WEAKLY CLOSED UNIPOTENT SUBGROUPS IN CHEVALLEY GROUPS

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Dedicated to Bernd Fischer on the occasion of his 70th birthday

ABSTRACT. The goal of this note is to classify the weakly closed unipotent subgroups in the split Chevalley groups. In an application we show under some mild assumptions on the characteristic that the Lie algebra of a connected simple algebraic group fails to be a so called 2F-module.

1. INTRODUCTION

Let G be a group and let $H \leq K$ be subgroups of G. The subgroup H is said to be *weakly closed in* K if H is the only G-conjugate of itself contained in K. The notion of weak closure has been quite important in finite group theory.

The aim of this note is to classify all weakly closed unipotent subgroups of a Borel subgroup in the split Chevalley groups and to obtain partial results for the finite twisted (or quasi-split) Chevalley groups. It is well known that unipotent radicals of parabolic subgroups are weakly closed (Lemma 2.1). Under some mild restrictions on the size of the underlying field, we show in our main result that the converse holds, i.e., that a weakly closed unipotent subgroup of a split Chevalley group is the unipotent radical of a parabolic subgroup (Theorem 2.3). However, even in split Chevalley groups defined over very small fields there are other examples (Example 2.5).

We obtain less complete results for the twisted groups (Theorem 3.9) and show that there are other examples of weakly closed subgroups no matter what the field size is in certain cases (Examples 3.6, 3.7). We only indicate examples for the Suzuki and Ree groups.

In analogy to the finite group case, for G a connected simple algebraic group we say that a non-trivial (irreducible) G-module V is a *2F-module* provided

$$2\dim X + \dim C_V(X) \geq \dim V, \tag{1.1}$$

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where X is a (closed) unipotent (but not necessarily abelian or connected) subgroup of G and $C_V(X)$ denotes the subspace of X-fixed points of V.

For the concept and relevance of 2F-modules in finite group theory, we refer the reader to [2], [9], and [10]. Here the original question is for a finite group G and a given absolutely irreducible faithful G-module V to find the maximum of the expression $|X|^2 \cdot |C_V(X)|$, when X is a non-trivial elementary abelian unipotent subgroup of G, cf. [9].

For the finite simple groups there are very few 2F-modules, see [9], [10]. Analogously, we briefly discuss the sparsity of 2F-modules for a simple algebraic group G (Remark 4.9). In particular, we apply our main theorem to show that the adjoint module $\mathfrak{g} = \operatorname{Lie} G$ of G is not a 2F-module (Corollary 4.7). This generalizes a result of Guralnick and Malle, [9], [10].

We assume throughout that the groups of Lie type considered are generated by unipotent elements.

Let k be a field. With the exception of Section 3, $G = G(k)$ denotes a split (adjoint) Chevalley group in the sense of [17].

Let T be a Cartan subgroup of G and B is a Borel subgroup of G containing T. Let $U \leq B$ be the unipotent radical of B. Let $\Psi = \Psi(G,T)$ be the root system of G with respect to T and let $\Pi = \Pi(B)$ be the set of simple roots of G and $\Psi^+ = \Psi(B)$ the set of positive roots of G defined by B. For $\gamma \in \Psi$ we denote the root subgroup defined by γ by U_γ. For a subgroup H of G we set $\Psi(H) = \{\beta \in \Psi \mid U_\gamma \leq H\}$. By W we denote the Weyl group of G with respect to T.

Let $P \geq B$ be a parabolic subgroup of G. Then P factors as $P = LP_u$ with some Levi complement L and unipotent radical P_u. In such a decomposition we always assume that L is *standard*, i.e., that L is generated by T along with the root subgroups of a subsystem of Ψ which is generated by a subset of the simple roots, e.g., see [6, §2.6].

We say that *p is a very bad prime for G* if p divides one of the structure constants of the Chevalley commutator relations for G, [17, p. 12]; that is 2 (resp. 3) is a very bad prime for G if G admits a simple factor of type B_n, C_n, for $n \geq 2$, F_4, or G_2 (resp. G_2); else there are no very bad primes for G.

As general references for Chevalley groups and algebraic groups we refer the reader to [3], [5], [6], and [17].

