GLOBAL DEFORMATIONS OF THE WITT ALGEBRA OF KRICHEVER-NOVIKOV TYPE

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Abstract. By considering non-trivial global deformations of the Witt (and the Virasoro) algebra given by geometric constructions it is shown that, despite their infinitesimal and formal rigidity, they are globally not rigid. This shows the need of a clear indication of the type of deformations considered. The families appearing are constructed as families of algebras of Krichever-Novikov type. They show up in a natural way in the global operator approach to the quantization of two-dimensional conformal field theory. In addition, a proof of the infinitesimal and formal rigidity of the Witt algebra is presented.

1. Introduction

Deformations of mathematical structures are important in most part of mathematics and its applications. Deformation is one of the tools used to study a specific object, by deforming it into some families of "similar" structure objects. This way we get a richer picture about the original object itself.

But there is also another question approached via deformations. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (maybe up to certain equivalences) with the structure of a topological or even geometric space. In other words, does there exists a moduli space for these structures. If so, then for a fixed object the deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

In this respect, a clear success story is the classification of complex analytic structures on a fixed topological manifold. Also in algebraic geometry one has well-developed results in this direction. One of these results is that the local situation at a point $[C]$ of the moduli space is completely governed by the cohomological properties of the geometric object C. As typical example recall that for the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g over $\mathbb{C}$ (or equivalently, compact Riemann surfaces of genus g) the tangent space $T_{[C]}\mathcal{M}_g$ can be naturally identified with $\mathrm{H}^1(C,T_C)$, where T_C is the sheaf of holomorphic vector fields over C. This extends to higher dimension. In particular, it turns out that for compact complex manifolds M, the condition $\mathrm{H}^1(M,T_M)$ implies that M is rigid, [16, Thm. 4.4]. Rigidity means that any differentiable family $\pi:\mathcal{M}\to B\subseteq\mathbb{R}$, $0\in B$ which contains M as the special member $M_0:=\pi^{-1}(0)$ is trivial in a neighbourhood of 0, i.e. for t small enough $M_t:=\pi^{-1}(t)\cong M$. Even more generally, for M a compact complex manifold and $\mathrm{H}^1(M,T_M)\ne\{0\}$ there exists a

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versal family which can be realized locally as a family over a certain subspace of $\mathrm{H}^1(M,T_M)$ such that every appearing deformation family is "contained" in this versal family (see also [18] for definitions, results, and further references).

These positive results lead to the impression that the vanishing of the relevant cohomology spaces will imply rigidity with respect to deformations also in the case of other structures.

The goal of this article is to shed some light on this in the context of deformations of infinite-dimensional Lie algebras. We consider the case of the Witt algebra $\mathcal{W}$ (see its definition further down). The cohomology "responsible" for deformations is $\mathrm{H}^2(\mathcal{W},\mathcal{W})$. It is known that $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = \{0\}$ (see Section 3). Hence, guided by the experience in the theory of deformations of complex manifolds, one might think that $\mathcal{W}$ is rigid in the sense that all families containing $\mathcal{W}$ as a special element will be trivial. But this is not the case as we will show. Certain natural families of Krichever-Novikov type algebras of geometric origin (see Section 4 for their definition) will appear which contain $\mathcal{W}$ as special element. But none of the other elements will be isomorphic to $\mathcal{W}$. In fact, from $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = \{0\}$ it follows that the Witt algebra is infinitesimally and formally rigid. But this condition does not imply that there are no non-trivial global deformation families. The main point to learn is that it is necessary to distinguish clearly the formal and the global deformation situation. The formal rigidity of the Witt algebra indeed follows from $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = \{0\}$, but no statement like that about global deformations.

How intricate the situation is, can be seen from the fact that for the subalgebra L_1 of $\mathcal{W}$, consisting of those vector fields vanishing at least of order two at zero, there exists a versal formal family consisting of three different families parameterized over a collection of three curves in $\mathrm{H}^2(L_1,L_1)$. Each family corresponds to a scalar multiple of the same cohomology class ω [6],[7], [9] which gives their infinitesimal deformations. It turns out that $\mathrm{H}^2(L_1,L_1)$ is 3-dimensional, but only the infinitesimal deformation corresponding to this ω can be extended to formal deformations.

The results of this article will show that the theory of deformations of infinite-dimensional Lie algebras is still not in satisfactory shape. Hopefully, the appearing features will be of help in a further understanding.

Clearly, what will be done here, can also be done in the case of associative algebras. In particular, there will be global deformations of the associative algebra of Laurent polynomials of Krichever-Novikov type obtained by the same process as the one presented here.

First, let us introduce the basic definitions. Consider the complexification of the Lie algebra of polynomial vector fields on the circle with generators

$$l_n := \exp(\mathrm{i}n\varphi)\frac{d}{d\varphi}, \qquad n \in \mathbb{Z},$$

where φ is the angular parameter. The bracket operation in this Lie algebra is

$$[l_n, l_m] = (m-n)l_{n+m}.$$

We call it the *Witt algebra* and denote it by $\mathcal{W}$. Equivalently, the Witt algebra can be described as the Lie algebra of meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are holomorphic outside the points 0 and ∞. In this presentation $l_n = z^{n+1}\frac{d}{dz}$, where z is the quasi-global coordinate on $\mathbb{P}^1(\mathbb{C})$.

The Lie algebra $\mathcal{W}$ is infinite dimensional and graded with the standard grading $\deg l_n = n$. By taking formal vector fields with the projective limit topology we get the completed topological Witt algebra $\widehat{\mathcal{W}}$. Throughout this paper we will consider its everywhere dense subalgebra $\mathcal{W}$.

It is well-known that $\mathcal{W}$ has a unique nontrivial one-dimensional central extension, the *Virasoro algebra* $\mathcal{V}$. It is generated by $l_n \quad (n \in \mathbb{Z})$ and the central element c, and its bracket operation is defined by

$$[l_n, l_m] = (m-n)l_{n+m} + \tfrac{1}{12}(m^3 - m)\delta_{n,-m}\, c, \qquad [l_n, c] = 0. \tag{1.1}$$

In Section 2 we recall the different concepts of deformation. There is a lot of confusion in the literature in the notion of a deformation. Several different (inequivalent) approaches exist. One of the aims of this paper is to clarify the difference between deformations of geometric origin and so called formal deformations. Formal deformation theory has the advantage of using cohomology. It is also complete in the sense that under some natural cohomology assumptions there exists a versal formal deformation, which induces all other deformations.

In this context see Theorem 2.5 and Corollary 2.6 which are quoted from [6],[9]. Formal deformations are deformations with a complete local algebra as base. A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the base of deformation – which is a commutative algebra of functions – with a smooth manifold, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations. As already indicated above, in infinite dimension, there is no tight relation between global and formal deformations, as we will show in this paper.

In Section 3, we supply a detailed proof of the infinitesimal and formal rigidity of the Witt algebra $\mathcal{W}$, by showing that $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = 0$.

In Section 4 we introduce and recall the necessary properties of the Krichever-Novikov type vector field algebras. They are generalizations of the Witt algebra (in its presentation as vector fields on $\mathbb{P}^1(\mathbb{C})$) to higher genus smooth projective curves.

In Section 5 we construct global deformations of the Witt algebra by considering certain families of algebras for the genus one case (i.e. the elliptic curve case) and let the elliptic curve degenerate to a singular cubic. The two points, where poles are allowed, are the zero element of the elliptic curve (with respect to its additive structure) and a 2-torsion point. In this way we obtain families parameterized over the affine line with the peculiar behaviour that every family is a global deformation of the Witt algebra, i.e. $\mathcal{W}$ is a special member, whereas all other members are mutually isomorphic but not isomorphic to $\mathcal{W}$, see Theorem 5.3. Globally these families are non-trivial, but infinitesimally and formally they are trivial. The construction can be extended to the centrally extended algebras, yielding a global deformation of the Virasoro algebra. Finally, we consider the subalgebra L_1 of $\mathcal{W}$ corresponding to the vector fields vanishing at least with order two at 0. This algebra is formally not rigid, and its formal versal deformation has been determined [6, 8]. We identify one of the appearing three families in our geometric context.

The results obtained do not have only relevance to the deformation theory of algebras but also to the theory of two-dimensional conformal fields and their quantization. It is well-known that the Witt algebra, the Virasoro algebra, and their representations are of fundamental importance for the local description of conformal field theory on the Riemann sphere (i.e. for genus zero), see [1]. Krichever and Novikov [17] proposed in the case of higher genus Riemann surfaces the use of global operator fields which are given by the Lie algebra of vector fields of Krichever-Novikov type and its representations (Section 4).

