A simple counterexample to the Fuchsian version of the Hilbert 21st problem.

[1]

D.V.Anosov

1. Standing notations, definitions and several basic facts. Content of this section is completely covered by the book [1]. Perhaps this especially concerns notations, because essential facts can be found in other places as well (but in [1] they are collected together and presented in the form appropriate to our present goals). We begin from some of the notations.

$S := \overline{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}$ is the punctured Riemann sphere, $\pi : \widetilde{S} \to S$ — its universal covering [2], Δ — the group of deck transformations. Points of $\widetilde{S}$ are denoted with "tilde": $\tilde{x}$.

Assume we are given a linear system of ordinary differential equations which in the standard vector-matrix notations looks as

$$\frac{dy}{dx} = A(x)y, \qquad y \in \mathbb{C}^p, \tag{1}$$

where the matrix $A(x)$ is holomorphic in S. Generally solutions of (1) are branching, but they are single-valued holomorphic functions on $\widetilde{S}$. As a linear combination of solutions (with constant complex coefficients) is again a solution, the set $\mathfrak{V}$ of all solutions [3] is a complex linear vector space (which is p-dimensional). Given (1), one can associate to it a matrix ODE

$$\frac{dY}{dx} = A(x)Y \quad \text{(where } Y \text{ is a } (p \times p) - \text{matrix).} \tag{2}$$

One deals mainly with those solutions of (2) for which the determinant $\det Y \neq 0$. (If it is $\neq 0$ at some point, it is $\neq 0$ everywhere.) Columns of such Y constitute a fundamental system of solutions of (1), i.e. a basis in $\mathfrak{V}$.

The system (1) is called Fuchsian if the entries of $A(x)$ have only simple poles at all a_i. It is called regular if its solutions grow or decrease no faster than some power of $|x - a_i|$ when $x = \pi(\tilde{x}) \to a_i$ provided that x remains inside some sector having the vertex in a_i. (The latter proviso is necessary, because the solution can branch logarithmically, and if in such case $x \to a_i$ along an appropriate spiral, one can get arbitrarily big growth of $|y(\tilde{x})|$.) Fuchsian systems are regular, the converse is not true.

To any $\sigma \in \Delta$ one associates the map

$$\sigma^* : \mathfrak{V} \to \mathfrak{V} \qquad \sigma^* y := y \circ \sigma^{-1} \tag{3}$$

[1]This work was partially supported by the Göran Gustaffson foundation, Royal Inst. of Technology (Stokholm), and by the RFBR – DFG grant 436 RUS 113/73/3. Author expresses his gratitude to the staff of the Royal Inst. of Technology and of the Max Planck Inst. for Math (Bonn) for hospitality.

Extended version of this paper is supposed to appear in the "Ulmer Seminare über Funktionalanalysis und Differentialgleichungen".

[2]In [1] the map π was denoted by p, but p is also the number of equations in the system which we shall consider. Of course π has also the different meaning, namely $\frac{22}{7}$, but here the danger of confusion is less.

[3]In [1] gothic letters were not used, but it was for a technical reason only.

(in other words, $(\sigma^* y)(\widetilde{x}) = y(\sigma\widetilde{x})$ or $(\sigma^* y)(\sigma\widetilde{x}) = y(\widetilde{x})$). The map

$$\Delta \to \mathrm{GL}(\mathfrak{V}) \qquad \sigma \mapsto \sigma^*$$

turns out to be a group representation. It is called the (abstract) monodromy representation for the system (1). We call it "abstract" because the space $\mathfrak{V}$ is a little bit abstract — a space of solutions; it is not so "concrete" as $\mathbb{C}^p$ (although they are isomorphic). If we choose a basis $Y = (y_1, \dots, y_p)$ in $\mathfrak{V}$ (speaking more classically, a fundamental system of solutions of (1)), the monodromy representation takes a more concrete form: we get a representation

$$\chi : \Delta \to \mathrm{GL}(p, \mathbb{C}) \qquad \sigma^* Y; = (\sigma^* y_1, \dots, \sigma^* y_p) = Y\chi(\sigma), \tag{4}$$

where the $(p \times p)$-matrix $\chi(\sigma)$ acts from the right in a standard way on a row $(y_1, \dots, y_p)$; this simply means that $\chi(\sigma)$ is the coefficient matrix of the linear transformation σ^* in terms of coordinates defined by the basis $(y_1, \dots, y_p)$. If we replace this basis by another one, the "concrete" monodromy representation χ is replaced by a conjugate representation. We say that χ arises from the system (1) or is realized by this system.