2. Weakly Closed Unipotent Subgroups

We maintain the notation and assumptions from the Introduction. In particular, in this section $G = G(k)$ denotes a split Chevalley group.

We first show that the unipotent radicals of parabolic subgroups are weakly closed. This is a well known fact; it is stated almost in this form in [8, Lem. 4.2]. This is also proved in [2, I.2.5], based on Alperin–Goldschmidt fusion.

Lemma 2.1. *Let $P \leq G$ be a parabolic subgroup of G. Then P_u is weakly closed in U.*

Proof. Suppose $P_u^g \leq U$ for $g \in G$. Let $w \in W$ be the minimal length double coset representative of the (P, B)-double coset in G containing g. Then $P_u^w \leq U$. Suppose $w \neq 1$. Then for some simple root $\alpha \in \Pi \backslash \Psi(L)$ the simple reflection s_α is a prefix of w, i.e., w has a reduced expression beginning with s_α. Since $U_\beta = U_\alpha^w \leq P_u^w \leq U$ and β is a negative root, this is a contradiction. Consequently, $w = 1$ and thus $g \in P$ and so $P_u^g = P_u$, as desired. □

Lemma 2.2. *Let $X \leq U$ be weakly closed in U. Set $P = N_G(X)$. Then P is a parabolic subgroup of G and X is contained in P_u.*

Proof. Since X is weakly closed in U, we have $B = N_G(U) \leq N_G(X) = P$ and so P is a parabolic subgroup of G. As P_u is the largest normal unipotent subgroup of P, we have $X \leq P_u$. □

If X is a weakly closed subgroup of U, it is normalized by $T \leq B$. If k is sufficiently large, then X is generated by the root groups contained in it, cf. [1], [3, Prop. 14.4.(2a)], [16], and [17, Lem. 17]. In order to ensure that, we make the following restrictions on k; e.g., see §5 in [18] and in particular the references therein.

(†) $k \neq \mathbb{F}_2, \mathbb{F}_4$ in case G is of type A_2;
$k \neq \mathbb{F}_2, \mathbb{F}_3$ in case G is of type A_3, B_n, C_n, for $n \geq 2$, D_n, for $n \geq 3$, F_4, or G_2;
$k \neq \mathbb{F}_2$ in case G is of type A_n, for $n \geq 4$, E_6, E_7, or E_8;
k is perfect if $\operatorname{char} k = 2$ and G is of type C_n, for $n \geq 1$.

Our main result gives a converse to Lemma 2.1 assuming (†).

Theorem 2.3. *Assume (†). Suppose $X \leq U$ is weakly closed in U. Set $P = N_G(X)$. Then $X = P_u$.*

Proof. By Lemma 2.2, P is a parabolic subgroup of G and $X \leq P_u$.

Since X is normalized by $T \leq B$, the restrictions on k in (†) ensure that X is generated by the root subgroups contained in X. Since X is normalized by B, it follows from the commutator relations of G (cf. [17, p. 30, Lem. 33]) that $\Psi(X)$ is a closed subset of Ψ^+, in the sense of [17, p. 24]. Consequently, we have $X = \prod U_\beta$, where the product is taken in any fixed order over $\Psi(X)$, cf. [17, Lem. 17].

Now suppose that there is a simple root $\alpha \in \Pi \cap \Psi(P_u)$ such that $U_\alpha \nleq X$. Then $U_\alpha \cap X = \{1\}$, since X is generated by root subgroups and distinct root subgroups intersect trivially; the latter follows from the uniqueness of factorization in the product decomposition $X = \prod U_\beta$, [17, Lem. 17]. Then $X^{s_\alpha} \leq U$, since s_α permutes $\Psi^+ \setminus \{\alpha\}$. Thus, as X is weakly closed in P_u, we have $s_\alpha \in N_G(X) = P$. But for a simple root α, we have $s_\alpha \in P$ if and only if $\alpha \in \Psi(L)$ if and only if $\alpha \notin \Psi(P_u)$, a contradiction. Consequently, $\Pi \cap \Psi(P_u) \subseteq \Psi(X)$. Since P_u is the normal closure in P of the root groups relative to the simple roots in $\Psi(P_u)$, e.g., see [12, Prop. 2.10, Rk. 2.13], we derive that $P_u \leq X$. □

We recall a well-known fact concerning regular unipotent elements.

