BIRATIONAL RIGIDITY IS NOT AN OPEN PROPERTY

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ABSTRACT. We construct an example of the birationally rigid complete intersection of a quadric and a cubic in $\mathbb{P}^5$ with an ordinary double point, which under a small deformation gives a non-rigid Fano variety. Thus we show that birational rigidity is not open in moduli.

1. INTRODUCTION.

In this article we discuss the question, which is closely related to the nature of the birational rigidity notion: whether birational rigidity is open in moduli, or not?

We consider all varieties to be defined over an algebraically closed field of characteristic 0 (e.g., over $\mathbb{C}$). We recall that a triple $\mu : V \to S$ is said to be a *Mori fibration* (also Mori fiber space) if V is a projective $\mathbb{Q}$-factorial terminal threefold, S is a projective normal variety with $\dim S < \dim X$, and μ is an extremal contraction of fibering type, i.e., the relative Picard number $\rho(V/S) = \mathrm{rk}\,\mathrm{Pic}(V) - \mathrm{rk}\,\mathrm{Pic}(S)$ is equal to 1 and $(-K_V)$ is μ-ample. Abusing the notation, we will denote Mori fibrations also $V \to S$, V/S, or simply V when the corresponding contractions or bases are clear. We have the following possibilities for Mori fibrations: Fano varieties ($\dim S = 0$), del Pezzo fibrations ($\dim S = 1$), and conic bundles ($\dim S = 2$).

We say that a birational map $\chi : X \dashrightarrow X'$ of two Mori fibrations $X \to S$ and $X' \to S$ is *square* if it fits into a commutative diagram

$$\begin{array}{ccc} X & \overset{\chi}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \\ S & \overset{\psi}{\longrightarrow} & S' \end{array}$$

where ψ is a birational map, and moreover, χ induces an isomorphism $X_\eta \xrightarrow{\simeq} X'_\eta$ of the fibers over the generic point η (or, which is the same, χ induces isomorphisms of general fibers).

DEFINITION 1.1. *A Mori fibration X/S is said to be birationally rigid, if for any birational map $\varphi : X \dashrightarrow X'$ to another Mori fibration X'/S' there is a birational map $\mu \in \mathrm{Bir}(X)$ such that the composition*

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$\varphi \circ \mu : X \dashrightarrow X'$ is square, and X/S is birationally superrigid, if μ can be chosen biregular (or simply an identity map).

Consider a Fano variety X and a birational map $\varphi : X \dashrightarrow X'$ to a Mori fibration X'/S'. If X is (birationally) rigid, the definition says that X' is isomorphic to X and φ can be viewed as a birational automorphism. If X is superrigid, then φ is an isomorphism itself.

One of the most common conjecture about birationally rigid varieties was that small deformations keep the rigidity (e.g., [3], conjecture 1.4): given any scheme T, and a flat family of Mori fibrations $\mathcal{X} \to \mathcal{S}$ parameterized by T, the set of all $t \in T$ such that the corresponding fiber $\mathcal{X}_t \to \mathcal{S}_t$ is birationally rigid, is open in T (possibly empty). In other words, birational rigidity is open in moduli. Up to this moment, all known examples of rigid varieties satisfy this conjecture.

Nevertheless, in this paper we show that the conjecture falls. Our counter-examples are based on degenerations of the complete intersections of quadrics and cubics in $\mathbb{P}^5$. It is known ([8], [9]) that a general non-singular Fano variety of this kind is rigid. We construct a special class of the singular complete intersections that are generally non-rigid, but it contains a sub-class of rigid varieties. Our consideration is based on ideas of V.A.Iskovskikh and A.V.Pukhlikov's works. At the last section we give counter-examples in the class of del Pezzo fibrations.

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2. Main result.

We construct the essential family of the singular complete intersections of quadrics and cubics as follows. Let $X \subset \mathbb{P}^4$ be a quartic given by the equation

$$ht - q_1 q_2 = 0, \tag{2.1}$$

where h, t, q_1, and q_2 are homogeneous polynomials of degree 1, 3, 2, and 2 respectively, in the coordinates $[y_0 : y_1 : \ldots : y_4]$ in $\mathbb{P}^4$. We assume that X has the only 12 different ordinary double points given by the equalities

$$h = t = q_1 = q_2 = 0.$$

It is easy to see that the hyperplane $\{h = 0\}$ cuts off two quadratic surfaces $S_1 = \{h = q_1 = 0\}$ and $S_2 = \{h = q_2 = 0\}$. We have

$$\begin{aligned} \mathrm{Pic}(V) &= \mathbb{Z}[-K_X], \\ \mathrm{Cl}(V) &= \mathbb{Z}[-K_X] \oplus \mathbb{Z}[S_1], \end{aligned}$$

thus X is not $\mathbb{Q}$-factorial, so it does not belong to a category of Mori fibrations. We can get two Mori fibrations that are birational to X, using two "unprojections" as follows. Let $[y_0 : y_1 : \ldots : y_5]$ be the coordinates in $\mathbb{P}^5$, and consider varieties V_1 and V_2 defined by the equations

$$(2.2) \qquad V_1 = \begin{cases} y_5 h &= q_1 \\ y_5 q_2 &= t \end{cases}, \quad V_2 = \begin{cases} y_5 h &= q_2 \\ y_5 q_1 &= t \end{cases}$$

Note that the polynomials h, t, q_1, and q_2 do not depend on y_5, and we get the equation (2.1) by excluding the variable y_5 from the equations for V_1 or V_2. In other words, we can get X from V_1 or V_2 by the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ from the point $\xi = (0 : \ldots : 0 : 1)$).

It is easy to check that under a general choice of the corresponding polynomials the varieties V_1 and V_2 are Fano varieties with a unique ordinary double point ξ. The birational maps $\varphi_1 : V_1 \dashrightarrow X$, $\varphi_2 : V_2 \dashrightarrow X$, and $\psi : V_1 \dashrightarrow V_2$ can be described as follows. Let $\alpha_1 : \tilde{V}_1 \to X$ be the blow-up of the quadric S_2. Thus α_1 is the small resolution of singularities of X with the exceptional lines $\tilde{l}_1, \ldots, \tilde{l}_{12}$. Denote $\tilde{S}_1$ and $\tilde{S}_2$ the strict transforms of the corresponding quadratic surfaces. The variety $\tilde{V}_1$ is non-singular, and the quadric $\tilde{S}_2$ has the normal sheaf $\mathcal{O}(-1)$. So there exists the divisorial contraction $\beta_1 : \tilde{V}_1 \to V_1$ of the exceptional divisor $\tilde{S}_2$. Now we set $\varphi_1 = \alpha_1 \circ \beta_1^{-1}$. The birational map φ_2 is constructed in the same way as φ_1, but first we blow up the quadric S_2. Finally, there exists a flop $\gamma : \tilde{V}_1 \dashrightarrow \tilde{V}_2$ centered at all 12 lines $\tilde{l}_1, \ldots, \tilde{l}_{12}$ simultaneously, and we have $\psi = \beta_2 \circ \gamma \circ \beta_1^{-1} : V_1 \dashrightarrow V_2$.

Denote $\mathcal{F}$ a family of all the complete intersections of quadrics and cubics in $\mathbb{P}^5$ that are constructed like V_1 and V_2. So we see that each variety $U_1 \in \mathcal{F}$ has its birational "counterpart" U_2 like V_1 and V_2, and U_1 is not isomorphic to U_2 in general. Indeed, any isomorphism of U_1 and U_2 is induced by an automorphism of $\mathbb{P}^5$ that keep fixed the singular point $(0 : \ldots : 0 : 1)$. Thus the projection from this point induces also an involution of $\mathbb{P}^4$. The equation of the corresponding quartic is invariant with respect to the involution, which change the places of q_1 and q_2 in (2.1). It is impossible in general case (e.g., we can choose an equation for h which is not invariant with respect to any such an involution).

