].

Let us recall the principal results of [S1] here. For any indices $n \geqslant 1$ and $i, j = \pm 1, \dots, \pm N$ denote by $F^{(n)}_{ij\,|\,N}$ the element of the algebra $\mathrm{U}(\mathfrak{q}_N)$

$$\sum_{k_1, \dots, k_{n-1}} (-1)^{\,\bar k_1 + \dots + \bar k_{n-1}} \, F_{i k_1} F_{k_1 k_2} \dots F_{k_{n-2} k_{n-1}} F_{k_{n-1} j} \tag{1.7}$$

where each of the indices $k_1, \dots, k_{n-1}$ runs through $\pm 1, \dots, \pm N$. Note that

$$F^{(n)}_{-i,-j\,|\,N} = (-1)^{\,n-1} \, F^{(n)}_{ij\,|\,N} \, . \tag{1.8}$$

Of course, here $F^{(1)}_{ij\,|\,N} = F_{ij}$. Observe that if $n > 1$, then by the definition (1.7)

$$F^{(n)}_{ij\,|\,N} \; = \; \sum_{k} (-1)^{\,\bar k} \, F_{ik} \, F^{(n-1)}_{kj\,|\,N} \tag{1.9}$$

where the index k runs through $\pm 1, \dots, \pm N$. Using this observation one proves by induction on $n = 1, 2, \dots$ the following generalization of (1.6): in the $\mathbb{Z}_2$-graded algebra $\mathrm{U}(\mathfrak{q}_N)$ the supercommutator

$$\begin{aligned} [\, F_{ij} \, , F^{(n)}_{kl\,|\,N} \,] \; = \; \delta_{kj} \, F^{(n)}_{il\,|\,N} - (-1)^{\,(\,\bar\imath + \bar\jmath\,)\,(\,\bar k + \bar l\,)} \, \delta_{il} \, F^{(n)}_{kj\,|\,N} \; + \\ \delta_{-k,j} \, F^{(n)}_{-i,l\,|\,N} - (-1)^{\,(\,\bar\imath + \bar\jmath\,)\,(\,\bar k + \bar l\,)} \, \delta_{i,-l} \, F^{(n)}_{k,-j\,|\,N} \, . \end{aligned} \tag{1.10}$$

For a more general formula, expressing the supercommutator $[\, F^{(m)}_{ij\,|\,N} \, , F^{(n)}_{kl\,|\,N} \,]$ for any m and n, see Proposition 3.1 and the remark after its proof. Now put

$$C^{(n)}_N = \sum_k F^{(n)}_{kk\,|\,N} \tag{1.11}$$

where the index k runs through $\pm 1, \dots, \pm N$. The relations (1.10) immediately imply that $C^{(n)}_N \in \mathrm{Z}(\mathfrak{q}_N)$. Note that $C^{(2)}_N = C^{(4)}_N = \ldots = 0$ due to (1.8). The following proposition has been stated in [S1] without proof.

Proposition 1.1. *The elements $C^{(1)}_N, C^{(3)}_N, \dots$ generate the centre $\mathrm{Z}(\mathfrak{q}_N)$.*

The dependence of the elements $C^{(n)}_N$ and $F^{(n)}_{ij\,|\,N}$ of $\mathrm{U}(\mathfrak{q}_N)$ on the index N has been indicated for the purposes of the next argument, which extends [S1].

For any integers $N \geqslant 0$ and $M \geqslant 1$ consider the Lie superalgebra $\mathfrak{q}_{N+M}$. Now let the indices i, j run through $-N-M, \dots, -1, 1, \dots, N+M$. Regard the Lie superalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_M$ as the subalgebras of $\mathfrak{q}_{N+M}$ spanned by the elements $F_{ij} = F^{(1)}_{ij\,|\,N+M}$ where $|i|, |j| \leqslant N$ and $|i|, |j| > N$ respectively. Denote by A^M_N the centralizer of $\mathfrak{q}_M$ in the associative superalgebra $\mathrm{U}(\mathfrak{q}_{N+M})$.

By definition, the centralizer A^M_N contains the centre $\mathrm{Z}(\mathfrak{q}_{N+M})$ of the $\mathrm{U}(\mathfrak{q}_{N+M})$. It also contains the subalgebra $\mathrm{U}(\mathfrak{q}_N) \subset \mathrm{U}(\mathfrak{q}_{N+M})$. Moreover, the relations (1.10) imply that the centralizer A^M_N contains the elements

$$F^{(1)}_{ij\,|\,N+M} \, , \, F^{(2)}_{ij\,|\,N+M} \, , \; \dots \quad \text{where} \quad |i|, |j| \leqslant N. \tag{1.12}$$

Theorem 1.2. *The elements* $C^{(1)}_{N+M}\,, C^{(3)}_{N+M}\,, \ldots$ *and* (1.12) *generate* A^M_N.

We prove this theorem in Section 2 of the present article. In the particular case $N=0$, we will then obtain Proposition 1.1.

Now take the Lie superalgebra $\mathfrak{q}_{N+M-1}$. As a subalgebra of $\mathfrak{q}_{N+M}$, it is spanned by the elements F_{ij} where $|i|,|j| < N+M$. In particular, the subalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_{M-1}$ of $\mathfrak{q}_{N+M-1}$ are spanned by the elements F_{ij} where $|i|,|j| \leqslant N$ and $N < |i|,|j| < N+M$ respectively. The enveloping algebra $\mathrm{U}(\mathfrak{q}_{N+M-1})$ and its subalgebra A^{M-1}_N will be also regarded as subalgebras in the associative algebra $\mathrm{U}(\mathfrak{q}_{N+M})$. We assume that $M \geqslant 1$ and $\mathrm{A}^0_N = \mathrm{U}(\mathfrak{q}_N)$.

Denote by I_{N+M} the right ideal in the algebra $\mathrm{U}(\mathfrak{q}_{N+M})$ generated by the elements

$$F_{N+M,\pm 1}\,, \ldots, \,F_{N+M,\pm(N+M)}\,. \tag{1.13}$$

Lemma 1.3. *a) the intersection* $\mathrm{I}_{N+M} \cap \mathrm{A}^M_N$ *is a two-sided ideal of* A^M_N;
b) there is a decomposition $\mathrm{A}^M_N = \mathrm{A}^{M-1}_N \oplus (\,\mathrm{I}_{N+M} \cap \mathrm{A}^M_N\,)$.

We prove this lemma in Section 3. Using Part (b) of the lemma, denote by α_M the projection of A^M_N to its direct summand A^{M-1}_N. Due to Part (a), the map $\alpha_M : \mathrm{A}^M_N \to \mathrm{A}^{M-1}_N$ is a homomorphism of associative algebras. The proof of the next proposition will also be given in Section 3.

Proposition 1.4. *For any* $n \geqslant 1$ *and any* $i\,,j$ *such that* $|i|,|j| \leqslant N$ *we have*

$$\alpha_M(F^{(n)}_{ij\,|\,N+M}) = F^{(n)}_{ij\,|\,N+M-1} \quad \textit{and} \quad \alpha_M(C^{(n)}_{N+M}) = C^{(n)}_{N+M-1}\,.$$

The standard filtration (2.9) on the enveloping algebra $\mathrm{U}(\mathfrak{q}_{N+M})$ defines a filtration on its subalgebra A^M_N. By definition, the map $\alpha_M : \mathrm{A}^M_N \to \mathrm{A}^{M-1}_N$ preserves that filtration. It also preserves the $\mathbb{Z}_2$-gradation, inherited from $\mathrm{U}(\mathfrak{q}_{N+M})$. Using the homomorphisms $\alpha_1, \alpha_2, \ldots$ define an algebra A_N as the inverse limit of the sequence $\mathrm{A}^0_N, \mathrm{A}^1_N, \mathrm{A}^2_N, \ldots$ in the category of associative filtered algebras. The main result of this article is an explicit description of the algebra A_N in terms of generators and relations.

By definition, an element of A_N is any sequence of elements $Z_0, Z_1, Z_2, \ldots$ of the algebras $\mathrm{A}^0_N, \mathrm{A}^1_N, \mathrm{A}^2_N, \ldots$ respectively, such that $\alpha_M(Z_M) = Z_{M-1}$ for each $M \geqslant 1$, and the filtration degrees of the elements in the sequence are bounded from above. Utilising Proposition 1.4, for any $n = 1, 2, \ldots$ and any $i, j = \pm 1, \ldots, \pm N$ define an element $F^{(n)}_{ij} \in \mathrm{A}_N$ as the sequence

$$F^{(n)}_{ij\,|\,N}\,, F^{(n)}_{ij\,|\,N+1}\,, F^{(n)}_{ij\,|\,N+2}\,, \ldots\,. \tag{1.14}$$

Further, for any $n = 1, 3, \ldots$ define an element $C^{(n)} \in \mathrm{A}_N$ as the sequence

$$C^{(n)}_N\,, C^{(n)}_{N+1}\,, C^{(n)}_{N+2}\,, \ldots\,. \tag{1.15}$$

The filtration degree of every element in (1.14) and (1.15) does not exceed n.

Note that the algebra A_N is unital, and comes with a $\mathbb{Z}_2$-gradation, such that all possible indices n and i,j we have

$$\deg C^{(n)} = 0 \quad \text{and} \quad \deg F_{ij}^{(n)} = \bar\imath + \bar\jmath\,. \tag{1.16}$$

By their definition, the elements $C^{(1)}, C^{(3)}, \ldots \in \mathrm{A}_N$ are central. Due to (1.8)

$$F_{-i,-j}^{(n)} = (-1)^{n-1} F_{ij}^{(n)}. \tag{1.17}$$

Theorem 1.5. *a) The algebra A_N is generated by the elements $C^{(1)}, C^{(3)}, \ldots$ and $F_{ij}^{(1)}, F_{ij}^{(2)}, \ldots$.*
b) The central elements $C^{(1)}, C^{(3)}, \ldots$ of A_N are algebraically independent.
c) Together with the centrality and algebraic independence of $C^{(1)}, C^{(3)}, \ldots$, the defining relations of the $\mathbb{Z}_2$-graded algebra A_N are (1.17) *and*

$$\begin{gathered}
[F_{ij}^{(m)}, F_{kl}^{(n)}] = F_{il}^{(m+n-1)} \delta_{kj} - (-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)} \delta_{il} F_{kj}^{(m+n-1)} + \\
(-1)^{m-1} (F_{-i,l}^{(m+n-1)} \delta_{-k,j} - (-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)} \delta_{i,-l} F_{k,-j}^{(m+n-1)}) + \\
(-1)^{\bar\jmath \bar k + \bar\jmath \bar l + \bar k \bar l} \sum_{r=1}^{\min(m,n)-1} (F_{il}^{(m+n-r-1)} F_{kj}^{(r)} - F_{il}^{(r)} F_{kj}^{(m+n-r-1)}) + \\
(-1)^{\bar\jmath \bar k + \bar\jmath \bar l + \bar k \bar l + \bar k + \bar l} \times \\
\sum_{r=1}^{\min(m,n)-1} (-1)^{m+r} (F_{-i,l}^{(m+n-r-1)} F_{-k,j}^{(r)} - F_{i,-l}^{(r)} F_{k,-j}^{(m+n-r-1)})
\end{gathered} \tag{1.18}$$

where $m,n = 1,2,\ldots$ and $i,j,k,l = \pm 1, \ldots, \pm N$.