The 21st Hilbert problem (or the Riemann-Hilbert problem) has two versions: the regular and the Fuchsian one. (See [1] on the historical discussion.) One inquires whether there exists a regular (Fuchsian) system realizing the given monodromy representation χ. (As we speak about S, we tacitly assume that singularities a_i are also prescribed.) J.Plemelj proved many years ago that the answer to the regular version is always affirmative. Bolibruch discovered that the answer to the Fuchsian version can be negative. The goal of this note is to describe one of his examples which seems to be the simplest one. (As it often happens, the first example was somewhat more complicated.) This description is somewhat simplified comparatively to his original exposition.

The above general information was given in order to explain the general goal of this paper. In order to describe the example mentioned and to prove that it is really a counterexample, we need more background.

First of all, we need some local theory, i.e. theory concerning the behavior of solutions of (1) near singular points a_i. Denote by U_i a small open disk centered at a_i that does not contain other a_j; on occasion we can demand a little bit more. (If $a_i = \infty$, "small disk" means "$\overline{\mathbb{C}} \setminus D$ where D is a big closed disk containing all other a_i".) Denote by U_i^* the punctured neighborhood $U_i \setminus \{a_i\}$ and by $\widetilde{U_i^*}$ its universal covering surface. The group Δ_i of deck transformations of the covering $\widetilde{U_i^*} \to U_i^*$ is the infinite cyclic group generated by the deck transformation σ_i corresponding to one turn around a_i counterclockwise (if $a_i = \infty$, we take a clockwise turn). Denote by $\mathfrak{V}_i$ the space of solutions of our system (1) considered only over U_i^*, so that they are holomorphic functions on $\widetilde{U_i^*}$. Analogously to what was said before, Δ_i acts on $\mathfrak{V}_i$, but in this case all reduces to the action of σ_i (and its iterates). If Y is a non-degenerate solution of (2), i.e. a fundamental system of solutions of (1), then

$$\sigma_i^* Y = Y G_i^{-1} \qquad \text{with some } G_i \in \mathrm{GL}(p, \mathbb{C}), \tag{5}$$

i.e. $G_i = \chi(\sigma_i^{*-1})$. It may seem more natural to take $G_i = \chi(\sigma_i^*)$, but the definition (5) leads to the formula $Y(\sigma_i \widetilde{x}) = Y G_i$ which is classical. Define $E_i := \frac{1}{2\pi i} \ln G_i$, where the choice of ln is normalized by demanding that if λ_i^j are eigenvalues of G_i and μ_i^j — of E_i, then

$$\mu_i^j = \frac{1}{2\pi i}\ln\lambda_i \quad \text{and } 0 \le \Re\mu_i^j < 1.$$

Besides this, for a regular system Levelt defines some integer numbers φ_i^j; $i = 1, \ldots, n; j = 1, \ldots, p$. Numbers $\varphi_i^1, \ldots, \varphi_i^p$ somehow characterize growth (or decreasing) of solutions near a_i or, to be more precise, the part of this growth which is described in terms of holomorphic functions while there can remain some growth related to fractional powers or logarithms which is not taken into consideration (so for $\tilde{x}^{3/2}$ the corresponding φ is 1). For a fixed i numbers φ_i^j are numbered in such a way that $\varphi_i^1 \ge \ldots \ge \varphi_i^p$ We shall refer to these φ_i^j, so to say, twice: they figure in I) below; numbers $\beta_i^j := \varphi_i^J + \mu_i^j$ which are defined in the case of a regular system and are called its exponents figure in II).

We shall use two theorems proved by Levelt (in the Fuchsian case which we only need first of them essentially was somewhat earlier proved by Gantmacher). One can find their proofs in [1].

I). For each a_i there exists such a basis Y_i in $\mathfrak{V}$ (i.e. such a non-degenerate solution of (2)) that in $\widetilde{U_i^*}$

$$Y_i(\tilde{x}) = V_i(x)(x - a_i)^{\Phi_i}(\widetilde{x - a_i})^{E_i'}, \tag{6}$$

where $V_i(x)$ is holomorphic in the (non-punctured!) neighborhood U_i of a_i, Φ_i is the diagonal matrix with the diagonal elements $\varphi_i^1, \ldots, \varphi_i^p$ and E_i' is an upper-triangular matrix similar to E_i (thus E_i' also is the coefficient matrices of the linear transformation $\frac{1}{2\pi i}\ln\sigma_i^*$, but in another basis than E_i, — in the basis Y_i). ($(\widetilde{x - a_i})^{E_i'}$ is interpreted in a natural way; see [1] where we wrote $(\tilde{x} - a_i)^{E_i}$.) Further, it turns out that Fuchsian case is just the case when $\det V_i(a_i) \ne 0$. We call Y_i a Levelt basis for (near) a_i.