Remark 2.4. Let G be a reductive algebraic group. A unipotent element u of G is called *regular* provided $\dim C_G(u)$ is minimal possible among unipotent elements in G. A regular unipotent element is contained in a unique Borel subgroup of G, see [6, Prop. 5.1.3]. Let F be a Frobenius endomorphism of G so that the subgroup of F-fixed points G^F of G is a finite group of Lie type. Let $u \in G^F$ be regular unipotent in G. Since any Borel subgroup of G^F is the fixed point subgroup of a unique F-stable Borel subgroup of G, the uniqueness result for G just quoted implies that u is in a unique Borel subgroup of G^F.

The following example shows that the hypothesis (†) of Theorem 2.3 is necessary.

Example 2.5. Let G be a split simple Chevalley group over the field of 2 elements of rank at least 2. Note that $U = B$. Let Y be the subgroup of U generated by the root subgroups U_γ relative to all the non-simple positive roots, i.e., $\gamma \in \Psi^+ \setminus \Pi$. So every subgroup of G between U and Y is normal in B. Let $u \in U$ be a regular unipotent element. Note that this determines the coset uY uniquely. Let X be the subgroup of G generated by u and Y. We claim that X is weakly closed in U. Since u is regular unipotent, it is contained in no other Borel subgroup of G, cf. Remark 2.4, and so the same is true for X. So if $X^g \leq U$, then $X \leq U^{g^{-1}}$ and so $g \in N_G(U) = U$, whence $X^g = X$. Thus X is weakly closed in U. Since $\operatorname{rank} G > 1$, it follows that X is not the unipotent radical of any parabolic subgroup of G.

One can construct in a similar way examples for all cases of split groups when (†) fails.

3. Weakly Closed Subgroups in Finite Twisted Groups

We note that Lemma 2.1 also holds for the finite twisted Chevalley groups and the proof goes through verbatim only involving the (B, N)-pair structure of the underlying group. We record this:

Lemma 3.1. *Let G be a finite simple Chevalley group and $P \leq G$ be a parabolic subgroup of G. Then P_u is weakly closed in U.*

This is also proved in [2] and [8]. We sketch a proof of Lemma 3.1 for classical groups in Lemma 3.3 below that is quite different from the other proofs mentioned. For groups of rank 1, there is nothing to prove. We do not complete the argument for the exceptional groups, but the proof of Lemma 3.3 below does show that it suffices to check the statement for unipotent radicals of the maximal parabolic subgroups. By a classical group, we mean a linear, unitary, symplectic or orthogonal group.

We first recall some general properties of weakly closed subgroups for finite groups (with obvious modifications Lemma 3.2 also applies to unipotent subgroups of algebraic groups).

Lemma 3.2. *Let G be a finite group with U a Sylow p-subgroup of G. Let $B = N_G(U)$. Let X be a normal subgroup of U. The following are equivalent:*

1. *X is weakly closed in B;*
2. *X is weakly closed in $P := N_G(X)$;*
3. *X has a unique fixed point on G/P.*

Moreover, if any of these conditions holds, then $P = N_G(P)$.

Proof. Assume that $X^g \leq P$ for some $g \in G$. Then the subgroup of G generated by X and X^g is a p-subgroup of P and so by conjugating, we may assume that $X^g \leq U$. Thus, (1) implies (2). Since $U \leq P$, (2) implies (1).

Note that $X^g \leq P$ if and only if X fixes the point gP in G/P. Thus, if X is weakly closed in P and if X fixes gP in G/P, then $X = X^g$, and so $g \in N_G(X) = P$. Thus (3) follows from (2).

Now assume that (3) holds. Since X fixes the point P in G/P, if $X^g \leq P$, we have $gP = P$ and so $g \in P$. Thus (2) holds.

Finally, if g normalizes P, then $X^g \leq P$, whence g normalizes X and so is in P. So, the last assertion follows. $\square$

Lemma 3.3. *Let G be a simple classical group over a finite field. If P is a parabolic subgroup of G, then P_u is weakly closed in B.*

Proof. We argue by induction on the rank of G. If $\operatorname{rank} G = 1$, then U is the unique Sylow p-subgroup of B, and hence is weakly closed in B.