The proof will be given in Section 3. In particular, Theorem 1.5 shows that the algebra A_N is isomorphic to the tensor product of its two subalgebras, generated by the elements $C^{(1)}, C^{(3)}, \ldots$ and by the elements $F_{ij}^{(1)}, F_{ij}^{(2)}, \ldots$ respectively. Denote the latter subalgebra by B_N, it is a $\mathbb{Z}_2$-graded subalgebra.

The algebra B_N appeared in [N3] in the following guise. Let us consider the associative unital $\mathbb{Z}_2$-graded algebra $\mathrm{Y}(\mathfrak{q}_N)$ over the field $\mathbb{C}$ with the countable set of generators $T_{ij}^{(n)}$ where $n = 1,2,\ldots$ and $i,j = \pm 1, \ldots, \pm N$. The $\mathbb{Z}_2$-gradation on the algebra $\mathrm{Y}(\mathfrak{q}_N)$ is determined by setting $\deg T_{ij}^{(n)} = \bar\imath + \bar\jmath$ for any $n \geqslant 1$. To write down the defining relations for these generators, put

$$T_{ij}(x) = \delta_{ij} \cdot 1 + T_{ij}^{(1)} x^{-1} + T_{ij}^{(2)} x^{-2} + \ldots$$

where x is a formal parameter, so that $T_{ij}(x) \in \mathrm{Y}(\mathfrak{q}_N)[[x^{-1}]]$. Then for all possible indices i,j,k,l we have the relations

$$T_{-i,-j}(x) = T_{ij}(-x) \tag{1.19}$$

and

$$(x^2-y^2)\cdot[\,T_{ij}(x),T_{kl}(y)]\cdot(-1)^{\bar\imath\,\bar k\,+\,\bar\imath\,\bar l\,+\,\bar k\,\bar l}\;=$$
$$(x+y)\cdot(\,T_{kj}(x)\,T_{il}(y)-T_{kj}(y)\,T_{il}(x)\,)\;-$$
$$(x-y)\cdot(\,T_{-k,\,j}(x)\,T_{-i,\,l}(y)-T_{k,\,-j}(y)\,T_{i,\,-l}(x)\,)\cdot(-1)^{\bar k\,+\,\bar l} \tag{1.20}$$

where y is a formal paramater independent of x, so that (1.20) is an equality in the algebra of formal Laurent series in x^{-1},y^{-1} with coefficients in $\mathrm{Y}(\mathfrak{q}_N)$.

The algebra $\mathrm{Y}(\mathfrak{q}_N)$ is called the *Yangian* of the Lie superalgebra $\mathfrak{q}_N$. Note that the centre of the associative superalgebra $\mathrm{Y}(\mathfrak{q}_N)$ with $N\geqslant 1$ is not trivial. For a description of the centre of $\mathrm{Y}(\mathfrak{q}_N)$ see [N3, Section 3]. In particular, all central elements of $\mathrm{Y}(\mathfrak{q}_N)$ have $\mathbb{Z}_2$-degree 0. In our Section 3 we will prove

Proposition 1.6. *The assignement $F_{ij}^{(n)}\mapsto(-1)^{\bar\imath}\,T_{ji}^{(n)}$ for any $n=1,2,\ldots$ and $i,j=\pm1,\ldots,\pm N$ extents to an anti-isomorphism of $\mathbb{Z}_2$-graded algebras*

$$\omega:\,\mathrm{B}_N\,\rightarrow\,\mathrm{Y}(\mathfrak{q}_N)\,.$$

Now denote by ω_{N+M} the principal anti-automorphism of the enveloping algebra $\mathrm{U}(\mathfrak{q}_{N+M})$. It preserves the subalgebra $\mathrm{U}(\mathfrak{q}_M)\subset\mathrm{U}(\mathfrak{q}_{N+M})$. Hence it also preserves the centralizer $\mathrm{A}_N^M\subset\mathrm{U}(\mathfrak{q}_{N+M})$ of that subalgebra. For any $M\geqslant 0$ let $\pi_M:\mathrm{A}_N\to\mathrm{A}_N^M$ be the canonical homomorphism. By definition,

$$\pi_M\,(F_{ij}^{(n)})\,=\,F_{ij\,|\,N+M}^{(n)} \tag{1.21}$$

for any $n=1,2,\ldots$ and $i,j=\pm1,\ldots,\pm N$. Using Proposition 1.6, we can define a homomorphism $\tau_M:\mathrm{Y}(\mathfrak{q}_N)\to\mathrm{A}_N^M$ by the equality

$$\tau_M\circ\omega\,=\,\omega_{N+M}\circ(\,\pi_M\,|\,\mathrm{B}_N)\,.$$

By (1.21),

$$\tau_M\,(T_{ij}^{(n)})\,=\,(-1)^{\bar\jmath}\;\omega_{N+M}\,(F_{ji\,|\,N+M}^{(n)})\,.$$

Using the homomorphisms τ_M for all $M=0,1,2,\ldots$, one can define a family of irreducible finite-dimensional $\mathrm{Y}(\mathfrak{q}_N)$-modules; see [N4, Section 1] and [P]. Another family of irreducible finite-dimensional $\mathrm{Y}(\mathfrak{q}_N)$-modules can be defined by using the results of [N1] and [N3, Section 5]. It should be possible to give a parametrization of all irreducible finite-dimensional $\mathrm{Y}(\mathfrak{q}_N)$-modules, similarly to the parametrization of the irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_N)$-modules as given by V. Drinfeld; see [D2, Theorem 2] and [M].

It was shown in [N3] that the associative $\mathbb{Z}_2$-graded algebra $\mathrm{Y}(\mathfrak{q}_N)$ has a natural Hopf superalgebra structure. In particular, the homomorphism of comultiplication $\mathrm{Y}(\mathfrak{q}_N)\to\mathrm{Y}(\mathfrak{q}_N)\otimes\mathrm{Y}(\mathfrak{q}_N)$ can be defined by

$$T_{ij}(x)\,\mapsto\,\sum_k\;T_{ik}(x)\otimes T_{kj}(x)\cdot(-1)^{(\,\bar\imath+\bar k\,)(\,\bar\jmath+\bar k\,)} \tag{1.22}$$

where the tensor product is over the subalgebra $\mathbb{C}[[x^{-1}]]$ of $\mathrm{Y}(\mathfrak{q}_N)[[x^{-1}]]$, and k runs through $\pm 1, \ldots, \pm N$. See [N3, Section 2] for the definitions of the the counit map $\mathrm{Y}(\mathfrak{q}_N) \to \mathbb{C}$ and the antipodal map $\mathrm{Y}(\mathfrak{q}_N) \to \mathrm{Y}(\mathfrak{q}_N)$.

There is a distinguished ascending $\mathbb{Z}$-filtration on the associative algebra $\mathrm{Y}(\mathfrak{q}_N)$. It is obtained by assigning to every generator $F_{ij}^{(n)}$ the degree $n-1$. The corresponding $\mathbb{Z}$-graded algebra will be denoted by $\operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$. Let $G_{ij}^{(n)}$ be the element of $\operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$ corresponding to the generator $F_{ij}^{(n)} \in \mathrm{Y}(\mathfrak{q}_N)$. The algebra $\operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$ inherits the $\mathbb{Z}_2$-gradation from the algebra $\mathrm{Y}(\mathfrak{q}_N)$, so that

$$\deg G_{ij}^{(n)} = \bar\imath + \bar\jmath\,.$$

Moreover, $\operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$ inherits from $\mathrm{Y}(\mathfrak{q}_N)$ the Hopf superalgebra structure. It follows from the defintion (1.22) that with respect to the homomorphism of comultiplication $\operatorname{gr}\mathrm{Y}(\mathfrak{q}_N) \to \operatorname{gr}\mathrm{Y}(\mathfrak{q}_N) \otimes \operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$, for any $n \geqslant 1$ we have

$$G_{ij}^{(n)} \mapsto G_{ij}^{(n)} \otimes 1 + 1 \otimes G_{ij}^{(n)}.$$

On the other hand, for arbitrary Lie superalgebra $\mathfrak{a}$, a comultiplication map $\mathrm{U}(\mathfrak{a}) \to \mathrm{U}(\mathfrak{a}) \otimes \mathrm{U}(\mathfrak{a})$ can be defined for $X \in \mathfrak{a}$ by $X \mapsto X \otimes 1 + 1 \otimes X$, and then extended to a homomorphism of $\mathbb{Z}_2$-graded associative algebras by using the convention (1.2). Let us now consider the enveloping algebra $\mathrm{U}(\mathfrak{g})$ of the *twisted polynomial current* Lie superalgebra

$$\mathfrak{g} = \{\, X(t) \in \mathfrak{gl}_{N|N}[t] \,:\, \eta(\,X(t)) = X(-t)\,\}\,.$$

Here we employ the automorphism (1.1) of the Lie superalgebra $\mathfrak{gl}_{N|N}$. As a vector space, $\mathfrak{g}$ is spanned by the elements

$$E_{ij}\, t^n + E_{-i,-j}\,(-t)^n \tag{1.23}$$

where $n = 0,1,2 \ldots$ and $i,j = \pm 1, \ldots, \pm N$. Note that the $\mathbb{Z}_2$-degree of the element (1.23) equals $\bar\imath + \bar\jmath$. The algebra $\mathrm{U}(\mathfrak{g})$ also has a natural $\mathbb{Z}$-gradation, such that the degree of the element (1.23) is n.

It turns out that $\mathrm{U}(\mathfrak{g})$ and $\operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$ are isomorphic as Hopf superalgebras. By [N3, Theorem 2.3] their isomorphism $\mathrm{U}(\mathfrak{g}) \to \operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$ can be defined by mapping the element (1.23) of the algebra $\mathrm{U}(\mathfrak{g})$ to the element $(-1)^{\bar\imath+1}\, G_{ji}^{(n+1)}$ of the algebra $\operatorname{gr}\mathrm{Y}(\mathfrak{q}_N)$. Moreover, $\mathrm{Y}(\mathfrak{q}_N)$ is a deformation of $\mathrm{U}(\mathfrak{g})$ as a Hopf superalgebra; see the end of [N3, Section 2] for details.

Let us finish this introductory section with a few remarks of a historical nature. Our construction of the algebra A_N follows a similar construction for the general linear Lie algebra $\mathfrak{gl}_N$ instead of the queer Lie superalgebra $\mathfrak{q}_N$, due to G. Olshanski [O1, O2]. It was him who first considered the inverse limit of the sequence of centralizers of $\mathfrak{gl}_M$ in the enveloping algebras $\mathrm{U}(\mathfrak{gl}_{N+M})$ for $M = 1,2, \ldots$. Following a suggestion of B. Feigin, he then described the inverse limit in terms of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ of the Lie algebra $\mathfrak{gl}_N$. The latter

Yangian is a deformation of the enveloping algebra of the polynomial current Lie algebra $\mathfrak{gl}_N[t]$ in the class of Hopf algebras [D1].

