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Snigdhayan Mahanta

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Snigdhayan Mahanta

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Mathematical Institute
University of Muenster
Einsteinstr. 62
48149 Muenster
Germany

MPIM 14-10

SYMMETRIC MONOIDAL NONCOMMUTATIVE SPACES AND SPECTRA AND (CO)LOCALIZATIONS WITH RESPECT TO STRONGLY SELF-ABSORBING C^*-ALGEBRAS

SNIGDHAYAN MAHANTA

ABSTRACT. Continuing our earlier work we construct symmetric monoidal ∞-categorical models for separable C^*-algebras $\mathtt{SC}^*_\infty$ and noncommutative spectra $\mathtt{NSp}$ using the framework of Higher Algebra due to Lurie. We study localizations of $\mathtt{SC}^*_\infty$ and colocalizations of $\mathtt{NSp}$ with respect to any strongly self-absorbing C^*-algebra. We analyse the homotopy categories of the localizations of $\mathtt{SC}^*_\infty$ and characterize them by a universal property. We also describe the colocalized subcategories of $\mathtt{hNSp}$ spanned by the stabilizations of C^*-algebras in the purely infinite case. As a consequence we compute the noncommutative stable cohomotopy of the $ax+b$-semigroup C^*-algebra arising from any number ring.

Introduction

In [15] we constructed a stable presentable ∞-category of noncommutative spectra $\mathtt{NSp}$. It is an ideal framework to carry out stable homotopy theory of noncommutative spaces. It was used to show that the triangulated noncommutative stable homotopy category $\mathtt{NSH}$ constructed by Thom [20] is a topological triangulated category as defined by Schwede [19]. Nevertheless, a very important part of the homotopy theory package, viz., the symmetric monoidal structure was left out of the discussion in [15]. In the present article we use Lurie's Higher Algebra [12] to construct a symmetric monoidal stable presentable ∞-category of noncommutative spectra $\mathtt{NSp}$ (see Theorem 1.5).

Toms–Winter introduced a class of simple C^*-algebras called *strongly self-absorbing C^*-algebras* [21], which play a pivotal role in Elliott's Classification Program. Prominent examples of such C^*-algebras, which are also purely infinite, are Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$, and tensor products of UHF algebras of infinite type with $\mathcal{O}_\infty$. In the sequel we construct *smashing localizations* of the ∞-category of separable C^*-algebras $\mathtt{SC}^*_\infty$ with respect to arbitrary strongly self-absorbing C^*-algebras. We describe the homotopy categories of the localized ∞-categories (see Proposition 2.8) and derive several useful results. At the level of homotopy categories we also obtain a univeral characterization in this setting (see Theorem 2.13).

In the stable setting we colocalize the stable ∞-category of noncommutative spectra $\mathtt{NSp}$ with respect to the stabilization of any strongly self-absorbing C^*-algebra. In the purely infinite case we describe the homotopy category of the colocalized ∞-category spanned by the stabilizations of C^*-algebras (cf. Theorems 3.4, 3.8, and 3.10). As a consequence we prove that the canonical map from noncommutative stable cohomotopy to topological K-theory is an isomorphism for $\mathcal{O}_\infty$-stable C^*-algebras. Using the results of [5, 11] this isomorphism

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enables us to complete the computation of noncommutative stable cohomotopy (see Disambiguation 3.11) of $ax + b$-semigroup C^*-algebras arising from number rings (see Theorem 3.12).

The $\mathcal{Z}$-stable situation, where $\mathcal{Z}$ is the Jiang–Su algebra, is the most interesting case from the viewpoint of classification but it is not covered by the results in this article. We also include an appendix on (potential) Quillen model structures in the context of C^*-algebras.

Notations and conventions: Throughout this article $\hat{\otimes}$ will denote the maximal C^*-tensor product. All C^*-algebras are assumed to be separable unless otherwise stated. For any ∞-category $\mathcal{C}$ we denote by $\mathtt{h}\mathcal{C}$ its homotopy category. A functor between ∞-categories will implicitly mean an ∞-functor, i.e., a map of underlying simplicial sets.

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1. The symmetric monoidal structure and localizations of $\mathtt{SC}^*_\infty$

Recall from [15] that there is an ∞-category of noncommutative pointed spaces $\mathtt{N}\mathcal{S}_*$ as well as a stable ∞-category of noncommutative spectra $\mathtt{NSp}$, which is obtained after a localization of the stabilization of the ∞-category $\mathtt{N}\mathcal{S}_*$. In this section we construct a symmetric monoidal structure on $\mathtt{N}\mathcal{S}_*$ (resp. $\mathtt{NSp}$) generalizing the smash product of pointed spaces (resp. spectra).

Let $\mathtt{Fin}_*$ denote the category, whose objects are pointed sets $\langle n\rangle = \{*, 1, \cdots, n\}$ with $*$ being the basepoint and whose morphisms are pointed maps. Let $\mathrm{N}(\mathtt{Fin}_*)$ denote its nerve. A *symmetric monoidal* ∞*-category* $\mathcal{C}^\otimes$ is a coCartesian fibration of simplicial sets $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathtt{Fin}_*)$ with the property: for each $n \geqslant 0$ there is an equivalence $\mathcal{C}^\otimes_{\langle n\rangle} \simeq (\mathcal{C}^\otimes_{\langle 1\rangle})^n$ induced by the maps $\{\rho^i : \langle n\rangle \to \langle 1\rangle\}_{1\leqslant i\leqslant n}$. One should regard $\mathcal{C} := \mathcal{C}^\otimes_{\langle 1\rangle}$ as the ∞-category, which is symmetric monoidal. It is customary to work with the underlying symmetric monoidal category $\mathcal{C}$, leaving out the rest of the structure as implicitly understood. A symmetric monoidal ∞-category can also be regarded as a commutative monoid object in $\mathtt{Cat}_\infty$, which is the ∞-category of ∞-categories. For further details the readers may consult [12].

Proposition 1.1. The categories $\mathtt{SC}^*_\infty$ and $\mathtt{N}\mathcal{S}_* := \mathrm{Ind}({\mathtt{SC}^*_\infty}^{\mathrm{op}})$ are symmetric monoidal ∞-categories. Moreover, the tensor product functor $\otimes : \mathtt{N}\mathcal{S}_* \times \mathtt{N}\mathcal{S}_* \to \mathtt{N}\mathcal{S}_*$ preserves small colimits in each variable separately and $j : {\mathtt{SC}^*_\infty}^{\mathrm{op}} \to \mathtt{N}\mathcal{S}_*$ is symmetric monoidal.

Proof. It is well-known that the topological category $\mathtt{SC}^*$ is symmetric monoidal under the maximal C^*-tensor product $\hat{\otimes}$. As a consequence its topological nerve $\mathtt{SC}^*_\infty$ is a symmetric monoidal ∞-category. The symmetric monoidal structure on $\mathtt{SC}^*_\infty$ endows ${\mathtt{SC}^*_\infty}^{\mathrm{op}}$ with a symmetric monoidal structure $\otimes$ that is uniquely defined up to a contractible space of choices (see Remark 2.4.2.7 of [12]). Since the symmetric monoidal structure extends to the Ind-completion $\mathtt{N}\mathcal{S}_* := \mathrm{Ind}({\mathtt{SC}^*_\infty}^{\mathrm{op}})$ and $\otimes$ commutes with finite colimits in ${\mathtt{SC}^*_\infty}^{\mathrm{op}}$, all other assertions follow from Corollary 6.3.1.13 of ibid.. □

Lemma 1.2. The stabilization $\mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ is a symmetric monoidal stable ∞-category and the ∞-functor $\Sigma^\infty : \mathtt{N}\mathcal{S}_* \to \mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ is symmetric monoidal.