Now we construct the subfamily $\mathcal{F}_r \subset \mathcal{F}$. Consider a reflection $\iota : \mathbb{P}^4 \to \mathbb{P}^4$ given by the following action:

$$(2.3) \qquad \begin{cases} y_0 \to -y_0 \\ y_i \to y_i, \qquad i = 1 \ldots 4 \end{cases}$$

We continue the action of ι assuming $y_5 \to y_5$. Let h and t be homogeneous polynomials of degree 1 and 3 in the variables $y_1, \ldots, y_4$, and q be a homogeneous polynomial of degree 2 in the variables $y_0, \ldots, y_4$. Thus h and t are invariant with respect to the reflection ι. Consider a quadric $X \subset \mathbb{P}^4$ given by the equation

$$ht - q\iota^*(q) = 0.$$

We can choose the birational modifications of X as before, and get Fano varieties

$$V_1 = \begin{cases} y_5 h &= q \\ y_5 \iota^*(q) &= t \end{cases}, \quad V_2 = \begin{cases} y_5 h &= \iota^*(q) \\ y_5 q &= t \end{cases} \tag{2.4}$$

We see immediately that $\iota(V_1) = V_2$, i.e., V_1 and V_2 are isomorphic, and the birational map $\psi : V_1 \dashrightarrow V_2$ is actually a birational automorphism.

Let us fix the reflection ι, and suppose $\mathcal{F}_r$ consisting of varieties of the kind (2.4). Obviously, $\mathcal{F}_r \subset \mathcal{F}$, and we can make an important observation that varieties from $\mathcal{F}$ can be viewed as small deformations of varieties from $\mathcal{F}_r$.

Now we describe some conditions of generality for varieties of these two families. Let Q and T be a quadric and a cubic in $\mathbb{P}^5$, and V their complete intersection with a unique ordinary double point ξ. We assume that V satisfies the following additional conditions of generality:

- $\mathcal{G}_1$: the quadric Q is non-singular;
- $\mathcal{G}_2$: if $l \subset V$ is a line and $P \in \mathbb{P}^5$ is a plane that contains l, then the scheme-theoretical intersection $V \cap P$ is reduced along l (in case $\xi \notin l$ it means that the normal sheaf $\mathcal{N}_{l|V} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$, see [9], chapter 3, proposition 1.1);
- $\mathcal{G}_3$: for any plane $P \subset \mathbb{P}^5$, the intersection $V \cap P$ is not three lines with a common point, and if $\xi \in P$, it does not consist of any three lines;
- $\mathcal{G}_4$: given $l \subset L \subset \mathbb{P}^5$, where $l \subset V$ is a line through the point ξ and L is a three-dimensional subspace such that $Q|_L$ consists of two planes (with the common line l), then L is not the tangent space to V for any point in $l \setminus \{\xi\}$;
- $\mathcal{G}_5$: let $l \subset V$ be a line, $\xi \in l$, then for each point $B \in l \setminus \{\xi\}$ there are not more than 3 lines passing through B (including l itself).

Our main result is the following:

THEOREM 2.1. *(i) A general variety from $\mathcal{F}_r$ or $\mathcal{F}$ satisfies the conditions $\mathcal{G}_1, \ldots, \mathcal{G}_5$.*
(ii) Let V_1 and V_2 be general varieties from $\mathcal{F}$ or $\mathcal{F}_r$ given by the equations (2.2). Then V_1 and V_2 are unique Mori fibrations in their class of birational equivalence, and they are isomorphic if they belong to the subfamily $\mathcal{F}_r$.

From this theorem we can deduce immediately:

COROLLARY 2.2. *Birational rigidity is not open in moduli.*

Indeed, by the part (ii) of theorem 2.1, general varieties from the family $\mathcal{F}_r$ are birationally rigid, but general varieties from $\mathcal{F}$ are non-rigid. So we conclude that birational rigidity is not kept by small deformations.

Thus, it is proved that a general $V \in \mathcal{F}$ has exactly 2 different models of Mori fibration, both of them are Fano varieties. The first example of this kind was constructed by A.Corti and M.Mella in [5]. From the viewpoint of their work, general varieties from $\mathcal{F}$ have the pliability to be equal to 2, and equal to 1 for varieties from $\mathcal{F}_r$.

The further exposition is organized as follows. We prove the part (i) of theorem 2.1 in section 3, and the part (ii) in sections 4, 5, and 6. We conclude with some further examples in section 7.

3. Generality conditions.

In this section we prove the part (i) of theorem (2.1) in lemmas 3.3 and 3.4. Is is clear that it is enough to prove the generality conditions $\mathcal{G}_1, \ldots, \mathcal{G}_5$ only for varieties from $\mathcal{F}_r$, the result for $\mathcal{F}$ will follow automatically.

Let $[y_0 : y_1 : \ldots : y_5]$ be the coordinates in $\mathbb{P}^5$. We assume the reflection ι to be defined by

$$\begin{cases} y_0 \to -y_0 \\ y_i \to y_i, \qquad i = 1, \ldots, 5. \end{cases}$$

The set of fixed points of ι are the plane $F = \{y_0 = 0\}$ and the point $\zeta = (1 : 0 : \ldots : 0)$. We choose the polynomials h, t, q as follows:

$$\begin{aligned} h(y_*) &= \textstyle\sum_{i=1}^4 \alpha_i y_i, \\ q(y_*) &= Ay_0^2 + y_0 \textstyle\sum_{i=1}^4 \beta_i y_i + \sum_{1\le i\le j\le 4} \gamma_{ij} y_i y_j, \\ t(y_*) &= \textstyle\sum_{1\le i\le j\le k\le 4} \delta_{ijk} y_i y_j y_k + y_0^2 \sum_{i=1}^4 \varepsilon_i y_i. \end{aligned}$$

Assuming a variety $V \in \mathcal{F}_r$ to be the complete intersection of a quadric Q and a cubic T, we have the following equations for these varieties:
(3.1)

$$\begin{cases} y_5 \sum \alpha_i y_i = Ay_0^2 + y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j; \\ y_5 \left(Ay_0^2 - y_0 \sum \beta_i y_i + \sum \gamma_{ij} y_i y_j\right) = \sum \delta_{ijk} y_i y_j y_k + y_0^2 \sum \varepsilon_i y_i; \end{cases}$$

where the coefficients A, α_i, β_i, γ_{ij}, δ_{ijk}, and ε_i are the parameters.

Notice that V has a singular point at $\xi = (0 : \ldots : 0 : 1)$. We will always assume that the coefficient A is not 0, thus V does not pass through the point ζ (the isolated fixed point of the reflection ι).

Lemma 3.1. *Let $l \subset V$ be a line that does not pass trough the singular point $\xi \in V$. Then l does not intersect the line $L \subset \mathbb{P}^5$ through the points ζ and ξ.*

Proof. Assume the converse, i.e., $l \cap L \neq \emptyset$. Then there exists a plane P such that $l, L \subset P$. We may suppose that l intersects the hyperplane F at the point $(0 : 1 : 0 : 0 : 0 : 0)$, and since $L = \{y_1 = y_2 = y_3 = y_4 = 0\}$, the plane P is defined by $\{y_2 = y_3 = y_4 = 0\}$. Notice that $A \neq 0$ by the assumption, so $P \not\subset Q$.

The restriction $Q|_P$ gives us a conic

$$\alpha_1 y_1 y_5 = Ay_0^2 + \beta_1 y_0 y_1 + \gamma_{11} y_1^2,$$

as it follows from the equation (3.1). On the other hand, $l \subset Q|_P$, so the equation before have to be a production of two linear forms. From this we deduce that $\alpha_1 = 0$, thus $\xi \in l$, which contradicts to the conditions. Lemma 3.1 is proved.

LEMMA 3.2. *Let V be a general variety in $\mathcal{F}_r$, and $l \subset V$ a line that does not pass through the singular point $\xi \in V$. Then $l \not\subset F$.*

Proof. We will argue by counting of dimensions. Since $\xi \notin l$, we can take the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ and prove the lemma the the quartic X, which is the image of V.