Let N be the natural module of G. First consider the case that P is a maximal parabolic subgroup of G. Then P is the stabilizer of a totally singular m-subspace M of N for some $m \leq \dim N/2$ and P_u is the subgroup of G acting trivially on both M and $M^\perp/M$ (for the linear case we have $M^\perp = N$). Note that M is precisely the set of fixed singular vectors for P_u.

We claim that M is the unique P_u-invariant totally singular subspace of dimension m (of the given type in the case of orthogonal groups). We show this by induction on m. The case $m = 1$ is clear. Let $m > 1$. Now let V be a P_u-invariant totally singular subspace of N of dimension m. Then P_u fixes some non-zero vector $v \in V$ and so $v \in M$. So V is contained in $\langle v\rangle^\perp$. By induction, P_u fixes a unique totally isotropic subspace of dimension $m-1$ (of the given type) in $\langle v\rangle^\perp/\langle v\rangle$, whence the claim.

Thus, P_u has a unique fixed point on G/P, whence P_u is weakly closed in B, thanks to Lemma 3.2.

Suppose that P is not maximal. Let Q be a maximal parabolic subgroup of G containing P. Then P_u/Q_u is the unipotent radical of P/Q_u in Q/Q_u and the latter is a central product of a torus and some number of smaller classical groups. Since P/P_u contains the torus and is a central product of parabolic subgroups in each factor, it follows by induction that P_u/Q_u is weakly closed in B/Q_u.

Observe that $P_u^g \leq B$ implies that $Q_u^g \leq B$. Since Q_u is weakly closed in B, by the case above, it follows that $g \in Q$. Thus, as $P_u^g \leq B$ implies that $g \in Q$ and since P_u/Q_u is weakly closed in B/Q_u, we have $(P_u/Q_u)^{gQ_u} = P_u/Q_u$ and thus we obtain $P_u^g = P_u$, as desired. □

Our next examples show that in the twisted groups there are additional instances of weakly closed unipotent subgroups for *all* finite fields. The proof of Lemma 3.3 shows that the critical case is that of a maximal parabolic subgroup.

Example 3.4. Let $G = \mathrm{U}_3(q)$. Take $X = Z(U)$. So $|U| = q^3$ and $|X| = q$. Note that X consists of all transvections in U and so clearly it is weakly closed in U.

Remark 3.5. Note that a minor variation of the previous example shows that for all the twisted rank 1 groups there are proper weakly closed subgroups of U. Similarly, this holds for Ree groups of type F_4.

Since $\mathrm{U}_3(q)$ is a Levi subgroup of a parabolic subgroup of $\mathrm{U}_{2m+1}(q)$ for all $m > 1$, we can use Example 3.4 to give other examples.

Example 3.6. Let $G = \mathrm{U}_{2m+1}(q)$ with $m > 1$. Let P be a minimal parabolic subgroup of G with unipotent radical R such that the derived subgroup of P/R is a 3-dimensional unitary group. Let X be the subgroup of P such that X/R is the center of U/R. Since R is weakly closed in U, by Lemma 3.1, it follows that $X^g \le U$ implies that $R^g \le U$ and so $R^g = R$, i.e. $g \in P$. By Example 3.4, this implies that $X^g = X$, as required.

In fact, there are additional examples of weakly closed subgroups X where $N_G(X)$ is even a maximal parabolic subgroup of G.

Example 3.7. Let $G = \mathrm{U}_{2m+1}(q)$ with $m \ge 1$. Let $P \ge U$ be the parabolic subgroup of G that is the stabilizer of a totally singular m-subspace of the natural module N of G. Let X be the derived subgroup of P_u. We claim that X is weakly closed in U and proper in P_u.

Since P_u is nilpotent, X is proper in P_u. Let V be the set of fixed points of P on N. Then V is m-dimensional and totally singular.