For our current purposes we can understand I) as the statement about the existence of Y_i, V_i, Φ_i and E_i' with the properties indicated. From this point of view the only thing which is stated about Φ_i is that it is a diagonal matrix with integer entries ordered in such a way that φ_i^j decrease when j increases and that Φ_i is defined in a unique way (while there can be some freedom in the choice of Y_i, V_i and E_i'). Then we consider φ_i^j as being by definition diagonal elements of Φ_i which existence is asserted in I). The fact that they coincide with some numbers defined in another way can be considered as a separate theorem (which we do not need).

II). For any regular system

$$\sum_{i,j}\beta_i^j \le 0 \;\text{ and }\; \sum_{i,j}\beta_i^j = 0 \Leftrightarrow \text{ the system is Fuchsian.} \tag{7}$$

The latter theorem is not of the purely local character — it involves β_i^j for various a_i, —- although its proof does not require the serious discussion of connections between local and global theories. However, for some other purposes this discussion is necessary, and we shall turn to it.

One has to tie $\widetilde{U_i^*}$ and $\tilde{S}$. If $n > 2$, $\pi^{-1}U_i^*$ (which is a subset of $\tilde{S}$) has infinitely many connected components which are all isomorphic (as a spaces covering U_i^*) to $\widetilde{U_i^*}$. After we identify one of these components with $\widetilde{U_i^*}$, we can consider for $y \in \mathfrak{V}$ their restrictions on $\widetilde{U_i^*}$ and in this way we get a map

$$\mathfrak{V} \to \mathfrak{V}_i \qquad y \mapsto y|\widetilde{U_i^*}$$

which, of course, is an isomorphism of two p-dimensional complex vector spaces. (This linear map is non-degenerate because the only holomorphic function on $\widetilde{S}$ which is identically zero on $\widetilde{U_i^*}$ is 0 everywhere.) After this we can consider σ_i^* as acting on $\mathfrak{V}$. Once more: this interpretation of σ_i^* as elements of $GL(\mathfrak{V})$ depends upon the choice of the identification of $\widetilde{U_i^*}$ with one of the connected components of $\pi^{-1}(U_i^*)$.

In order to describe various identifications in more detail [4], it is useful to appeal to the continuous paths in S. The product $\alpha\beta$ of two such paths is defined in a well-known way, if the end of α coincides with the beginning of β. (We simply pass first α, then β.) If α and β vary continuously, but their endpoints remain fixed, $\alpha\beta$ also varies continuously with the fixed endpoints, thus the multiplication is correctly defined for some classes each of which consists of paths which are mutually homotopy equivalent as paths with fixed endpoints; one can speak about the product $[\alpha][\beta]$ of such classes if the end of the paths from $[\alpha]$ (this end is the same for all paths from this class, that is why we say "end", not "ends") coincides with the beginning of the paths from $[\beta]$. Fix a base point $b \in S$ and consider classes of all paths beginning at b. It is well known that one can introduce a topology in this set so that this set together with its evident projection on S (the projection maps every class simply to the end of its paths) becomes the universal covering space $\widetilde{S}$. In these terms the fundamental group $\pi_1(S, b)$ acts on the $\widetilde{S}$ from the left as

$$([\alpha], [\beta]) \mapsto [\alpha\beta].$$

It turns out that in this way one gets all the deck transformations thus establishing an isomorphism $\pi_1(S, b) \approx \Delta$. Identification of $\widetilde{S}$ with the space of classes of paths to some extent depends on the choice of b and for this reason generally one does not get a unique isomorphism $\pi_1(S, b) = \Delta$ which would be the same for all b. It is well known that various isomorphisms defined by this way differ by the conjugation: if via our construction we get isomorphisms

$$j_1 : \pi_1(S, b_1) \to \Delta, \quad j_2 : \pi_1(S, b_2) \to \Delta,$$

then there exists such path α from b_1 to b_2 that for all $[\beta] \in \pi_1(S, b_1)$

$$j_2([\beta]) = j_1([\alpha][\beta][\alpha]^{-1}).$$

Analogously, if we fix a base point $b_i \in U_i^*$, then we get an isomorphism $\pi_1(U_i, b_i) = \Delta_i$. Because this group is commutative, now the isomorphism does not depend on various choices made in the process of construction. Now, if we choose a class of paths α_i beginning in b and ending at b_i, we get an inclusion

$$\widetilde{U_i^*} \to \widetilde{S} \qquad [\beta] \mapsto [\alpha_i][\beta].$$