The elements (1.7) of $\mathrm{U}(\mathfrak{q}_N)$ were initially considered by A. Sergeev [S1], in order to describe the centre of the superalgebra $\mathrm{U}(\mathfrak{q}_N)$. The homomorphisms $\alpha_M : \mathrm{A}_N^M \to \mathrm{A}_N^{M-1}$ for $M = 1, 2, \ldots$ and the elements $F_{ij}^{(1)}, F_{ij}^{(2)}, \ldots$ of the inverse limit algebra A_N were introduced by M. Nazarov following [O1, O2]. He then identified the algebra *defined* by the relations (1.19) and (1.20), as a deformation of the enveloping algebra $\mathrm{U}(\mathfrak{g})$ in the class of Hopf superalgebras [N3]. It was also explained in [N3] why this deformation should be called the Yangian of $\mathfrak{q}_N$. However, our Theorems 1.2 and 1.5 were only conjectured by M. Nazarov. The purpose of the present article is to prove these conjectures.

We hope these remarks indicate importance of the role that G. Olshanski played at various stages of our work. We are very grateful to him for friendly advice. This work was finished while M. Nazarov stayed at the Max Planck Institute of Mathematics in Bonn. He is grateful to the Institute for hospitality. M. Nazarov has been also supported by the EC grant MRTN-CT-2003-505078.

2. Proof of Theorem 1.2

We will use basic properties of complex semisimple associative superalgebras and their modules [J]. Most of these properties were first established in [W], in a generality greater than we need in the present article. We will also use the following simple lemma. Its proof carries over almost verbatim from the ungraded case, but we shall include the proof for the sake of completeness. Let A be any finite dimensional $\mathbb{Z}_2$-graded associative algebra over the complex field $\mathbb{C}$. Let G be a finite group of automorphisms of A. The crossed product algebra $G \ltimes \mathrm{A}$ is also $\mathbb{Z}_2$-graded: for any $g \in G$ we have $\deg g = 0$ in $G \ltimes \mathrm{A}$.

Lemma 2.1. *Suppose the superalgebra* A *is semisimple. Then the superalgebra* $G \ltimes A$ *is also semisimple.*

Proof. We will write $G \ltimes \mathrm{A} = \mathrm{B}$. Let V be any module over the superalgebra B, and let $\rho : \mathrm{B} \to \mathrm{End}(V)$ be the corresponding homomorphism. Here we assume that the homomorphism ρ preserves $\mathbb{Z}_2$-gradation. Let $U \subset V$ be any B-submodule. Since A is semisimple, we have the decomposition $V = U \oplus U'$ into a direct sum of A-modules for some $U' \subset V$. Let $P \in \mathrm{End}(V)$ be the projection onto U along U'. Put

$$S \,=\, \frac{1}{|G|} \sum_{g \in G} \rho(g)\, P\, \rho(g)^{-1} \in \mathrm{End}(V)\,.$$

For any element $X \in \mathrm{A}$ let X^g be its image under the automorphism g. Then

$$\rho(X)\, S \,=\, \frac{1}{|G|} \sum_{g \in G} \rho(X)\, \rho(g)\, P\, \rho(g)^{-1} \,=\, \frac{1}{|G|} \sum_{g \in G} \rho(g)\, \rho(X^g)\, P\, \rho(g)^{-1}$$

$$= \frac{1}{|G|} \sum_{g \in G} \rho(g)\, P\, \rho(X^g)\, \rho(g)^{-1} \;=\; \frac{1}{|G|} \sum_{g \in G} \rho(g)\, P\, \rho(g)^{-1} \rho(X) \;=\; S\rho(X)\,.$$

For any $h \in G$ we also have $\rho(h)\,S = S\,\rho(h)$ by the definition of S. Note that $S \in \operatorname{End}(V)$ is of $\mathbb{Z}_2$-degree 0, as well as P is. So $\ker S \subset V$ is a B-submodule.

Since U is a B-submodule, we have the equalities $P\rho(g)\,P = \rho(g)\,P$ for all $g \in G$. Using the definition of S, these equalities imply that $S\,P = P$ and $P\,S = S$. The latter pair of equalities guarantees that $\operatorname{im} S = \operatorname{im} P = U$ and $S^2 = S$. So $V = U \oplus \ker S$. Since V is an arbitrary module over the superalgebra B, this superalgebra is semisimple by [J, Proposition 2.4] $\square$

We will also need the following general "double centralizer theorem". Let V be any $\mathbb{Z}_2$-graded complex vector space. The associative algebra $\operatorname{End}(V)$ is then also $\mathbb{Z}_2$-graded. Take any subalgebra B in the superalgebra $\operatorname{End}(V)$. Here we assume that B as a vector space splits into the direct sum of its subspaces of $\mathbb{Z}_2$-degrees 0 and 1. Denote by B' the centralizer of B in the superalgebra $\operatorname{End}(V)$.

Proposition 2.2. *Suppose that the superalgebra* B *is finite dimensional and semisimple. Also suppose that the* B'*-module* V *is finitely generated. Then* $\mathrm{B} = \mathrm{B}''$ *in* $\operatorname{End}(V)$.

Proof. We shall prove that for any homogeneous $v_1, \ldots, v_n \in V$ and $X \in \mathrm{B}''$, there exists $Y \in \mathrm{B}$ such that $Xv_r = Yv_r$ for any index $r = 1, \ldots, n$. Then we will choose $v_1, \ldots, v_n$ to be homogeneous generators of V over B'. By writing any vector $v \in V$ as the sum $Z_1v_1 + \ldots + Z_nv_n$ for some homogeneous $Z_1, \ldots, Z_n \in \mathrm{B}'$, we will get

$$\begin{aligned} Xv \;&=\; \sum_{r=1}^{n} XZ_r v_r \;=\; \sum_{r=1}^{n} (-1)^{\deg X \deg Z_r}\, Z_r X v_r \\ &=\; \sum_{r=1}^{n} (-1)^{\deg X \deg Z_r}\, Z_r Y v_r \;=\; \sum_{r=1}^{n} Y Z_r v_r \;=\; Yv\,. \end{aligned}$$

Along with the obvious embedding $\mathrm{B} \subset \mathrm{B}''$, this will prove Proposition 2.2.

Recall that the $\mathbb{Z}_2$-graded vector space $\overline{V}$ *opposite* to V is obtained from V by changing the $\mathbb{Z}_2$-gradation deg to deg+1. Define the action of the algebra B in $\overline{V}$ as the pullback its action in V via the involutive automorphism $Y \mapsto (-1)^{\deg Y}\, Y$, where Y is any homogeneous element of the $\mathbb{Z}_2$-graded algebra B. Now consider the direct sum of B-modules

$$W \;=\; \bigoplus_{r=1}^{n} V_r\,,$$

where the B-module V_r equals V or $\overline{V}$ depending on whether $\deg v_r$ in V is 0 or 1. For each index $r = 1, \ldots, n$ we have an embedding of vector spaces

$A_r : V \to W$, and a projection $B_r : W \to V$. Note that the $\mathbb{Z}_2$-degrees of the linear maps A_r and B_r coincide with that of the vector v_r. We also have the equality $B_p A_q = \delta_{pq} \cdot \mathrm{id}$ in $\mathrm{End}(V)$, and the equality

$$\sum_{r=1}^{n} A_r B_r = \mathrm{id}$$

in $\mathrm{End}(W)$. Any homogeneous element $Y \in \mathrm{B}$ acts in W as the linear operator

$$\widetilde{Y} = \sum_{r=1}^{n} (-1)^{\deg v_r \deg Y} A_r Y B_r \, .$$

Given $X \in \mathrm{B}''$, put

$$\widetilde{X} = \sum_{r=1}^{n} (-1)^{\deg v_r \deg X} A_r X B_r \, .$$

Any homogeneous element $Z \in \mathrm{End}(W)$ can be written as

$$Z = \sum_{p,q=1}^{n} A_p Z_{pq} B_q$$

for some homogeneous $Z_{pq} \in \mathrm{End}(V)$, where $\deg Z_{pq} = \deg Z + \deg v_p + \deg v_q$. Then

$$\widetilde{Y} Z = \sum_{r=1}^{n} (-1)^{\deg v_r \deg Y} A_r Y B_r \cdot \sum_{p,q=1}^{n} A_p Z_{pq} B_q \tag{2.1}$$
$$= \sum_{p,q=1}^{n} (-1)^{\deg v_p \deg Y} A_p Y Z_{pq} B_q \, ,$$
$$Z \widetilde{Y} = \sum_{p,q=1}^{n} A_p Z_{pq} B_q \cdot \sum_{r=1}^{n} (-1)^{\deg v_r \deg Y} A_r Y B_r \tag{2.2}$$
$$= \sum_{p,q=1}^{n} (-1)^{\deg v_q \deg Y} A_p Z_{pq} Y B_q \, .$$

Suppose that the element $Z \in \mathrm{End}(W)$ belongs to the centralizer of the image of the superalgebra B in $\mathrm{End}(W)$. Due to (2.1) and (2.2), the assumption $[\widetilde{Y}, Z] = 0$ is equivalent to the collection of equalities $[Y, Z_{pq}] = 0$ for all $p, q = 1, \ldots, n$. Since here $Y \in \mathrm{B}$ is arbitrary and $X \in \mathrm{B}''$, we then have $[X, Z_{pq}] = 0$ for all $p, q = 1, \ldots, n$. A calculation similar to (2.1) and (2.2) then shows that $[\widetilde{X}, Z] = 0$ in $\mathrm{End}(W)$.

Now take the vector

$$w = \sum_{r=1}^{n} A_r v_r \in W ,$$

it has $\mathbb{Z}_2$-degree 0. Let U be the cyclic span of the vector w under the action of B. Since the superalgebra B is finite dimensional semisimple, we have the decomposition $W = U \oplus U'$ into direct sum of B-modules for some $U' \subset W$; see [J, Proposition 2.4]. Choose the element $Z \in \mathrm{End}(W)$ from the centralizer of the image of B, to be the projector onto U' along U. Here $\deg Z = 0$. Then $Z\widetilde{X}w = \widetilde{X}Zw = 0$, and $\widetilde{X}w \in U$. So there exists $Y \in \mathrm{B}$ such that $\widetilde{X}w = \widetilde{Y}w$. The last equality means that $Xv_r = Yv_r$ for each $r = 1, \ldots, n$. □

In the notation of Proposition 2.2, we have the following corollary.

Corollary 2.3. *Suppose that the vector space V is finite dimensional, and that the superalgebra* $\mathrm{B} \subset \mathrm{End}(V)$ *is semisimple. Then* $\mathrm{B} = \mathrm{B}''$ *in* $\mathrm{End}(V)$.

Now for any integer $n \geqslant 1$ consider the tensor product $(\mathrm{End}(V))^{\otimes n}$ of n copies of the $\mathbb{Z}_2$-graded algebra $\mathrm{End}(V)$. This tensor product is a $\mathbb{Z}_2$-graded associative algebra defined using the conventions (1.2) and (1.3). The proof of the following lemma is also included for the sake of completeness.

