Visible contours of cubic surfaces in $\mathbb{R}P^3$

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VISIBLE CONTOURS OF CUBIC SURFACES IN $\mathbb{R}P^3$

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ABSTRACT. The paper describes the pictures that we can see if we look at cubic surfaces from different points of view in $\mathbf{R}P^3$. The paper lists 47 different isotopy types of plane curves which appear as stable visible contours of cubic surfaces in $\mathbf{R}P^3$ The question whether the 7 isotopy types pictured on fig.7, fig.8 and fig.9 appear as visible contours of cubic surfaces is left open. For cubic surfaces diffeomorphic to $\mathbf{R}P^2 \sqcup S^2$ and $\mathbf{R}P^2$ the paper gives the complete classification of stable visible contours up to diffeomorphism of $\mathbf{R}P^2$. The proof is omitted in the preprint.

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a (nonsingular) homogeneous polynomial of degree 3. The surface

$$\mathbb{R}C = \{y \in \mathbb{R}P^3 \mid f(y) = 0\}$$

is called a (nonsingular) *cubic surface* in $\mathbb{R}P^3$. A point $x \in \mathbb{R}P^3$ defines a projection

$$\mathbb{R}P^3 - \{x\} \to \mathbb{R}P^2.$$

If $x \in \mathbb{R}P^3 - \mathbb{R}C$ then the restriction of this projection to $\mathbb{R}C$ gives a map

$$p : \mathbb{R}C \to \mathbb{R}P^2.$$

The singular values of p form a curve $\mathbb{R}A \subset \mathbb{R}P^2$, this is the curve where p folds (see fig.1). The curve $\mathbb{R}A$ is what we see if we look at $\mathbb{R}C$ from p and it is called the *visible contour* of $\mathbb{R}C$.

We are interested in topological pictures of generic (or *stable*) visible contours. Thus we assume that the cubic $\mathbb{R}C$ is nonsingular and the point x (the point of view) is generic. But even under these assumptions the visible contour $\mathbb{R}A$ may be singular. Its complexification $\mathbb{C}A$ which is formed by singular values of the complexified projection $p_{\mathbb{C}} : \mathbb{C}C \to \mathbb{C}P^2$ is always singular. The singularities of $\mathbb{C}A$ are six cusps sitting on the same conic in $\mathbb{C}P^2$ and some of these cusps (an even number of them) may be real (see fig.1).

The diffeomorphism type of the nonsingular cubic $\mathbb{R}C$ is one of the following:

$$\mathbb{R}P^2 \sqcup S^2,\ \mathbb{R}P^2,\ \mathbb{R}P^2\#T,\ \mathbb{R}P^2\#2T,\ \mathbb{R}P^2\#3T,$$

where $\#nT$ denotes the attaching of n copies of the handle $T \approx S^1 \times S^1$ to $\mathbb{R}P^2$. By isotopy type of a (possibly singular) curve in $\mathbb{R}P^2$ we mean the diffeomorphism type of the pair which consists of $\mathbb{R}P^2$ and the curve.

Theorem 1. *If a cubic surface in $\mathbb{R}P^3$ is diffeomorphic to $\mathbb{R}P^2 \sqcup S^2$ then its stable visible contour is of the isotopy type pictured on fig.2. Each of the isotopy types of fig.2 appears as a visible contour of a cubic surface in $\mathbb{R}P^3$ diffeomorphic to $\mathbb{R}P^2 \sqcup S^2$.*