Note that σ_i corresponds to a path β_i in U_i^* from b_i to b_i going around a_i and making one turn in positive direction (counterclockwise for $a_i \neq \infty$ and clockwise for $a_i = \infty$) or, to

[4] See also [1].

be more precise, to the class $[\beta_i]$. Namely, for $[\gamma] \in \widetilde{U_i^*}$ (which means that γ lies in U_i^* and begins in b_i)

$$\sigma_i[\gamma] = [\beta_i][\gamma].$$

Now the natural way to make σ_i acting on $\widetilde{S}$ is to pass to the element $[\alpha_i][\beta_i][\alpha_i]^{-1} \in \pi_1(S,b)$ and to consider its action on $\widetilde{S}$. In any case, for elements of $\widetilde{U_i^*}$ considered as being inclosed to $\widetilde{S}$ (with the aid of the same $[\alpha_i]$, of course) this leads to the same result as before. After we replaced σ_i by elements of Δ which we shall still denote by σ_i, we can take usual σ_i^*. They turn out to be the same σ_i^* as considered earlier.

Various inclusions $\widetilde{U_i^*} \hookrightarrow \widetilde{S}$ constructed in this way (with various $[\alpha_i]$) can be obtained from a fixed inclusion by deck transformations. Namely, if the previous inclusion (obtained via using $[\alpha_i]$) is j and another inclusion (obtained via using $[\alpha_i']$) is j', then for all $\widetilde{x} \in \widetilde{U_i^*}$

$$j'(\widetilde{x}) = \delta j(\widetilde{x}), \quad \text{where } \delta = [\alpha_i'][\alpha_i]^{-1}$$

(expression for δ is written having in mind the previous identification of Δ with $\pi_1(S,b)$). If we use a third path $\alpha_i'' = \delta'\alpha_i$ and get the inclusion j'', then

$$j'\widetilde{U_i^*} = j''\widetilde{U_i^*} \quad \Leftrightarrow \delta' \in \delta\Delta_i$$

(having in mind the previous $\Delta_i \hookrightarrow \Delta$), so the connected components of $\pi^{-1}U_i^*$ can be "numbered" by the elements of Δ/Δ_i. Various inclusions $\Delta_i \hookrightarrow \Delta$ constructed in this way can be obtained from each other by conjugations in the group Δ.

Geometrically a rather pictorial description of Δ and $\pi_1(S,b)$ is the following. Let first all $a_i \neq \infty$. Take $b \in \mathbb{C}$ outside of all direct lines a_ia_j. Then one can number our a_i in such a way that if we take a ray with the end in b and passing through b_1 and rotate it counterclockwise making one turn around a_i, then it meets points a_2, …,a_n in the same order in which they are numbered. Take b_i on this ray when it passes through a_i between b and a_i, and take α_i to be the path from b to a_i along the interval ba_i. If one of a_i is ∞, this is modified as follows. We can assume that ∞ is a_n. Choice of b, the numbering of remaining points $a_1, \dots, a_{n-1}$, points $b_1, \dots, b_{n-1}$ and paths $\alpha_1, \dots, \alpha_{n-1}$ are as before with $n-1$ instead of n. Take a ray with the vertex at b lying in the angle which is bounded by the rays ba_1 and ba_{n-1} and which does not contain other points a_i. Take a point b_n lying on this ray sufficiently far (outside the disk D figuring in U_i‘$n^* = \overline{\mathbb{C}} \setminus D$) and take α_n to be a path from b to b_n along bb_n. Under such choice of the objects figuring in our construction, $\pi_1(S,b)$ can be described as the group generated by σ_1,…,σ_n which satisfy the only relation

$$\sigma_1 \dots \sigma_n = 1$$

(thus Δ is the free group with $n-1$ generators). In order to define its representation in $\mathrm{GL}(p,\mathbb{C})$, one has to choose n non-degenerate matrices $G_1, \dots, G_n$ satisfying

$$G_n \dots G_1 = I \tag{8}$$

(where I is the identity matrix; the identity transformation will be also denoted by I) and let $\chi(\sigma_i) := G_i^{-1}$.

A representation is called a B-representation (in honor of Bolibruch; this was suggested by Yu.S.Il'yashenko), if

a) χ is reducible;

b) $\chi(\sigma_i)$ are similar to one Jordan block.