Lemma 2.4. *Suppose that* B *contains the identity* $1 \in \mathrm{End}(V)$. *Then the centralizer of* $\mathrm{B}^{\otimes n}$ *in the superalgebra* $(\mathrm{End}(V))^{\otimes n}$ *coincides with* $(\mathrm{B}')^{\otimes n}$.

Proof. We will use the induction on n. If $n = 1$, the statement of Lemma 2.4 is tautological. Suppose that $n > 1$. The centralizer of $\mathrm{B}^{\otimes n}$ contains $(\mathrm{B}')^{\otimes n}$ due to the conventions (1.2) and (1.3). Now suppose that for some homogeneous elements $X_1, \ldots, X_l \in \mathrm{End}(V)$ and $Y_1, \ldots, Y_l \in (\mathrm{End}(V))^{\otimes (n-1)}$ the sum $X_1 Y_1 + \ldots + X_l Y_l$ belongs to the centralizer of $\mathrm{B}^{\otimes n}$. In particular, then for any homogeneous $Y \in \mathrm{B}^{\otimes (n-1)}$ we have

$$\begin{aligned}
\sum_{k=1}^{l} X_k \otimes (Y_k Y) &= \sum_{k=1}^{l} (X_k \otimes Y_k)(1 \otimes Y) \\
&= \sum_{k=1}^{l} (-1)^{(\deg X_k + \deg Y_k)\deg Y} (1 \otimes Y)(X_k \otimes Y_k) \\
&= \sum_{k=1}^{l} (-1)^{\deg Y_k \deg Y} X_k \otimes (Y Y_k) .
\end{aligned}$$

We may assume that the elements $X_1, \ldots, X_l$ are linearly independent, then the above equalities imply that for any k the element $Y_k \in (\mathrm{End}(V))^{\otimes (n-1)}$ belongs to the centralizer of $\mathrm{B}^{\otimes (n-1)}$. Then $Y_k \in (\mathrm{B}')^{\otimes (n-1)}$ by the induction assumption. A similar argument shows that $X_k \in \mathrm{B}'$ for any index k. □

For any $N \geqslant 0$ and $M \geqslant 1$ take the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N+M \,|\, N+M}$. Identify the $\mathbb{Z}_2$-graded vector spaces $\mathbb{C}^{N|N}$ and $\mathbb{C}^{M|M}$ with the subspaces in

$\mathbb{C}^{N+M\,|\,N+M}$ spanned by the vectors e_i where $i=-N,\ldots,-1,1,\ldots,N$ and $i=-N-M,\ldots,-N-1,N+1,\ldots,N+M$ respectively. The decomposition

$$\mathbb{C}^{N|N} = \mathbb{C}^{N|N}\oplus\mathbb{C}^{M|M} \tag{2.3}$$

determines the embeddings of the Lie superalgebras $\mathfrak{gl}_{N|N}$ and $\mathfrak{gl}_{M|M}$ into $\mathfrak{gl}_{N+M\,|\,N+M}$, and of their subalgebras $\mathfrak{q}_N$ and $\mathfrak{q}_M$ into the Lie algebra $\mathfrak{q}_{N+M}$.

Let us now regard $\mathrm{End}\,(\mathbb{C}^{N|N})$ and Q_M as subalgebras in the associative superalgebra $\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$, using the decomposition (2.3). The elements

$$\sum_{|i|>N} E_{ii} \quad\text{and}\quad \sum_{|i|>N}(-1)^{\bar\imath}\,E_{i,-i}$$

of $\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$ span a subalgebra, isomorphic to the associative algebra Q_1. Using this isomorphism, a direct calculation shows that the centralizer of Q_M in $\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$ coincides with $\mathrm{End}\,(\mathbb{C}^{N|N})\oplus\mathrm{Q}_1$. This centralizer is a semisimple associative $\mathbb{Z}_2$-graded algebra, denote it by C.

For any integer $n\geqslant 1$ consider the tensor product $V=(\mathbb{C}^{N+M\,|\,N+M})^{\otimes n}$ of n copies of the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N+M\,|\,N+M}$. Identify the algebras

$$(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n} \quad\text{and}\quad \mathrm{End}\,((\mathbb{C}^{N+M\,|\,N+M})^{\otimes n}) \tag{2.4}$$

so that for any homogeneous elements $X_1,\ldots,X_n\in\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$ and any homogeneous vectors $u_1,\ldots,u_n\in\mathbb{C}^{N+M\,|\,N+M}$

$$(X_1\otimes\ldots\otimes X_n)\,(u_1\otimes\ldots\otimes u_n) = (-1)^d\,X_1u_1\otimes\ldots\otimes X_nu_n$$

where

$$d \;=\; \sum_{1\leqslant p<q\leqslant n}\deg u_p\,\deg X_q\,.$$

By identifying the two algebras we determine an action on V of the subalgebra

$$\mathrm{C}^{\otimes n}\subset(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}. \tag{2.5}$$

The symmetric group S_n acts on V so that for any adjacent transposition $\sigma_p=(p,p+1)$

$$\sigma_p\,(u_1\otimes\ldots\otimes u_p\otimes u_{p+1}\otimes\ldots\otimes u_n) =$$
$$(-1)^{\deg u_p\,\deg u_{p+1}}\,u_1\otimes\ldots\otimes u_{p+1}\otimes u_p\otimes\ldots\otimes u_n\,.$$

The action of S_n and $\mathrm{C}^{\otimes n}$ on V extends to that of $S_n\ltimes\mathrm{C}^{\otimes n}$. Here the group S_n acts by automorphisms of the algebra $(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ so that

$$\sigma_p\,(X_1\otimes\ldots\otimes X_p\otimes X_{p+1}\otimes\ldots\otimes X_n) =$$
$$(-1)^{\deg X_p\,\deg X_{p+1}}\,X_1\otimes\ldots\otimes X_{p+1}\otimes X_p\otimes\ldots\otimes X_n\,. \tag{2.6}$$

Now consider the action of enveloping algebra $\mathrm{U}(\mathfrak{q}_N)$ on the vector space V, as of a subalgebra of the $\mathbb{Z}_2$-graded associative algebra $\mathrm{U}(\mathfrak{gl}_{N+M\,|\,N+M})$. We use the comultiplication

$$\mathrm{U}(\mathfrak{gl}_{N+M\,|\,N+M}) \to \mathrm{U}(\mathfrak{gl}_{N+M\,|\,N+M})^{\otimes n} \tag{2.7}$$

along with the identification of the two algebras in (2.4).

Proposition 2.5. *The centralizer of the image of* $\mathrm{U}(\mathfrak{q}_M)$ *in the associative superalgebra* $\mathrm{End}\,(V)$ *coincides with the image of* $S_n \ltimes \mathrm{C}^{\otimes n}$.

Proof. The $\mathbb{Z}_2$-graded algebra $\mathrm{C}^{\otimes n}$ is semisimple, see [J, Proposition 2.10]. So is the crossed product $S_n \ltimes \mathrm{C}^{\otimes n}$, see our Lemma 2.1. Let B be the image of the crossed product in $\mathrm{End}\,(V)$. The $\mathbb{Z}_2$-graded algebra B is semisimple too. By Corollary 2.3 it suffices to prove that the centralizer of B in $\mathrm{End}\,(V)$ coincides with the image of $\mathrm{U}(\mathfrak{q}_M)$.

Let us describe the latter image. Consider the span in $\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$ of $\mathfrak{q}_M$ and of the identity element 1. It is a subalgebra in $\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$, denote this subalgebra by D. We will prove that the invariant subalgebra

$$(\mathrm{D}^{\otimes n})^{S_n} \subset (\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$$

coincides with the image of $\mathrm{U}(\mathfrak{q}_M)$. Here we use the definition (2.6), and the comultiplication (2.7). There is no need to identify the two algebras (2.4) here.

Due to the convention (1.2) and the definition of the comultiplication (2.7), the image of $\mathrm{U}(\mathfrak{q}_M)$ in $(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ is contained in $(\mathrm{D}^{\otimes n})^{S_n}$. Now for any n elements $X_1, \ldots, X_n \in \mathrm{D}$ consider their *symmetrized* tensor product

$$\langle X_1, \ldots, X_n \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma\,(X_1 \otimes \ldots \otimes X_n)\,. \tag{2.8}$$

Suppose that for some $p \in \{1, \ldots, n\}$ we have $X_q \in \mathbb{C}\,1$ if and only if $p < q$. By induction on p, let us prove that $\langle X_1, \ldots, X_p, 1, \ldots, 1\rangle$ belongs to the image of $\mathrm{U}(\mathfrak{q}_M)$ in $(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$. This is evident if $p = 1$. If $p > 1$, then

$$\langle X_1, \ldots, X_{p-1}, 1, \ldots, 1\rangle\,\langle X_p, 1, \ldots, 1\rangle$$

equals

$$\frac{n-p+1}{n}\,\langle X_1, \ldots, X_p, 1, \ldots, 1\rangle$$

plus a sum of certain symmetrized tensor products in $(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ which belong to the image of $\mathrm{U}(\mathfrak{q}_M)$ by the induction assumption.

To prove that the centralizer of B in $\mathrm{End}\,(V)$ coincides with the image of $\mathrm{U}(\mathfrak{q}_M)$, it now suffices to show that the centralizer of the subalgebra (2.5) coincides with $\mathrm{D}^{\otimes n}$. But we have $\mathrm{C}' = \mathrm{D}$ in $\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$ by definition. Using Lemma 2.4 we now complete the proof of Proposition 2.5. $\square$

Consider the symmetric algebra $\mathrm{S}(\mathfrak{q}_{N+M})$ of the Lie superalgebra $\mathfrak{q}_{N+M}$. The standard filtration

$$\mathbb{C} = \mathrm{U}^0(\mathfrak{q}_{N+M}) \subset \mathrm{U}^1(\mathfrak{q}_{N+M}) \subset \mathrm{U}^2(\mathfrak{q}_{N+M}) \subset \ldots \tag{2.9}$$

on the algebra $\mathrm{U}(\mathfrak{q}_{N+M})$ determines for each $n=1,2,\ldots$ a linear map

$$\mathrm{U}^n(\mathfrak{q}_{N+M})\,/\,\mathrm{U}^{n-1}(\mathfrak{q}_{N+M})\;\to\;\mathrm{S}^n(\mathfrak{q}_{N+M})\,, \tag{2.10}$$

which is bijective by the Poincaré-Birkhoff-Witt theorem for Lie superalgebras [MM, Theorem 5.15]. Using (2.10), for all indices $i,j=\pm1,\,\ldots\,,\pm(N+M)$ define an element $f^{(n)}_{ij\,|\,N+M}\in\mathrm{S}^n(\mathfrak{q}_{N+M})$ as the image of the element

$$F^{(n)}_{ij\,|\,N+M}\in\mathrm{U}^n(\mathfrak{q}_{N+M})\,;$$

the latter element is the sum (1.7) where each of the indices $k_1\,,\,\ldots\,,k_{n-1}$ runs through $\pm1,\,\ldots\,,\pm(N+M)$. Define $c^{(n)}_{N+M}\in\mathrm{S}^n(\mathfrak{q}_{N+M})$ as the image of

$$C^{(n)}_{N+M}\in\mathrm{U}^n(\mathfrak{q}_{N+M})\,;$$

see (1.11). Note that if n is even, then $C^{(n)}_{N+M}=0$ and hence $c^{(n)}_{N+M}=0$.