In the process of quantization of the conformal fields one has to consider families of algebras and representations over the moduli space of compact Riemann surfaces (or equivalently, of smooth projective curves over $\mathbb{C}$) of genus g with N marked points. See [28] for a global operator version, and [29] for a sheaf version. In passing to the boundary of the moduli space one obtains the limit objects which are defined over the normalization of curves of lower genus. Assuming good behaviour of the examined objects under deformation also in the limit (e.g. "factorization"), the degeneration is an important technique to obtain via induction from results for lower genus results for all genera. See Tsuchiya, Ueno, and Yamada's proof of the Verlinde formula [29] as an application of this principle.

By a maximal degeneration a collection of $\mathbb{P}^1(\mathbb{C})$'s will appear. For the vector field algebras (with or without central extension) one obtains families of algebras which are related to the Witt or Virasoro algebra or certain subalgebras respectively. Indeed, the examples considered in this article are exactly of this type. They appear as families which are naturally defined over the moduli space of complex one-dimensional tori (i.e. of elliptic curves) with two marked points. The full geometric picture behind it was discussed in [25]. In special cases the Witt and Virasoro algebra appear as degenerations of the Krichever-Novikov algebras. Considered from the opposite point of view, in the sense of this article, the Krichever-Novikov algebras are global deformations of the Witt and Virasoro algebra. Nevertheless, as we show here, the structure of these algebras are not determined by the Witt algebra, despite the formal rigidity of the latter.

2. Deformations and formal deformations

2.1. **Intuitively.** Let us start with the intuitive definition. Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$. A *deformation* of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket

$$\mu_t = \mu_0 + t\phi_1 + t^2\phi_2 + \ldots \tag{2.1}$$

where the ϕ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\bigwedge^2 \mathcal{L}, \mathcal{L}) = C^2(\mathcal{L};\mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. (see [5, 11]). Two deformations $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\hat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + \ldots$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C^1(\mathcal{L},\mathcal{L})$, such that

$$\mu'_t(x,y) = \hat{\psi}_t^{-1}(\mu_t(\hat{\psi}_t(x),\hat{\psi}_t(y)). \tag{2.2}$$

The Jacobi identity for the algebra $\mathcal{L}_t$ implies that the 2-cochain ϕ_1 is indeed a cocycle, i.e. it fulfills $d_2\phi_1 = 0$ with respect to the Lie algebra cochain complex of $\mathcal{L}$ with values in $\mathcal{L}$ (see [10] for the definitions). If ϕ_1 vanishes identically, the first nonvanishing ϕ_i will be a cocycle. If μ'_t is an equivalent deformation (with cochains ϕ'_i) then

$$\phi'_1 - \phi_1 = d_1\psi_1. \tag{2.3}$$

Hence, every equivalence class of deformations defines uniquely an element of $\mathrm{H}^2(\mathcal{L},\mathcal{L})$. This class is called the *differential* of the deformation. The differential of a family which is equivalent to a trivial family will be the zero cohomology class.

2.2. **Global deformations.** Consider now a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra A over $\mathbb{K}$ with identity as base of a deformation.

In the following we will assume that A is a commutative algebra over the field $\mathbb{K}$ of characteristic zero which admits an augmentation $\epsilon : A \to \mathbb{K}$. This says that ϵ is a $\mathbb{K}$-algebra homomorphism, e.g. $\epsilon(1_A) = 1$. The ideal $m_\epsilon := \operatorname{Ker}\epsilon$ is a maximal ideal of A. Vice versa, given a maximal ideal m of A with $A/m \cong \mathbb{K}$, then the natural factorization map defines an augmentation.

In case that A is a finitely generated $\mathbb{K}$-algebra over an algebraically closed field $\mathbb{K}$ then $A/m \cong \mathbb{K}$ is true for every maximal ideal m. Hence in this case every such A admits at least one augmentation and all maximal ideals are coming from augmentations.

Let us consider a Lie algebra $\mathcal{L}$ over the field $\mathbb{K}$, ϵ a fixed augmentation of A, and $m = \operatorname{Ker}\epsilon$ the associated maximal ideal.

Definition 2.1. A *global deformation* λ of $\mathcal{L}$ with the base (A, m) or simply with the base A, is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[.,.]_\lambda$ such that

$$\epsilon{\otimes}\mathrm{id} : A{\otimes}\mathcal{L} \to \mathbb{K}{\otimes}\mathcal{L} = \mathcal{L} \tag{2.4}$$

is a Lie algebra homomorphism (see [9]). Specifically, it means that for all $a, b \in A$ and $x, y \in \mathcal{L}$,

(1) $[a \otimes x, b \otimes y]_\lambda = (ab \otimes \mathrm{id})[1 \otimes x, 1 \otimes y]_\lambda$,
(2) $[.,.]_\lambda$ is skew-symmetric and satisfies the Jacobi identity,
(3) $\epsilon \otimes \mathrm{id}\,([1 \otimes x, 1 \otimes y]_\lambda) = 1 \otimes [x, y]$.

By Condition (1) to describe a deformation it is enough to give the elements $[1 \otimes x, 1 \otimes y]_\lambda$ for all $x, y \in \mathcal{L}$. By condition (3) follows that for them the Lie product has the form

$$[1{\otimes}x, 1{\otimes}y]_\lambda = 1{\otimes}[x, y] + \textstyle\sum_i a_i{\otimes}z_i, \tag{2.5}$$

with $a_i \in m$, $z_i \in \mathcal{L}$.

A deformation is called *trivial* if $A \otimes_{\mathbb{K}} \mathcal{L}$ carries the trivially extended Lie structure, i.e. (2.5) reads as $[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y]$. Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\epsilon \otimes \mathrm{id}$.

We say that a deformation is *local* if A is a local $\mathbb{K}$-algebra with unique maximal ideal m_A. By assumption $m_A = \operatorname{Ker}\epsilon$ and $A/m_A \cong \mathbb{K}$. In case that in addition ${m_A}^2 = 0$, the deformation is called *infinitesimal.*

Example. If $A = \mathbb{K}[t]$, then this is the same as an algebraic 1-parameter deformation of $\mathcal{L}$. In this case we sometimes use simply the expression "deformation over the affine line." This can be extended to the case where A is the algebra of regular functions on an affine variety X. In this way we obtain algebraic deformations over an affine variety. These deformations are non-local, and will be the objects of our study in Section 5.

Let A' be another commutative algebra over $\mathbb{K}$ with a fixed augmentation $\epsilon' : A' \to \mathbb{K}$, and let $\phi : A \to A'$ be an algebra homomorphism with $\phi(1) = 1$ and $\epsilon' \circ \phi = \epsilon$. If a deformation λ of $\mathcal{L}$ with base $(A, \operatorname{Ker}\epsilon = m)$ is given, then the *push-out* $\lambda' = \phi_* \lambda$ is the deformation of $\mathcal{L}$ with base $(A', \operatorname{Ker}\epsilon' = m')$, which is the Lie algebra structure

$$[a_1' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]_{\lambda'} := a_1' a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda,$$

$(a_1', a_2' \in A', a_1, a_2 \in A, l_1, l_2 \in \mathcal{L})$ on $A' \otimes \mathcal{L} = (A' \otimes_A A) \otimes \mathcal{L} = A' \otimes_A (A \otimes \mathcal{L})$. Here A' is regarded as an A-module with the structure $aa' = a'\phi(a)$.