Proof. Thanks to the previous Lemma, one way to argue is via the identification of stable ∞-categories $\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \simeq \mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} := \mathrm{Fun}^{\mathrm{R}}({\mathtt{N}\mathcal{S}_*}^{\mathrm{op}}, \mathtt{Sp})$ (see Example 6.3.1.22 of [12]; here the tensor product is taken in the category $\mathtt{Pr}^{\mathrm{L}}$). Using the stabilization $\Sigma^\infty : \mathcal{S}_* \to \mathtt{Sp}$ of pointed spaces, the stabilization of noncommutative pointed spaces can be regarded as

$$\mathtt{N}\mathcal{S}_* \simeq \mathtt{N}\mathcal{S}_* \otimes \mathcal{S}_* \to \mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \simeq \mathtt{Sp}(\mathtt{N}\mathcal{S}_*).$$

□

Recall from [15] that there is an ∞-functor $\Pi^{\mathrm{op}} : {\mathtt{SC}^*_\infty}^{\mathrm{op}} \to \mathtt{NSp}$. This arises as a composition of the following ∞-functors

$${\mathtt{SC}^*_\infty}^{\mathrm{op}} \xrightarrow{j} \mathtt{N}\mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \xrightarrow{L} S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) =: \mathtt{NSp}.$$

Here S is a strongly saturated collection generated by the image of the set of morphisms $S_0 = \{\mathrm{C}(f) \to \ker(f) \,|\, f : A \to B \text{ surjective in } \mathtt{SC}^*\}$ in ${\mathtt{SC}^*_\infty}^{\mathrm{op}}$ under $\Sigma^\infty \circ j$. Let T be the strongly saturated collection generated by $j(S_0)$ inside $\mathtt{N}\mathcal{S}_*$. Thus we obtain an accessible localization $L_T : \mathtt{N}\mathcal{S}_* \to T^{-1}\mathtt{N}\mathcal{S}_*$ with respect to T.

Proposition 1.3. The localization functor $L_T : \mathtt{N}\mathcal{S}_* \to T^{-1}\mathtt{N}\mathcal{S}_*$ is a symmetric monoidal ∞-functor between symmetric monoidal ∞-categories.

Proof. By Proposition 2.2.1.9 and Example 2.2.1.7 of [12] we need to verify that for any L_T-equivalence $g : X \to Y$ and any $Z \in \mathtt{N}\mathcal{S}_*$ the induced map $g \otimes \mathrm{id}_Z : X \otimes Z \to Y \otimes Z$ is also an L_T-equivalence. Since T is by construction a strongly saturated collection, the L_T-equivalences precisely coincide with T (see Proposition 5.5.4.15 of [13]). Using the exactness of the maximal C^*-tensor product one can check that if $\theta(f) : \ker(f) \to \mathrm{C}(f)$ is the canonical map in $\mathtt{SC}^*_\infty$ for any surjection $f : A \to B$ in $\mathtt{SC}^*$, then for any $C \in \mathtt{SC}^*$ the map $\theta(f) \otimes \mathrm{id}_C : \ker(f)\hat{\otimes}C \to \mathrm{C}(f)\hat{\otimes}C$ is the same as $\theta(f \otimes \mathrm{id}_C) : \ker(f \otimes \mathrm{id}_C) \to \mathrm{C}(f \otimes \mathrm{id}_C)$. Thus we have shown that for any $\theta(f)^{\mathrm{op}} \in j(S_0)$ and any $C \in {\mathtt{SC}^*_\infty}^{\mathrm{op}}$ the map $\theta(f)^{\mathrm{op}} \otimes \mathrm{id}_C \in j(S_0) \subset T$. Since $\otimes$ commutes with small colimits in $\mathtt{N}\mathcal{S}_*$ the same holds for all $Z \in \mathtt{N}\mathcal{S}_*$, i.e., for any $g \in j(S_0)$ and any $Z \in \mathtt{N}\mathcal{S}_*$ the map $g \otimes \mathrm{id}_Z \in T$. The rest follows from the explicit construction of the strongly saturated collection T from $j(S_0)$. □

Corollary 1.4. The stable ∞-category $T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$ is symmetric monoidal and the canonical ∞-functor $\mathtt{N}\mathcal{S}_* \otimes \mathcal{S}_* \to \mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \to T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$ is symmetric monoidal.

Theorem 1.5. There is an equivalence of stable ∞-categories $\mathtt{NSp} \simeq T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$.

Proof. Using Corollary 1.4.2.23 of [12] one obtains the dotted ∞-functor F (unique up to equivalence) making the following diagram commute

$$\begin{array}{ccc} \mathtt{N}\mathcal{S}_* & \xleftarrow{\ \Omega^\infty\ } & \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \\ \downarrow{\scriptstyle L_T} & & \downarrow{\scriptstyle F} \\ T^{-1}\mathtt{N}\mathcal{S}_* & \xleftarrow{\ \Omega^\infty\ } & \mathtt{Sp}(T^{-1}\mathtt{N}\mathcal{S}_*). \end{array}$$

Using the characterization of localization (see Proposition 5.2.7.12 of [13]) one concludes that the there is a factorization $\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \xrightarrow{L_S} S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \xrightarrow{\overline{F}} \mathtt{Sp}(T^{-1}\mathtt{N}\mathcal{S}_*) \simeq T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$ so that $F = \overline{F} \circ L_S$ with $\overline{F}$ exact. Using the same characterization of localization one obtains the dotted ∞-functor G (unique up to equivalence) making the following diagram commute

$$\begin{array}{ccc} \mathtt{N}\mathcal{S}_* & \xrightarrow{\Sigma^\infty} & \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \\ \downarrow{\scriptstyle L_T} & & \downarrow{\scriptstyle L_S} \\ T^{-1}\mathtt{N}\mathcal{S}_* & \overset{G}{\dashrightarrow} & S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*). \end{array}$$

Now we obtain an ∞-functor $\overline{G} = G \otimes \mathrm{id} : T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \to S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \otimes \mathtt{Sp} \simeq S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ using the fact that $\mathtt{Sp}$ is the tensor unit in $\mathtt{Pr}^{\mathrm{L}}$. Observe that $S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) = \mathtt{NSp}$ and the ∞-functors $\overline{F}$ and $\overline{G}$ are inverse equivalences of stable ∞-categories. □

Remark 1.6. Using the above identification $\mathtt{NSp} \simeq T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$ we are going to regard $\mathtt{NSp}$ as a symmetric monoidal stable ∞-category.

Corollary 1.7. The homotopy category of noncommutative spectra $\mathtt{hNSp}$ is a tensor triangulated category, containing $\mathtt{NSH}^{\mathrm{op}}$ (the opposite of the noncommutative stable homotopy category) as a full tensor triangulated subcategory. It also contains $(\mathtt{NSH}^f)^{\mathrm{op}}$ (the opposite of the category of noncommutative finite spectra $\mathtt{NSH}^f$; see Definition 2.1 of [16]) as a full tensor triangulated subcategory.