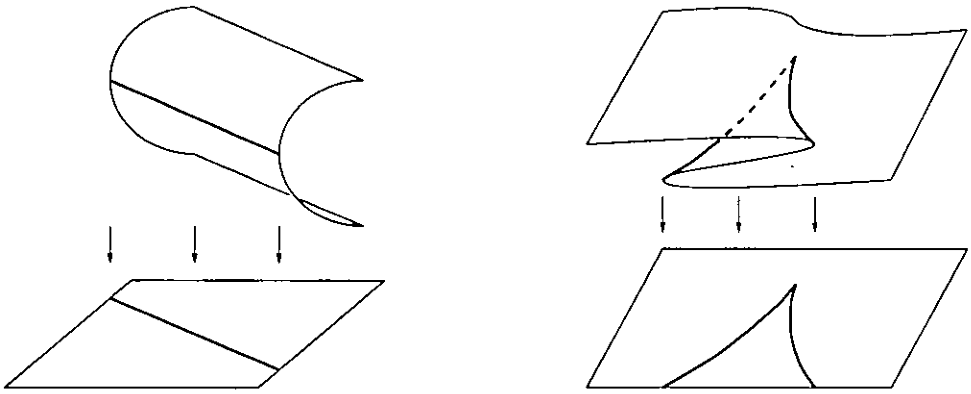

FIGURE 1. a folding curve and a cusp

Theorem 2. *If a cubic surface in $\mathbb{R}P^3$ is diffeomorphic to $\mathbb{R}P^2$ then its stable visible contour is of the isotopy type pictured on fig.3. Each of the isotopy types of fig.3 appears as a visible contour of a cubic surface in $\mathbb{R}P^3$ diffeomorphic to $\mathbb{R}P^2$.*

Theorem 3. *If a cubic surface in $\mathbb{R}P^3$ is diffeomorphic to $\mathbb{R}P^2\#T$ then its stable visible contour is of the isotopy type pictured on fig.4 or fig.7. Each of the isotopy types of fig.4 appears as a visible contour of a cubic surface in $\mathbb{R}P^3$ diffeomorphic to $\mathbb{R}P^2\#T$.*

Theorem 4. *If a cubic surface in $\mathbb{R}P^3$ is diffeomorphic to $\mathbb{R}P^2\#2T$ then its stable visible contour is of the isotopy type pictured on fig.5 or fig.8. Each of the isotopy types of fig.5 appears as a visible contour of a cubic surface in $\mathbb{R}P^3$ diffeomorphic to $\mathbb{R}P^2\#2T$.*

Theorem 5. *If a cubic surface in $\mathbb{R}P^3$ is diffeomorphic to $\mathbb{R}P^2\#3T$ then its stable visible contour is of the isotopy type pictured on fig.6 or fig.9. Each of the isotopy types of fig.6 appears as a visible contour of a cubic surface in $\mathbb{R}P^3$ diffeomorphic to $\mathbb{R}P^2\#3T$.*

Remark. There are only two isotopy types of curves in $\mathbb{R}P^2$ which appear as stable visible contours of non-diffeomorphic cubics in $\mathbb{R}P^3$. These are the emptyset and the isotopy type of a circle in $\mathbb{R}P^2$. They appear as stable visible contours of a cubic diffeomorphic to $\mathbb{R}P^2 \sqcup S^2$ (**7.** and **4.** of fig.2) and a cubic diffeomorphic to $\mathbb{R}P^2$ (**12.** and **8.** of fig.3). In all other cases the diffeomorphism type of a cubic surface is determined by the isotopy type of its stable visible contour.

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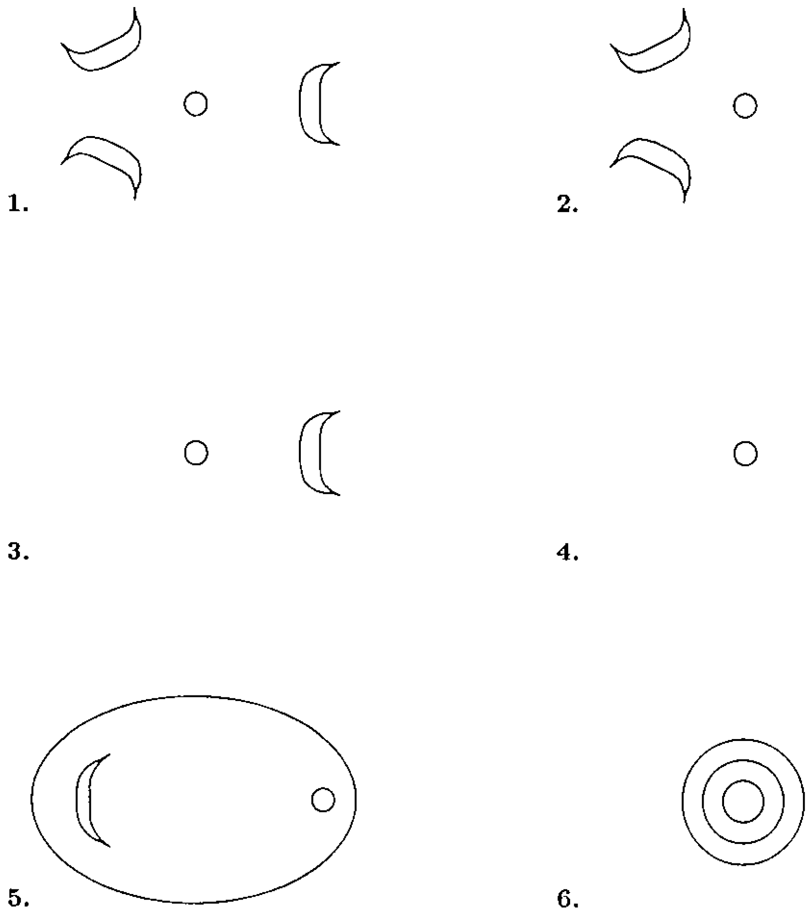