Remark. For non-local algebras there exist more than one maximal ideal, and hence in general many different augmentations ϵ. Let $\mathcal{L}$ be a $\mathbb{K}$-vector space and assume that there exists a Lie A-algebra structure $[.,.]_A$ on $A \otimes_{\mathbb{K}} \mathcal{L}$. Given an augmentation $\epsilon : A \to \mathbb{K}$ with associated maximal ideal $m_\epsilon = \operatorname{Ker}\epsilon$, one obtains a Lie $\mathbb{K}$-algebra structure $\mathcal{L}^\epsilon = (\mathcal{L}, [.,.]_\epsilon)$ on the vector space $\mathcal{L}$ by

$$\epsilon \otimes \mathrm{id}\,([1 \otimes x, 1 \otimes y]_A) = 1 \otimes [x, y]_\epsilon. \tag{2.6}$$

Comparing this with Definition 2.1 we see that by construction the Lie A-algebra $A \otimes_{\mathbb{K}} \mathcal{L}$ will be a global deformation of the Lie $\mathbb{K}$-algebra $\mathcal{L}^\epsilon$. On the level of structure constants the described construction corresponds simply to the effect of "reducing the structure constants of the algebra modulo m_ϵ." In other words, for $x, y, z \in \mathcal{L}$ basis elements, let the Lie A-algebra structure be given by

$$[1 \otimes x, 1 \otimes y]_A = \sum_z C_{x,y}^z (1 \otimes z), \qquad C_{x,y}^z \in A. \tag{2.7}$$

Then $\mathcal{L}^\epsilon$ is defined via

$$[x, y]_\epsilon := \sum_z (C_{x,y}^z \bmod m_\epsilon)\, z. \tag{2.8}$$

In general, the algebras $\mathcal{L}^\epsilon$ will be different for different ϵ. The Lie A-algebra $A \otimes_{\mathbb{K}} \mathcal{L}$ will be a deformation of different Lie $\mathbb{K}$-algebras L^ϵ.

Example. For a deformation of the Lie algebra $\mathcal{L} = \mathcal{L}_0$ over the affine line, the Lie structure $\mathcal{L}_\alpha$ in the fiber over the point $\alpha \in \mathbb{K}$ is given by considering the augmentation corresponding to the maximal ideal $m_\alpha = (t - \alpha)$. This explains the picture in the geometric interpretation of the deformation.

2.3. **Formal deformations.** Let A be a complete local algebra over $\mathbb{K}$, so $A = \overleftarrow{\lim\limits_{n\to\infty}}\,(A/m^n)$, where m is the maximal ideal of A and we assume that $A/m \cong \mathbb{K}$.

Definition 2.2. A *formal deformation* of $\mathcal{L}$ with base A is a Lie algebra structure on the completed tensor product $A \widehat{\otimes} \mathcal{L} = \overleftarrow{\lim\limits_{n\to\infty}}\,((A/m^n) \otimes \mathcal{L})$ such that

$$\epsilon \widehat{\otimes} \mathrm{id} : A \widehat{\otimes} \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Example. If $A = \mathbb{K}[[t]]$, then a formal deformation of $\mathcal{L}$ with base A is the same as a formal 1-parameter deformation of $\mathcal{L}$ (see [11]).

There is an analogous definition for equivalence of deformations parameterized by a complete local algebra.

2.4. **Infinitesimal and versal formal deformations.** In the following let the base of the deformation be a local $\mathbb{K}$-algebra (A, m) with $A/m \cong \mathbb{K}$. In addition we assume that $\dim(m^k/m^{k+1}) < \infty$ for all k.

Proposition 2.3. *([9]) With the assumption* $\dim \mathrm{H}^2(\mathcal{L};\mathcal{L}) < \infty$*, there exists a universal infinitesimal deformation* $\eta_{\mathcal{L}}$ *of the Lie algebra* $\mathcal{L}$ *with base* $B = \mathbb{K} \oplus H^2(\mathcal{L};\mathcal{L})'$*, where the second summand is the dual of* $\mathrm{H}^2(\mathcal{L};\mathcal{L})$ *equipped with zero multiplication, i.e.*

$$(\alpha_1, h_1) \cdot (\alpha_2, h_2) = (\alpha_1\alpha_2, \alpha_1 h_2 + \alpha_2 h_1).$$

This means that for any infinitesimal deformation λ of the Lie algebra $\mathcal{L}$ with finite dimensional (local) algebra base A there exists a unique homomorphism $\phi : \mathbb{K} \oplus H^2(\mathcal{L};\mathcal{L})' \to A$ such that λ is equivalent to the push-out $\phi_*\eta_{\mathcal{L}}$.

Although in general it is impossible to construct a universal formal deformation, there is a so-called versal element.

Definition 2.4. A formal deformation η of $\mathcal{L}$ parameterized by a complete local algebra B is called *versal* if for any deformation λ, parameterized by a complete local algebra (A, m_A), there is a morphism $f : B \to A$ such that

1) The push-out $f_*\eta$ is equivalent to λ.
2) If A satisfies ${m_A}^2 = 0$, then f is unique (see [5, 9]).

Remark. A versal formal deformation is sometimes called miniversal.

Theorem 2.5. *([6],[9, Thm. 4.6]) Let the space* $\mathrm{H}^2(\mathcal{L};\mathcal{L})$ *be finite dimensional.*
(a) There exists a versal formal deformation of $\mathcal{L}$*.*
(b) The base of the versal formal deformation is formally embedded into $\mathrm{H}^2(\mathcal{L};\mathcal{L})$*, i.e. it can be described in* $\mathrm{H}^2(\mathcal{L};\mathcal{L})$ *by a finite system of formal equations.*

A Lie algebra $\mathcal{L}$ is called (infinitesimally, formally, or globally) rigid if every (infinitesimal, formal, global) family is equivalent to a trivial one. Assume $\mathrm{H}^2(\mathcal{L},\mathcal{L}) < \infty$ in the following. By Proposition 2.3 the elements of $H^2(\mathcal{L};\mathcal{L})$ correspond bijectively to the equivalence classes of infinitesimal deformations, as equivalent deformations up to order 1 differ from each other only in coboundary. Together with Theorem 2.5, Part (b), follows

Corollary 2.6.
(a) $\mathcal{L}$ *is infinitesimally rigid if and only if* $\mathrm{H}^2(\mathcal{L},\mathcal{L}) = \{0\}$*.*
(b) $H^2(\mathcal{L};\mathcal{L}) = \{0\}$ *implies that* $\mathcal{L}$ *is formally rigid.*

Let us stress the fact, that $\mathrm{H}^2(\mathcal{L},\mathcal{L}) = \{0\}$ does not imply that every global deformation will be equivalent to a trivial one. Hence, $\mathcal{L}$ is in this case not necessarily globally rigid. In Section 5 we will see plenty of nontrivial global deformations of the Witt algebra $\mathcal{W}$. Hence, the Witt algebra is not globally rigid. In the next section we will present the proof of $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = \{0\}$, which implies infinitesimal and formal rigidity of $\mathcal{W}$.

3. Formal rigidity of the Witt and Virasoro algebras

As we pointed out in the Introduction, in formal deformation theory cohomology is an important tool.

The Lie algebras considered in this paper are infinite dimensional. Such Lie algebras possess a topology with respect to which all algebraic operations are continuous. In this situation, in a cochain complex it is natural to distinguish the sub-complex formed by the continuous cohomology of the Lie algebra (see [3]).

It is known (see [7]) that the Witt and the Virasoro algebra are formally rigid in sense introduced in Section 2. The statement follows from a general result of Tsujishita [30], combined with results of Goncharova [14]. The goal of this section is to explain the relation in more detail.

First we recall the result of Tsujishita. Recall that in this article the Witt algebra $\mathcal{W}$ is defined to be the complexification of the Lie algebra of polynomial vector fields on S^1. They constitute a dense subalgebra of the algebra $\operatorname{Vect} S^1$ of all smooth vector fields $\operatorname{Vect} S^1$. The results of Tsujishita concerns the continuous cohomology of $\operatorname{Vect} S^1$ with values in formal tensor fields.

In fact he deals with the cohomology of the algebra of vector fields on general smooth compact manifold, but we only need his result in case of the unit circle S^1. To formulate his results we have to introduce the space $Y(S^1)$ which is defined as follows. Let us consider the trivial principal $U(1)$-bundle $u(S^1)$, associated with the complexification of the real tangent bundle of S^1, and let $U(S^1)$ be its total space. Denote by $x(S^1)$ the trivial principal bundle $S^1 \times S^3 \to S^1$ with structural group $SU(2)$ and base S^1 and let ΩS^3 be the loop space of S^3. The space of sections $\operatorname{Sec} x(S^1)$ of the bundle $x(S^1)$ is the space $\operatorname{Map}(S^1, S^3) = S^3 \times \Omega S^3$. Consider $u(S^1)$ as a subbundle of $x(S^1)$. Now $Y(S^1)$ is the space

$$Y(S^1) := \{(y,s) \in S^1 \times \operatorname{Sec} x(S^1) \mid s(y) \in U(S^1)\}. \tag{3.1}$$

The space $Y(S^1)$ is homeomorphic to $S^1 \times S^1 \times \Omega S^3$, as can be seen as follows. We note that $s(y) \in y \times U(S^1)$, so we can write the section in the form $s(u) = (u, f(u))$, $u \in S^1$, where $f : S^1 \to S^3$, $f(u) \in U(1) \subset S^3$. Now let h be f right translated by $f(1)^{-1}$, i.e. $h(u) = f(u)f(1)^{-1}$. Then h takes 1 to 1 in S^3 and we get the required mapping from $Y(S^1)$ to $S^1 \times S^1 \times \Omega S^3$. On the other hand, take $y \in S^1$, $z \in S^1 = U(1)$ and a loop $h \in \Omega S^3$ such that $h : S^1 \to S^3$, $h(1) = 1$. Then the section $s(y) = (y, h(y)[h(y)^{-1}z])$ defines an element of $Y(S^1)$.