2. Localizations of $\mathtt{SC}^*_\infty$

A separable unital C^*-algebra $\mathcal{D}$ ($\mathcal{D} \neq \mathbb{C}$) is called *strongly self-absorbing* if the there is an isomorphism $\phi : \mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{D}$ that is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$ [21]. In ibid. the authors introduced and conducted an elaborate study of strongly self-absorbing C^*-algebras mainly with applications to the Elliott's Classification Program in mind. We are going to use these C^*-algebras to construct interesting (co)localizations of noncommutative spaces and spectra.

Remark 2.1. In [7] the authors showed the for any strongly self-absorbing C^*-algebra $\mathcal{D}$ the map $\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$ is homotopic to an isomorphism $\phi : \mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{D}$. In ibid. the result was asserted under the K_1-injectivity condition, which later turned out to be redundant (see Remark 3.3. of [23]).

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. Then a map $e : \mathbf{1} \to E$ exhibits E as an *idempotent object* if $\mathrm{id}_E \otimes e : E \simeq E \otimes \mathbf{1} \to E \otimes E$ is an equivalence in $\mathcal{C}$ (see, for instance, Definition 6.3.2.1 of [12]). We immediately observe

Lemma 2.2. Any strongly self-absorbing C^*-algebra $\mathcal{D}$ is an idempotent object in $\mathtt{SC}^*_\infty$. The same assertion holds for $\mathbb{K}$.

Proof. For a strongly self-absorbing C^*-algebra $\mathcal{D}$ the canonical unital $*$-homomorphism $\mathbb{C} \to \mathcal{D}$ exhibits it as an idempotent object in $\mathtt{SC}^*_\infty$ (see Remark 2.1). For $\mathbb{K}$ the map $\mathbb{C} \to \mathbb{K}$ sending $1 \mapsto e_{11}$ exhibits $\mathbb{K}$ as an idempotent object in $\mathtt{SC}^*_\infty$. □

Remark 2.3. If $E \in \mathcal{C}$ is an idempotent object, then $L_E : \mathcal{C} \to \mathcal{C}$ of the form $L_E(X) = - \otimes E$ is a localization. In [8] the authors called localizations $L_E : \mathcal{C} \to \mathcal{C}$ of the form $L_E(X) = - \otimes E$ for some $E \in \mathcal{C}$ *smashing localizations* in keeping with the terminology prevalent in stable homotopy theory. Any smashing localization $L_E : \mathcal{C} \to \mathcal{C}$ is compatible with the symmetric monoidal structure on $\mathcal{C}$ and, in fact, $L_E\mathcal{C}$ inherits a symmetric monoidal structure from $\mathcal{C}$, such that $L_E : \mathcal{C} \to L_E\mathcal{C}$ becomes symmetric monoidal (see Proposition 2.2.1.9 and

Proposition 6.3.2.7 of [12]). By abuse of notation we are sometimes going to drop the object E from the smashing localization L_E and denote it simply by L.

Example 2.4. Smashing localizations of the ∞-category of separable C^*-algebras $\mathtt{SC}^*_\infty$ produces interesting results. By definition $\mathtt{SC}^*_\infty$ is opposite to the ∞-category of noncommutative pointed compact Hausdorff spaces. We present a few pertinent examples here.

(1) If $L(A) = A \otimes \mathbb{K}$, then we denote the smashing localization $L\mathtt{SC}^*_\infty$ by $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$. It is the ∞-category of *C^*-stable C^*-algebras*. For finite pointed CW complexes (X, x) and (Y, y) the homotopy set $\mathtt{hSC}^*_\infty[\mathbb{K}^{-1}](L(\mathrm{C}(X, x)), L(\mathrm{C}(Y, y)))$ is the connective E-theory group denoted by $\mathrm{kk}((Y, y),(X, x))$ in [6] (see Remark 2.11 below).

(2) If $L(A) = A \otimes \mathcal{D}$, where $\mathcal{D}$ is a strongly self-absorbing C^*-algebra, then we denote the smashing localization $L\mathtt{SC}^*_\infty$ by $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$. We refer to it as the ∞-category of *$\mathcal{D}$-stable C^*-algebras*. From the perspective of Elliott's Classification Program the ∞-category $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$ would be the most interesting localization, where $\mathcal{Z}$ is the Jiang–Su algebra. We call it the ∞-category of *$\mathcal{Z}$-stable C^*-algebras*.

(3) If $\mathcal{D} = \mathcal{O}_\infty$ we call $\mathtt{SC}^*_\infty[\mathcal{O}_\infty^{-1}]$ the ∞-category of *strongly purely infinite C^*-algebras*. The suspension stable version of this category will be analysed in the next section.

Proposition 2.5. Let us suppose that there is a unital embedding $\iota_\mathcal{D} : \mathcal{D} \to \mathcal{D}'$ of strongly self-absorbing C^*-algebras. Then $\mathcal{D}'$ is an idempotent object in $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$.

Proof. Consider the following commutative diagram in $\mathtt{SC}^*$

$$\begin{array}{ccc} \mathcal{D}' & \xrightarrow{\mathrm{id}_{\mathcal{D}'}\otimes \mathbf{1}_{\mathcal{D}'}} & \mathcal{D}'\hat{\otimes}\mathcal{D}' \\ & {\scriptstyle \mathrm{id}_{\mathcal{D}'}\otimes \mathbf{1}_{\mathcal{D}}}\searrow \quad \nearrow{\scriptstyle \mathrm{id}_{\mathcal{D}'}\otimes \iota_{\mathcal{D}}} & \\ & \mathcal{D}'\hat{\otimes}\mathcal{D}. & \end{array}$$

Since $\mathcal{D}'$ is strongly self-absorbing $\mathrm{id}_{\mathcal{D}'} \otimes \mathbf{1}_{\mathcal{D}'}$ is homotopic to an isomorphism $\mathcal{D}' \to \mathcal{D}'\hat{\otimes}\mathcal{D}'$. It follows from Proposition 5.12 of [21] that $\mathrm{id}_{\mathcal{D}'} \otimes \mathbf{1}_\mathcal{D}$ is homotopic to an isomorphism $\mathcal{D}' \to \mathcal{D}'\hat{\otimes}\mathcal{D}$ demonstrating that $\mathcal{D}'$ is $\mathcal{D}$-stable. It follows that $\mathrm{id}_{\mathcal{D}'} \otimes \iota_\mathcal{D}$ is a homotopy equivalence. Observe that the unit object in $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$ is $\mathcal{D}$. Thus the unital embedding $\iota_\mathcal{D} : \mathcal{D} \to \mathcal{D}'$ exhibits $\mathcal{D}'$ as an idempotent object in $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$. ☐

Corollary 2.6. In the localized ∞-category $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$ every strongly self-absorbing C^*-algebra is an idempotent object.

Proof. The assertion follows from the characterization of $\mathcal{Z}$ as the initial object in the homotopy category of strongly self-absorbing C^*-algebras with unital $*$-homomorphisms (see Corollary 3.2 of [23]). ☐

Remark 2.7. In view of the above Corollary one may construct $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$ for any strongly self-absorbing C^*-algebra $\mathcal{D}$ as a localization of $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$. Thus isomorphisms in $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$ contain the most refined information amongst all smashing localizations with respect to strongly self-absorbing C^*-algebras.