(Condition b) does not depend upon a concrete choice of inclusion $\Delta_i \hookrightarrow \Delta$: different inclusions are conjugate to each other and so matrices $\chi(\sigma_i)$ for two different inclusions are similar, thus they have the same Jordan normal form.) In this case we can write λ_i instead of λ_i^j.

A regular system is called B-system if its monodromy representation is a B-representation. We shall prove

Theorem. *For a Fuchsian B-system*

$$\lambda_1 \dots \lambda_n = 1. \tag{9}$$

This theorem is due to A.A.Bolibruch [2]. The proof given below is essentially the same, but its exposition seems to be somewhat simplified.

Then in section 3 we shall give an example of a B-representation for which this product is $\neq 1$. Such representation cannot be realized by a Fuchsian system, independently of concrete choice of a_i.

Remarks. 1). There exist other examples where some representation can be realized by a Fuchsian system for one choice of a_i and cannot for another.

2). This theorem gives a necessary condition for a B-representation to be realizable as a monodromy representation for a Fuchsian system. This condition is not sufficient. If $p = 3$, then any B-representation satisfies this condition, while there exist non-realizable B-representations with $p = 3$. Indeed, such was the first counterexample discovered.

3). In the case $p = 3$, all the representations $\Delta \to \mathrm{GL}(p, \mathbb{C})$ which cannot be realizable as monodromy representations of Fuchsian systems are B-representations. In this case necessary and sufficient conditions for realizability are known [1].

4). There exist representations (with $p > 3$) which are non-realizable and are not B-representations. For $p = 4$ such example was given in [3]. It can be easily extended to any bigger p.

2. Proof of the theorem. Assume we have a B-representation χ which is realized by a Fuchsian system (1) (under some choice of the fundamental system of solutions Y of the latter). Then in the space $\mathfrak{V}$ we have the equivalent representation $\sigma \mapsto \sigma^*$ (see (3) and (4)).

a) means that in the space $\mathfrak{V}$ there exists a subspace $\mathfrak{Y}$ which is invariant with respect to the representation $\sigma \mapsto \sigma^*$. Let $l := \dim \mathfrak{Y}$. Let $y_1, \dots, y_p$ be a basis in $\mathfrak{V}$ such that $y_1, \dots, y_l \in \mathfrak{Y}$. $Y^1 := (y_1, \dots, y_l)$ is a row consisting of l elements which are vectors from $\mathbb{C}^p$, i.e. columns consisting of p elements (the latter being some functions of $\tilde{x}$; $Y^1(\tilde{x}) = (y_1(\tilde{x}), \dots, y_l(\tilde{x}))$ is a $p \times l$-matrix consisting of numbers). In this basis χ has the form

$$\chi(\sigma) = \begin{pmatrix} \chi_1(\sigma) & * \\ 0 & * \end{pmatrix}$$

where χ_1 is also a representation of the deck transformations group — this times a representation by $(p \times p)$-matrices. Then

$$Y^1(\sigma\tilde{x}) = Y^1(\tilde{x})\chi_1(\sigma^{-1}). \tag{10}$$

b) implies that for any i there exists a filtration in $\mathfrak{V}$ invariant with respect to σ_i^*:

$$\{0\} = \mathfrak{V}_i^0 \subset \mathfrak{V}_i^1 \subset \ldots \subset \mathfrak{V}_i^p = \mathfrak{V}$$

such that dim $\mathfrak{V}_i = i$. Namely,

$$\mathfrak{V}_i^j = \text{Ker } (\sigma_i^* - \lambda_i I)^j.$$

The following statement is a general statement for linear operators $\mathcal{A} : \mathfrak{V} \to \mathfrak{V}$ which are similar to one Jordan block. Let $\mathfrak{Z}$ be a k-dimensional invariant subspace for $\mathcal{A}$. Then $\mathfrak{Z} = \text{Ker } (\mathcal{A} - \alpha I)^k$, where α the the eigenvalue of $\mathcal{A}$. Indeed, $(\lambda - \alpha)^k$ is the characteristic polynomial of $\mathcal{A}|\mathfrak{Z}$. (It must divide the characteristic polynomial $(\lambda - \alpha)^p$ of $\mathcal{A}$ and be of degree k.) Thus

$$(\mathcal{A} - \alpha I)^k \mathfrak{Z} = 0 \quad \text{and so } \mathfrak{Z} \subset \text{Ker } (\mathcal{A} - \alpha I)^k.$$

But the latter spaces are both k-dimensional.