Now consider adjoint action of the Lie superalgebra $\mathfrak{q}_{N+M}$ on $\mathrm{S}(\mathfrak{q}_{N+M})$. In particular, consider the action of $\mathfrak{q}_M$ on $\mathrm{S}(\mathfrak{q}_{N+M})$ as that of a subalgebra of $\mathfrak{q}_{N+M}$. Then take the subalgebra of invariants $\mathrm{S}(\mathfrak{q}_{N+M})^{\mathfrak{q}_M}\subset\mathrm{S}(\mathfrak{q}_{N+M})$.

Proposition 2.6. *The subalgebra* $\mathrm{S}(\mathfrak{q}_{N+M})^{\mathfrak{q}_M}$ *is generated by the elements* $c^{(1)}_{N+M}\,,\,c^{(3)}_{N+M}\,,\,\ldots$ *and* $f^{(1)}_{ij\,|\,N+M}\,,\,f^{(2)}_{ij\,|\,N+M}\,,\,\ldots$ *where* $|i|,|j|\leqslant N$.

Proof. Consider $(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ as a vector space. Define a linear map φ_n from this space to the nth symmetric power $\mathrm{S}^n(\mathfrak{q}_{N+M})$ by setting

$$\varphi_n\,(E_{i_1j_1}\otimes\ldots\otimes E_{i_nj_n})=F_{i_1j_1}\ldots F_{i_nj_n} \tag{2.11}$$

for any indices $i,j=\pm1,\,\ldots\,,\pm(N+M)$. The map φ_n commutes with the natural action of the Lie superalgebra $\mathfrak{q}_{N+M}$ on $(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ and $\mathrm{S}^n(\mathfrak{q}_{N+M})$. The map φ_n has a right inverse linear map ψ_n which commutes with the action of $\mathfrak{q}_{N+M}$ as well. Namely, using (2.8) we set

$$\psi_n\,(F_{i_1j_1}\ldots F_{i_nj_n})=2^{-n}\,\langle F_{i_1j_1}\,,\,\ldots\,,F_{i_nj_n}\rangle\,.$$

It follows that the subspace of $\mathfrak{q}_M$-invariants of the $\mathbb{Z}$-degree n in $\mathrm{S}(\mathfrak{q}_{N+M})$,

$$\mathrm{S}^n(\mathfrak{q}_{N+M})^{\mathfrak{q}_M}=\varphi_n\,(((\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n})^{\mathfrak{q}_M})=\varphi_n(\mathrm{B})$$

by Proposition 2.5; here B is the image in of the crossed product $S_n\ltimes\mathrm{C}^{\otimes n}$ in

$$(\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}=\mathrm{End}\,((\mathbb{C}^{N+M\,|\,N+M})^{\otimes n})\,.$$

The vector subspace $\mathrm{C}\subset\mathrm{End}\,(\mathbb{C}^{N+M\,|\,N+M})$ is spanned by the identity element 1, the elements E_{ij} where $|i|,|j|\leqslant N$ and by the element

$$J=\sum_i\,(-1)^{\bar\imath}\,E_{i,-i}$$

where the summation index i runs through $\pm 1, \dots, \pm(N+M)$. For each $p = 1, \dots, n$ introduce the elements of the algebra $(\mathrm{End}(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$,

$$E_{ij}^{(p)} = 1^{\otimes (p-1)} \otimes E_{ij} \otimes 1^{\otimes (n-p)} \quad \text{and} \quad J_p = 1^{\otimes (p-1)} \otimes J \otimes 1^{\otimes (n-p)}.$$

The vector subspace $\mathrm{B} \subset (\mathrm{End}(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ is spanned by the products of the form

$$J_{q_1} \dots J_{q_b} E_{i_1 j_1}^{(p_1)} \dots E_{i_a j_a}^{(p_a)} H \tag{2.12}$$

where

$$1 \leqslant p_1 < \dots < p_a \leqslant n, \quad 1 \leqslant q_1 < \dots < q_b \leqslant n,$$
$$\{p_1, \dots, p_a\} \cap \{q_1, \dots, q_b\} = \varnothing,$$
$$|i_1|, |j_1|, \dots, |i_a|, |j_a| \leqslant N$$

and H is the image in $(\mathrm{End}(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ of some permutation from S_n.

By the definintion of the symmetric algebra $\mathrm{S}(\mathfrak{q}_{N+M})$ and due to (2.6), we have the identity $\varphi_n \circ \sigma = \varphi_n$ for any $\sigma \in S_n$. Therefore it suffices to compute the φ_n-image of the element (2.12) where the factor H corresponds to a permutation of the form

$$(1, 2, \dots, r_1)(r_1+1, r_1+2, \dots, r_2) \dots (r_c+1, r_c+2, \dots, n)$$

where $c \geqslant 0$ and $0 < r_1 < r_2 < \dots < r_c < n$. But for any $p = 1, \dots, n-1$ we have the relation

$$\varphi_p(X)\, \varphi_{n-p}(Y) = \varphi_n(X \otimes Y), \tag{2.13}$$

$$X \in (\mathrm{End}(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes p} \quad \text{and} \quad Y \in (\mathrm{End}(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes (n-p)}.$$

Hence it suffices to consider only the case where H corresponds to the cyclic permutation $(1, 2, \dots, n)$. Suppose this is the case. Then we may assume that $p_1 = 1$ or $a = 0$. Here we use the identity $\varphi_n \circ \sigma = \varphi_n$ for $\sigma = (1, 2, \dots, n)$.

For the cyclic permutation $(1, 2, \dots, n)$ we have

$$H = \sum_{k_1, \dots, k_n} (-1)^{\bar{k}_1 + \dots + \bar{k}_{n-1}} E_{k_n k_1}^{(1)} E_{k_1 k_2}^{(2)} \dots E_{k_{n-1} k_n}^{(n)} \tag{2.14}$$

where each of the indices $k_1, \dots, k_n$ runs through $\pm 1, \dots, \pm(N+M)$. Hence

$$E_{ij}^{(1)} H = \sum_{k_1, \dots, k_{n-1}} (-1)^{\bar{k}_1 + \dots + \bar{k}_{n-1}} E_{k_n k_1}^{(1)} E_{k_1 k_2}^{(2)} \dots E_{k_{n-1} k_n}^{(n)}. \tag{2.15}$$

Further, for any $p = 1, \dots, n-1$ we have the equality

$$E_{ij}^{(1)} E_{kl}^{(p+1)} H = (-1)^{\bar{\jmath}\bar{k} + \bar{\jmath}\bar{l} + \bar{k}\bar{l}}\, E_{il}^{(1)} E_{kj}^{(p+1)} H' H''$$

where the factors H' and H'' are the images in $(\mathrm{End}(\mathbb{C}^{N+M\,|\,N+M}))^{\otimes n}$ of the cyclic permutations $(1, \dots, p)$ and $(p+1, \dots, n)$. Using this equality together with the relation (2.13), we reduce the case $a > 1$ to the case $a = 1$.

Suppose that $a=1$ and $p_1=1$. In this case, the φ_n-image of (2.12) equals

$$\sum_{k_1,\ldots,k_{n-1}} (-1)^{\,\bar k_1+\ldots+\bar k_{n-1}+b\,\bar\imath+q_1+\ldots+q_b}\,F_{(-1)^b\,i,k_1}F_{k_1k_2}\ldots F_{k_{n-1}j}$$
$$=(-1)^{\,b\,\bar\imath+q_1+\ldots+q_b}\,f^{\,(n)}_{(-1)^b\,i,\,j\,|\,N+M}\,;$$

we use the equality (2.15), the definition (2.11) and the identity $F_{-k,-l}=F_{kl}$.

It remains to consider the case $a=0$. Then the φ_n-image of (2.12) equals

$$(-1)^{\,q_1+\ldots+q_b}\sum_{k_1,\ldots,k_n}(-1)^{\,\bar k_1+\ldots+\bar k_{n-1}+b\,\bar k_n}\,F_{(-1)^b\,k_n,k_1}F_{k_1k_2}\ldots F_{k_{n-1}k_n}\,;$$

we use the equality (2.14), the definition (2.11) and the identity $F_{-k,-l}=F_{kl}$. Denote by f the sum over $k_1,\ldots,k_n$ in the above display. If the number b is even, then $f=c^{(n)}_{N+M}$. Now suppose that the number b is odd, so that f is

$$\sum_{k_1,\ldots,k_n}(-1)^{\,\bar k_1+\ldots+\bar k_{n-1}+\bar k_n}\,F_{-k_n,k_1}F_{k_1k_2}\ldots F_{k_{n-1}k_n}\,. \tag{2.16}$$

By changing the signs of the indices $k_1,\ldots,k_n$ in (2.16) and then using the identity $F_{-k,-l}=F_{kl}$ one shows that $f=(-1)^n f$. So $f=0$ if n is odd. Since

$$\deg F_{-k_n,k_1}=\bar k_1+\bar k_n+1=\deg\,(F_{k_1k_2}\ldots F_{k_{n-1}k_n})+1\,,$$

the element (2.16) of the symmetric algebra $\mathrm{S}(\mathfrak{q}_{N+M})$ can be also written as

$$\sum_{k_1,\ldots,k_n}(-1)^{\,\bar k_1+\ldots+\bar k_{n-1}+\bar k_n}\,F_{k_1k_2}\ldots F_{k_{n-1}k_n}\,F_{-k_n,k_1}\,. \tag{2.17}$$

By changing the signs of the indices $k_2,\ldots,k_n$ in (2.17) and then using the identity $F_{-k,-l}=F_{kl}$ one shows that $f=(-1)^{n-1}f$. So $f=0$ if n is even. □

By using the fact that for each $n\geqslant 1$ the linear map (2.10) commutes with the adjoint action of the Lie superalgebra $\mathfrak{q}_{N+M}$ on $\mathrm{U}^n(\mathfrak{q}_{N+M})$ and $\mathrm{S}^n(\mathfrak{q}_{N+M})$, we can now complete the proof of Theorem 1.2. Take any element $X\in\mathrm{A}_N^M$. We have $X\in\mathrm{U}^n(\mathfrak{q}_{N+M})$ for some $n\geqslant 0$. Let us demonstrate by induction on n that X belongs to the subalgebra in $\mathrm{U}(\mathfrak{q}_{N+M})$ generated by the elements $C^{(1)}_{N+M},C^{(3)}_{N+M},\ldots$ and (1.12). Since $\mathrm{U}^0(\mathfrak{q}_{N+M})=\mathbb{C}$, here we can take $n\geqslant 1$ and make the induction assumption. By Proposition 2.6, the image of X in $\mathrm{S}^n(\mathfrak{q}_{N+M})$ under the map (2.10) is a linear combination of the products of the elements $c^{(1)}_{N+M}$, $c^{(3)}_{N+M}$, … and $f^{\,(1)}_{ij\,|\,N+M}$, $f^{\,(2)}_{ij\,|\,N+M}$, … where $|i|,|j|\leqslant N$. By replacing these elements of $\mathrm{S}(\mathfrak{q}_{N+M})$ respectively by $C^{(1)}_{N+M},C^{(3)}_{N+M},\ldots$ and (1.12) in the linear combination, we obtain a certain element $Y\in\mathrm{U}(\mathfrak{q}_{N+M})$ such that $X-Y\in\mathrm{U}^{(n-1)}(\mathfrak{q}_{N+M})$. We also have $Y\in\mathrm{A}_N^M$. By applying the induction assumption to the difference $X-Y$, we complete the the proof.