For any $A, B \in \mathtt{SC}^*$ we denote by $[A, B]$ the homotopy classes of $*$-homomorphisms $A \to B$.

Proposition 2.8. For any $A, B \in \mathtt{SC}^*$ and any strongly self absorbing C^*-algebra $\mathcal{D}$ there is a natural isomorphism

$$\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L(A), L(B)) \cong [A, B\hat{\otimes}\mathcal{D}].$$

Proof. Let us first observe that there is an identification

$$\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L(A), L(B)) \cong \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}).$$

There is an element $\theta_A = \mathrm{id}_A \otimes \mathbf{1}_\mathcal{D} \in \mathtt{SC}^*(A, A\hat{\otimes}\mathcal{D})$ sending $a \mapsto a \otimes \mathbf{1}_\mathcal{D}$. This induces a map

$$K : \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D})$$

by precomposing with $[\theta_A]$ (here $[-]$ denotes the homotopy class). Using the fact that $\mathrm{id}_\mathcal{D} \otimes \mathbf{1}_\mathcal{D} : \mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{D}$ is homotopic to an isomorphism $\gamma : \mathtt{SC}^*(\mathcal{D}, \mathcal{D}\hat{\otimes}\mathcal{D})$, we deduce that the map $\mathrm{id}_B \otimes \mathrm{id}_\mathcal{D} \otimes \mathbf{1}_\mathcal{D}$ is homotopic to an isomorphism $\gamma_B \in \mathtt{SC}^*(B\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D})$. Now we define a map

$$L : \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D})$$

as follows: $L([\phi]) = [\gamma_B^{-1} \circ (\phi \otimes \mathrm{id}_\mathcal{D})]$. Observe that $K \circ L([\phi]) = [\gamma_B^{-1} \circ (\phi \otimes \mathrm{id}_\mathcal{D})] \circ [\theta_A] = [\gamma_B^{-1} \circ (\mathrm{id}_B \otimes \mathrm{id}_\mathcal{D} \otimes \mathbf{1}_\mathcal{D}) \circ \phi]$. Since $[\mathrm{id}_B \otimes \mathrm{id}_\mathcal{D} \otimes \mathbf{1}_\mathcal{D}] = [\gamma_B]$ the composition $K \circ L = \mathrm{id} : \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D})$.

Now $L \circ K([\psi]) = L([\psi \circ \theta_A]) = [\gamma_B^{-1} \circ ((\psi \circ \theta_A) \otimes \mathrm{id}_\mathcal{D})]$. Let $\tau_\mathcal{D} : \mathcal{D} \to \mathcal{D}$ denote the tensor flip map, which is also homotopic to the identity. A verification on the simple tensors demonstrates that $[(\mathrm{id}_B \otimes \tau_\mathcal{D}) \circ ((\psi \circ \theta_A) \otimes \mathrm{id}_\mathcal{D})] = [\gamma_B \circ \psi]$. It follows that $L \circ K = \mathrm{id} : \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D})$. It remains to observe that $\mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D}) \cong [A, B\hat{\otimes}\mathcal{D}]$ (see [15]). □

Observe that the subset $\{s_i s_j^* \,|\, i, j \in \mathbb{N}\} \subset \mathcal{O}_\infty$ generates a copy of the compact operators $\mathbb{K}$ inside $\mathcal{O}_\infty$. Let $\iota : \mathbb{K} \to \mathcal{O}_\infty$ denote the canonical inclusion.

Proposition 2.9. In the C^*-stable ∞-category $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$ the map $\iota : \mathbb{K} \to \mathcal{O}_\infty$ exhibits $\mathcal{O}_\infty$ as an idempotent object.

Proof. Consider the diagram $\mathcal{O}_\infty \xrightarrow{\theta} \mathcal{O}_\infty\hat{\otimes}\mathbb{K} \xrightarrow{\phi} \mathcal{O}_\infty\hat{\otimes}\mathcal{O}_\infty$ in $\mathtt{SC}^*$. The map θ sends $a \mapsto a \otimes e_{11}$ and the map $\phi = \mathrm{id}_{\mathcal{O}_\infty} \otimes \iota$. The composite $\phi \circ \theta$ is homotopic to $\mathrm{id}_{\mathcal{O}_\infty} \otimes \mathbf{1}_{\mathcal{O}_\infty} : \mathcal{O}_\infty \to \mathcal{O}_\infty\hat{\otimes}\mathcal{O}_\infty$, whence it is an equivalence in $\mathtt{SC}^*_\infty$. The map θ is an equivalence in $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$. It follows that $\phi = \mathrm{id}_{\mathcal{O}_\infty} \otimes \iota$ is an equivalence in $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$. □

Corollary 2.10. The ∞-category $\mathtt{SC}^*_\infty[\mathcal{O}_\infty^{-1}]$ can be obtained as a localization of $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$.

Remark 2.11. It is well-known that $\mathtt{hSC}^*_\infty[\mathbb{K}^{-1}](A, B) \cong [A, B\hat{\otimes}\mathbb{K}]$. Isomorphisms in $\mathtt{hSC}^*_\infty[\mathbb{K}^{-1}]$ between C^*-algebras of the form $\mathrm{C}(X, x)\hat{\otimes}\mathbb{K}$, where (X, x) is a finite pointed CW complex, can be detected in terms of connective kk-theory (see Theorem 2.4 of [6]). The connective kk-theory should not be confused with Cuntz kk-theory for m-algebras (or locally convex algebras).

Corollary 2.12. Consider the following problem: Given two finite pointed CW complexes (X, x) and (Y, y) are the C^*-algebras $\mathrm{C}(X, x)\hat{\otimes}\mathcal{O}_\infty$ and $\mathrm{C}(Y, y)\hat{\otimes}\mathcal{O}_\infty$ homotopy equivalent? In view of the above Remark 2.11 a sufficient criterion can be obtained in terms of connective kk-theory. Homotopy equivalences of matrix bundles can also be detected by connective E-theory [20].

Now we demonstrate that the homotopy category of the smashing localization $\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}]$ admits a universal characterization much like KK-theory. The localization ∞-functor $L_\mathcal{D} : \mathtt{SC}^*_\infty \to \mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$ induces a canonical (ordinary) functor $L_\mathcal{D} : \mathtt{SC}^* \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}]$. Recall that a functor $F : \mathtt{SC}^* \to \mathcal{C}$ ($\mathcal{C}$ an ordinary category) is called $\mathcal{D}$-*stable* if F sends the morphism $A \to A\hat{\otimes}\mathcal{D}$ mapping $a \mapsto a \otimes \mathbf{1}_\mathcal{D}$ to an isomorphism in $\mathcal{C}$ for all $A \in \mathtt{SC}^*$.

Theorem 2.13. The functor $L_\mathcal{D} : \mathtt{SC}^* \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}]$ is the universal homotopy invariant and $\mathcal{D}$-stable functor on $\mathtt{SC}^*$.