In particular, $\mathfrak{Y} = \mathfrak{V}_i^l$, — both are l-dimensional subspaces invariant for σ_i^*.

Let $y_{i1}, \ldots, y_{ip}$ be a Levelt basis for a_i. Then $\sum_{j=1}^l \oplus \mathbb{C} y_{ij}$ is an l-dimensional invariant subspace, so

$$\sum_{j=1}^{l} \oplus \mathbb{C} y_{ij} = \mathfrak{V}_i^l = \mathfrak{Y} = \sum_{j=1}^{l} \oplus \mathbb{C} y_j\,,$$

and for all i

$$(y_1, \ldots, y_l) = (y_{i1}, \ldots, y_{il}) C_i \quad \text{with some constant matrix } C_i.$$

For a polyvector $w \in \Lambda^l(\mathbb{C}^p)$ denote by $w^{i_1 \ldots i_l}$ its coordinates in the basis

$$\{e_{j_1} \wedge \ldots \wedge e_{j_l}\}, \qquad 1 \le j_1 < \ldots < j_l \le p$$

of the space $\Lambda^l(\mathbb{C}^p)$; here $e_1, \ldots, e_p$ is the standard basis of $\mathbb{C}^p$. If the polyvector is a simple (decomposable) one, say $w = w_1 \wedge \ldots \wedge w_l$, then the coordinates

$$w^{i_1 \ldots i_l} = \det\, (w_j^{i_h}) j, h = 1, \ldots, l$$

(where w_i^j are the coordinates of w_i) are the Plücker coordinates of the space $\sum \oplus \mathbb{C} w_i$ (spanned by $w_1, \ldots, w_l$). For any $\tilde{x} \in \widetilde{S}$ we have already introduces some $Y^1(\tilde{x}) = (y_1(\tilde{x}), \ldots, y_l(\tilde{x}))$; introduce now the polyvector

$$w(\tilde{x}) := y_1(\tilde{x}) \wedge \ldots \wedge y_l(\tilde{x})$$

with coordinates $w^{i_1 \ldots i_l}(\tilde{x})$. Note that

$$w^{j_1 \ldots j_l}(\sigma \tilde{x}) = w^{j_1 \ldots j_l}(\tilde{x}) \det \chi_1(\sigma^{-1}).$$

Let $w^{j_1 \ldots j_l} \not\equiv 0$. If $w^{j_1 \ldots j_l}(\tilde{x}) = 0$, then all $w^{j_1 \ldots j_l}(\sigma \tilde{x}) = 0$. Thus singular points and zeroes for $w^{j_1 \ldots j_l}$ are: possibly $a_1, \ldots, a_n$ and perhaps some other points

$$\underbrace{b_1, \ldots, b_m}_{only\ zeroes}.$$

(We write b_j, not $\widetilde{b}_j$, because zeroes are in all points of $\widetilde{S}$ covering b_j.) Now

$$d\ln w^{j_1\ldots j_l}(\sigma\widetilde{x}) = d\ln w^{j_1\ldots j_l}(\widetilde{x}) + \underbrace{d\ \mathrm{const}}_{=0},$$

and we see that the form

$$\omega := d\ln w^{j_1\ldots j_l}(\widetilde{x}) = \text{ 1-form on } S' := S\setminus\{b_1,\ldots,b_m\}.$$

ω is a form with only finite number of singularities (all of them being poles). Sum of all residues of such form equals to zero:

$$\sum \mathrm{res}_{a_i}\omega + \sum \mathrm{res}_{b_j}\omega = 0.$$

Let $k_j := \mathrm{res}_{b_j}\omega$; this number equals to the order of zero of $w^{j_1\ldots j_l}$ at the point b_j. All $k_j \geq 0$. Now consider $\mathrm{res}_{a_i}\omega$. Since some fundamental system of solutions (the system which is Levelt system at a_i) has the form (6) with E_i' having all diagonal elements equal to μ_i, we have

$$(y_{i1}(\widetilde{x}),\ldots,y_{il}(\widetilde{x})) = (v_{i1}(x),\ldots,v_{il}(x))T,$$

where T is an upper-triangular matrix with diagonal elements $t_{jj} = (\widetilde{x-a_i})^{\varphi_i^j+\mu_i}$. Then

$$y_{i1}\wedge\ldots\wedge y_{il} = t_{11}\ldots t_{ll}v_{i1}\wedge\ldots\wedge v_{il}.$$

Consequently, in $\widetilde{U}^*$

$$\begin{gathered} y_1\wedge\ldots\wedge y_l = cy_{i1}\wedge\ldots\wedge y_{il} = \\ = c(\widetilde{x-a_i})^{\varphi_i^1+\ldots+\varphi_i^l+l\mu_i}v_{i1}\wedge\ldots\wedge v_{il} \end{gathered}$$

(with $c =$ const.)