3. Proof of Theorem 1.5

Here we prove Theorem 1.5 along with Lemma 1.3 and Propositions 1.4 ,1.6.

Proof of Lemma 1.3. Together with the right ideal I_{N+M} generated by the elements (1.13), consider the left ideal J_{N+M} in $\mathrm{U}(\mathfrak{q}_{N+M})$ generated by the elements

$$F_{\pm 1,\,N+M}\;,\,\ldots\,,\;F_{\pm(N+M),\,N+M}\;.$$

By the Poincaré-Birkhoff-Witt theorem for Lie superalgebras, every element $X\in\mathrm{U}(\mathfrak{q}_{N+M})$ can be uniquely written as a sum of the products of the form

$$\begin{gathered}F_{N+M,1}^{\,p_1}\,F_{N+M,-1}^{\,p_{-1}}\,\cdots\,F_{N+M,N+M-1}^{\,p_{N+M-1}}\,F_{N+M,-N-M+1}^{\,p_{-N-M+1}}\;\times\\
F_{N+M,N+M}^{\,p_{N+M}}\,F_{N+M,-N-M}^{\,p_{-N-M}}\,Y\;\times\\
F_{1,N+M}^{\,q_1}\,F_{-1,N+M}^{\,q_{-1}}\,\cdots\,F_{N+M-1,N+M}^{\,q_{N+M-1}}\,F_{-N-M+1,N+M}^{\,q_{-N-M+1}}\end{gathered}\tag{3.1}$$

where each of the exponents p_k and q_k runs through $0\,,1\,,2\,,\ldots$ or through $0\,,1$ if $k>0$ or $k<0$ respectively, whereas the factor $Y\in\mathrm{U}(\mathfrak{q}_{N+M-1})$ depends on these exponents. Note that here

$$[\,F_{N+M,\,N+M}\,,Y\,]\,=\,0\quad\text{and}\quad[\,F_{-N-M,\,N+M}\,,Y\,]\,=\,0\,.\tag{3.2}$$

Now suppose that $X\in\mathrm{A}_N^M$. In particular, then we have

$$[\,F_{N+M,\,N+M}\,,X\,]\,=\,0\tag{3.3}$$

since $M\geqslant 1$ by our assumption. Note that if $|\,k\,|<N+M$, then due to (1.6)

$$\begin{aligned}[\,F_{N+M,\,N+M}\,,F_{N+M,\,k}\,]\,&=\,F_{N+M,k}\;,\\
[\,F_{N+M,\,N+M}\,,F_{k,\,N+M}\,]\,&=\,-\,F_{N+M,k}\;.\end{aligned}$$

If $|\,k\,|=N+M$, then

$$[\,F_{N+M,\,N+M}\,,F_{k,\,N+M}\,]\,=\,0\,.$$

Hence the condition (3.3) implies that X is a sum of the products (3.1) where

$$\begin{aligned}&p_1+p_{-1}+\ldots+p_{N+M-1}+p_{-N-M+1}=\\
&q_1+q_{-1}+\ldots+q_{N+M-1}+q_{-N-M+1}\;.\end{aligned}\tag{3.4}$$

The intersection $\mathrm{I}_{N+M}\cap\mathrm{A}_N^M$ consists of those elements $X\in\mathrm{A}_N^M$ which are sums of the products (3.1) where

$$p_1+p_{-1}+\ldots+p_{N+M}+p_{-N-M}>0\,.$$

Due to the equality (3.4), the latter inequality is equivalent to

$$q_1+q_{-1}+\ldots+q_{N+M-1}+q_{-N-M+1}+p_{N+M}+p_{-N-M}>0\,.$$

So by using (3.2),
$$\mathrm{I}_{N+M} \cap \mathrm{A}_N^M = \mathrm{J}_{N+M} \cap \mathrm{A}_N^M\,. \tag{3.5}$$
In particular, the intersection $\mathrm{I}_{N+M} \cap \mathrm{A}_N^M$ is a two-sided ideal of A_N^M. Thus we get Part (a) of Lemma 1.3.

Furthemore, due to (3.4) there is only one summand (3.1) of $X \in \mathrm{A}_N^M$ with
$$p_1 + p_{-1} + \ldots + p_{N+M} + p_{-N-M} = 0\,, \tag{3.6}$$
this summand has the form of $Y \in \mathrm{U}(\mathfrak{q}_{N+M-1})$. Note that the right ideal I_{N+M} of $\mathrm{U}(\mathfrak{q}_{N+M})$ is stable under the adjoint action of the subalgebra $\mathfrak{q}_{N+M-1} \subset \mathfrak{q}_{N+M}$. Indeed, if $|i|, |j| < N+M$ then by (1.6) we have
$$[\, F_{ij}\,, F_{N+M,l}\,] \,=\, -\,(-1)^{\,\bar\imath\,\bar l + \bar\jmath\,\bar l}\,\delta_{il}\,F_{N+M,j} \,-\, (-1)^{\,\bar\imath\,\bar l + \bar\jmath\,\bar l}\,\delta_{i,-l}\,F_{N+M,-j}$$
for any index $l = \pm 1, \ldots, \pm(N+M)$. In particular, I_{N+M} is stable under the adjoint action of $\mathfrak{q}_{M-1}$. So the condition $[\,Z\,, X\,] = 0$ on X for any $Z \in \mathfrak{q}_{M-1}$ implies the condition $[\,Z, Y\,] = 0$ on the summand Y of X corresponding to (3.6). Therefore $Y \in \mathrm{A}_N^{M-1}$, and we get Part (b) of Lemma 1.3. $\square$

Proof of Proposition 1.4. Suppose that $|i|\,, |j| \leqslant N$. Let us prove by induction on $n = 1, 2, \ldots$ that the differences
$$F_{ij\,|\,N+M}^{(n)} - F_{ij\,|\,N+M-1}^{(n)} \quad \text{and} \quad C_{N+M}^{(n)} - C_{N+M-1}^{(n)} \tag{3.7}$$
belong to the left ideal J_{N+M} in $\mathrm{U}(\mathfrak{q}_{N+M})$. Due to (3.5), Proposition 1.4 will then follow. Neither of the elements $F_{ij\,|\,N+M}^{(n)}$ and $C_{N+M}^{(n)}$, nor the ideal J_{N+M} depend on the partition of the number $N+M$ into N and M. Hence it suffices to consider the case $M = 1$. Note that according to the definition (1.11)
$$C_{N+1}^{(n)} - C_N^{(n)} \,=\, \sum_{|k| \leqslant N} (\, F_{kk\,|\,N+1}^{(n)} - F_{kk\,|\,N}^{(n)}\,) \,+$$
$$F_{N+1,N+1\,|\,N+1}^{(n)} \,+\, F_{-N-1,-N-1\,|\,N+1}^{(n)}$$
where the last two summands belong to J_{N+1}, by their definition. Therefore it suffices to consider only the first of the differences (3.7), where $M = 1$.

If $n = 1$, that difference is zero. Now suppose that $n > 1$, and make the induction assumption. Using the relation (1.9),
$$F_{ij\,|\,N+1}^{(n)} - F_{ij\,|\,N}^{(n)} \,=\, \sum_{|k| \leqslant N} (-1)^{\bar k}\, F_{ik}\,(\, F_{kj\,|\,N+1}^{(n-1)} - F_{kj\,|\,N}^{(n-1)}\,) \,+$$
$$F_{i\,,N+1}\,F_{N+1,j\,|\,N+1}^{(n-1)} - F_{i\,,-N-1}\,F_{-N-1,j\,|\,N+1}^{(n-1)}\,.$$
At the right hand side of the last equality, the summands corresponding to $|k| \leqslant N$ belong to the left ideal J_{N+1} by the induction assumption. Using (1.10) and (1.8), the remainder of the right hand side is equal to the sum

$$(-1)^{\bar\imath\bar\jmath}\,F^{(n-1)}_{N+1,j\,|\,N+1}\,F_{i,N+1}-(-1)^{(\bar\imath+1)(\bar\jmath+1)}\,F^{(n-1)}_{-N-1,j\,|\,N+1}\,F_{i,-N-1}+$$

$$(-1)^{\bar\imath\bar\jmath+1}\,(\,1+(-1)^{\bar\imath+\bar\jmath+n}\,)\,\delta_{ij}\,F^{(n-1)}_{N+1,N+1\,|\,N+1}+$$

$$(-1)^{(\bar\imath+1)(\bar\jmath+1)}\,(\,1+(-1)^{\bar\imath+\bar\jmath+n}\,)\,\delta_{i,-j}\,F^{(n-1)}_{-N-1,N+1\,|\,N+1}\,.$$

But in this sum, every summand evidently belongs to the left ideal J_{N+1}. □

Let us now prove Theorem 1.5. Firstly, we will verify the formula (1.18) for the supercommutator $[\,F^{(m)}_{ij},\,F^{(n)}_{kl}\,]$ in the algebra A_N. We will use

Proposition 3.1. *In* $\mathrm{U}(\mathfrak{q}_N)$ *for* $m,n=1,2,\,\ldots$ *and* $i,j,k,l=\pm1,\,\ldots\,,\pm N$

$$[\,F^{(m)}_{ij\,|\,N},F^{(n)}_{kl\,|\,N}\,]\,=\,F^{(m+n-1)}_{il\,|\,N}\,\delta_{kj}-(-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)}\,\delta_{il}\,F^{(m+n-1)}_{kj\,|\,N}+$$

$$(-1)^{m-1}\,(\,F^{(m+n-1)}_{-i,l\,|\,N}\,\delta_{-k,j}\,-(-1)^{(\bar\imath+\bar\jmath)(\bar k+\bar l)}\,\delta_{i,-l}\,F^{(m+n-1)}_{k,-j\,|\,N}\,)\,+$$

$$(-1)^{\bar\jmath\bar k+\bar\jmath\bar l+\bar k\bar l}\,\sum_{r=1}^{m-1}\,(\,F^{(n+r-1)}_{il\,|\,N}\,F^{(m-r)}_{kj\,|\,N}-F^{(m-r)}_{il\,|\,N}\,F^{(n+r-1)}_{kj\,|\,N}\,)\,+$$

$$(-1)^{\bar\jmath\bar k+\bar\jmath\bar l+\bar k\bar l+\bar k+\bar l}\,\sum_{r=1}^{m-1}\,(-1)^{r}\,(\,F^{(n+r-1)}_{-i,l\,|\,N}\,F^{(m-r)}_{-k,j\,|\,N}-F^{(m-r)}_{i,-l\,|\,N}\,F^{(n+r-1)}_{k,-j\,|\,N}\,)\,.$$