Theorem 3.1. *(Tsujishita* [30]*, Reshetnikov* [19]*)*
The cohomology ring $\mathrm{H}^*(\operatorname{Vect} S^1; C^\infty(S^1))$ *is isomorphic to* $\mathrm{H}^*(Y(S^1);\mathbb{R})$.

The real (topological) cohomology ring $\mathrm{H}^*(Y(S^1);\mathbb{R})$ of the space $Y(S^1)$ is known to be the free skew-symmetric $\mathbb{R}$-algebra $S(t,\theta,\xi)$, where $\deg t = \deg\theta = 1, \deg\xi = 2$. Hence $\mathrm{H}^*(Y(S^1);\mathbb{R}) \cong S(t,\theta,\xi)$ as graded algebra.

Theorem 3.2. *(Tsujishita* [30]*) For an arbitrary tensor* $\mathfrak{gl}(n,\mathbb{R})$*-module* A *and the space* $\mathcal{A}$ *of the corresponding formal tensor fields,* $\mathrm{H}^*(\operatorname{Vect} S^1; \mathcal{A})$ *is isomorphic to the tensor product of the the ring* $\mathrm{H}^*(Y(S^1);\mathbb{R})$ *and* $\operatorname{Inv}_{\mathfrak{gl}(n,\mathbb{R})}(\mathrm{H}^*(L_1)\otimes A)$*, where* L_1 *denotes the subalgebra of* $\mathcal{W}$ *with basis* $(l_1, l_2, ...)$.

See the book of Fuchs [10] concerning this form of the theorem and for related results.

Hence, for computing the cohomology ring, we need to know the cohomology (with trivial coefficients) of the Lie algebra L_1. This is computed by Goncharova [14]. She computed the cohomology spaces for all Lie algebras L_k with basis $(l_k, l_{k+1}, ...)$, but we will only state the result we need now. We point out that her computation is carried-out for graded cohomology.

Let $\mathrm{H}^q_{(s)}$ be the s-homogenous part of the cohomology space H^q where the grading is induced by the grading of $\mathcal{W}$, i.e. by $\deg l_n = n$.

Theorem 3.3. *(Goncharova [14]) For $q \geq 0$, the dimension of the graded cohomology spaces is:*

$$\dim \mathrm{H}^q_{(s)}(L_1) = \begin{cases} 1, & s = \frac{3q^2 \pm q}{2}, \\ 0, & s \neq \frac{3q^2 \pm q}{2}. \end{cases} \tag{3.2}$$

For the manifold S^1, all $\mathfrak{gl}(1,\mathbb{R})$-modules of formal tensor fields are of the form $C^\infty(S^1)d\varphi^s$ for some integer s, where φ is the angular coordinate on the circle. Using Goncharova's and Tsujishita's result we obtain

Theorem 3.4. *For $q \geq 0$*

$$\mathrm{H}^q(\mathrm{Vect}\, S^1; C^\infty(S^1)d\varphi^s) = \begin{cases} \mathrm{H}^{q-r}(Y(S^1);\mathbb{R}), & s = \frac{3r^2 \pm r}{2}, \\ \{0\}, & s \neq \frac{3r^2 \pm r}{2}. \end{cases} \tag{3.3}$$

In particular,

Corollary 3.5. *In case $s = -1$, we have*

$$\mathrm{H}^*(\mathrm{Vect}\, S^1, \mathrm{Vect}\, S^1) = 0.$$

Especially, $\mathrm{H}^2(\mathrm{Vect}\, S^1, \mathrm{Vect}\, S^1) = 0$, *so the algebra* $\mathrm{Vect}\, S^1$ *is formally rigid.*
Consequently, for the algebra of complexified vector fields $\mathrm{Vect}\, S^1 \otimes \mathbb{C}$ *we have*
$\mathrm{H}^2(\mathrm{Vect}\, S^1 \otimes \mathbb{C}, \mathrm{Vect}\, S^1 \otimes \mathbb{C}) = 0$, *and hence* $\mathrm{Vect}\, S^1 \otimes \mathbb{C}$ *is formally rigid as well.*

Corollary 3.6.
(a) For the Witt algebra $\mathcal{W}$ we have $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = 0$, hence the Witt algebra is formally rigid.
(b) For the Virasoro algebra $\mathcal{V}$ we have $\mathrm{H}^2(\mathcal{V},\mathcal{V}) = 0$, hence the Virasoro algebra is formally rigid

Proof. The algebra $\mathcal{W}$ is the subalgebra of complexified polynomial vector fields of $\mathrm{Vect}\, S^1 \otimes \mathbb{C}$. By density arguments $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = 0$ in the graded sense and the formal rigidity follows from Corollary 2.6. The algebra $\mathcal{V}$ is a one-dimensional central extension of $\mathcal{W}$. Using the Serre-Hochschild spectral sequence we obtain that $\mathcal{V}$ as a $\mathcal{V}$-module is an extension of $\mathcal{W}$ as a $\mathcal{W}$-module. Statement (b) then follows form the long exact cohomology sequence. □

4. Krichever-Novikov algebras

4.1. **The algebras with their almost-grading.** Algebras of Krichever-Novikov types are generalizations of the Virasoro algebra and all its related algebras. In this section we only recall the definitions and facts needed here. Let M be a compact Riemann surface of genus g, or in terms of algebraic geometry, a smooth projective curve over $\mathbb{C}$. Let $N, K \in \mathbb{N}$ with $N \geq 2$ and $1 \leq K < N$ be numbers. Fix

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_{N-K})$$

disjoint ordered tuples of distinct points ("marked points", "punctures") on the curve. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets and set $A = I \cup O$ as a set.

Denote by $\mathcal{L}$ the Lie algebra consisting of those meromorphic sections of the holomorphic tangent line bundle which are holomorphic outside of A, equipped with the Lie bracket $[.,.]$ of vector fields. Its local form is

$$[e,f]_{|} = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] := \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz} \ . \tag{4.1}$$

To avoid cumbersome notation we will use the same symbol for the section and its representing function.

For the Riemann sphere ($g = 0$) with quasi-global coordinate z, $I = \{0\}$ and $O = \{\infty\}$, the introduced vector field algebra is the Witt algebra. We denote for short this situation as the *classical situation.*

For infinite dimensional algebras and modules and their representation theory a graded structure is usually of importance to obtain structure results. The Witt algebra is a graded Lie algebra. In our more general context the algebras will almost never be graded. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure, will be enough to develop an interesting theory of representations (Verma modules, etc.).

Definition 4.1. Let $\mathcal{A}$ be an (associative or Lie) algebra admitting a direct decomposition as vector space $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$. The algebra $\mathcal{A}$ is called an *almost-graded* algebra if (1) $\dim \mathcal{A}_n < \infty$ and (2) there are constants R and S with

$$\mathcal{A}_n \cdot \mathcal{A}_m \quad \subseteq \bigoplus_{h=n+m+R}^{n+m+S} \mathcal{A}_h, \qquad \forall n,m \in \mathbb{Z} \ . \tag{4.2}$$

The elements of $\mathcal{A}_n$ are called *homogeneous elements of degree n.*

For the 2-point situation for M a higher genus Riemann surface and $I = \{P\}$, $O = \{Q\}$ with $P, Q \in M$, Krichever and Novikov [17] introduced an almost-graded structure of the vector field algebras $\mathcal{L}$ by exhibiting a special basis and defining their elements to be the homogeneous elements. In [21, 22, 23, 24] its multi-point generalization was given, again by exhibiting a special basis. Essentially, this is done by fixing their order at the points in I and O in a complementary way. For every $n \in \mathbb{Z}$, and $i = 1, \ldots, K$ a certain element $e_{n,p} \in \mathcal{L}$ is exhibited. The $e_{n,p}$ for $p = 1, \ldots, K$ are a basis of a subspace $\mathcal{L}_n$ and it is shown that $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$.