7. (emptyset)

FIGURE 2. $\mathbb{R}P^2 \sqcup S^2$

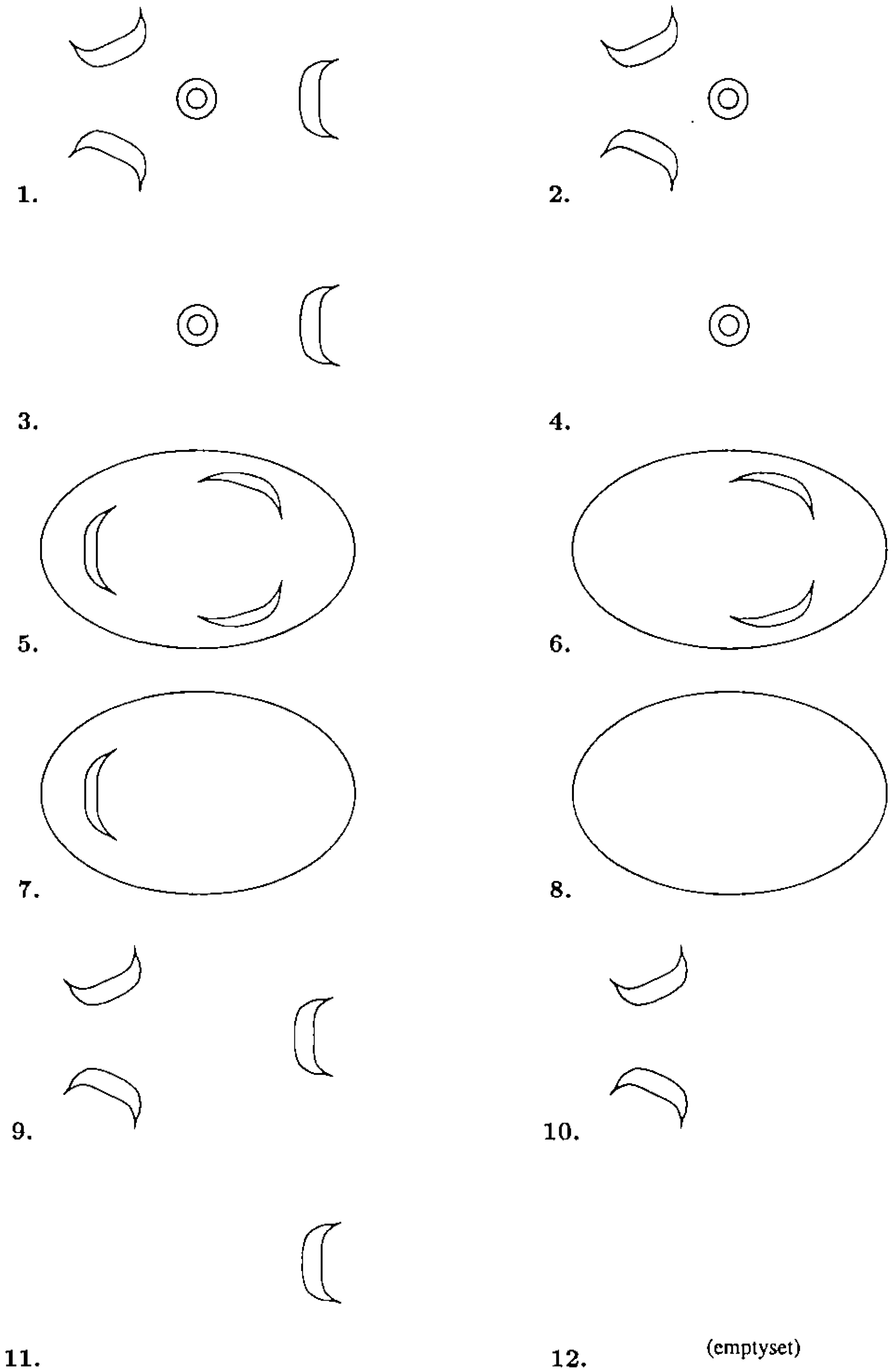

FIGURE 3. $\mathbb{R}P^2$