Suppose $l' \subset F'$, where the line l' and the hyperplane $F' = \{y_0 = 0\}$ are the images of l and F respectively. Then the restriction $X' = X|_{F'}$ has the equation

$$\left(\sum \alpha_i y_i\right)\left(\sum \delta_{ijk} y_i y_j y_k\right) + \left(\sum \gamma_{ij} y_i y_j\right)^2 = 0.$$

We may suppose that l' is defined by the equations $y_3 = y_4 = 0$ in F', so from $l' \subset X$ we have the following conditions:

$$\begin{aligned}
&\alpha_1\delta_{111} + \gamma_{11}^2 = 0,\\
&\alpha_1\delta_{112} + \alpha_2\delta_{111} + 2\gamma_{11}\gamma_{12} = 0,\\
&\alpha_1\delta_{122} + \alpha_2\delta_{112} + 2\gamma_{11}\gamma_{22} + \gamma_{12}^2 = 0,\\
&\alpha_1\delta_{222} + \alpha_2\delta_{122} + 2\gamma_{12}\gamma_{22} = 0,\\
&\alpha_2\delta_{222} + \gamma_{22}^2 = 0.
\end{aligned}$$

It is not very difficult to check that we they gives 5 independent conditions. Let $\mathcal{X}'$ be the set of all two-dimensional quartics that are defined by equations of the same form as X'. Consider the production $G(2,4)\times\mathcal{X}'$ with the corresponding projections p and q onto $G(2,4)$ and $\mathcal{X}'$ respectively, and a subvariety $I = \{(l', X') : l' \subset X'\} \subset G(2,4)\times\mathcal{X}'$. Now we have

$$\dim I = \dim\mathcal{X}' - 5 + \dim G(2,5) = \dim\mathcal{X}' - 1 < \dim\mathcal{X}'.$$

Thus a general variety X, and hence a general $V \in \mathcal{F}_r$, has no lines with the indicated conditions. Lemma 3.2 is proved.

LEMMA 3.3. *A general variety $V \in \mathcal{F}_r$ satisfies condition $\mathcal{G}_1$.*

Proof. The lemma is obvious: the quadric Q is non-singular a for general choice of the parameters in the equation (3.1).

In what follows, instead of fixing the reflection ι, we will fix the hyperplane F, the point ξ, which will be the singular point for our varieties, and the line $l \not\subset F$ (see lemma 3.2), and then will move in $\mathbb{P}^5$ the fixed point ζ of the reflection. Notice also that any linear transformation of the coordinates $y_1, \ldots, y_4$ keeps the look of the equations (3.1).

Thus we fix a new coordinates $[x_0 : \ldots : x_5]$ in $\mathbb{P}^5$ and assume $F = \{x_0 = 0\}$, $l = \{x_2 = \ldots = x_5 = 0\}$, $\xi = (0 : \ldots : 0 : 1)$,

$\zeta = (1 : -a_1 : -a_2 : -a_3 : -a_4 : -a_5)$. The relation between the two system of coordinates is given by

$$\begin{cases} y_0 = x_0; \\ y_i = x_i + a_i x_0, \quad i = 1, \dots, 4; \\ y_5 = x_5 + bx_1 + (a_5 + ba_1)x_0; \end{cases} \tag{3.2}$$

where b and a_i are some numbers. The reflection ι acts now as follows:

$$\begin{aligned} &x_0 \to -x_0; \\ &x_i \to x_i + 2a_i x_0, \quad i = 1, \dots, 5. \end{aligned}$$

LEMMA 3.4. *A general variety $V \in \mathcal{F}_r$ satisfies conditions $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, and $\mathcal{G}_5$.*

Proof. In all cases, we argue by counting the dimensions. First we consider condition $\mathcal{G}_2$, the case $\xi \notin l$. Let $l \subset V$ be a line, and $L \subset \mathbb{P}^5$ be a line that passes trough the points ξ and ζ. By lemmas 3.1 and 3.2, for any l we may assume $l \not\subset F$ and $l \cap L = \emptyset$. Consider an open subset of a variety of (1,2)–flags $T = \{(l, P) : l \subset P, l \not\subset F, l \cap L = \emptyset\}$, and a closed subset $S = \{(l, P) : P \subset \langle l, L \rangle\} \subset T$, where $\langle l, L \rangle$ denote a unique 3-dimensional linear subspace that contains both the lines. It easy to compute that $\dim T = 11$ and $\dim S = 6$.

Consider a flag $(l, P) \in T \setminus S$. Then we choose the coordinates $[x_0 : \ldots : x_5]$ as before this lemma, and we can assume $l = \{x_2 = x_3 = x_4 = x_5 = 0\}$ and $P = \{x_3 = x_4 = x_5 = 0\}$. It is important to observe that the numbers a_3 and a_4 can not vanish simultaneously because $P \not\subset \langle l, L \rangle$. Suppose $l \subset V$ and $V \cap P$ is not reduced along l. Then we substitute the coordinates $[y_*]$ for $[x_*]$ in the equation (3.1) using the system (3.2), and look at the restrictions $Q|_P$ and $T|_P$. By the assumption, these restrictions are not reduced along l. It is not very difficult to see that this gives 12 independent linear conditions for the coefficients in (3.1). Consider a subvariety $I \subset T \setminus S \times \mathcal{F}_r$ consisting of all pairs $((l, P), V)$ such that $l \subset V$ and $V \cap P$ is not reduced along l. Then $\dim I = \dim T \setminus S + \dim \mathcal{F}_r - 12 = \dim \mathcal{F}_r - 1$, thus I can not cover $\mathcal{F}_r$ under the projection $T \setminus S \times \mathcal{F}_r \to \mathcal{F}_r$. We argue by the same way for the situation $(l, P) \in S$, with a few modifications. Exactly, here $a_3 = a_4 = 0$, but $a_2 \neq 0$ since $l \cap L = \emptyset$. Depending on the mutual location of ξ and P, we obtain 9 or 10 linear conditions for the coefficients in (3.1). It remains to take into account that $\dim S = 6$. So finally we see that the condition holds for a general V and lines l such that $\xi \notin l$.

The case $\xi \in l \subset V$ for the condition $\mathcal{G}_2$ and the conditions $\mathcal{G}_3$, $\mathcal{G}_4$, and $\mathcal{G}_5$ can be proved in the same way by counting the dimensions. Lemma 3.4 is proved.

4. Maximal singularities centered at smooth points.

Consider a general variety $V \in \mathcal{F}$. We assume that V satisfies the conditions $\mathcal{G}_1$–$\mathcal{G}_5$. In this section we prove the following result:

Proposition 4.1. *Let $\mathcal{D} \subset |n(-K_V)|$ be a linear system on V without fixed components. Suppose $\mathcal{D}$ has no maximal singularities centered at curves. Then $\mathcal{D}$ has no maximal singularities centered at points in $V \setminus \{\xi\}$.*

We use the method of maximal singularities ([12]). Suppose that $\mathcal{D}$ has a maximal singularity centered at the point $B_0 \in V \setminus \{\xi\}$. It means that there exists a discrete valuation $\mathfrak{v}$ centered at B_0 such that the Nöther-Fano inequality for $\mathcal{D}$ with respect to $\mathfrak{v}$ holds: $\mathfrak{v}(\mathcal{D}) > n\delta_{\mathfrak{v}}$. Here $\delta_{\mathfrak{v}}$ means the canonical multiplicity with respect to $\mathfrak{v}$. Then we know ([11]) that there exists a discrete valuation $\mathfrak{v}_{div}$ that is centered at B_0 and can be realized by a weighted blow-up with the weights $(1, L, N)$, where $1 \leq L < N$ or $N = L = 1$. In its turn, the weighted blow-up can be realized as a chain of usual blow-ups

$$V_N \xrightarrow{\varphi_N} V_{N-1} \xrightarrow{\varphi_{N-1}} \ldots \xrightarrow{\varphi_2} V_1 \xrightarrow{\varphi_1} V_0 = V$$

with centers $B_{i-1} \subset V_{i-1}$ and exceptional divisors $E_i \subset V_i$, where:

- $B_0, \ldots, B_{L-1}$ are points and $B_L, \ldots, B_{N-1}$ are curves;
- B_L is a line in $E_L \cong \mathbb{P}^2$, B_i for $i > L$ is a section of the corresponding linear surface $E_i \cong \mathbb{F}_{i+1-L}$ that does not intersect the minimal section;
- in all cases $B_i \cap E^i_{i-1} = \emptyset$ (upper indices denote the strict transform of a curve or a divisor on the corresponding floor of the chain of blow-ups).