Proposition 4.2. [21, 24] *With respect to the above introduced grading the Lie algebras $\mathcal{L}$ are almost-graded. The almost-grading depends on the splitting $A = I \cup O$.*

In the following we will have explicit description of the basis elements for certain genus zero and one situation. Hence, we will not recall their general definition.

4.2. **Central extensions.** To obtain the equivalent of the Virasoro algebra we have to consider central extensions of the algebras. Central extensions are given by elements of $\mathrm{H}^2(\mathcal{L}, \mathbb{C})$. The usual definition of the Virasoro cocycle is not coordinate independent. We have to introduce a projective connection R.

Definition 4.3. Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates, with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$. A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) projective connection if it transforms as

$$R_\beta(z_\beta)\cdot(f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{4.3}$$

the Schwartzian derivative. Here $'$ denotes differentiation with respect to the coordinate z_α.

Every Riemann surface admits a holomorphic projective connection R [15]. From (4.3) it follows that the difference of two projective connections will be a quadratic differential. Hence, after fixing one projective connection all others are obtained by adding quadratic differentials.

For the vector field algebra $\mathcal{L}$ the 2-cocycle

$$\gamma_{S,R}(e,f) := \frac{1}{24\pi\mathrm{i}}\int_{C_S}\left(\frac{1}{2}(e'''f - ef''') - R\cdot(e'f - ef')\right)dz\ . \tag{4.4}$$

defines a central extension. Here C_S is a cycle separating the in-points from the out-points. In particular, C_S can be taken to be $C_S = \sum_{i=1}^K C_i$ where the C_i are deformed circles around the points in I. Recall that we use the same letter for the vector field and its local representing function.

Theorem 4.4. [26, 21]
(a) The cocycle class of $\gamma_{S,R}$ does not depend on the chosen connection R.
(b) The cocycle $\gamma_{S,R}$ is cohomologically non-trivial.
(c) The cocycle $\gamma_{S,R}$ is local, i.e. there exists an $M \in \mathbb{Z}$ such that

$$\forall n,m: \quad \gamma(\mathcal{L}_n,\mathcal{L}_m) \neq 0 \implies M \leq n+m \leq 0.$$

(d) Every local cocycle for $\mathcal{L}$ is either a coboundary or a scalar multiple of (4.4) with R a meromorphic projective connection which is holomorphic outside A.

The central extension $\widehat{\mathcal{L}}$ can be given via $\widehat{\mathcal{L}} = \mathbb{C}\oplus\mathcal{L}$ with Lie structure (using the notation $\widehat{e} = (0,e)$, $c = (1,0)$)

$$[\widehat{e},\widehat{f}] = \widehat{[e,f]} + \gamma_{S,R}\,c, \quad [c,\mathcal{L}] = 0. \tag{4.5}$$

Using the locality, by defining $\deg c := 0$, the almost-grading can be extended to the central extension $\widehat{\mathcal{L}}$.

Note that Theorem 4.4 does not claim that there is only one non-trivial cocycle class (which in general is not true). It only says that there is up to multiplication with a scalar only one class such that it contains cocycles which are local with respect to the almost-grading. Recall that the almost-grading is given by the splitting of A into $I \cup O$.

5. The algebra for the elliptic curve case

5.1. **The family of elliptic curves.** We consider the genus one case, i.e. the case of one-dimensional complex tori or equivalently the elliptic curve case. We have degenerations in mind. Hence it is more convenient to use the purely algebraic picture. Recall that the elliptic curves can be given in the projective plane by

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3, \quad g_2, g_3 \in \mathbb{C}, \quad \text{with } \Delta := {g_2}^3 - 27{g_3}^2 \neq 0. \tag{5.1}$$

The condition $\Delta \neq 0$ assures that the curve will be nonsingular. Instead of (5.1) we can use the description

$$Y^2 Z = 4(X - e_1 Z)(X - e_2 Z)(X - e_3 Z) \tag{5.2}$$

with

$$e_1 + e_2 + e_3 = 0, \quad \text{and} \quad \Delta = 16(e_1 - e_2)^2 (e_1 - e_3)^2 (e_2 - e_3)^2 \neq 0. \tag{5.3}$$

These presentations are related via

$$g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3), \quad g_3 = 4(e_1 e_2 e_3). \tag{5.4}$$

The elliptic modular parameter classifying the elliptic curves up to isomorphy class is given as

$$j = 1728\, \frac{g_2^3}{\Delta} \tag{5.5}$$

We set

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0, \quad e_i \neq e_j \text{ for } i \neq j\}. \tag{5.6}$$

In the product $B \times \mathbb{P}^2$ we consider the family of elliptic curves $\mathcal{E}$ over B defined via (5.2). The family can be extended to

$$\widehat{B} := \{e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0\}. \tag{5.7}$$

The fibers above $\widehat{B} \setminus B$ are singular cubic curves. Resolving the one linear relation in $\widehat{B}$ via $e_3 = -(e_1 + e_2)$ we obtain a family over $\mathbb{C}^2$.

Consider the complex lines in $\mathbb{C}^2$

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\}, \quad s \in \mathbb{C}, \qquad D_\infty := \{(0, e_2) \in \mathbb{C}^2\}. \tag{5.8}$$

Set also

$$D_s^* = D_s \setminus \{(0,0)\} \tag{5.9}$$

for the punctured line. Now

$$B \cong \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}). \tag{5.10}$$

Note that above D_1^* we have $e_1 = e_2 \neq e_3$, above $D_{-1/2}^*$ we have $e_2 = e_3 \neq e_1$, and above D_{-2}^* we have $e_1 = e_3 \neq e_2$. In all these cases we obtain the nodal cubic. The nodal cubic E_N can be given as

$$Y^2 Z = 4(X - eZ)^2 (X + 2eZ) \tag{5.11}$$

where e denotes the value of the coinciding $e_i = e_j$ ($-2e$ is then necessarily the remaining one). The singular point is the point $(e : 0 : 1)$. It is a node. It is up to isomorphy the only singular cubic which is stable in the sense of Mumford-Deligne.

Above the unique common intersection point $(0,0)$ of all D_s there is the cuspidal cubic E_C

$$Y^2 Z = 4X^3. \tag{5.12}$$

The singular point is $(0 : 0 : 1)$. The curve is not stable in the sense of Mumford-Deligne. In both cases the complex projective line is the desingularisation, In all cases (non-singular or singular) the point $\infty = (0:1:0)$ lies on the curves. It is the only intersection with the line at infinity, and is a non-singular point. In passing to an affine chart in the following we will loose nothing.

For the curves above the points in D_s^* we calculate $e_2 = se_1$ and $e_3 = -(1+s)e_1$ (resp. $e_3 = -e_2$ if $s = \infty$). Due to the homogeneity, the modular parameter j for the curves above D_s^* will be constant along the line. In particular, the curves in the family lying above D_s^* will be isomorphic. For completeness let us write down

$$j(s) = 1728\,\frac{4(1+s+s^2)^3}{(1-s)^2(2+s)^2(1+2s)^2},\quad j(\infty) = 1728. \tag{5.13}$$

5.2. **The family of vector field algebras.** We have to introduce the points where poles are allowed. For our purpose it is enough to consider two marked points. More marked points are considered in [25, 20]. One marking we will always put to $\infty = (0:1:0)$ and the other one to the point with the affine coordinate $(e_1, 0)$. This marking defines two section of the family $\mathcal{E}$ over $\widehat{B} \cong \mathbb{C}^2$. With respect to the group structure on the elliptic curves given by ∞ as the neutral element (the first marking) the second marking chooses a two-torsion point. All other choices of two-torsion points will yield isomorphic situations.

In [25] for this situation (and for a three-point situation) a basis of the Krichever-Novikov type vector field algebras were given.