Proof. Let us first show that functor $L_\mathcal{D}$ is homotopy invariant and $\mathcal{D}$-stable. It is easy to verify that it is homotopy invariant. It follows from the arguments in the proof of Proposition 2.8 that the map $\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_\mathcal{D}(A\hat{\otimes}\mathcal{D}), L_\mathcal{D}(B)) \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_\mathcal{D}(A), L_\mathcal{D}(B))$ induced by $A \to A\hat{\otimes}\mathcal{D}$ is an isomorphism for all $B \in \mathtt{SC}^*$. For any $B \in \mathtt{SC}^*$ the map

$$\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_\mathcal{D}(B), L_\mathcal{D}(A)) \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_\mathcal{D}(B), L_\mathcal{D}(A\hat{\otimes}\mathcal{D}))$$

is equivalent to that map $[B, A\hat{\otimes}\mathcal{D}] \to [B, A\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D}]$ once again by Proposition 2.8. This map is induced by $A\hat{\otimes}\mathcal{D} \to A\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D}$ sending $a \otimes d \mapsto a \otimes \mathbf{1}_\mathcal{D} \otimes d$. Since $\mathcal{D}$ is strongly self-absorbing one easily sees $[B, A\hat{\otimes}\mathcal{D}] \to [B, A\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D}]$ is an isomorphism. Since $L_\mathcal{D}$ is surjective on objects we conclude that $L_\mathcal{D}$ is $\mathcal{D}$-stable.

Let $F_i : \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}] \to \mathcal{C}$ with $i = 1, 2$ be two functors making the following diagram commute

(1)
$$\begin{array}{ccc} \mathtt{SC}^* & \xrightarrow{L_\mathcal{D}} & \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}] \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle F_i} & \\ & \mathcal{C}. & \end{array}$$

On objects they are both determined by $\mathcal{D}$-stability $F_i(A\hat{\otimes}\mathcal{D}) \cong F(A\hat{\otimes}\mathcal{D}) \cong F(A)$. Similarly, on each morphism $\phi : A\hat{\otimes}\mathcal{D} \to B\hat{\otimes}\mathcal{D}$ the value of $F_i(\phi)$ is uniquely determined by the following diagram:

$$\begin{array}{ccc} F_i(A\hat{\otimes}\mathcal{D}) & \xrightarrow{F_i(\phi)} & F_i(B\hat{\otimes}\mathcal{D}) \\ \cong\uparrow & & \uparrow\cong \\ F(A) & \xrightarrow{F(\phi)} & F(B). \end{array}$$

For the existence note that for any homotopy invariant and $\mathcal{D}$-stable functor $F : \mathtt{SC}^* \to \mathcal{C}$ there is a functor $\overline{F} : \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}] \to \mathcal{C}$ sending $A\hat{\otimes}\mathcal{D}$ to $F(A\hat{\otimes}\mathcal{D}) \cong F(A)$ that makes the above diagram (1) commute (up to a natural isomorphism). □

3. Coidempotent objects and colocalizations of NSp

Let us remind the readers that the ∞-functor $\Pi^{\mathrm{op}} : \mathtt{SC}^{*\,\mathrm{op}}_\infty \to \mathtt{NSp}$ arises as a composition of the following ∞-functors

$$\mathtt{SC}^{*\,\mathrm{op}}_\infty \xrightarrow{j} \mathtt{N}\mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \xrightarrow{L} S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) =: \mathtt{NSp}.$$

For any separable C^*-algebra A one ought to regard $\Pi^{\rm op}(A)$ as its suspension spectrum after localization with respect to S. Hence we are going to reset $\Sigma^\infty_S A := \Pi^{\rm op}(A)$. Owing to the symmetric monoidal structure on $\mathtt{NSp}$ that we established earlier, one may consider the endofunctor $-\otimes \Sigma^\infty_S A : \mathtt{NSp} \to \mathtt{NSp}$ for any $A \in {\mathtt{SC}^*_\infty}^{\rm op}$.

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. We say that a map $e : E \to \mathbf{1}$ exhibits E as a *coidempotent object* in $\mathcal{C}$ if the dual map $e^{\rm op} : \mathbf{1} \to E$ exhibits E as an idempotent object in $\mathcal{C}^{\rm op}$. Recall that the symmetric monoidal structure on $\mathcal{C}$ endows $\mathcal{C}^{\rm op}$ with a symmetric monoidal structure that is uniquely defined up to a contractible space of choices.

Lemma 3.1. If $\mathcal{D}$ is a strongly self-absorbing C^*-algebra, then $j(\mathcal{D})$ is a coidempotent object in $\mathtt{N}\mathcal{S}_*$. The same assertion holds for $\mathbb{K}$, i.e., $j(\mathbb{K})$ is a coidempotent object in $\mathtt{N}\mathcal{S}_*$.

Proof. Let X stand for $\mathcal{D}$ or $\mathbb{K}$. Since X is an idempotent object in $\mathtt{SC}^*_\infty$, it becomes a coidempotent object in ${\mathtt{SC}^*_\infty}^{\rm op}$. Consequently, $j(X)$ becomes a coidempotent object in $\mathtt{N}\mathcal{S}_*$ (since $j : {\mathtt{SC}^*_\infty}^{\rm op} \to \mathtt{N}\mathcal{S}_*$ is a fully faithful symmetric monoidal ∞-functor). □

Lemma 3.2. For any strongly self-absorbing C^*-algebra $\mathcal{D}$, the stabilization $\Sigma^\infty_S \mathcal{D}$ is a coidempotent object in $\mathtt{NSp}$. The same assertion holds for $\mathbb{K}$, i.e., $\Sigma^\infty_S \mathbb{K}$ is a coidempotent object in $\mathtt{NSp}$.

Proof. Since $\Sigma^\infty : \mathtt{N}\mathcal{S}_* \to \mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ and $L : \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \simeq \mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \to T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \simeq S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ are both symmetric monoidal ∞-functors, the assertion follows from the previous Lemma. □

Recall that an ∞-functor $R : \mathcal{C} \to \mathcal{C}$ is called a *colocalization* if $R : \mathcal{C} \to R\mathcal{C}$ is the right adjoint to the inclusion $R\mathcal{C} \subset \mathcal{C}$; in particular, the inclusion is the left adjoint to R and hence preserves all small colimits.

Proposition 3.3. Let A be a strongly self-absorbing C^*-algebra or $\mathbb{K}$. The ∞-functors $R_1 : \mathtt{N}\mathcal{S}_* \to \mathtt{N}\mathcal{S}_*$ and $R_2 : \mathtt{NSp} \to \mathtt{NSp}$ given by $R_1(X) = X \otimes j(A)$ and $R_2(X) = X \otimes \Sigma^\infty_S A$ are colocalization functors.