We claim that V is the only totally singular m-space left invariant by X. We prove this by induction on m. The case $m = 1$ is clear. Let $m > 1$. Note that the subspace of fixed points of X on N is $V^\perp$ and has dimension $m+1$. Moreover every vector in $V^\perp \setminus V$ is non-singular. So if X leaves invariant a totally singular m-space V', then X is trivial on some 1-space $V_1 \le V' \cap V_1^\perp$. Now view X acting on $V_1^\perp/V_1$. This gives a homomorphism of X into $\mathrm{U}_{2m-1}(q)$. The image of X is precisely the derived subgroup of the unipotent radical of the maximal parabolic subgroup stabilizing V/V_1, whence by induction $V'/V_1 = V/V_1$ and so $V = V'$, as claimed.

So X has a unique fixed point on G/P, whence X is weakly closed in P and thus also in B, by Lemma 3.2.

Remark 3.8. One might wonder why the previous example does not extend to odd dimensional orthogonal groups (or twisted orthogonal groups). The problem is that the first step in the induction fails.

Examples 3.6 and 3.7 show that there are additional weakly closed unipotent subgroups in the odd-dimensional unitary groups (over any finite field). However, our main result of this section shows that for the remaining families of twisted groups, we have the same result as Theorem 2.3, as long as the field size is sufficiently large.

Theorem 3.9. *Assume that k is a finite field with $|k| > 5$. Let $G = {}^2A_{2m+1}(k)$, for $m \ge 1$, ${}^2D_n(k)$ for $n \ge 4$, ${}^3D_4(k)$, or ${}^2E_6(k)$. Let U be the unipotent radical of a Borel subgroup B of G. If $X \le U$ is weakly closed in B, then $X = P_u$ for some parabolic subgroup P of G.*

Proof. Let T be a maximal torus contained in B. So X is normalized by T. The assumption on $|k|$ guarantees that X is a product of root subgroups, see [16, Lem. 3]. Moreover, in all the cases considered, the root subgroups are abelian (and can be identified with k or the quadratic extension of k or, in the case of ${}^3D_4(k)$, the cubic extension of k) and the intersection of any two distinct root subgroups is trivial. Let $P = N_G(X)$, a parabolic subgroup of G. So $X \leq P_u$. As in the previous section, in this case, we see that X must contain all the root subgroups corresponding to simple roots in $\Psi(P_u)$ (this uses the fact that root subgroups relative to distinct roots intersect trivially and that the simple reflections s_α in the Weyl group of G preserve the set of positive roots other than the simple root α). As in the split case, we see that the normal closure of the simple root subgroups contained in P_u is all of P_u. Thus, $X = P_u$. □

4. Centralizers of Weakly Closed Unipotent Groups

For the remainder of the note we assume that G is a connected, simple algebraic group and that k is algebraically closed. For a (closed) subgroup H of G we denote the identity component of H by H^0 and note that $\dim H = \dim H^0$ meaning dimension as an algebraic variety.

In this section we show that the adjoint module $\mathfrak{g}$ of G is not a 2F-module, in the sense of (1.1). This extends a result due to Guralnick and Malle [9]. Moreover, we show that there are very few possibilities for a 2F-module for dimension reasons.

We only prove the results for algebraic groups. We leave it to the reader to prove the same statements for the split Chevalley groups satisfying (†).

Lemma 4.1. *Let P be a proper parabolic subgroup of G. Then $C_G(P_u) \leq P_u Z(G)$. So $C_G(P_u)^0 \leq P_u$ and $C_G(P_u)^0 = Z(P_u)$.*

Proof. Let $L \geq T$ be the standard Levi complement of P. We first show that $C_G(P_u)^0$ contains no semisimple elements. Suppose that there is a non-trivial torus S centralizing P_u. Then S is conjugate to a subtorus of T in P and since $C_G(P_u)$ is normal in P, there is no loss in assuming that $S \leq T$. Then S centralizes each root subgroup $U_\alpha \leq P_u$ and so also $U_{-\alpha}$. However, as G is simple, it is generated by P_u and the unipotent radical of the parabolic subgroup of G opposite to P (with respect to L), e.g. see [4, Prop. 4.11]. Thus, $S \leq Z(G)$; a contradiction. It follows that $C_G(P_u)^0$ is a normal unipotent subgroup of P and so contained in P_u. Thus, $C_G(P_u)^0 = Z(P_u)$. The only finite normal subgroups of L are contained in $Z(L) \leq T$ and arguing as above, we see that $C_T(P_u) = Z(G)$. Thus we have $C_G(P_u) \leq B$. Moreover, since $C_G(P_u)$ is normal in P, we have $C_G(P_u) \leq P_u Z(L)$. By the action of