Theorem 5.1. *For any elliptic curve $E_{(e_1,e_2)}$ over $(e_1,e_2) \in \mathbb{C}^2 \setminus (D_1^* \cup D_{-1/2}^* \cup D_{-2}^*)$ the Lie algebra $\mathcal{L}^{(e_1,e_2)}$ of vector fields on $E_{(e_1,e_2)}$ has a basis $\{V_n,\ n \in \mathbb{Z}\}$ such that the Lie algebra structure is given as*

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \textit{ odd},\\ (m-n)\big(V_{n+m} + 3e_1V_{n+m-2} & \\ \qquad +(e_1-e_2)(e_1-e_3)V_{n+m-4}\big), & n, m \textit{ even},\\ (m-n)V_{n+m} + (m-n-1)3e_1V_{n+m-2} & \\ \qquad +(m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4}, & n \textit{ odd},\ m \textit{ even}. \end{cases} \tag{5.14}$$

By defining $\deg(V_n) := n$, *we obtain an almost-grading.*

Proof. This is proved in [25, Prop.3,Prop.4]. Our generators are

$$V_{2k+1} := (X-e_1)^k Y \frac{d}{dX},\quad V_{2k} := 1/2 f(X)(X-e_1)^{k-2}\frac{d}{dX}, \tag{5.15}$$

with $f(X) = 4(X-e_1)(X-e_2)(X-e_3)$. Note that V_n is the V_{n-1} given in [25]. □

The algebras of Theorem 5.1 defined with the structure (5.14) make sense also for the points $(e_1,e_2) \in D_1 \cup D_{-1/2} \cup D_{-2}$. Altogether this defines a two-dimensional family of Lie algebras parameterized over $\mathbb{C}^2$. In particular, note that we obtain for $(e_1,e_2) = 0$ the Witt algebra.

Let us remark that this two-dimensional family of geometric origin can also be written just with the symbols p and q instead of $3e_1$ and $(e_1-e_2)(e_1-e_3)$. In this form it was algebraically found by Deck [2], (see also Ruffing, Deck and Schlichenmaier [20]) as a two-dimensional family of Lie algebra. Guerrini [12, 13] related it later (again in a purely algebraic manner) to deformations of the Witt algebra over certain spaces of polynomials. Due to its geometric interpretation we prefer to use the parameterization (5.14). Further higher-dimensional families of geometric origins can be obtained if we consider the multi-point situation for the elliptic curve and degenerate the curve to the cuspidal cubic and let the marked points (beside the point at ∞) move to the singularity. But no new effects will appear.

We consider now the family of algebras obtained by taking as base variety the line D_s (for any s). First consider $s \neq \infty$. We calculate $(e_1 - e_2)(e_1 - e_3) = e_1^2(1-s)(2+s)$ and can rewrite for these curves (5.14) as

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \qquad + e_1^2(1-s)(2+s)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \qquad + (m-n-2)e_1^2(1-s)(2+s)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.16}$$

For D_∞ we have $e_3 = -e_2$ and $e_1 = 0$ and obtain

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} - e_2^2 V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} - (m-n-2)e_2^2 V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.17}$$

If we take $V_n^* = (\sqrt{e_1})^{-n} V_n$ (for $s \neq \infty$) as generators we obtain for $e_1 \neq 0$ always the algebra with $e_1 = 1$ in our structure equations. For $s = \infty$ a rescaling with $(\sqrt{e_2})^{-n} V_n$ will do the same (for $e_2 \neq 0$).

Hence we see that in all cases the algebras will be isomorphic above every point in D_s as long as we are not above $(0,0)$.

Proposition 5.2. *For $(e_1, e_2) \neq (0,0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are not isomorphic to the Witt algebra.*

Proof. Assume that we have an Lie isomorphism $\Phi : \mathcal{W} = \mathcal{L}^{(0,0)} \to \mathcal{L}^{(e_1,e_2)}$. Denote the generators of the Witt algebra by $\{l_n, \, n \in \mathbb{Z}\}$. In particular, we have $[l_0, l_n] = n l_n$ for every n. We assign to every l_n numbers $m(n) \leq M(n)$ such that $\Phi(l_n) = \sum_{k=m(n)}^{M(n)} \alpha_k(n) V_k$ with $\alpha_{m(n)}(n), \alpha_{M(n)}(n) \neq 0$. From the relation in the Witt algebra we obtain

$$[\Phi(l_0), \Phi(l_n)] = \sum_{k=m(0)}^{M(0)} \sum_{l=m(n)}^{M(n)} \alpha_k(0)\alpha_l(n)[V_k, V_l] = n \cdot \sum_{l=m(n)}^{M(n)} \alpha_l(n) V_l.$$

We can choose n in such a way that the structure constants in the expression of $[V_k, V_l]$ at the boundary terms will not vanish. Using the almost-graded structure we obtain $M(0) + M(n) = M(n)$ which implies $M(0) = 0$, and $m(0) + m(n) - 4 = m(n)$ or $m(0) + m(n) - 2 = m(n)$ (for $s = 1$ or $s = -2$) which implies $2 \leq m(0) \leq M(0) = 0$ which is a contradiction. $\square$

It is necessary to stress the fact, that in our approach the elements of the algebras are only finite linear combinations of the basis elements V_n.

In particular, we obtain a family of algebras over the base D_s, which is always the affine line. In this family the algebra over the point $(0,0)$ is the Witt algebra and the isomorphy type above all other points will be the same but different from the special element, the Witt algebra. This is a phenomena also appearing in algebraic geometry. There it is related to non-stable singular curves (which is for genus one only the cuspidal cubic). Note that it is necessary to consider the two-dimensional family introduced above to "see the full behaviour" of the cuspidal cubic E_C.

Let us collect the facts:

Theorem 5.3. *For every $s \in \mathbb{C} \cup \{\infty\}$ the families of Lie algebras defined via the structure equations (5.16) for $s \neq \infty$ and (5.17) for $s = \infty$ define global deformations $\mathcal{W}_t^{(s)}$ of the Witt algebra $\mathcal{W}$ over the affine line $\mathbb{C}[t]$. Here t corresponds to the parameter e_1 and e_2 respectively. The Lie algebra above $t = 0$ corresponds always to the Witt algebra, the algebras above $t \neq 0$ belong (if s is fixed) to the same isomorphy type, but are not isomorphic to the Witt algebra.*

If we denote by $g(s) := (1-s)(2+s)$ the polynomial appearing in the structure equations (5.16), we see that the algebras over D_s will be isomorphic to the algebras over D_t if $g(s) = g(t)$. This is the case if and only if $t = -1-s$. Under this map the lines D_∞ and $D_{-1/2}$ remain fixed. Geometrically this corresponds to interchanging the role of e_2 and e_3.

5.3. **The degenerations and the three-point algebras for genus zero.** Next we want to identify the algebras corresponding to the singular cubic situation. We have three different possibilities:

(I) All three e_1, e_2 and e_3 come together. This implies necessarily that $e_1 = e_2 = e_3 = 0$. We obtain the cuspidal cubic. The pole at $(e_1, 0)$ moves to the singular point $(0,0)$. This appears if we approach in our two-dimensional family the point $(0,0)$.

(II) If 2 but not 3 of the e_i come together we obtain the nodal cubic and we have to distinguish 2 subcases with respect to the marked point:

(IIa) $e_1 \neq e_2 = e_3$, then the point of a possible pole will remain non-singular. This appears if we approach a point of $D^*_{-1/2}$.

(IIb) Either $e_1 = e_2 \neq e_3$ or $e_1 = e_3 \neq e_2$, then the singular point (the node) will become a possible pole. This situation occurs if we approach points from $D_1^* \cup D_{-2}^*$. In the cases (IIa) and (IIb) we obtain the algebras by specializing the value of s in (5.16).

We want to identify these exceptional algebras above D_s for $s = 1, -1/2$ and -2.