Proof. The assertions follow from the dual of Proposition 6.3.2.4 of [12]. □

3.1. **Colocalizations and purely infinite strongly self absorbing C^*-algebras.** The list of known examples of strongly self-absorbing C^*-algebras is rather limited. The list includes Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the Jiang–Su algebra $\mathcal{Z}$, UHF algebras of infinite type, and tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type. It follows from the results of Kirchberg that strongly self-absorbing C^*-algebras are either stably finite or purely infinite. In the purely infinite case Toms–Winter completely classified all strongly self-absorbing C^*-algebras satisfying UCT (Corollary page 4022 [21]), viz., they are $\mathcal{O}_2$, $\mathcal{O}_\infty$ and tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type. We are particularly interested in the purely infinite ones since $ax+b$-semigroup C^*-algebras of number rings are all purely infinite (Corollary 8.2.11 of [5]). Among the strongly self-absorbing purely infinite C^*-algebras $\mathcal{O}_\infty$ plays a distinguished role in the classification program. The C^*-algebra $A\hat{\otimes}\mathcal{O}_\infty$ is purely infinite for any $A \in \mathtt{SC}^*$ [10]. Deviating slightly from the predictable pattern the colocalization of $\mathtt{NSp}$ by the ∞-functor $R_{\Sigma^\infty_S \mathcal{D}}(-) = -\otimes \Sigma^\infty_S \mathcal{D}$ is denoted by $\mathtt{NSp}[\mathcal{D}^{-1}]$ (and not by $\mathtt{NSp}[(\Sigma^\infty_S \mathcal{D})^{-1}]$). In what follows we are going to drop the object $\Sigma^\infty_S \mathcal{D}$ from the colocalization functor $R_{\Sigma^\infty_S \mathcal{D}}$ and denote it simply by R.

Thanks to Proposition 3.3 above one can study colocalizations of both $\mathtt{N}\mathcal{S}_*$ and $\mathtt{NSp}$ with respect to a strongly self-absorbing C^*-algebra $\mathcal{D}$ or $\mathbb{K}$. We are mostly interested in the (suspension) stable situation. Let us call the full ∞-subcategory of $\mathtt{NSp}$ (or of its colocalization) spanned by $\Sigma^\infty_S A$ for all $A \in \mathtt{SC}^*_\infty$ (or $\Sigma^\infty_S A$ followed by the the colocalization functor) the *C^*-core*. In the sequel we describe the homotopy category of the C^*-core of the colocalization of $\mathtt{NSp}$ when $\mathcal{D}$ is a purely infinite strongly self-absorbing C^*-algebra. We leave out the cases involving the stably finite ones for a future project.

Theorem 3.4. For any $A, B \in \mathtt{SC}^*$ there is a natural isomorphism

$$\mathtt{hNSp}[\mathcal{O}_\infty^{-1}](R(\Sigma^\infty_S A), R(\Sigma^\infty_S B)) \cong \mathrm{E}_0(B, A).$$

Proof. By construction there is a natural identification

$$\mathtt{hNSp}[\mathcal{O}_\infty^{-1}](R(\Sigma^\infty_S A), R(\Sigma^\infty_S B)) \cong \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)),$$

where we used the fact that $\Sigma^\infty_S : \mathtt{hSC}^{*\,\mathrm{op}}_\infty \to \mathtt{hNSp}$ is symmetric monoidal (see Corollary 1.4). Now consider the canonical composition of $*$-homomorphisms $\mathbb{K} \hookrightarrow \mathcal{O}_\infty \xrightarrow{\theta} \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$. Here $\theta : \mathcal{O}_\infty \to \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ is the corner embedding $a \mapsto a \otimes e_{11}$. We ought to view this as a diagram $\mathbb{K} \leftarrow \mathcal{O}_\infty \leftarrow \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ in $\mathtt{SC}^{*\,\mathrm{op}}_\infty$. Tensoring the diagram with A and applying $\Sigma^\infty_S(-)$ leads to the following diagram in $\mathtt{hNSp}$

$$\Sigma^\infty_S(A\hat{\otimes}\mathbb{K}) \leftarrow \Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty) \xleftarrow{\Theta} \Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}).$$

Now we apply the functor $\mathtt{hNSp}(-, \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty))$ to this diagram and use Theorem 4.25 of [15] to obtain

$$\begin{array}{ccc}
\mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathbb{K}), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \xrightarrow{\cong} & \mathtt{NSH}^{\mathrm{op}}(A\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty) \\
\downarrow & & \downarrow \\
\mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \xrightarrow{\cong} & \mathtt{NSH}^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \\
\downarrow \Theta & & \downarrow \\
\mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \xrightarrow{\cong} & \mathtt{NSH}^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty).
\end{array}$$

Observe that for any $E, F \in \mathtt{SC}^*$ there is a natural map $\mathtt{NSH}(E, F) \to \mathrm{E}_0(E, F)$, which becomes an isomorphism as soon as F is stable (see Theorem 4.1.1. of [20]). Thus we may modify the above diagram as follows:

$$\begin{array}{ccc}
\mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathbb{K}), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \xrightarrow{\cong} & \mathrm{E}^{\mathrm{op}}_0(A\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty) \\
\downarrow & & \downarrow \\
\mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \longrightarrow & \mathrm{E}^{\mathrm{op}}_0(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \\
\downarrow \Theta & & \downarrow \\
\mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \xrightarrow{\cong} & \mathrm{E}^{\mathrm{op}}_0(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty).
\end{array}$$

Since the diagram $\mathbb{K} \hookrightarrow \mathcal{O}_\infty \xrightarrow{\theta} \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ produces a E-equivalence, the right vertical composition is an isomorphism. It follows that the left vertical composition is also an isomorphism, i.e., the natural map

$$\Theta : \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) \to \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty))$$

induced by $\theta : \mathcal{O}_\infty \to \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ is split surjective.

Now consider the composition of $*$-homomorphisms $\mathcal{O}_\infty \xrightarrow{\theta} \mathcal{O}_\infty\hat{\otimes}\mathbb{K} \xrightarrow{\kappa} \mathcal{O}_\infty$ with $\kappa(a\otimes e_{ij}) = s_i a s_j^*$. Since $\kappa\circ\theta$ is homotopic to an isomorphism in $\mathtt{SC}^*$, the composition in the induced diagram in $\mathtt{hNSp}$ (after tensoring with A and applying $\Sigma^\infty_S(-)$)

$$\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty) \leftarrow \Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}) \leftarrow \Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty)$$

is an isomorphism in $\mathtt{hNSp}$. Applying the functor $\mathtt{hNSp}(-, \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty))$ we see that the dotted composite

$$\begin{array}{ccc} \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \xrightarrow{\Theta} & \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) \\ & \searrow & \downarrow \\ & & \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) \end{array}$$

must be an isomorphism. It follows that Θ is split injective and consequently an isomorphism. Now in the commutative diagram

$$\begin{array}{ccc} \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \longrightarrow & \mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \\ \downarrow \Theta & & \downarrow \\ \mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}), \Sigma^\infty_S(B\hat{\otimes}\mathcal{O}_\infty)) & \xrightarrow{\cong} & \mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty) \end{array}$$

the right vertical arrow is an isomorphism due to the C^*-stability of E-theory, whence the top horizontal arrow must also be an isomorphism. Finally, we observe that

$$\mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{E}_0(B, A)$$

due to the $\mathcal{O}_\infty$-stability of E-theory in both variables and all the identifications made thus far were natural. □

Remark 3.5. The above Theorem demonstrates that the colocalized ∞-category $\mathtt{NSp}[\mathcal{O}_\infty^{-1}]$ produces an ∞-categorical model for an enlarged version of the opposite of bivariant E-theory category. Of course, if the separable C^*-algebras in sight are nuclear, then one can replace E-theory by KK-theory.

Remark 3.6. An inspection of the proof of Theorem 3.4 demonstrates that actually a stronger result holds, viz.,

$$\mathtt{hNSp}(\Sigma^\infty_S(A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_S B) \cong \mathtt{NSH}(B, A\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{E}_0(B, A)$$

for any $A, B \in \mathtt{SC}^*$.