Denote $\nu_i = \mathrm{mult}_{B_{i-1}} \mathcal{D}^{i-1}$, $i > 0$. Then the Nöther-Fano inequality looks as follows:

$$\nu_1 + \ldots + \nu_N > n(L + N). \tag{4.1}$$

We consider the following possible cases:

- $N > L > 1$ – the general infinitely near case;
- $N > L = 1$ – the special infinitely near case (we blow up the point B_0 and then the curves $B_1, \ldots, B_{N-1}$);
- $N = L = 1$ – the "point" case ($\mathfrak{v}_{div}$ is realized by a single blow-up of the point B_0, and then $\nu_1 > 2n$).

Suppose that there is no lines on V that pass through the point B_0 or all such lines does not contain the singular point $\xi \in V$. Then we can use the argumentation of the paper [9]. So in what follows, we assume that l is one of the 12 lines on V that pass through the singular point ξ, and $B_0 \in l$.

The general infinitely near case. Let D_1 and D_2 be general elements of the linear system $\mathcal{D}$. Then we can put

$$D_1 \circ D_2 = \alpha l + C,$$

where C is the residual curve, $l \not\subset \operatorname{Supp} C$, and general $H \in |-K_V|$ we have

$$\deg(\alpha l + C) = \alpha + \deg C = (\alpha l + C) \circ H = 6n^2.$$

We introduce the number k by

$$k = \max\left\{i \le L : B_{i-1} \in l^{i-1}\right\}.$$

Clearly, $k > 0$, and either $k = L$ or $B_k \notin l^k$. Then, we have the so-called quadratic inequality for the cycle $D_1 \circ D_2$ (see [12]):

$$\sum_{i=1}^{L} \operatorname{mult}_{B_{i-1}} (D_1 \circ D_2)^{i-1} \ge \sum_{i=1}^{N} \nu_i^2 > n^2 \frac{(N+L)^2}{N},$$

or, denoting $m_i = \operatorname{mult}_{B_{i-1}} C^{i-1}$,

$$k\alpha + \sum_{i=1}^{L} m_i > n^2 \frac{(N+L)^2}{N}.$$

Assume $k = 1$, i.e., $B_1 \notin l^1$. Then the linear system $|-K_V - B_0 - B_1|$ (those elements from $|-K_V|$ that pass through B_0 and the infinitely near point B_1) has no basic curves on V. For a general $H \in |-K_V - B_0 - B_1|$ we have

$$m_1 + m_2 \le C \circ H = 6n^2 - \alpha,$$

and from the quadratic inequality we get a contradiction:

$$3n^2 L \ge \alpha + (3n^2 - \frac{1}{2}\alpha)L \ge \alpha + \sum_{i=1}^{L} m_i > n^2 \frac{(N+L)^2}{N} > 4n^2 L.$$

So we assume $k \ge 2$. Using a general element H of the linear system $|-K_V - B_0 - \ldots - B_{k-1}|$, we have

$$m_1 + \ldots + m_k \le C \circ H = 6n^2 - \alpha,$$

and from the quadratic inequality we get

$$k\alpha + 6n^2 - \alpha + \frac{6n^2 - \alpha}{k}(L - k) > n^2 \frac{(N+L)^2}{N},$$

or an even more rough estimation,

$$(k-1)\alpha + 6n^2 \frac{L}{k} > n^2 \frac{(N+L)^2}{N}. \tag{4.2}$$

Now one needs to get an upper estimation of α. Denote $\mu = \operatorname{mult}_l \mathcal{D}$ and $\nu_0 = \operatorname{mult}_\xi \mathcal{D}$. Clearly, $\nu_0 \ge \frac{1}{2}\mu$. Consider the birational morphisms

$$\tilde{V} \xrightarrow{\tilde{\varphi}} V' \xrightarrow{\varphi'} V_{k-1},$$

where φ' is the blow-up of the singular point ξ with the exceptional divisor $E' \cong \mathbb{F}_0$, and $\tilde{\varphi}$ the blow-up of the strict transform of the line l. Denote $\tilde{E}$ the exceptional divisor of $\tilde{\varphi}$, and $\tilde{\mathcal{H}}$ the strict transform of the linear system $|H - B_0 - \ldots - B_k - l|$. It is easy to see that $\tilde{\mathcal{H}}$ has no basic curves, and moreover, this linear system is ample on $\tilde{E} \cong \mathbb{F}_1$ (indeed, $\tilde{\mathcal{H}}|_{\tilde{E}} \subset |s + 2f|$, where s and f are the minimal section and a fiber of $\tilde{E}$ respectively). We observe two important things. First,

$$\tilde{\mathcal{D}}|_{\tilde{E}} \subset |\mu s + (n + (k+1) - \nu_1 - \ldots - \nu_k - \nu_0)f|, \tag{4.3}$$

and denoting $\theta = \frac{1}{k}(\nu_1 + \ldots + \nu_k)$, we get

$$(k + \frac{1}{2})\mu + n \geq k\theta. \tag{4.4}$$

Second, we have

$$\tilde{D}_1 \circ \tilde{D}_2 \circ \tilde{H} = 6n^2 - k\theta^2 - 2\mu(n - k\theta) - \nu_0^2 - (\mu - \nu_0)^2 - (k+1)\mu^2,$$

hence

$$\alpha \leq \mu_2 + \tilde{D}_1 \circ \tilde{D}_2 \circ \tilde{H} < 6n^2 - k\theta^2 - 2\mu(n - k\theta) - k\mu^2.$$

This quadratic gets its maximal value at

$$\mu = \frac{k\theta - n}{k},$$

and we have

$$\alpha < 6n^2 - k\theta^2 + \frac{(k\theta - n)^2}{k}. \tag{4.5}$$

Notice that if $\theta \leq \frac{5}{4}n$, from the Nöther-Fano inequality (4.1) we obtain $N > 4L$, and we get a contradiction with (4.2). So we assume $\theta > \frac{5}{4}n$, and taking into account that $\alpha \leq 6n^2$, from (4.5) we have

$$\alpha < \frac{23}{6}n^2. \tag{4.6}$$

On the other hand, from (4.4) we obtain

$$\theta \leq \frac{(k + \frac{1}{2})\mu + n}{k} \leq \frac{3}{2}n,$$

and then the Nöther-Fano inequality (4.1) yields

$$N > 2L.$$

This estimation and the inequality (4.2) give

$$L\left(\frac{k-1}{L}\alpha + \frac{6n^2}{k}\right) > \frac{9}{2}n^2 L. \tag{4.7}$$

Suppose $L \geq k + 1$. Then from (4.6) and (4.7) we get $k < 2$, a contradiction.