T on the root subgroups of U and by the commutator relations we see that

$$C_G(P_u) = C_G(P_u) \cap P_u Z(L) = (C_G(P_u) \cap P_u)(C_G(P_u) \cap Z(L)).$$

Note that $C_G(P_u) \cap Z(L) = \cap_{\alpha \in \Pi} \ker \alpha = Z(G)$ and since G is simple, the latter is finite. So $C_G(P_u) \leq P_u Z(G)$, and $C_G(P_u)^0 \leq P_u$, as claimed. □

Remark 4.2. Let P be a parabolic subgroup of G. There is a natural bound for $\dim C_G(P_u)$ stemming from Richardson's Dense Orbit Theorem, e.g., see [6, §5.2]. There is a conjugacy class C of P in P_u which is open dense in P_u. It turns out that for any x in C we get $C_G(x)^0 = C_P(x)^0$, cf. [6, Cor. 5.2.2], and thus $\dim C_G(x) = \dim C_P(x)$. For any $x \in C$ we clearly have $C_G(P_u) \leq C_G(x)$ and thus, since $\dim C + \dim C_G(x) = \dim P$ and $\dim C = \dim P_u$, we obtain

$$\dim P_u + \dim C_G(P_u) \leq \dim P.$$

The existence of such a dense P-orbit is part of a fundamental theorem due to R.W. Richardson [14]. The proof relies on the fact that the number of unipotent classes of G is finite. This was first proved also by Richardson under some mild restrictions on the characteristic of the ground field [13]; these were removed subsequently by G. Lusztig in [11].

We can improve the bound from Remark 4.2 as follows.

Proposition 4.3. *Let P be a proper parabolic subgroup of G. Then*

$$\dim P_u + \dim C_G(P_u) \leq \dim B.$$

Proof. By Lemma 4.1, $C_G(P_u)^0$ is a connected normal abelian subgroup of U. Thus, by [15, Thm. 1.1], there are only finitely many B-orbits on $C_G(P_u)^0$, and consequently there is a dense such orbit. Thus $\dim C_G(P_u) + \dim C_B(x) = \dim B$ for some $x \in C_G(P_u)^0$. Finally, since $P_u \leq C_B(x)$, the desired inequality follows. □

If H is a (closed) subgroup of G, we define

$$f(H) = 2 \dim H + \dim C_G(H).$$

Our next result is a restatement of Lemma 2.1 from [9] in our context and the proof is identical, see also [7].

Lemma 4.4. *Let $H \leq M \leq G$ and $K \leq G$ be subgroups of G. Suppose that $f(H)$ is maximal among all the subgroups of M. Let D be the (algebraic) subgroup of G generated by H and K. Then $f(D) \geq f(K)$.*

Proof. Note that $\dim D \geq \dim HK = \dim H + \dim K - \dim(H \cap K)$. Here HK need not be a subgroup of G. Since $C_G(H \cap K)$ contains both $C_G(H)$ and $C_G(K)$, and $C_G(H) \cap C_G(K) = C_G(D)$, we similarly obtain $\dim C_G(D) \geq \dim C_G(H) + \dim C_G(K) - \dim C_G(H \cap K)$. Thus, we have

$$\begin{aligned} f(D) &= 2\dim D + \dim C_G(D) \\ &\geq 2(\dim H + \dim K) + \dim C_G(H) + \dim C_G(K) \\ &\qquad - 2\dim(H \cap K) - \dim C_G(H \cap K) \\ &= f(H) + f(K) - f(H \cap K) \end{aligned}$$

and since $f(H) \geq f(H \cap K)$, by maximality of $f(H)$ among the subgroups of M, the lemma follows. $\square$

For X a subgroup of U let $\widehat{X}$ denote the *weak closure of X in U*, that is the smallest weakly closed subgroup of U containing X (so $\widehat{X}$ is the subgroup of U generated by all conjugates of X contained in U).