Proof. The formula for the supercommutator $[\,F^{(m)}_{ij\,|\,N},F^{(n)}_{kl\,|\,N}\,]$ in the $\mathbb{Z}_2$-graded algebra $\mathrm{U}(\mathfrak{q}_N)$ displayed above is easy to verify by using the induction on m. When $m=1$, this formula coincides with (1.10). Now make the induction assumption. Let the index h run through $\pm1,\,\ldots\,,\pm N$. Then due to (1.9),

$$[\,F^{(m+1)}_{ij\,|\,N},F^{(n)}_{kl\,|\,N}\,]\,=\,\sum_h\,(-1)^{\bar h}\,[\,F_{ih}\,F^{(m)}_{hj\,|\,N},F^{(n)}_{kl\,|\,N}\,]\,=$$

$$\sum_h\,(-1)^{\bar h}\,F_{ih}\,[\,F^{(m)}_{hj\,|\,N},F^{(n)}_{kl\,|\,N}\,]\,+\sum_h\,(-1)^{\bar h+(\bar h+\bar\jmath)(\bar k+\bar l)}\,[\,F_{ih}\,,F^{(n)}_{kl\,|\,N}\,]\,F^{(m)}_{hj\,|\,N}$$

$$=\,\sum_h\,(-1)^{\bar h}\,F_{ih}\,(\,F^{(m+n-1)}_{hl\,|\,N}\,\delta_{kj}-(-1)^{(\bar h+\bar\jmath)(\bar k+\bar l)}\,\delta_{hl}\,F^{(m+n-1)}_{kj\,|\,N}\,+$$

$$(-1)^{m-1}\,(\,F^{(m+n-1)}_{-h,l\,|\,N}\,\delta_{-k,j}\,-(-1)^{(\bar h+\bar\jmath)(\bar k+\bar l)}\,\delta_{h,-l}\,F^{(m+n-1)}_{k,-j\,|\,N}\,)\,+$$

$$(-1)^{\bar\jmath\bar k+\bar\jmath\bar l+\bar k\bar l}\,\sum_{r=1}^{m-1}\,(\,F^{(n+r-1)}_{hl\,|\,N}\,F^{(m-r)}_{kj\,|\,N}-F^{(m-r)}_{hl\,|\,N}\,F^{(n+r-1)}_{kj\,|\,N}\,)\,+$$

$$(-1)^{\bar\jmath\bar k+\bar\jmath\bar l+\bar k\bar l+\bar k+\bar l}\,\sum_{r=1}^{m-1}\,(-1)^{r}(\,F^{(n+r-1)}_{-h,l\,|\,N}\,F^{(m-r)}_{-k,j\,|\,N}-F^{(m-r)}_{h,-l\,|\,N}\,F^{(n+r-1)}_{k,-j\,|\,N}\,))$$

$$+\,\sum_h\,(-1)^{\bar h+(\bar h+\bar\jmath)(\bar k+\bar l)}\,(\,\delta_{kh}\,F^{(n)}_{il\,|\,N}-(-1)^{(\bar\imath+\bar h)(\bar k+\bar l)}\,\delta_{il}\,F^{(n)}_{kh\,|\,N}\,+$$

$$\delta_{-k,h}\,F^{(n)}_{-i,l\,|\,N} - (-1)^{\,(\,\bar\imath+\bar h\,)\,(\,\bar k+\bar l\,)}\,\delta_{i,-l}\,F^{(n)}_{k,-h\,|\,N}\,)\,F^{(m)}_{hj\,|\,N}\,.$$

Here we used the induction assumption with the index i replaced by h, and the relation (1.10) with the index j replaced by h. Using the relation (1.9) repeatedly, the right hand side of the above displayed equalities equals

$$F^{(m+n)}_{il\,|\,N}\,\delta_{kj} - (-1)^{\,(\,\bar\imath+\bar\jmath\,)\,(\,\bar k+\bar l\,)}\,\delta_{il}\,F^{(m+n)}_{kj\,|\,N}\ +$$

$$(-1)^{m}\,(\,F^{(m+n)}_{-i,l\,|\,N}\,\delta_{-k,j}\ - (-1)^{\,(\,\bar\imath+\bar\jmath\,)\,(\,\bar k+\bar l\,)}\,\delta_{i,-l}\,F^{(m+n)}_{k,-j\,|\,N}\,)\ +$$

$$(-1)^{\,\bar\jmath\bar k\,+\bar\jmath\bar l\,+\bar k\bar l}\ (\,F^{(n)}_{il\,|\,N}\,F^{(m)}_{kj\,|\,N}\ -\ F^{(1)}_{il\,|\,N}\,F^{(m+n-1)}_{kj\,|\,N}\,)\ +$$

$$(-1)^{\,\bar\jmath\bar k\,+\bar\jmath\bar l\,+\bar k\bar l\,+\bar k\,+\bar l}((-1)^{m+1}\,F^{(1)}_{i,-l\,|\,N}\,F^{(m+n-1)}_{k,-j\,|\,N}\ - F^{(n)}_{-i,l\,|\,N}\,F^{(m)}_{-k,j\,|\,N}\,)\ +$$

$$(-1)^{\,\bar\jmath\bar k\,+\bar\jmath\bar l\,+\bar k\bar l}\ \sum_{r=1}^{m-1}\,(\,F^{(n+r)}_{il\,|\,N}\,F^{(m-r)}_{kj\,|\,N}\ - F^{(m-r+1)}_{il\,|\,N}\ F^{(n+r-1)}_{kj\,|\,N}\,)\ +$$

$$(-1)^{\,\bar\jmath\bar k\,+\bar\jmath\bar l\,+\bar k\bar l\,+\bar k\,+\bar l}\ \sum_{r=1}^{m-1}\,(-1)^{r+1}(\,F^{(n+r)}_{-i,l\,|\,N}\,F^{(m-r)}_{-k,j\,|\,N} + F^{(m-r+1)}_{i,-l\,|\,N}\ F^{(n+r-1)}_{k,-j\,|\,N}\,)\,.$$

But the last displayed sum can also be obtained by replacing m by $m+1$ at right hand side of the equality in Proposition 3.1. Thus we have made the induction step. $\square$

If $m>n$ then at the right hand side of the equality in Proposition 3.1, the summands corresponding to the indices $r=1,\ldots,m-n$ cancel in each of the two sums over $r=1,\ldots,m-1$. In the first of the two sums, this is obvious. To cancel these summands in the second sum, one utilises the relations (1.8). Hence if $m>n$, the summation over $r=1,\ldots,m-1$ in Proposition 3.1 can be replaced by the summation over $r=m-n+1,\ldots,m-1$. Thus if we change the running index r to $m-r$, the latter index should run through $1,\ldots,\min(m,n)-1$. Using this remark, the relation (1.18) in Theorem 1.5 follows from Proposition 3.1.

In the remainder of the proof of Theorem 1.5, we will also make use of the next proposition. For any integers $M\geqslant 0$ and $n\geqslant 1$ consider the elements

$$c^{(n)}_{N+M}\quad\text{and}\quad f^{(n)}_{ij\,|\,N+M} \tag{3.8}$$

of the algebra $\mathrm{S}(\mathfrak{q}_{\,N+M})^{\,\mathfrak{q}_M}$, see Proposition 2.6. Fix any positive integer s.

Proposition 3.2. *Take the elements $f^{(n)}_{ij\,|\,N+M}$ where*

$$1\leqslant n\leqslant s\,,\quad 1\leqslant i\leqslant N\,,\quad 1\leqslant |j|\leqslant N\,. \tag{3.9}$$

Along with these elements, take the elements $c^{(n)}_{N+M}$ where $1\leqslant n\leqslant s$ and n is odd. For any sufficiently large number M, all these elements are algebraically independent in the supercommutative algebra $\mathrm{S}(\mathfrak{q}_{\,N+M})$.

Proof. We will use arguments from [MO, Subsection 2.11]. By the Poincaré-Birkhoff-Witt theorem for Lie superalgebras, the elements

$$F_{kl} = f^{(1)}_{kl\,|\,N+M}$$

where $k = 1, \ldots, N+M$ and $l = \pm 1, \ldots, \pm N+M$, are free generators of the supercommutative algebra $\mathrm{S}(\mathfrak{q}_{N+M})$. Let X_s be the quotient algebra of $\mathrm{S}(\mathfrak{q}_{N+M})$, defined by imposing the following relations on these free generators.

For every triple (i, j, n) satisfying the conditions (3.9), choose a subset

$$\mathcal{O}^{(n)}_{ij} \subset \{N+1, N+2, \ldots\}$$

of cardinality $n-1$ in such a way, that all these subsets are disjoint. Let M be so large that all these subsets are contained in $\{N+1, \ldots, N+M-s\}$. If

$$\mathcal{O}^{(n)}_{ij} = \{l_1, \ldots, l_{n-1}\},$$

then put

$$F_{i\,l_1} = F_{l_1 l_2} = \ldots = F_{l_{n-2}\,l_{n-1}} = 1.$$

Denote by $x^{(n)}_{ij}$ the image of the element $F_{l_{n-1}\,j} \in \mathrm{S}(\mathfrak{q}_{N+M})$ in the algebra X_s. Having done this for every triple (i, j, n) satisfying the conditions (3.9), for every $r = 1, \ldots, s$ denote by x_r the image in X_s of the element

$$F_{N+M-s+r\,,\,N+M-s+r} \in \mathrm{S}(\mathfrak{q}_{N+M}).$$

Finally, put $F_{kl} = 0$ if $k > 0$ and (k, l) is not one of the pairs

$$(i, l_1)\,,\,(l_1, l_2)\,,\,\ldots\,,\,(l_{n-2}, l_{n-1})\,,\,(l_{n-1}, j)$$

for any triple (i, j, n) satisfying (3.9), and not one of the pairs

$$(N+M-s+r, N+M-s+r) \quad \text{where} \quad r = 1, \ldots, s\,.$$

The elements $x_1, \ldots, x_s$ and the elements $x^{(n)}_{ij}$ for all the triples (i, j, n) satisfying (3.9), are free generators of the algebra X_s. For any of these triples, the image in X_s of

$$f^{(n)}_{ij\,|\,N+M} \in \mathrm{S}(\mathfrak{q}_{N+M})$$

equals $x^{(n)}_{ij}$ plus a certain linear combination of products of the elements $x^{(m)}_{kl}$ where $1 \leqslant m < n$. For any odd n, the image in X_s of the element

$$c^{(n)}_{N+M} \in \mathrm{S}(\mathfrak{q}_{N+M})$$

equals

$$2^{\,n}(x^n_1 + \ldots + x^n_s)$$

plus a linear combination of products of elements $x^{(m)}_{kl}$ where $1 \leqslant m \leqslant n$. Hence all these images are algebraically independent in the quotient X_s of the supercommutative algebra $\mathrm{S}(\mathfrak{q}_{N+M})$. $\square$

Let us show that the associative algebra A_N is generated by the elements $C^{(1)}, C^{(3)}, \ldots$ and $F_{ij}^{(1)}, F_{ij}^{(2)}, \ldots$. Take any element $Z \in \mathrm{A}_N$, and consider its canonical image $Z_M = \pi_M(Z) \in \mathrm{A}_N^M$ for any $M \geqslant 0$. By Theorem 1.2, the element Z_M is a linear combination of the products of the elements $C_{N+M}^{(n)}$ where $n = 1, 3, \ldots$ and of the elements $F_{ij\,|\,N+M}^{(n)}$ where $n = 1, 2, \ldots$ whereas $i = 1, \ldots, N$ and $j = \pm 1, \ldots, \pm N$; see (1.8). Choose any linear ordering of all these elements. Applying Proposition 3.1 to the algebra $\mathrm{U}(\mathfrak{q}_{N+M})$ instead of $\mathrm{U}(\mathfrak{q}_N)$, we will assume that any of the products in the linear combination Z_M is an ordered monomial in these elements. If $\bar\imath + \bar\jmath = 1$, then the element $F_{ij\,|\,N+M}^{(n)}$ may appear in any of these monomials only with the degree 1.