So we assume $L = k \geq 2$. Let us note that there are two lines on the exceptional divisor $E' \cong \mathbb{F}_0$ that pass through the point $l' \cap E'$, and for at least one of them, say p, its strict transform $\tilde{p}$ on $\tilde{V}$ does not intersect the minimal section of the divisor $\tilde{E}$. Denoting by ε the multiplicity of $\mathcal{D}'$ along p, we see easily

$$\varepsilon \geq \mu - \nu_0. \tag{4.8}$$

Then, since $L = k$, the strict transform $\tilde{B}_k$ of the line B_k intersects the divisor $\tilde{E}$ at a point that is different from $\tilde{p} \cap \tilde{E}$. Let h be a general element of the linear system $|s + 2f|$ on $\tilde{E}$ that passes through both the points $\tilde{p} \cap \tilde{E}$ and $\tilde{B}_k \cap \tilde{E}$, where s and f are the minimal section and a fiber of $\tilde{E}$. Taking into account (4.3), for a general $D \in \mathcal{D}$ we obtain

$$\tilde{D}|_{\tilde{E}} \circ h = (k+2)\mu + n - \nu_1 - \ldots - \nu_{k+1} - \nu_0 - \varepsilon.$$

Denote $\theta = \frac{1}{k+1}(\nu_1 + \ldots + \nu_{k+1})$. Then, if $\nu_0 \geq \mu$, we get

$$\theta \leq \frac{k+2}{k+1} n.$$

We have the same estimation even if $\nu_0 < \mu$, using $\varepsilon \geq \mu - \nu_0$. Anyway, since $k \geq 2$, we find $\theta \leq \frac{4}{3} n$, and from the Nöther-Fano inequality (4.1) we see that $N > 3L$. Combining this estimation with (4.2), one gets

$$L\left(\frac{k-1}{k}\alpha + \frac{6n^2}{k}\right) > \frac{16}{3} n^2 L,$$

and we have a contradiction with the estimation (4.6). The general infinitely near case is dealt.

The special infinitely near case. In this case $L = 1$, and we can always assume $N \leq 3$. Indeed, if $N \geq 4$, we immediately get a contradiction with the quadratic inequality (4.2).

The case can be dealt exactly in the same way as in [9], so here we only give a couple of remarks. Consider a unique plane $P \subset \mathbb{P}^5$ that contains the point B_0 and the infinitely near line B_1. If $\xi \notin P$, we do not need any changes with respect to [9]. Suppose $\xi \in P$, and let M be a general hyperplane in $\mathbb{P}^5$ that contains P. Then the surface $H = M \cap V$ is a K3 surface with a unique singular (double) point at ξ. We follow the original explanations in [9], but first we blow up the singular point of H. It is not difficult to observe that we get even more strong estimations. The unique case that can not be dealt is the case of three different lines $P \cap V$. Two of them must have their common point at ξ. But this situation is prohibited by the condition $\mathcal{G}_3$.

The "point" case. Let the linear system $\mathcal{D}$ has the multiplicity $\nu_1 > 2n$ at the point $B_0 \neq \xi$, and there is a line $l \subset V$ that contains B_0 and ξ. Our consideration follows to the ones in [9], with simplifications.

Let T be a three-linear subspace in $\mathbb{P}^5$ that is tangent to V at B_0. From the condition $\mathcal{G}_4$ it follows that $Q|_T$ is a non-degenerate quadratic cone with the vertex at B_0. The restriction $V|_T$ consists of the lines $l = l_1, \ldots, l_k$ that are generators of the cone, and the residual curve

C. The important observation is that the scheme-theoretic intersection $T \cap V$ is reduced along any of the lines $l_1, \ldots, l_k$ by the condition $\mathcal{G}_2$. Notice also that either $k \leq 3$ by the condition $\mathcal{G}_5$. The residual curve C is reduced and irreducible (if we blow up the vertex of the cone $Q|_T$, it becomes a section of the corresponding ruled surface of type $\mathbb{F}_2$).

Denote $\mu_i = \operatorname{mult}_{l_i} \mathcal{D}$, $i = 1, \ldots, k$, and $\alpha = \operatorname{mult}_C \mathcal{D}$. Consider a general hyperplane $M \supset T$, its restriction $H = M \cap V$, and a general element $D \in \mathcal{D}$. Then:

- H is a K3 surface with double (du Val) points at B_0 and ξ;
- $D|_H = \alpha C + \mu_1 l_1 + \ldots + \mu_k l_k + R$, where $\operatorname{Supp} R$ does not contain any of the curves $C, l_1, \ldots, l_k$.

The last fact is important: the multiplicities of these curves in $\mathcal{D}|_H$ coincide with the multiplicities of the linear system $\mathcal{D}$ along the corresponding curves. It follows from the indicated important observation.

Let $V' \to V$ be the blow up of the point B_0, and H' the strict transform of H. We may assume that the strict transforms (we mark it by $'$) of all the curves l_i and C lie at the non-singular part of H'. Denote $R' = D'|_{H'} - \alpha C' - \mu_1 l_1' - \ldots - \mu_l l_k'$. We have

$$\begin{aligned}
R' \circ C' &= (6-k)n - (4-k)\nu_1 + \tfrac{3}{2}\alpha - \tfrac{1}{2}\mu_1 - \mu_2 - \ldots - \mu_k \geq 0, \\
R' \circ l_1' &= n - \nu_1 + \tfrac{3}{2}\mu_1 - \tfrac{1}{2}\alpha \geq 0, \\
R' \circ l_2' &= n - \nu_1 + 2\mu_2 - \alpha \geq 0, \\
&\ldots \\
R' \circ l_k' &= n - \nu_1 + 2\mu_k - \alpha \geq 0.
\end{aligned}$$

Take the sum of all this equations,and we obtain

$$6n + \mu_1 + \ldots + \mu_k \geq (k-2)\alpha + 4\nu_1 > (k-2)\alpha + 8n,$$

and if $k \leq 2$, we get a contradiction with the condition $\mu_i \leq n$ for all i (we suppose there are no maximal singularities along curves). Assume $k = 3$, then we see that

$$\frac{1}{2}\mu_1 + \mu_2 + \mu_3 \geq \frac{7}{6}\alpha + \frac{4}{3}\nu_1 - \frac{4}{3}n,$$

and combining it with the first equation, we obtain

$$\frac{13}{3}n + \frac{1}{3}\alpha \geq \frac{7}{3}\nu_1 > \frac{14}{3}n,$$

and this yields a contradiction since $\alpha \leq n$.

Notice that we assume $k \leq 3$ by the condition $\mathcal{G}_5$ (the method does not work for $k = 4$, the reader can check it by himself). Lemma 4.1 is proved.

5. Maximal singularities centered at curves.

Let $V \in \mathcal{F}$ be the complete intersection of a quadric Q and a cubic T with an ordinary double point ξ, as before. We suppose all the conditions $\mathcal{G}_1, \ldots, \mathcal{G}_5$ are satisfied.

REMARK 5.1. Let Λ be a three-dimensional linear subspace in $\mathbb{P}^5$. Then $V|_\Lambda$ is reduced along every its non-plane component. Indeed, the intersection $V|_\Lambda$ does not contain non-plane curves in the case when $\Lambda\cap Q$ is reducible. On the other hand, a cubic surface in $\mathbb{P}^3$ does not intersect an irreducible quadric surface by a double twisted cubic.

We fix a linear system $\mathcal{D}\subset|-nK_V|$ that has no fixed components.

Let B an irreducible curve on V. A linear span of a curve B is denoted by $\langle B\rangle$ and considered as a linear subspace in $\mathbb{P}^5$. Put $\nu=\mathrm{mult}_B(\mathcal{D})$.

PROPOSITION 5.2. *Suppose that $\nu>n$, i.e., $\mathcal{D}$ has a maximal singularity along B. Then one of the following holds:*

(1) *B is a line;*
(2) *B is a conic, and $\langle B\rangle\subset Q$;*
(3) *B is a conic, and $\xi\in B$.*

Proof. The proof of the proposition is similar to the proof of lemma 3.6 in [9] but we do our calculation on V (not on its blow ups), because the claim of the proposition is simpler than the claim of lemma 3.6 in [9], which is used not only to exclude curves on the non-singular complete intersection of a quadric and a quartic but also to find relations between birational involutions.

We may assume that $\xi\in B$ due to [9]. The proof consists of several lemmas.

LEMMA 5.3. *The curve B is not a plane cubic.*

Proof. Suppose that B is a plane cubic. Then $\langle B\rangle\subset Q$. Let Λ be a general a three-dimensional subspace containing B. Then $\Lambda\cap V=B\cup\bar{B}$, where $\bar{B}$ is a plane cubic.

The curves B and $\bar{B}$ intersect in three distinct points, different from ξ. Hence, for a general surface $D\in\mathcal{D}$ we have

$$3n=D\cdot\bar{B}\geq 3\nu,$$

which is a contradiction. The lemma is proved.