Note that since weakly closed unipotent subgroups are unipotent radicals of parabolic subgroups of G, they in fact are closed. So we could define the weak closure of X to be the algebraic group generated by the conjugates of X contained in U.

Corollary 4.5. *Let $X \leq U$ with $f(X)$ maximal among all subgroups of U, then $f(X) = f(\widehat{X})$, where $\widehat{X}$ is the weak closure of X in U.*

Proof. Note that there are finitely many conjugates X_i, $1 \leq i \leq m$, of X so that the group generated by $X_1, \dots, X_m$ has the same centralizer as $\widehat{X}$. Let Y_i be the group generated by $X_1, \dots, X_i$. We show that $f(Y_i) = f(X)$ for all i and this proves the corollary.

This is clear for $i = 1$, as f is constant on conjugates. Let $i > 1$. Inductively, we have $f(Y_{i-1}) = f(X)$ and since this is maximal among all the subgroups of U, we can apply the previous lemma with $H = Y_{i-1}$, $M = U$, and $K = X_i$ to conclude that $f(Y_i) \geq f(X)$ which, by maximality, gives $f(X) = f(Y_i)$. Thus, $f(\widehat{X}) = f(Y_m) = f(X)$. $\square$

Theorem 4.6. *Let G be a simple algebraic group of rank $r \geq 2$. In case* $\operatorname{char} k$ *is a very bad prime for G assume that $r \geq 3$. Let X be a non-trivial unipotent subgroup of G. Then*

$$2\dim X + \dim C_G(X) < \dim G.$$

Proof. Choose $X \leq U$ with $f(X)$ maximal. Choose X to be non-trivial if possible. If this is not possible, then $f(X) < f(\{1\}) = \dim G$ and the result follows.

It follows from Corollary 4.5 that the maximum is achieved on the weak closure of X in U. Thus, thanks to Theorem 2.3, we may choose a parabolic subgroup P of G with $X \le P_u$ and $f(X) = f(P_u)$. Since P is proper, it follows from Proposition 4.3 that

$$2\dim P_u + \dim C_G(P_u) \le \dim P_u + \dim B < \dim G,$$

unless $P = B$. If $\operatorname{char} p$ is not a very bad prime of G, then $\dim Z(U) = 1$, else $\dim Z(U) = 2$. Thus, by the hypotheses on r, we have

$$2\dim U + \dim Z(U) = |\Psi| + \dim Z(U) < |\Psi| + r = \dim G,$$

giving the desired strict inequality also for $P = B$. $\square$

We next prove an analogue of Theorem 4.6 for $\mathfrak{g} = \operatorname{Lie} G$, the Lie algebra of G; this says that $\mathfrak{g}$ is not a 2F-module of G, cf. (1.1).

For H a (closed) subgroup of G, let $\mathfrak{c}_{\mathfrak{g}}(H) := \{y \in \mathfrak{g} \mid \operatorname{Ad}(h)y = y \text{ for all } h \in H\}$ denote the subspace of $\mathfrak{g}$ of $\operatorname{Ad}(H)$-fixed points of $\mathfrak{g}$.

We define a function similar to f above: for H a (closed) subgroup of G set

$$f_{\mathfrak{g}}(H) := 2\dim H + \dim \mathfrak{c}_{\mathfrak{g}}(H),$$

where we use the centralizer in the Lie algebra instead of the group. In general, we have $\operatorname{Lie} C_G(H) \le \mathfrak{c}_{\mathfrak{g}}(H)$ and thus $f(H) \le f_{\mathfrak{g}}(H)$ for any subgroup H of G. We have equality precisely when the scheme-theoretic centralizer of H in G is smooth.

Using dimension arguments of subalgebras instead of subgroups, one readily checks that the proof of Lemma 4.4 also applies for $f_{\mathfrak{g}}$ in place of f (with essentially no change). Further, since $\mathfrak{c}_{\mathfrak{g}}(H^0) \ge \mathfrak{c}_{\mathfrak{g}}(H)$, the proof of Corollary 4.5 is also valid for $f_{\mathfrak{g}}$ in place of f.