Corollary 3.7. The nonconnective algebraic K-theory of $\mathcal{O}_\infty$-stable separable C^*-algebras factors through $\mathtt{hNSp}[\mathcal{O}_\infty^{-1}]$.

Proof. It was shown in [14] that the nonconnective algebraic K-theory of $\mathcal{O}_\infty$-stable C^*-algebras agrees naturally with their topological K-theory. The assertion now follows since topological K-theory factors through $\mathtt{hNSp}[\mathcal{O}_\infty^{-1}]$. □

Now let $\mathcal{Q}$ denote any UHF algebra of infinite type, so that $\mathcal{O}_\infty \hat{\otimes} \mathcal{Q}$ is a purely infinite strongly self-absorbing C^*-algebra.

Theorem 3.8. For any $A, B \in \mathtt{SC}^*$ there is a natural isomorphism

$$\mathtt{hNSp}[(\mathcal{O}_\infty \hat{\otimes} \mathcal{Q})^{-1}](R(\Sigma^\infty_S A), R(\Sigma^\infty_S B)) \cong \mathrm{E}_0(B \hat{\otimes} \mathcal{Q}, A \hat{\otimes} \mathcal{Q}).$$

Proof. As before we first observe that

$$\mathtt{hNSp}[(\mathcal{O}_\infty \hat{\otimes} \mathcal{Q})^{-1}](R(\Sigma^\infty_S A), R(\Sigma^\infty_S B)) \cong \mathtt{hNSp}(\Sigma^\infty_S (A \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} \mathcal{Q}), \Sigma^\infty_S (B \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} \mathcal{Q})).$$

Arguing as in the previous Theorem one then proves that

$$\mathtt{hNSp}(\Sigma^\infty_S (A \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} \mathcal{Q}), \Sigma^\infty_S (B \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} \mathcal{Q})) \cong \mathrm{E}_0(B \hat{\otimes} \mathcal{Q}, A \hat{\otimes} \mathcal{Q}).$$

□

Example 3.9. If $\mathcal{Q}$ is the universal UHF algebra, then the C^*-core of the colocalization of $\mathtt{NSp}$ by the ∞-functor $- \otimes \Sigma^\infty_S (\mathcal{O}_\infty \hat{\otimes} \mathcal{Q})$ produces an ∞-categorical model for the opposite of rationalized bivariant E-theory category. Indeed, it is well known that tensoring with the universal UHF algebra rationalizes E-theory, e.g., it follows from the Theorem in Section 3 of [7] that

$$\mathrm{E}_i(\mathcal{O}_\infty \hat{\otimes} \mathcal{Q}, A \hat{\otimes} \mathcal{Q}) \cong \mathrm{E}_i(A) \otimes_\mathbb{Z} \mathbb{Q} \quad \text{for } i = 0, 1.$$

Now we show that the colocalization of $\mathtt{NSp}$ by $-\hat{\otimes} \Sigma^\infty_S \mathcal{O}_2$ annihilates its C^*-core.

Theorem 3.10. For any $A, B \in \mathtt{SC}^*$ there is a natural isomorphism

$$\mathtt{hNSp}[\mathcal{O}_2^{-1}](R(\Sigma^\infty_S A), R(\Sigma^\infty_S B)) \cong 0.$$

Proof. Once again we first observe that

$$\mathtt{hNSp}[\mathcal{O}_2^{-1}](R(\Sigma^\infty_S A), R(\Sigma^\infty_S B)) \cong \mathtt{hNSp}(\Sigma^\infty_S (A \hat{\otimes} \mathcal{O}_2), \Sigma^\infty_S (B \hat{\otimes} \mathcal{O}_2)).$$

We also know from Theorem 4.25 of [15] that

$$\mathtt{hNSp}(\Sigma^\infty_S (A \hat{\otimes} \mathcal{O}_2), \Sigma^\infty_S (B \hat{\otimes} \mathcal{O}_2)) \cong \mathtt{NSH}(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2).$$

Since $\mathcal{O}_2$ is properly infinite one can again find a diagram in $\mathtt{SC}^*$

$$\mathcal{O}_2 \to \mathcal{O}_2 \hat{\otimes} \mathbb{K} \to \mathcal{O}_2,$$

such that the composition is homotopic to an isomorphism (see Proposition 1.1.2 of [18]). Tensoring the diagram with A we get another one

$$A \hat{\otimes} \mathcal{O}_2 \to A \hat{\otimes} \mathcal{O}_2 \hat{\otimes} \mathbb{K} \to A \hat{\otimes} \mathcal{O}_2,$$

such that the composition is again homotopic to an isomorphism. Applying the homotopy functor $\mathtt{NSH}(B \hat{\otimes} \mathcal{O}_2, -)$ to the above diagram we find that $\mathtt{NSH}(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2)$ is a summand of $\mathtt{NSH}(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2 \hat{\otimes} \mathbb{K}) \cong \mathrm{E}_0(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2 \hat{\otimes} \mathbb{K}) \cong \mathrm{E}_0(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2)$. It suffices to show that $\mathrm{E}_0(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2)$ vanishes. Since $\mathcal{O}_2$ is KK-contractible, so is $B \hat{\otimes} \mathcal{O}_2$ and hence it satisfies UCT. Thus one may identify $\mathrm{E}_0(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2) \cong \mathrm{KK}_0(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2)$ and the group $\mathrm{KK}_0(B \hat{\otimes} \mathcal{O}_2, A \hat{\otimes} \mathcal{O}_2)$ evidently vanishes. □

3.2. **Noncommutative stable cohomotopy of $ax+b$-semigroup C^*-algebras of number rings.** Number rings are central objects of study in number theory. To any number ring one can associate an $ax+b$-semigroup C^*-algebra that possesses very intriguing structure [4]. It is an important task to ascertain (co)homological invariants of these C^*-algebras. We begin with a disambiguation.

Disambiguation 3.11. In [15, 16] the author decided to call the groups $\mathtt{NSH}(\mathbb{C},-)$ (resp. $\mathtt{NSH}(-,\mathbb{C})$) the noncommutative stable homotopy (resp. noncommutative stable cohomotopy) groups. The terminology was motivated by the fact that $\mathtt{NSH}(\mathbb{C},-)$ is covariant and $\mathtt{NSH}(-,\mathbb{C})$ is contravariant. However, it was observed in ibid. that $\mathtt{NSH}(\mathbb{C},-)$ generalizes stable cohomotopy, whereas $\mathtt{NSH}(-,\mathbb{C})$ generalizes stable homotopy of finite pointed CW complexes. In order to align the theory with the terminology familiar to topologists, we rename as follows:

$$\begin{aligned} \mathtt{NSH}(\mathbb{C},-) &= \text{noncommutative stable cohomotopy} \\ \mathtt{NSH}(-,\mathbb{C}) &= \text{noncommutative stable homotopy} \end{aligned}$$

We also extend the terminology predictably to their graded versions.