We may assume that $\dim\langle B\rangle\geq 3$. The inequality $\nu>n$ implies that $\deg B\leq 5$.

LEMMA 5.4. *The following cases are impossible:*

- *the curve B is a rational normal curve of degree 4 such that $\langle B\rangle=\mathbb{P}^4$;*
- *the curve B is a rational normal curve of degree 5 such that $\langle B\rangle=\mathbb{P}^5$;*
- *the curve B is an elliptic normal curve of degree 5 such that $\langle B\rangle=\mathbb{P}^4$.*

Proof. Suppose B is smooth. Put $d=\deg B$. Take the smallest natural number $m\leq d$ such that the following conditions hold:

- the curve B is cut out on the threefold V in a set-theoretic sense by surfaces of the linear system $|-mK_V|$ that pass through the curve B;
- the scheme-theoretic intersection of two sufficiently general surfaces of the linear system $|-mK_V|$ passing through B is reduced in the general point of B.

Let $\psi\colon V'\to V$ be the extremal blow up of the curve B (see [13]), E be the exceptional divisor of ψ, and $H'=\psi^*(-K_V)$. Then

$$\big(mH'-E\big)\cdot\big(\mu H'-\nu E\big)^2\geq 0,$$

because the proper transform of the linear system $\mathcal{D}$ on the threefold V' does not have fixed components, but $mH'-E$ is nef (see Lemma 5.2.5 in [4]). Hence, the inequality

$$6mn^2-dm\nu^2-2d\nu n-n^2\Big(2-2g(B)-d-\frac{1}{2}\Big)\geq 0 \tag{5.1}$$

holds (see Lemma 15 in [10], and the proof of Lemma 2 in §3 in [9]).

Putting $m=2$ in the inequality 5.1, we conclude the proof. The lemma is proved.

Thus, we see that either $\langle B\rangle=\mathbb{P}^3$, or B is a curve of degree 5 such that $\langle B\rangle=\mathbb{P}^4$.

LEMMA 5.5. *Either $\langle B\rangle=\mathbb{P}^3$, or the curve B is singular.*

Proof. Suppose that the curve B is a smooth rational curve of degree 5 such that $\langle B\rangle=\mathbb{P}^4$, which implies that B is an image of a smooth rational curve of degree 5 in $\mathbb{P}^3$ via general projection. The curve B is smooth. Thus, we can use the assumptions and notations of the proof of Lemma 5.4. Putting $m=3$ in the inequality 5.1, we obtain a contradiction. The lemma is proved.

LEMMA 5.6. *Either $\langle B\rangle=\mathbb{P}^3$, or the curve B is smooth at ξ.*

Proof. We may assume that the curve B is a curve of degree 5 such that $\langle B\rangle=\mathbb{P}^4$, and the curve B is singular at the point ξ. Let W be a sufficiently general surface of the linear system $|-3K_V|$ that contains the curve B. Put $\nu_0=\mathrm{mult}_\xi\,\mathcal{D}$.

Let $g_0\colon\bar{V}\to V$ be a blow up of the singular point ξ, $\bar{W}$ be the proper transform of the surface W on $\bar{V}$, $\bar{B}$ be a proper transform of the curve B on $\bar{V}$, and E_0 be the exceptional divisor of g_0. Then $\deg\mathcal{N}_{\bar{B}|\bar{V}}=1$.

Let $g\colon\tilde{V}\to\bar{V}$ be a blow up of the curve $\bar{B}$. Then the linear system

$$\Big|(g_0\circ g)^*\big(-nK_V\big)-\nu_0 g^*(E_0)-\nu E\Big|$$

does not have fixed components, where E is the g-exceptional divisor, but the complete linear system $|g^*(\bar{W})-E|$ does not have base curves. Thus, we have

$$\Big((g_0\circ g)\big(-nK_V\big)-\nu_0g^*(E_0)-\nu E\Big)^2\Big((g_0\circ g)\big(-3K_V\big)-g^*(E_0)-E\Big)\geq 0,$$

which implies that

$$0 \leq 18n^2 - 10n\nu - 12\nu^2 + 4\nu\nu_0 - \nu_0^2 = (18n^2 - 10n\nu - 8\nu^2) - (2\nu - \nu_0)^2 < 0,$$

which is a contradiction. The lemma is proved.

LEMMA 5.7. *The curve B is not a curve of degree 5 that is smooth at ξ such that $\langle B\rangle = \mathbb{P}^4$.*

Proof. Suppose that the curve B is a singular rational curve of degree 5 that is singular at some point $p_1 \in B$ such that $p_1 \neq \xi$ and $\langle B\rangle = \mathbb{P}^4$. Let us use the arguments of the proof of Lemma 5.6 to derive a contradiction[1].

Let $g_0 : \bar{V} \to V$ be a blow up of the points ξ and p_1, $\bar{W}$ be the proper transform of the surface W on $\bar{V}$, $\bar{B}$ be a proper transform of the curve B on $\bar{V}$, and E_0 and E_1 be the exceptional divisors over the points ξ and p_1. Then $\deg \mathcal{N}_{\bar{B}|\bar{V}} = -2$.

Let $g : \tilde{V} \to \bar{V}$ be a blow up of $\bar{B}$. Then the linear system

$$\Big|(g_0 \circ g)^*\big(-nK_V\big) - \nu_0 g^*(E_0) - \nu_1 g^*(E_1) - \nu E\Big|$$

has no fixed components, where E is the g-exceptional divisor, $\nu_0 = \mathrm{mult}_\xi \mathcal{D}$ and $\nu_1 = \mathrm{mult}_{p_1} \mathcal{D}$.

The linear system $|g^*(\bar{W}) - E|$ does not have base curves. Thus, taking the intersection index of a general element of this linear system with two general elements of the system above, we have

$$\begin{aligned} 0 \leq 18n^2 - 10n\nu - 14\nu^2 + 2\nu\nu_0 - \nu_0^2 + 4\nu\nu_1 - \nu_1^2 = \\ = (18n^2 - 10n\nu - 9\nu^2) - (\nu - \nu_0)^2 - (2\nu - \nu_1)^2 < 0, \end{aligned}$$

which is a contradiction. The lemma is proved.

Thus, we proved that $\langle B\rangle = \mathbb{P}^3$. Put $\Lambda = \langle B\rangle$. Then

$$V|_\Lambda = B \cup \bar{B},$$

where $\bar{B}$ is a curve on the complete intersection V such that $\bar{B} \neq B$ (see Remark 5.1).

REMARK 5.8. We consider only the case when B is a smooth rational curve of degree 3. The other cases are much simpler and left to the reader.

Let H be a sufficiently general hyperplane section of the threefold V that passes through the curve B, and $\pi\colon S \to H$ be the minimal resolution of singularities of the surface H.

LEMMA 5.9. *The point ξ is a singular point of type $\mathbb{A}_k$ on the surface H, where $k \leq 4$.*

[1]In fact, we can apply here use the arguments of the proof of Lemma 5.4. The inequality 5.2 holds in the case when B is smooth at ξ, but instead of genus $g(B)$ we must plug in the arithmetic genus of the curve B, which gives a contradiction, because the curve B is cut out by cubic hypersurfaces.

Proof. The birational morphism π can be factorized through a blow up of the point ξ on the threefold V, which immediately implies that ξ is an isolated Du Val singular point of the surface H of type $\mathbb{A}_k$. Easy local calculations imply that $k \leq 4$. The lemma is proved.

Note, that S is a smooth $K3$ surface.

LEMMA 5.10. *The cycle $\bar{B}$ is not a triple line.*

Prof. Suppose $\bar{B} = 3L$, where L is a line on H. Put $\bar{Q} = Q|_\Lambda$ and $\bar{T} = T|_\Lambda$. Then

$$\bar{Q} \cdot \bar{T} = 3L + B,$$

and $\bar{Q}$ is irreducible, because B is not a plane curve.

Suppose that $\bar{Q}$ is smooth. Then $\bar{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, and B must be a divisor of type $(3, 0)$ on the quadric $\bar{Q}$, which is impossible, because B is irreducible and reduced.