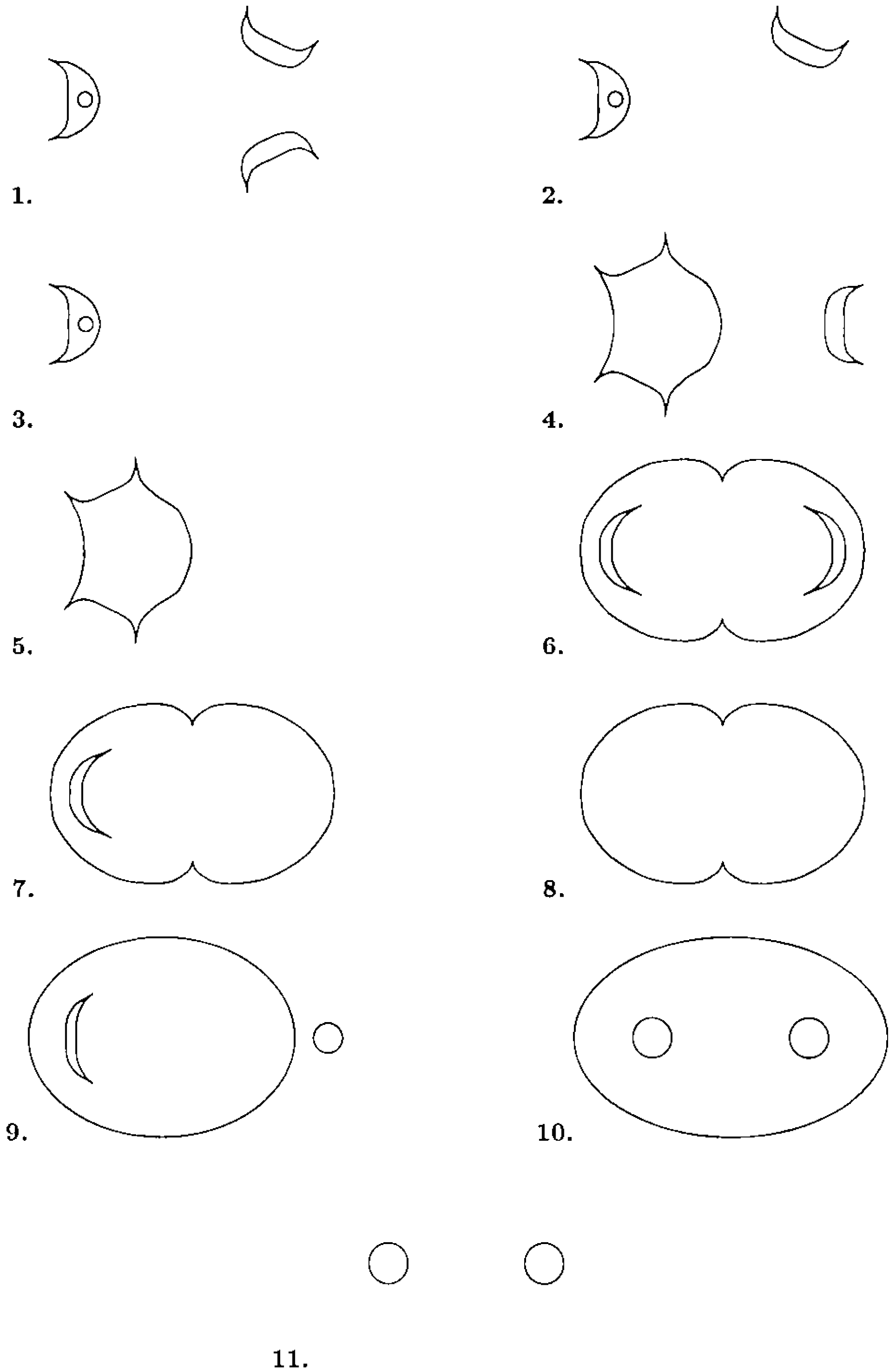

FIGURE 4. $\mathbb{R}P^2 \# T$

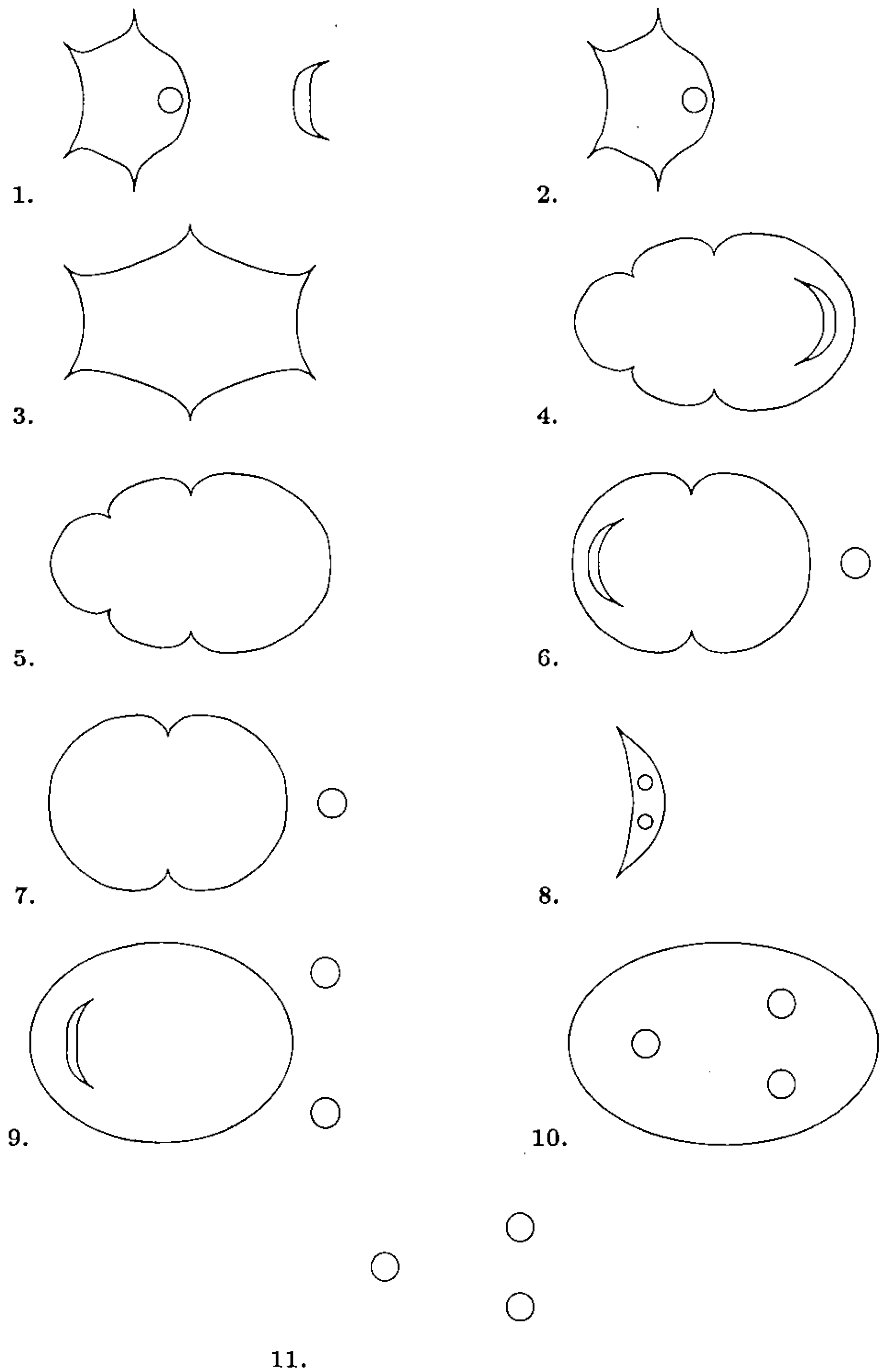