Recently Li showed that for a countable integral domain R with vanishing Jacobson radical (which is, in addition, not a field) the left regular $ax+b$-semigroup C^*-algebra $C^*_\lambda(R\rtimes R^\times)$ is $\mathcal{O}_\infty$-stable, i.e., $C^*_\lambda(R\rtimes R^\times)\hat{\otimes}\mathcal{O}_\infty \cong C^*_\lambda(R\rtimes R^\times)$ (see Theorem 1.3 of [11]). Cuntz–Echterhoff–Li computed the topological K-theory of such $ax+b$-semigroup C^*-algebras in [5] as follows:

$$\mathrm{K}_*(C^*_\lambda(R\rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash \mathcal{J}} \mathrm{K}_*(C^*(G_X)), \tag{2}$$

where $\mathcal{J}$ is the set of fractional ideal of R, $G = K\rtimes K^\times$, and G_X is the stabilizer of X under the G-action on $\mathcal{J}$. The orbit space $G\setminus\mathcal{J}$ can be identified with the ideal class group of K.

Theorem 3.12. The noncommutative stable cohomotopy of the left regular $ax+b$-semigroup C^*-algebra of the ring of integers R of a number field K is 2-periodic and explicitly given by

$$\mathtt{NSH}(\mathbb{C}, C^*_\lambda(R\rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash \mathcal{J}} \mathrm{K}_0(C^*(G_X)).$$

and

$$\mathtt{NSH}(\mathbb{C}, \Sigma C^*_\lambda(R\rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash \mathcal{J}} \mathrm{K}_1(C^*(G_X)).$$

Proof. Since $C^*_\lambda(R\rtimes R^\times)$ is $\mathcal{O}_\infty$-stable, there is an identification of noncommutative stable cohomotopy $\mathtt{NSH}(\mathbb{C}, C^*_\lambda(R\rtimes R^\times)) \cong \mathtt{NSH}(\mathbb{C}, C^*_\lambda(R\rtimes R^\times)\hat{\otimes}\mathcal{O}_\infty)$. By Remark 3.6 we conclude that $\mathtt{NSH}(\mathbb{C}, C^*_\lambda(R\rtimes R^\times)\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{E}_0(\mathbb{C}, C^*_\lambda(R\rtimes R^\times))$. One may identify the E-theory of $C^*_\lambda(R\rtimes R^\times)$ naturally with its topological K-theory (of course, $C^*_\lambda(R\rtimes R^\times)$ is itself nuclear). The results now follow from Equation (2) (the second one after suspension). □

4. Appendix

The symmetric monoidal presentable ∞-category of noncommutative spaces $\mathtt{NS}_*$ constructed here is well-suited for unstable homotopy theory of noncommutative pointed spaces. Indeed, by a general result one can produce a combinatorial simplicial model category, whose underlying ∞-category is equivalent to $\mathtt{N}\mathcal{S}_*$ (see Proposition A.3.7.6 of [13]). This model category will definitely contain all separable C^*-algebas as objects. However, it will also contain many other objects, which are needed to have a category large enough on which a model structure can be constructed. Although these extra objects are quite meaningful from the viewpoint of homotopy theory, they can be rather cumbersome for operator algebraists.

The above comment also applies to the model structure constructed by Østvær for unstable homotopy theory of C^*-algebras [17]. Andersen–Grodal had earlier observed that one cannot expect a model category structure that conforms with some reasonable expectations by restricting one's attention only to the category of C^*-algebras [1] (see also Appendix of [22]). One natural approach is to consider the category of pro C^*-algebras if one wants to stay close to analysis. Joachim–Johnson adopted a strategy roughly along these lines to produce a model structure, whose homotopy category (restricted to the subcategory spanned by separable C^*-algebras) is equivalent to Kasparov's KK-category [9]. Their work was motivated by the Cuntz picture of KK-theory [3]; however, to the best of the author's knowledge a model structure on objects similar to pro C^*-algebras modelling the unstable homotopy theory is still absent in the literature. This issue will be addressed in our upcoming work [2].

References

[1] K. K. S. Andersen and J. Grodal. A Baues fibration category structure on Banach and C^*-algebras. *http://www.math.ku.dk/ jg/papers/fibcat.pdf*, 1997.

[2] I. Barnea, M. Joachim, and S. Mahanta. Model structure on projective systems of C^*-algebras and bivariant homology theories. *in preparation.*

[3] J. Cuntz. A new look at KK-theory. *K-Theory*, 1(1):31–51, 1987.

[4] J. Cuntz, C. Deninger, and M. Laca. C^*-algebras of Toeplitz type associated with algebraic number fields. *Math. Ann.*, 355(4):1383–1423, 2013.

[5] J. Cuntz, S. Echterhoff, and X. Li. On the K-theory of the C^*-algebra generated by the left regular representation of an Ore semigroup. *arXiv:1201.4680, to appear in J. Eur. Math. Soc.*

[6] M. Dadarlat and J. McClure. When are two commutative C^*-algebras stably homotopy equivalent? *Math. Z.*, 235(3):499–523, 2000.

[7] M. Dadarlat and W. Winter. On the KK-theory of strongly self-absorbing C^*-algebras. *Math. Scand.*, 104(1):95–107, 2009.

[8] D. Gepner, M. Groth, and T. Nikolaus. Universality of multiplicative infinite loop space machines. *arXiv:1305.4550.*

[9] M. Joachim and M. W. Johnson. Realizing Kasparov's KK-theory groups as the homotopy classes of maps of a Quillen model category. In *An alpine anthology of homotopy theory*, volume 399 of *Contemp. Math.*, pages 163–197. Amer. Math. Soc., Providence, RI, 2006.

[10] E. Kirchberg and M. Rørdam. Non-simple purely infinite C^*-algebras. *Amer. J. Math.*, 122(3):637–666, 2000.

[11] X. Li. Semigroup C^*-algebras of $ax+b$-semigroups. *arXiv:1306.5553.*

[12] J. Lurie. Higher Algebra. *http://www.math.harvard.edu/∼lurie/.*

[13] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2009.

[14] S. Mahanta. Algebraic K-theory, K-regularity, and $\mathbb{T}$-duality of $\mathcal{O}_\infty$-stable C^*-algebras. *arXiv:1311.4720.*

[15] S. Mahanta. Noncommutative stable homotopy, stable infinity categories, and semigroup C^*-algebras. *arXiv:1211.6576.*

[16] S. Mahanta. On the Generating Hypothesis in noncommutative stable homotopy. *to appear in Math. Scand.; arXiv:1302.2051.*

[17] P. A. Østvær. *Homotopy theory of C^*-algebras*. Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2010.
[18] M. Rørdam. Classification of nuclear, simple C^*-algebras. In *Classification of nuclear C^*-algebras. Entropy in operator algebras*, volume 126 of *Encyclopaedia Math. Sci.*, pages 1–145. Springer, Berlin, 2002.
[19] S. Schwede. The p-order of topological triangulated categories. *J. Topol.*, 2013.
[20] A. Thom. Connective E-theory and bivariant homology for C^*-algebras. *thesis, Münster*, 2003.
[21] A. S. Toms and W. Winter. Strongly self-absorbing C^*-algebras. *Trans. Amer. Math. Soc.*, 359(8):3999–4029, 2007.
[22] O. Uuye. Homotopy Theory for C^*-algebras. *arXiv:1011.2926, to appear in JNCG*.
[23] W. Winter. Strongly self-absorbing C^*-algebras are $\mathcal{Z}$-stable. *J. Noncommut. Geom.*, 5(2):253–264, 2011.

E-mail address: `s.mahanta@uni-muenster.de`

Mathematical Institute, University of Muenster, Einsteinstrasse 62, 48149 Muenster, Germany.