The quadric $\bar{Q}$ is a cone, and L is on of its rulings. Then either the cubic $\bar{T}$ is singular along L, or the cubic $\bar{T}$ is tangent to the quadric $\bar{Q}$ along L. Hence, there is a two-dimensional linear subspace $\Omega \subset \mathbb{P}^5$ that is tangent to both $\bar{T}$ and $\bar{Q}$ along L.

The sub-scheme $V|_\Omega$ is not reduced along L, which contradicts the condition $\mathcal{G}_2$. The lemma is proved.

COROLLARY 5.11. *The surface H has at most isolated ordinary double points outside of ξ.*

The birational morphism π contracts a chain or smooth rational curves $C_1, \ldots, C_k$ to the point ξ such that $C_i^2 = -2$ on the surface S. We may assume that

$$C_1 \cdot C_2 = C_k \cdot C_{k-1} = 1$$

and $C_1 \cdot C_i = C_k \cdot C_j = 0$ for $i \neq 2$ and $j \neq k - 1$.

LEMMA 5.12. *The curve $\bar{B}$ is reducible.*

Proof. Suppose that $\bar{B}$ is irreducible. Then $\bar{B}$ is a smooth rational cubic curve. Let us restrict our linear system $\mathcal{D}$ to the surface H. We have

$$\mathcal{D}|_H = \nu B + \mathrm{mult}_{\bar{B}}(\mathcal{D})\bar{B} + \mathcal{B} \equiv n\Big(B + \bar{B}\Big),$$

where $\mathcal{B}$ is a linear system on H that has no fixed components. Thus, we have

$$0 \leq \Big((\nu - \mu)B + \mathcal{B}\Big) \cdot \bar{B} = \Big(n - \mathrm{mult}_{\bar{B}}(\mathcal{D})\Big)\bar{B}^2,$$

which implies that $\mathrm{mult}_{\bar{B}}(\mathcal{D}) > n$ in the case when $\bar{B}^2 < 0$. It is easy to check that the inequality $\mathrm{mult}_{\bar{B}}(\mathcal{D}) > n$ is impossible, because

$$(\nu - n)B + \mathcal{B} = \Big(n - \mathrm{mult}_{\bar{B}}(\mathcal{D})\Big)\bar{B}$$

on the surface H. The point ξ is an intersection point of B and $\bar{B}$, and the linear subspace Λ is not a tangent subspace to V, because multiplicity of $\bar{B}\cup B$ in every point is at most two. Thus, we have

$$\bar{B}^2 = -2 + \frac{k-1}{k} < 0,$$

because $\bar{B}$ on H can be contracted to a Du Val singular point of type $\mathbb{A}_{k+1}$. The lemma is proved.

It follows from Lemma 1 in §4 in [9] that Λ is a tangent linear subspace to V in at most two points outside of the singular point ξ. In fact, the subspace Λ is a tangent linear subspace to V in at most one point, because otherwise the quadric $Q|_{\Lambda}$ is a union of two planes, which is impossible because $\langle B\rangle = \mathbb{P}^3$.

LEMMA 5.13. *The curve $\bar{B}$ is not reduced.*

Proof. Suppose that $\bar{B}$ is reduced. Then $\bar{B}$ is either a union of three lines, or a union of a line and a conic. We consider only the former case. Thus, we have

$$\bar{B} = L_1 \cup L_2 \cup L_3,$$

where L_1, L_2 and L_3 are different lines.

The lines L_1, L_2, L_3 do not pass through a single smooth point of the surface H, because otherwise they must lie on a plane in $\mathbb{P}^5$, which contradicts to $\langle B\rangle = \mathbb{P}^3$. Thus, there are three possible subcases:

- the lines L_1, L_2, L_3 pass through the point ξ;
- the lines L_1, L_2, L_3 pass through a singular point of H that is different from ξ;
- the lines L_1, L_2, L_3 do not intersect in one point.

It follows from the proof of Lemma 5.12 that to conclude the proof it is enough to prove that the intersection form of the irreducible components of the lines L_1, L_2 and L_3 on the surface H is negatively defined.

Suppose that L_1, L_2, L_3 pass through the point ξ. Let $\bar{L}_i$ be the proper transform of the line L_i on the surface S. Then the curves $\bar{L}_1, \bar{L}_2, \bar{L}_3, C_1, \ldots, C_k$ forms a Du Val graph of type $\mathbb{D}_{k+3}$, which implies that they can be contracted to a Du Val singular point of type $\mathbb{D}_{k+3}$. Therefore, the lines L_1, L_2 and L_3 can be contracted on the surface H, which implies that they intersection form on the surface H is negatively defined (see [1]).

Suppose that the lines L_1, L_2, L_3 pass through a singular point of the surface H that is different from the point ξ. Then they can be contracted either to a Du Val singular point of type $\mathbb{D}_{k+4}$ in the case when $\xi \in L_i$, or to a Du Val singular point of type $\mathbb{D}_4$ otherwise, which implies that the intersection form of L_1, L_2 and L_3 is negatively defined.

In the case when L_1, L_2, L_3 do not intersect in one point, the surface H is smooth outside of the point ξ, and the previous arguments imply that L_1, L_2 and L_3 can be contracted on H, which implies that their intersection form is negatively defined. The lemma is proved.

Thus, we may assume that $\bar{B}$ is not reduced. Then

$$\bar{B} = 2L + L',$$

where L and L' are different lines. Moreover, it follows from the proof of Lemma 5.13 that to conclude the proof of Theorem 5.2 it is enough to prove the existence of a birational morphism $H \to \bar{H}$ that contracts both lines L and L'.

The surface H is smooth outside of $L \cup \xi$, and the line L passes through at most three singular points of the surface H, because H has isolated singularities. We assume $\xi \in L$, the other cases is dealt in [9].

Lemma 5.14. *The surface H has at most two singular points.*

Proof. Suppose that the surface H has exactly three singular points. Let p_1 and p_2 be singular points of the surface H that are different from the singular point ξ, and Ξ be a hyperplane in $\mathbb{P}^5$ such that $H = V \cap \Xi$. Put $\check{Q} = Q|_{\Xi}$ and $\check{T} = T|_{\Xi}$.

Let Γ be any hyperplane in $\Xi \cong \mathbb{P}^4$ that is tangent to the quadric $\check{Q}$ at any point of the line L. Then the cycle $V|_{\Gamma}$ is not reduced along L, because $V|_{\Gamma}$ is a complete intersection of a quadric $\check{Q}$ and a cubic $\check{T}$ that has at least 4 singular point on the line L.

Suppose that $\check{Q}$ is smooth. Then $\check{Q}|_{\Gamma}$ is a quadric cone. Hence, the arguments of the proof of Lemma 5.10 imply that there is a two-dimensional linear subspace $\Omega \subset \mathbb{P}^5$ such that the subschema $V|_{\Omega}$ is not reduced along L, which is impossible by the condition $\mathcal{G}_2$.

Suppose that $\check{Q}$ is a cone that is singular at some point of L. Then $Q|_{\Lambda}$ is an irreducible quadric cone. Hence, the arguments of the proof of Lemma 5.10 imply that there is a two-dimensional linear subspace $\Omega \subset \mathbb{P}^5$ such that the subschema $V|_{\Omega}$ is not reduced along L, which is impossible by the condition $\mathcal{G}_2$.

The quadric $\check{Q}$ must be a cone that is smooth along the line L. Then $\check{Q}|_{\Gamma} = \Omega_1 \cup \Omega_2$, where Ω_i is a two-dimensional linear subspace in $\Xi \cong \mathbb{P}^4$. We may assume that

$$\Omega_1 \cap \Omega_2 \neq L \subset \Omega_1,$$

but the cycle $V|_{\Gamma}$ is not reduced along L, which implies that the subschema $V|_{\Omega_1}$ is not reduced along L, which is impossible by the condition $\mathcal{G}_2$. The lemma is proved.