FIGURE 5. $\mathbb{R}P^2\#2T$

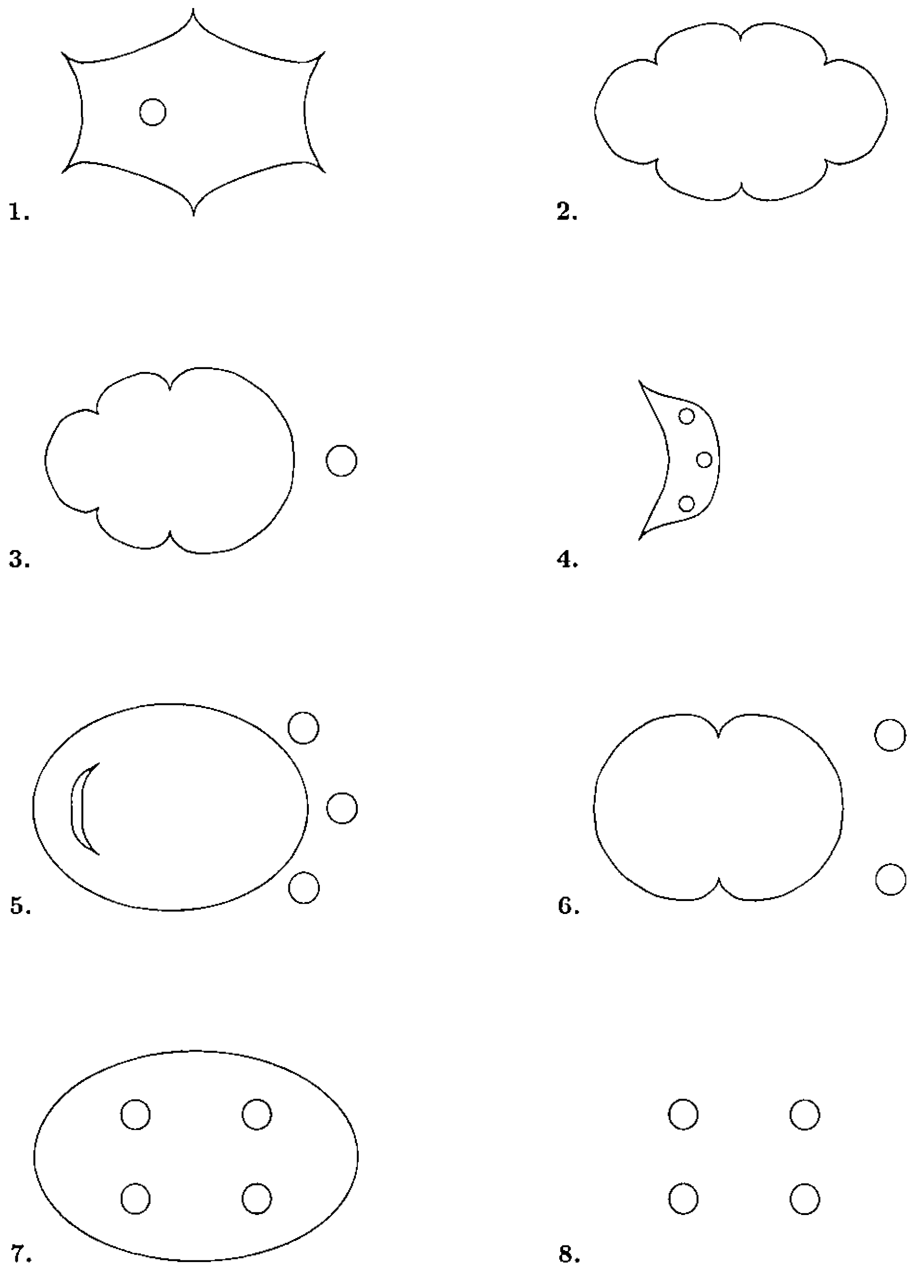

FIGURE 6. $\mathbb{R}P^2\#3T$

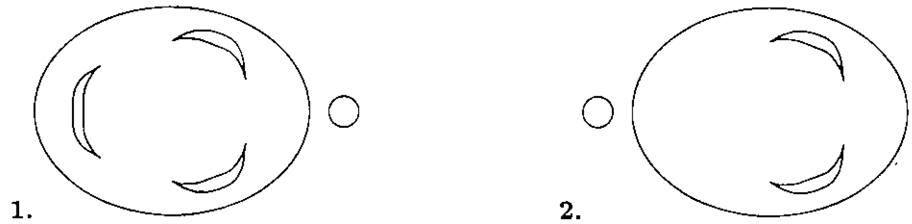