We will assume that for any $M \geqslant 1$, the map α_M preserves the ordering. Then for every monomial Y_M appearing in the linear combination Z_M, the monomial $\alpha_M(Y_M)$ may appear in the linear combination $\alpha_M(Z_M) = Z_{M-1}$.

The filtration degrees of all the elements $Z_0, Z_1, Z_2, \ldots$ are bounded from above. Hence for any factor $C_{N+M}^{(n)}$ or $F_{ij\,|\,N+M}^{(n)}$ of the monomials appearing in linear combination Z_M, we have $n \leqslant s$ for a certain integer s which does not depend on M. Then for a sufficiently large number M, the coefficients of the monomials appearing in the linear combinations $Z_M, Z_{M+1}, Z_{M+2}, \ldots$ are determined uniquely. The uniqueness follows from Proposition 3.2.

Now fix a sufficiently large number M, as above. Let Y_M be any monomial appearing in the linear combination Z_M, say with a coefficient $z \in \mathbb{C}$. We assume that $z \neq 0$. Then for any integer $L > M$, the linear combination Z_L contains the summand zY_L where Y_L is a monomial and

$$(\alpha_{M+1} \circ \ldots \circ \alpha_L)(Y_L) = Y_M\,.$$

For any non-negative integer $L \leqslant M$, define $Y_L = (\alpha_L \circ \ldots \circ \alpha_M)(Y_M)$. The sequence $Y_0, Y_1, Y_2, \ldots$ determines an element $Y \in \mathrm{A}_N$, which is a monomial in $C^{(1)}, C^{(3)}, \ldots$ and $F_{ij}^{(1)}, F_{ij}^{(2)}, \ldots$. Here $1 \leqslant i \leqslant N$ and $1 \leqslant |j| \leqslant N$. The element $Z \in \mathrm{A}_N$ is then a sum of the products of the form zY. This sum is finite, because any such product corresponds to a summand zY_M in the linear combination Z_M. Thus we have proved Part (a) of Theorem 1.5.

Let us now prove Parts (b) and (c). By definition, the algebra A_N comes with an ascending $\mathbb{Z}$-filtration, such that the generators $C^{(n)}$ and $F_{ij}^{(n)}$ of A_N have the degree n. Denote the corresponding $\mathbb{Z}$-graded algebra by $\mathrm{gr}\,\mathrm{A}_N$. Let $c^{(n)}$ and $f_{ij}^{(n)}$ be the generators of the algebra $\mathrm{gr}\,\mathrm{A}_N$ corresponding to $C^{(n)}$ and $F_{ij}^{(n)}$. We always assume that the index n in $C^{(n)}$ and $c^{(n)}$ is odd. We also assume that $|i|, |j| \leqslant N$ in $F_{ij}^{(n)}$ and $f_{ij}^{(n)}$. It follows from (1.17) that for any $n = 1, 2, \ldots$

$$f_{-i,-j}^{(n)} = (-1)^{n-1} f_{ij}^{(n)}.$$

The algebra $\mathrm{gr}\,\mathrm{A}_N$ also inherits from A_N a $\mathbb{Z}_2$-gradation, such that

$$\deg c^{(n)} = 0 \quad \text{and} \quad \deg f_{ij}^{(n)} = \bar\imath + \bar\jmath\,,$$

see (1.16). The relation (1.18) demonstrates that the $\mathbb{Z}_2$-graded algebra $\mathrm{gr}\,\mathrm{A}_N$ is supercommutative. To complete the proof of Theorem 1.5, it suffices to show that the elements $c^{(1)}, c^{(3)}, \ldots$ together with the elements $f_{ij}^{(1)}, f_{ij}^{(2)}, \ldots$ where $i = 1, \ldots, N$ and $j = \pm 1, \ldots, \pm N$, are algebraically independent in the supercommutative algebra $\mathrm{gr}\,\mathrm{A}_N$.

The algebra $\mathrm{gr}\,\mathrm{A}_N$ can also be obtained as an inverse limit of the sequence of the supercommutative algebras $\mathrm{S}(\mathfrak{q}_{N+M})^{\mathfrak{q}_M}$ where $M = 0, 1, 2, \ldots$. The limit is taken in the category of $\mathbb{Z}$-graded algebras. We assume that if $M = 0$, then $\mathrm{S}(\mathfrak{q}_{N+M})^{\mathfrak{q}_M} = \mathrm{S}(\mathfrak{q}_N)$. The definition of the surjective homomorphism

$$\mathrm{S}(\mathfrak{q}_{N+M})^{\mathfrak{q}_M} \to \mathrm{S}(\mathfrak{q}_{N+M-1})^{\mathfrak{q}_{M-1}}$$

for any $M \geqslant 1$ is similar to the definition of the surjective homomorphism $\alpha_M : \mathrm{A}_N^M \to \mathrm{A}_N^{M-1}$, see Lemma 1.3. Here we omit the details, but notice that the elements $c^{(n)}$ and $f_{ij}^{(n)}$ of $\mathrm{gr}\,\mathrm{A}_N$ correspond to the sequences of elements (3.8) of the algebras $\mathrm{S}(\mathfrak{q}_{N+M})^{\mathfrak{q}_M}$ where $M = 0, 1, 2, \ldots$. Proposition 3.2 now guarantees the algebraic independence of the elements $c^{(1)}, c^{(3)}, \ldots$ together with the elements $f_{ij}^{(1)}, f_{ij}^{(2)}, \ldots$ where $i = 1, \ldots, N$ and $j = \pm 1, \ldots, \pm N$.

Proof of Proposition 1.6. Under the correspondence $F_{ij}^{(n)} \mapsto (-1)^{\bar\imath}\, T_{ji}^{(n)}$, the collection of relations (1.17) in the algebra B_N for the indices $n = 1, 2, \ldots$ corresponds to the equality (1.19) in $\mathrm{Y}(\mathfrak{q}_N)[[u^{-1}]]$. Put $T_{ij}^{(0)} = \delta_{ij}$ for any $i, j = 1, \ldots, \pm N$. Using (1.4), the relation (1.18) in B_N then corresponds to

$$(-1)^{\bar\imath\bar\jmath + \bar\imath\bar k + \bar\jmath\bar k}\, [\, T_{ji}^{(m)}, T_{lk}^{(n)}\,] =$$
$$\sum_{r=0}^{m-1} (\, T_{jk}^{(m+n-r-1)}\, T_{li}^{(r)} - T_{jk}^{(r)}\, T_{li}^{(m+n-r-1)}\,) \;+\; (-1)^{\bar\imath + \bar\jmath + 1} \;\times$$
$$\sum_{r=0}^{m-1} (-1)^{m+r}\, (\, T_{-j,k}^{(m+n-r-1)}\, T_{-l,i}^{(r)} - T_{j,-k}^{(r)}\, T_{l,-i}^{(m+n-r-1)}\,)\,. \tag{3.10}$$

Here we also used a remark on the summation over $r = 1, \ldots, m-1$ similar to that made immediately after the proof of Proposition 3.1.

Put $T_{ij}^{(-1)} = 0$ for any $i, j = 1, \ldots, \pm N$. The collection of relations (3.10) for $m, n = 1, 2, \ldots$ is equivalent to the collection of relations

$$(-1)^{\bar\imath\bar\jmath + \bar\imath\bar k + \bar\jmath\bar k}\, (\, [\, T_{ji}^{(m+1)}, T_{lk}^{(n-1)}\,] - [\, T_{ji}^{(m-1)}, T_{lk}^{(n+1)}\,]\,) =$$
$$T_{jk}^{(n-1)}\, T_{li}^{(m)} - T_{jk}^{(m)}\, T_{li}^{(n-1)} + T_{jk}^{(n)}\, T_{li}^{(m-1)} - T_{jk}^{(m-1)}\, T_{li}^{(n)} \;+\; (-1)^{\bar\imath + \bar\jmath} \;\times$$
$$(\, T_{-j,k}^{(n-1)}\, T_{-l,i}^{(m)} - T_{j,-k}^{(m)}\, T_{l,-i}^{(n-1)} - T_{-j,k}^{(n)}\, T_{-l,i}^{(m-1)} + T_{j,-k}^{(m-1)}\, T_{l,-i}^{(n)}\,) \tag{3.11}$$

for $m, n = 0, 1, 2, \ldots$. Multiplying the relation (3.11) by $x^{1-m}\, y^{1-n}$ and taking the sum of resulting relations over $m, n = 0, 1, 2, \ldots$ we get the relation

$$(x^2-y^2)\cdot[\,T_{ji}(x),T_{lk}(y)]\cdot(-1)^{\,\bar\imath\bar\jmath+\bar\imath\bar k+\bar\jmath\bar k}\;=$$
$$(x+y)\cdot(T_{jk}(y)\,T_{li}(x)-T_{jk}(x)\,T_{li}(y))\;+$$
$$(x-y)\cdot(T_{-j,\,k}(y)\,T_{-l,\,i}(x)-T_{j,\,-k}(x)\,T_{l,\,-i}(y))\cdot(-1)^{\bar\imath+\bar\jmath}\,. \tag{3.12}$$

Using (1.5), we can rewrite the left hand side of the relation (3.12) as

$$(y^2-x^2)\cdot[\,T_{lk}(y)\,,T_{ji}(x)]\cdot(-1)^{\,\bar\imath\bar\jmath+\bar\imath\bar l+\bar\jmath\bar l}\,.$$

Replacing in the resulting relation the indices $i\,,j\,,k\,,l$ and the parameters $x\,,y$ by $l\,,k\,,j\,,i$ and $y\,,x$ respectively, we obtain exactly the relation (1.20). Thus the defining relations of the subalgebra $\mathrm{B}_N\subset\mathrm{A}_N$ correspond to the defining relations of the algebra $\mathrm{Y}(\mathfrak{q}_N)$, see Theorem 1.5. $\square$

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