We may assume that the surface H has exactly two singular because the other case is simpler. Let p_1 be the singular point of H that is different from ξ, and Z be the curve on the surface S such that $\pi(Z) = p_1$. Then Z is a smooth rational curve such that $Z^2 = -2$.

Let $\bar{L}$ and $\bar{L}'$ be the proper transforms of the curves L and L' on the surface S, respectively. To conclude to proof of proposition 5.2, it is enough to show that the curves

$$\bar{L}, \bar{L}', Z, C_1, \ldots, C_k$$

form a graph of Du Val type. The curves $\bar{L}, \bar{L}', Z, C_1, \ldots, C_k$ form the following graphs:

- a graph of type $\mathbb{E}_7$ in the case when $k = 4$ and $\xi \in L'$;
- a graph of type $\mathbb{E}_6$ in the case when $k = 3$ and $\xi \in L'$;
- a graph of type $\mathbb{D}_5$ in the case when $k = 2$ and $\xi \in L'$;
- a graph of type $\mathbb{A}_4$ in the case when $k = 1$ and $\xi \in L'$;
- a graph of type $\mathbb{A}_{k+3}$ in the case when $p_1 \in L'$;
- a graph of type $\mathbb{D}_{k+3}$ in the case when $\xi \notin L'$ and $p_1 \notin L'$.

Thus, the claim of proposition 5.2 is proved.

6. Finishing the proof.

Let $V \in \mathcal{F}$ be the complete intersection of a quadric Q and a cubic T in $\mathbb{P}^5$ with an ordinary double point $\xi \in V$. We suppose V satisfies the conditions $\mathcal{G}_1, \ldots, \mathcal{G}_5$.

LEMMA 6.1. *Let $B \subset V$ be either a line, or a conic such that $\langle B \rangle \subset Q$, or a conic such that $\langle B \rangle \not\subset Q$ and $\xi \in B$. Consider a linear system $\mathcal{D} \subset |n(-K_X) - \nu B|$ without fixed components. Then there is a birational automorphism $\tau : V \dashrightarrow V$ such that:*

- $\tau_*^{-1}\mathcal{D} \subset |(4n - 3\nu)(-K_V) - (5n - 4\nu)B|$, *if B is a line;*
- $\tau_*^{-1}\mathcal{D} \subset |(13n - 12\nu)(-K_V) - (14n - 13\nu)B|$, *if B is a conic and $\langle B \rangle \subset Q$;*
- $\tau_*^{-1}\mathcal{D} \subset |(15n - 14\nu)(-K_V) - (16n - 15\nu)B|$, *if B is a conic, $\langle B \rangle \not\subset Q$, and $\xi \in B$.*

Proof. The first two cases are the same as in the non-singular case, even if B contains the singular point, and they are dealt completely in [9].

Let B be a conic, $\xi \in B$, and $\langle B \rangle \not\subset Q$. Then we define τ as follows. Consider a general 3-dimensional linear subspace $L \supset \langle B \rangle$. Then $V|_L = B + C$, where C is an elliptic curve that intersects B at the point ξ and at another 3 points. Let $V' \to V$ be the blow-up of ξ and then the strict transform of the curve B with the exceptional divisors E and F respectively. We see that there is a Zariski open subset $\bar{V}' \subset V'$ that is fibred into elliptic curves over an open subset $\bar{E} \in E$ (really we have to throw out only a finitely many points in E). The there is a reflection on $\bar{V}'$ that is on each fiber $s \subset \bar{V}'$ a usual reflection on elliptic curve with respect to the point $s \cap \bar{E}$. Notice that F becomes a three-section of the fibration. On V this reflection gives the birational automorphism τ. The reader can check that the action of τ is described as above. The lemma is proved.

LEMMA 6.2. *Let V_1 and V_2 be varieties from $\mathcal{F}$ that are birational to each other, and $\psi : V_1 \dashrightarrow V_2$ is the corresponding birational map (it is described in section 2). Let $\mathcal{D} \subset |n(-K_{V_1}) - \nu\xi|$ be a linear system without fixed components. Then*

$$\psi_*\mathcal{D} \subset |(2n-\nu)(-K_{V_2}) - (3n-2\nu)\xi|.$$

Proof. It can be checked immediately using the construction of ψ. Notice that on V_1 there is a unique element in $|-K_{V_1}|$ that has the multiplicity 3 at the point ξ. The lemma is proved.

Proof of the part *(ii)* **of theorem 2.1.** Let $\rho : U \to S$ be a Mori fibration, V_1 is a variety from $\mathcal{F}$ with its birational "counterpart" V_2, and $\chi : V_1 \dashrightarrow U$ a birational map. Consider a very ample linear system

$$\mathcal{D}_U = |n'(-K_U) + \rho^*(A)|,$$

where A is an ample divisor on S. Denote $\mathcal{D}$ the strict transform on V_1 of $\mathcal{D}_U$ by means of χ.

It only remains to collect the data of propositions 4.1 and 5.2 and lemmas 6.1 and 6.2. If $\mathcal{D}$ has no maximal singularities, then χ must be an isomorphism by theorem 4.2 in [2]. If $\mathcal{D}$ has maximal singularities, we apply finitely many times the birational automorphisms from lemma 6.1 or the birational map from lemma 6.2 in any order. Each time we decrease the degree of the strict transforms of $\mathcal{D}$, and finally we get the result by propositions 5.2 and 4.1. The theorem is proved.

7. Final remarks.

Birational rigidity may fall under small deformations not only for Fano varieties. We can give at least two examples for del Pezzo fibrations.

The first example. Let $Q \subset \mathbb{P}^4$ be a non-degenerate quadratic cone, and Q_S its non-singular section by a general quartic hypersurface F. Consider a double cover $V' \to Q$ branched along Q_S. The variety V' is not $\mathbb{Q}$-factorial and has two double points over the vertex of the cone. There are two small resolutions V_1 and V_2 of V' that are related by means of flop $\psi : V_1 \dashrightarrow V_2$. It is easy to see that these varieties are actually fibrations $V_1/\mathbb{P}^1$ and $V_2/\mathbb{P}^1$ into del Pezzo surfaces of degree 2, and the flop ψ is not square. The fibers of these fibrations arise from two pencils of planes in Q. It is proved ([6]) that, in general case, these fibrations are unique Mori fibrations in their class of birational equivalence, up to square birational maps. It is easy to see that for a general choice of F the varieties V_1 and V_2 are not isomorphic to each other, thus they are non-rigid. But they become isomorphic if the section Q_S is invariant with respect to the natural involution on Q that exchange the pencils of planes, and in this case, $V_1/\mathbb{P}^1$ (or, which is the same, $V_{@}/\mathbb{P}^1$) is birationally rigid. Clearly, small deformations may break the symmetry of Q_S, and we obtain the general, i.e., non-rigid, case.

The second example. This is the case $(\varepsilon, n_1, n_2, n_3) = (0, 2, 2, 2)$ in [7]. Consider a $\mathbb{P}^3$-fibration $X \to \mathbb{P}^1$ that is defined by the projectivization of the bundle $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(2)$. Let $Q \subset X$ be a threefold that is fibred over $\mathbb{P}^1$ into non-degenerate quadratic cones. The vertices of the cones lie on the minimal section t of X (i.e., that corresponds to surjection $\mathcal{E} \to \mathcal{O}$), $Q \sim 2M - 4L$, where M is the tautological divisor, L is a fiber of X. Consider a double cover $V \to X$ that is branched over a non-singular section $Q_R = Q \cap R$, where $R \sim 3M$. Notice that $t \circ R = 0$. We see that V is a fibration $V/\mathbb{P}^1$ into del Pezzo surfaces of degree 1. It has a section s that lies over t, and there is a flop $\psi : V \dashrightarrow U$ centered at s, onto a fibration $U/\mathbb{P}^1$ into del Pezzo surfaces of degree 1 with the same construction. It is proved that, up to square birational maps, these fibrations are unique Mori fibrations in their class of birational equivalence. As before, they are not isomorphic and non-rigid in general case, but for specially chosen divisors R, they becomes isomorphic, and thus rigid by definition 1.1.

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