FIGURE 7. $\mathbb{R}P^2\#T$, contours in question

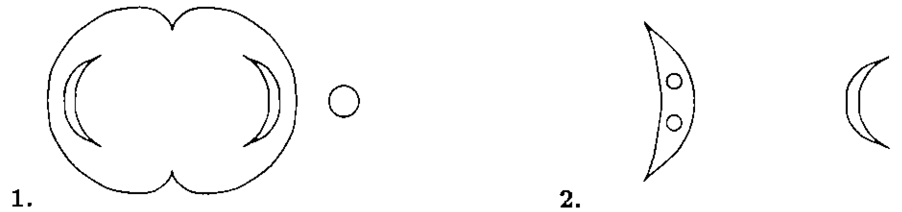

FIGURE 8. $\mathbb{R}P^2\#2T$, contours in question

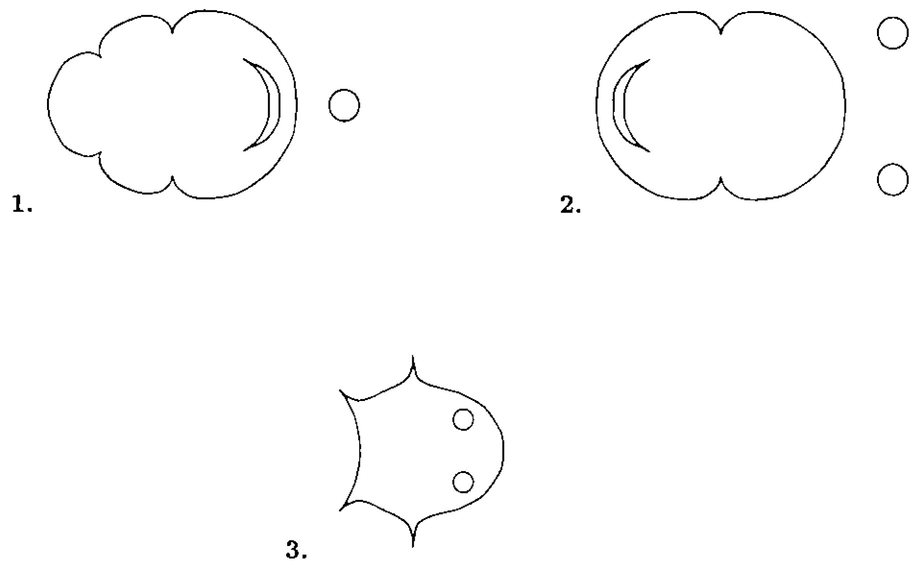

FIGURE 9. $\mathbb{R}P^2\#3T$, contours in question