First, clearly above $(0,0)$ there is always the Witt algebra corresponding to meromorphic vector fields on the complex line holomorphic outside $\{0\}$ and $\{\infty\}$. This corresponds to situation (I)

Next we consider the geometric situation $M = \mathbb{P}^1(\mathbb{C})$, $I = \{\alpha, -\alpha\}$ and $O = \{\infty\}$, $\alpha \neq 0$. As shown in [25] a basis of the corresponding Krichever-Novikov algebra is given by

$$V_{2k} := z(z-\alpha)^k(z+\alpha)^k \frac{d}{dz}, \qquad V_{2k+1} := (z-\alpha)^{k+1}(z+\alpha)^{k+1}\frac{d}{dz}, \qquad k \in \mathbb{Z}. \tag{5.18}$$

Here z is the quasi-global coordinate on $\mathbb{P}^1(\mathbb{C})$. The grading is given by $\deg(V_n) := n$. One calculates

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + \alpha^2 V_{n+m-2}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)\alpha^2 V_{n+m-2}, & n \text{ odd},\ m \text{ even}. \end{cases} \tag{5.19}$$

If we set $\alpha = \sqrt{e_1}$ we get exactly the structure for the algebras obtained in the degeneration (IIb). Hence,

Proposition 5.4. *The algebras $\mathcal{L}^\lambda$ for $\lambda \in D_1^* \cup D_{-2}^*$ are isomorphic to the algebra of meromorphic vector fields on $\mathbb{P}^1$ which are holomorphic outside $\{\infty, \alpha, -\alpha\}$.*

Finally, we consider the subalgebra of the Witt algebra defined by the basis elements

$$\begin{aligned} V_{2k} &= z^{2k-3}(z^2-\alpha^2)^2\frac{d}{dz} = l_{2k} - 2\alpha^2 l_{2k-2} + \alpha^4 l_{2k-4}, \\ V_{2k+1} &= z^{2k}(z^2-\alpha^2)\frac{d}{dz} = l_{2k+1} - \alpha^2 l_{2k-1}. \end{aligned} \tag{5.20}$$

One calculates

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n,m \text{ odd}, \\ (m-n)\big(V_{n+m} - 2\alpha V_{n+m-2} + \alpha^2 V_{n+m-4}\big), & n,m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)(-2\alpha)V_{n+m-2} & \\ \qquad +(m-n-2)\alpha^2 V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.21}$$

This is the algebra obtained by the degeneration (IIa) if we set $\alpha = \mathrm{i}\sqrt{3e_1/2}$. Hence,

Proposition 5.5. *The algebras $\mathcal{L}^\lambda$ for $\lambda \in D_{-1/2}^*$ are isomorphic to the subalgebra of the Witt algebra generated by the above basis elements.*

This subalgebra can be described as the subalgebra of meromorphic vector fields vanishing at α and $-\alpha$, with possible poles at 0 and ∞ and such that in the representation of $V(z) = f(z)(z^2-\alpha^2)\frac{d}{dz}$ the function f fulfills $f(\alpha) = f(-\alpha)$.

Clearly, as explained above, as long as $\alpha \neq 0$, by rescaling, the case $\alpha = 1$ can be obtained. Hence for $\alpha \neq 0$ the algebras are all isomorphic.

If we choose a line E in $\mathbb{C}^2$ not passing through the origin, by restricting our two-dimensional family to those algebras above E we obtain a family of algebras over an affine line. A generic line will meet D_1, D_{-2} and $D_{-1/2}$. In this way we obtain global deformations of these special algebras.

5.4. **Geometric interpretation of the deformation results.** The identification of the algebras obtained in the last subsection is not a pure coincidence. There is a geometric scheme behind, which was elaborated in [25]. To put the results in the right context we want to indicate the relation. In both cases of the singular cubic the desingularisation (which will be also the normalization) will be the projective line. The vector fields given in (5.15) make sense also for the degenerate cases. Vector fields on the singular cubic will correspond to vector fields on the normalization which have at the points lying above the singular points an additional zero.

In case of the cuspidal degeneration the possible pole moves to the singular point. Hence we will obtain the full Witt algebra. In the case of the nodal cubics we have to distinguish the two cases. If $(e_1, 0)$ will not be the singular point one obtains the subalgebra of the Witt algebra consisting of vector fields which have a zero at α and $-\alpha$ (where α is the point lying above the singular point) and fulfill the additional condition on f (see above). If $(e_1, 0)$ becomes a singular point, a pole at $(e_1, 0)$ will produce poles at the two points α and $-\alpha$ lying above $(e_1, 0)$. Hence we end up with the 3-point algebra for genus zero.

5.5. **Cohomology classes of the deformations.** Let $\mathcal{W}_t$ be a one-parameter deformation of the Witt algebra $\mathcal{W}$ with Lie structure

$$[x,y]_t = [x,y] + t^k\omega_0(x,y) + t^{k+1}\omega_1(x,y) + \cdots , \tag{5.22}$$

such that $\omega = \omega_0$ is a non-vanishing bilinear form. The form ω will be a 2-cocycle in $C^2(\mathcal{W},\mathcal{W})$. The element $[\omega] \in \mathrm{H}^2(\mathcal{W},\mathcal{W})$ will be the cohomology class characterizing the family infinitesimally. Recall that a class ω is per definition a coboundary if

$$\omega(x,y) = (d_1\Phi)(x,y) := \Phi([x,y]) - [\Phi(x),y] - [x,\Phi(y)] \tag{5.23}$$

for a linear map $\Phi : \mathcal{W} \to \mathcal{W}$. For the global deformation families $\mathcal{W}_t^{(s)}$ appearing in Theorem 5.3 we obtain with respect to the parameterization by e_1 and e_2 respectively, as first nontrivial contribution the following two cocycles.

$$\omega(l_n,l_m) = \begin{cases} 0, & n,m \text{ odd}, \\ (m-n)3\,l_{n+m-2}, & n,m \text{ even} \\ (m-n-1)3\,l_{n+m-2}, & n \text{ odd },m \text{ even}, \end{cases}$$

and (5.24)

$$\omega(l_n,l_m) = \begin{cases} 0, & n,m \text{ odd}, \\ -(m-n)l_{n+m-4}, & n,m \text{ even}, \\ -(m-n-1)l_{n+m-4}, & n \text{ odd },m \text{ even}. \end{cases}$$

From Section 3 we know that the Witt algebra is infinitesimally and formally rigid. Hence, the cohomology classes $[\omega]$ in $\mathrm{H}^2(\mathcal{W},\mathcal{W})$ must vanish. For illustration we will verify this directly. By a suitable ansatz one easily finds that $\omega = d_1\Phi$ for

$$\Phi(l_n) = \begin{cases} -3\,l_{n-2}, & n \text{ even}, \\ -3/2\,l_{n-2}, & n \text{ odd}, \end{cases} \quad \text{resp.} \quad \Phi(l_n) = \begin{cases} l_{n-4}, & n \text{ even}, \\ 1/2\,l_{n-4}, & n \text{ odd}. \end{cases} \tag{5.25}$$

From the formal rigidity of $\mathcal{W}$ we can conclude that the family $\mathcal{W}_t^s$ considered as a formal family over $\mathbb{C}[[t]]$ is equivalent to the trivial family. Hence on the formal level there is an isomorphism φ given by

$$\varphi_t(l_n) = V_n + \sum_{k=1}^{\infty} \alpha_k t^k V_{n-k}. \tag{5.26}$$

Here t is a formal variable. The formal sum (5.26) will not terminate, and even if we specialize t to a non-zero number, the element $\varphi_t(l_n)$ will not live in our Krichever-Novikov algebra.

5.6. **Families of the centrally extended algebras.** In all families considered above it is quite easy to incorporate a central term as defined via the local cocycle (4.4). In the genus one case with respect to the standard flat coordinates $(z-a)$ the projective connection $R \equiv 0$ will do. The difference of two projective connections will be a quadratic differential. Hence, we obtain that any local cocycle is either a coboundary or can be obtained as a scalar multiple of (4.4) with a suitable meromorphic quadratic differential R which has only poles at A. The integral (4.4) is written in the complex analytic picture. But the integration over a separating cycle can be given by integration over circles around the points where poles are allowed. Hence, it is given as sum of residues and the cocycle makes perfect sense in the purely algebraic picture. For the explicit calculation of the residue it is useful to use the fact that for tori $T = \mathbb{C}/\Lambda$ with lattice $\Lambda = \langle 1, \tau\rangle_{\mathbb{Z}}$, $\operatorname{Im}\tau > 0$, the complex analytic picture is isomorphic to the algebraic elliptic curve picture via

$$\bar{z} = z \bmod \Lambda \to \begin{cases} (\wp(z) : \wp'(z) : 1), & \bar{z} \neq 0, \\ (0:1:0) = \infty, & \bar{z} = 0. \end{cases} \tag{5.27}$$

Here $\wp$ denotes the Weierstraß $\wp$-function. Recall that the points where poles are allowed in the algebraic picture are ∞ and $(e_1 : 0 : 1)$. They correspond to $\bar{0}$ and $\overline{\frac{1}{2}}$ respectively in the analytic picture. The point $\overline{\frac{1}{2}}$ is a 2-torsion point. Replacing $\frac{1}{2}$ by any one of the other 2-torsion points $\frac{\tau}{2}$ and $\frac{\tau+1}{2}$ respectively, yields isomorphic structures.