Assume $(v_{i1}\wedge\ldots\wedge v_{il})^{j_1\ldots j_l}$ has a zero of order l_j at a_i. Then

$$d\ln(y_{i1}\wedge\ldots\wedge y_{il})^{j_1\ldots j_l} = (l_i+\varphi_i^1+\ldots+\varphi_i^l+l\mu_i)\frac{dx}{x}+\ldots,$$

$$\mathrm{res}_{a_i}\omega = l_i+\varphi_i^1+\ldots+\varphi_i^l+l\mu_i, \tag{11}$$

If at least one of the numbers k_i or l_j is >0, then

$$0 > \sum(\varphi_i^1+\ldots+\varphi_i^l+l\mu_i).$$

But in the Fuchsian case $\sum_{i,j}\beta_i^j = 0$, and as $\beta_i^j = \varphi_j+\mu_i$, it follows that

$$\frac{1}{p}\sum(\varphi_i^1+\ldots+\varphi_i^p+p\mu_i) = 0. \tag{12}$$

Left hand side of (12) is $\leq \frac{l}{p}\sum(\varphi_i^1+\ldots+\varphi_i^l+l\mu_i)$, as φ_i^j decrease as j increases. Here strict inequality holds, if there exists such i that $\varphi_i^1 \neq \varphi_i^p$. Indeed, in order to prove that

$$\varphi_i^1 \neq \varphi_i^p \Rightarrow \text{ left hand side } < 0,$$

write the equivalent formulas

$$p(\varphi_i^1 + \ldots + \varphi_i^l) > l(\varphi_i^1 + \ldots + \varphi_i^p), \tag{?}$$

$$(p-l)(\varphi_i^1 + \ldots + \varphi_i^l) > l(\varphi_i^{l+1} + \ldots + \varphi_i^p), \tag{?}$$

$$\sum_{j=1}^{l} [(p-l)\varphi_i^j - \varphi_i^{l+1} - \ldots - \varphi_i^p] > 0, \tag{?}$$

All summands are ≥ 0 and the first summand $[\ldots]$ is > 0.

So, if either one of the numbers k_i or l_j is > 0 or one of $\varphi_i^1 \neq \varphi_i^p$, we arrive at the contradiction. We have proved that in the Fuchsian case all $k_i = 0$ (i.e. ω has no zeroes outside of the set of singular points), all $l_i = 0$ (i.e.

$$(u_{i1} \wedge \ldots \wedge u_{il})^{j_1 \ldots j_l} \neq 0 \quad \text{at } a_i)$$

and all φ_i^j depend only on i. We shall denote the latter numbers simply by φ_i. We already know that $\mu_i^j = \mu_i$ depend only on i, thus the same is correct for β_i^j and we can write simply β_i.

Now (7) reduces to $\sum p(\varphi_i + \mu_i) = 0$, i.e. $\sum(\varphi_i + \mu_i) = 0$. As φ_i are integers, $\sum \mu_i \in \mathbb{Z}$. But $\lambda_i = e^{2\pi i \mu_i}$, and we arrive at (9).

Remark. Considering the Plücker coordinates $w^{j_1 \ldots j_l}(\tilde{z})$ of the linear subspace

$$\mathfrak{Y}(\tilde{x}) := \sum_{i=1}^{l} \oplus \mathbb{C} y_i(\tilde{x})$$

one can also prove that in our case $\mathfrak{Y}(\tilde{x}) =$ const (does not depend on $\tilde{x}$). This is not needed here, but this is an interesting general property of Fuchsian B-systems, and it is used in the proofs of the non-realizability of some B-representations.