Corollary 4.7. *Let G be a simple algebraic group of rank $r \ge 2$. Let X be a non-trivial unipotent subgroup of G. Then*

$$2\dim X + \dim \mathfrak{c}_{\mathfrak{g}}(X) < \dim \mathfrak{g}.$$

Proof. We argue as in the proof of Theorem 4.6. Choose X with $f_{\mathfrak{g}}(X)$ maximal with X non-trivial (if this is not possible, then $f_{\mathfrak{g}}(X) < f_{\mathfrak{g}}(\{1\}) = \dim \mathfrak{g}$ for every non-trivial X and the result holds). It follows from the $f_{\mathfrak{g}}$-analogue of Corollary 4.5 that the maximum is achieved on a (non-trivial) weakly closed subgroup of U. By Theorem 2.3, we have $X = P_u$ for some proper parabolic subgroup P of G.

We show that under our assumptions $\operatorname{Lie} C_G(P_u) = \mathfrak{c}_{\mathfrak{g}}(P_u)$, i.e., that the scheme-theoretic centralizer of P_u in G is smooth. In particular, we then obtain that $\dim C_G(P_u) = \dim \mathfrak{c}_{\mathfrak{g}}(P_u)$ and the desired result is immediate from Theorem 4.6.

Since $P \neq G$, it follows from Lemma 4.1 that $C_G(P_u)^0 \leq P_u \leq U$. And since P_u is T-stable, so is $C_G(P_u)^0$. It thus follows from [3, Prop. 14.4(2a)] that $C_G(P_u)^0 = \prod U_\gamma$, where the product is taken over the set of roots $\Gamma := \Psi(C_G(P_u))$. Thanks to the commutator relations and our restrictions on the characteristic, $\Gamma = \{\gamma \in \Psi(P_u) \mid \gamma + \beta \notin \Psi^+ \ \forall\ \beta \in \Psi(P_u)\}$. This is a closed subset of Ψ^+, in the sense of [17, p. 24]. Likewise, $\mathfrak{c}_{\mathfrak{g}}(P_u)$ is $\mathrm{Ad}(T)$-stable and thus, $\mathfrak{c}_{\mathfrak{g}}(P_u)$ is a sum of root spaces in $\mathfrak{g}$, cf. [3, Prop. 13.20]. Because of the restrictions on $\operatorname{char} k$, there are no degeneracies in the adjoint action of root elements on root spaces of $\mathfrak{g}$, cf. [5, §4.3]. Consequently, $\mathfrak{c}_{\mathfrak{g}}(P_u) = \oplus \mathfrak{g}_\gamma$, where the sum is over the same set Γ defined above. in particular, we have $\operatorname{Lie} C_G(P_u) = \mathfrak{c}_{\mathfrak{g}}(P_u)$, as claimed. □

Remark 4.8. If $\operatorname{rank} G = 1$, then the inequalities in Theorem 4.6 and Corollary 4.7 are clearly still valid provided X is a non-trivial finite unipotent subgroup; else, of course, we get equality.

Remark 4.9. We can consider the same question for any rational G-module V. Assume that V is irreducible. For any (closed) subgroup H of G, define

$$f_V(H) := 2\dim H + \dim C_V(H).$$

The question is when there exists a non-trivial unipotent subgroup X of G with $f_V(X) \geq \dim V$. As for the Lie algebra case, it is straightforward to check that the f_V-analogues of Lemma 4.4 and Corollary 4.5 also hold with essentially identical proofs. This then shows that $f_V(X) \leq f_V(G) = 2\dim G$, since V is irreducible. So if V is a 2F-module for G, i.e., if $f_V(X) \geq \dim V$ (cf. (1.1)), then necessarily $\dim V \leq 2\dim G$. On the other hand, since V is irreducible, we obtain $f_V(U) = 2\dim U + 1 = \dim G - r + 1$. So the existence of such an X is only open for the case

$$\dim G - r + 1 < \dim V \leq 2\dim G.$$

There are very few irreducible G-modules with dimension in this range (see [10]). We have dealt with the adjoint module above. By the weak closure result, i.e., the f_V-analogue of Corollary 4.5 and Theorem 2.3, we just need to compute $f_V(P_u)$ for each parabolic subgroup P of G for the few remaining cases for V. We leave the details to the reader.

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