Let $\mathcal{W}_t^{(s)}$ be any of the above considered families parameterised over $D_s^* \cong \mathbb{C}\setminus\{0\}$ with parameter t such that $t = 0$ corresponds to $(0,0)$. Then $\widehat{\mathcal{W}_t^{(s)}} = \mathbb{C} \oplus \mathcal{W}_t^{(s)}$ with $\widehat{x} = (0, x)$, $c = (1, 0)$ with

$$[\widehat{x}, \widehat{y}] = \widehat{[x, y]} + \widetilde{c}(t)\gamma_{S,R_t}(x, y)\cdot c, \qquad [\widehat{x}, c] = 0, \tag{5.28}$$

for $\widetilde{c} : \mathbb{C} \to \mathbb{C}$ a non-vanishing algebraic function, and R_t a family of quadratic differentials varying algebraically with respect to t, will define a family of centrally extended algebras.

As shown above the non-extended algebras for fixed s are mutually isomorphic. Recall from Section 4.2 that choosing a different R_t gives only a different cocycle in the same cohomology class and that $\widetilde{c}(t)$ as long as $\widetilde{c}(t) \neq 0$ is just a rescaling of the central element. Hence, we obtain that also the centrally extended algebras are mutually isomorphic over D_s^*.

The cocycle (4.4) expressed as residue and calculated at the point ∞ makes perfect sense also for $t = 0$. For $t = 0$ it will yield the Virasoro cocycle. In this way we obtain a nontrivial deformation family for the Virasoro algebra. Clearly, to obtain examples we might directly take $c \equiv 1$ and $R_t \equiv 0$.

Remark. A typical appearing in 2-dimensional conformal field theory of centrally extended vector field algebras is via the Sugawara representation, i.e. by the modes of the energy-momentum tensor in representations of affine algebras (gauge algebras) or of algebras of $b-c$ systems. The classical constructions extend also to the higher genus multi-point situation, [27, 21], i.e. if the 2-dimensional conformal field theory is considered for higher genus Riemann surfaces. If one studies families of such systems varying with the moduli parameters, corresponding to deformation of the complex structure and moving the "insertions points", one obtains in a natural way families of centrally extended algebras. In these cases $\widetilde{c}(t)$ and R_t might vary. In [2] and [20] for $b-c$-systems explicit formulas for the central term are given.

5.7. **Deformations of the Lie algebra L_1.** Let L_1 be the subalgebra of the Witt algebra consisting of those vector fields which vanish of order ≥ 2 at 0, i.e. $L_1 = \langle l_n \mid n \geq 1\rangle$. It was shown by Fialowski in [4] that this algebra is not formally rigid, and that there are three independent formal one-parameter deformations. They correspond to pairwise non-equivalent deformations. Indeed any formal one-parameter deformation of L_1 can be reduced by a formal parameter change to one of these deformations (see also [8]):

$$\begin{aligned}
[l_n,l_m]_t^{(1)} &:= (m-n)(l_{n+m}+tl_{n+m-1});\\
[l_n,l_m]_t^{(2)} &:= \begin{cases}(m-n)l_{n+m}, & n\neq 1, m\neq 1\\ (m-1)l_{m+1}+tml_m, & n=1, m\neq 1;\end{cases}\\
[l_n,l_m]_t^{(3)} &:= \begin{cases}(m-n)l_{n+m}, & n\neq 2, m\neq 2\\ (m-2)l_{m+2}+tml_m, & n=2, m\neq 2.\end{cases}
\end{aligned} \tag{5.29}$$

The cocycles representing the infinitesimal deformations are given by

$$\begin{aligned}
\beta^{(1)}(l_n,l_m) &:= (m-n)l_{n+m-1};\\
\beta^{(2)}(l_n,l_m) &:= \begin{cases}ml_m, & n=1, m\neq 1\\ 0, & n\neq 1, m\neq 1;\end{cases}\\
\beta^{(3)}(l_n,l_m) &:= \begin{cases}ml_m, & n=2, m\neq 2\\ 0, & n\neq 2, m\neq 2.\end{cases}
\end{aligned} \tag{5.30}$$

It is shown in the above cited articles that the cohomology classes $[\beta^{(1)}]=[\beta^{(2)}]=0$ and $[\beta^{(3)}]\neq 0$.

In our geometric situation we consider the algebra $W_{1,\alpha^2} := \langle V_n \mid n\geq 1\rangle$ with the structure equations (5.19). If we vary α we obtain a family $\mathcal{W}_{1,\alpha^2}$. These algebras correspond to the algebra of vector fields on $\mathbb{P}^1(\mathbb{C})$ which have a pole at most at the point ∞ and zeros of order at least 1 at the points α and $-\alpha$. By Proposition 5.4 we know that as long as $\alpha\neq 0$ it is isomorphic to the corresponding subalgebra of the vector field algebra $\mathcal{L}^\lambda$ for $\lambda\in D_1^*\cup D_{-2}^*$. As long as $\alpha\neq 0$ we can rescale and obtain that all these subalgebras belong to the same isomorphy type.

Proposition 5.6. *The algebras $\mathcal{W}_{1,\alpha^2}$ for $\alpha\neq 0$ are not isomorphic to the algebra L_1.*

Proof. The proof again uses the almost-graded (respectively, graded) structure as in the proof of Proposition 5.2. By the missing of l_0 some additional steps are needed. We will only sketch them. Assume that there is an isomorphism $\phi : L_1 \to \mathcal{W}_{1,\alpha^2}$. From the structure (5.19) we conclude that $M(n) = nM(1)$ (notation as in the above-mentioned proof) with $M(1)\in\mathbb{N}$. If we assume $M(1)>1$, the basis element V_1 will not be in $\phi(L_1)$. So ϕ cannot be an isomorphism. Hence, $M(1)=1$. Now $\phi(l_1)=\alpha_1V_1$, $\phi(l_2)=\alpha_2V_2+\alpha_{2,2}V_1$ and in further consequence from $[l_1,l_2]=2l_3$ and $[l_1,l_3]=3l_4$ it follows that $\phi(l_3)=\alpha_3V_3$ and $\phi(l_4)=\alpha_4V_4$. The relations $[l_1,l_4]=3l_5$ and $[l_3,l_2]=-l_5$ in L_1 lead under ϕ to two relations in $\mathcal{W}_{1,\alpha^2}$ which are in contradiction. Hence there is no such ϕ.
Note that an alternative way to see the statement is to use Proposition 5.8 further down. □

In this way we obtain a non-trivial global deformation family $\mathcal{W}_{1,t}$ for the algebra L_1. For its 2-cocycle we calculate

$$\omega(l_n,l_m) = \begin{cases}0, & n,m \text{ odd},\\ (m-n)l_{n+m-2}, & n,m \text{ even},\\ (m-n-1)l_{n+m-2}, & n \text{ odd},\ m \text{ even}.\end{cases} \tag{5.31}$$

Again with a suitable ansatz we find with

$$\Phi(l_m) := \begin{cases} -\frac{m+8}{6}\, l_{m-2}, & m \text{ even},\ m \geq 4, \\ -\frac{m+5}{6}\, l_{m-2}, & m \text{ odd},\ m \geq 3, \\ 0, & m = 1, m = 2. \end{cases} \tag{5.32}$$

that $\omega - d_1\Phi = \frac{1}{3}\beta^{(3)}$.

From the structure equations (5.19) we can immediately verify

Lemma 5.7. *For $\alpha \neq 0$ the commutator ideal calculates to*

$$[W_{1,\alpha^2}, W_{1,\alpha^2}] = \langle V_n \mid n \geq 3\rangle, \tag{5.33}$$

and we have

$$\dim W_{1,\alpha^2}/[W_{1,\alpha^2}, W_{1,\alpha^2}] = 2. \tag{5.34}$$

Proposition 5.8. *The family $\mathcal{W}_{1,\alpha^2}$ is formally equivalent to the first family $[.,.]^1_t$ in (5.29).*

Proof. Clearly $\mathcal{W}_{1,\alpha^2}$ defines a formal family. From the above calculation we obtain $[\omega] = 1/3[\beta^{(3)}]$. But $[\beta^{(3)}] \neq 0$, hence considered as formal family it is also non-trivial. By the results about the versal family of L_1 it has to be equivalent to one of the three families of (5.29). Only the first family $L_1^{(1)}$ has $\dim L_1^{(1)}/[L_1^{(1)}, L_1^{(1)}] = 2$, for the other two this dimension equals one. By Lemma 5.7 we obtain the claim. □

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