If we are going to prove that $\mathfrak{Y}(\tilde{x}) =$ const, it is natural to use the Plücker coordinates of this space and (considering them as functions of $\tilde{x}$) to pay attention to their zeroes and singularities. In the proof of our theorem it does not seem to be so natural. It seems as if the proof arised as a by-product of the proof of the statement $\mathfrak{Y}(\tilde{x}) =$ const (although actually the history was more complicated). However, if we take the statement $\mathfrak{Y}(\tilde{x}) =$ const as granted, we can prove our theorem in a somewhat different way, just appealing to the "$\Leftarrow$" part of (7) which is applied to our original system (Fuchsian system realizing χ) and also to the l-th order subsystem which after an appropriate change of coordinates in $\mathbb{C}^p$ allowing to consider $\mathfrak{Y}(\tilde{x})$ as the space $y_{l+1} = \ldots = y_p = 0$ splits off from this system and which realizes the representation χ_1. But in the proof of the "$\Leftarrow$" part of (7) one considers the differential 1-form $d \ln \det Y(\tilde{x})$ which for the new (smaller) system is just $d \ln w^{1 \ldots l}(\tilde{x})$; so actually in this argument we also use our $d \ln w^{i_1 \ldots i_l}(\tilde{x})$, only this is hidden in the proof of (7) and only one special array of $i_1, \ldots, i_l$ is taken.

3. An example. Finally, we shall give an example of the B-representation violating (9). In this example $n = 3, p = 4$. χ is completely defined by giving three matrices G_1, G_2, G_3

such that $G_3G_2G_1 = I$. All three matrices will have the following block structure with (2×2)-blocks:

$$\begin{pmatrix} A & * \\ 0 & A \end{pmatrix}.$$

So representation will be reducible with $\mathfrak{Y} = \mathbb{C}e_1 \bigoplus \mathbb{C}e_2$. For G_1 we write A, for G_2 — B instead of A and for G_3 — C. A, B, C will be similar to two-dimensional Jordan blocks with eigenvalue, correspondingly, -1, 1, 1. G_i will have the same eigenvalues, and (9) certainly will be violated. For "generic" blocks $*$ matrix $G_i - \lambda_i I$ will have rank 3, so we have a good chance to find suitable blocks $*$. Finally, we must ascertain that

$$G_3G_2G_1 = I. \tag{13}$$

We have

$$\begin{pmatrix} C & * \\ 0 & C \end{pmatrix}\begin{pmatrix} B & * \\ 0 & B \end{pmatrix}\begin{pmatrix} A & D \\ 0 & A \end{pmatrix} = \begin{pmatrix} CBA & CBD + * \\ 0 & CBA \end{pmatrix},$$

where we need not write explicitly $*$'s; we need only to observe that in $CBD + *$ the $*$ does not depend on F. Thus (13) is equivalent to

$$CBA = I \tag{14}$$

and

$$CBD + * = 0. \tag{15}$$

If the first of them is satisfied, we can satisfy the second one taking $D = -(CB)^{-1}*$. Of course one has also to check whether G_i will be equivalent to one Jordan block, but, as it was already said, other cases which can happen seem to be exceptional, so more or less any choice of $*$ in G_1 and G_2 has a good chance to be successful.

The problem of finding (2×2)-matrices A, B, C satisfying the above-mentioned conditions is an exercise on linear algebra which is perhaps different from the exercises usually given to the students and which in this sense is unusual, but which is not a very difficult one. And the choice of the upper right blocks of two matrices can be more or less accidental (for the third matrix the corresponding block will be then uniquely defined). I shall not dwell on it and simply write the matrices G_i which were suggested by Bolibruch:

$$G_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(so G_3 is already a Jordan block with the eigenvalue 1),

$$G_2 = \begin{pmatrix} 3 & 1 & 15 & 4 \\ -4 & -1 & -19 & -5 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -4 & -1 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & -1 \end{pmatrix}.$$

Thus we take

$$A = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}, B = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is almost evident that A, B, C have the desired Jordan normal form; it is easy to check that (14) holds; checking of (15) needs more calculations, but of course they are made automatically. Finally, it is easy to see that all Rk $(G_i - \lambda_i I) = 3$.

References

[1] D.V.Anosov, A.A.Bolibruch, The Riemann-Hilbert Problem. — A Publication from the Steklov Institute of Mathematics. Vieweg, Braunschweig; Wiesbaden, 1994, pp.1-190.

[2] A.A.Bolibruch, Holomorphic Bundles, Associated with Linear Differential Equations and the Riemann–Hilbert Problem, in : B.L.J.Braaksma, G.K.Immink, M.van der Put (editors), The Stokes Phenomenon and Hilbert's 16th Problem, Groningen, The Netherlands 31 May–3 June 1995. World Scientific, Singapore-New Jersey-London-Hong Kong, p. 51–70.

[3] A.A.Bolibruch, On the problem of Existence of Fuchsian Systems with Given Asymptotics, in: Proc. of the Steklov Inst. of Math., v. 216, 1996, p. 26–37.