ON GRADED BETTI NUMBERS FOR FINITE SUBSETS OF CURVES

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1. Introduction

When a computer graphics program draws a curve on the screen, it replaces the curve by its finite subset, and this subset serves as a good enough approximation. The present paper grew out of an attempt to make this idea of approximation precise: what could an algebraic geometer mean by saying that curves are approximated by their finite subsets?

To be more specific, consider a projective curve $C \subset \mathbb{P}^n$ and various finite subsets $X \subset C$. Let us compare the defining equations of C and X; do the equations of X "converge" to those of C as the subset $X \subset C$ grows? Actually we work not only with the equations, but with relations among these equations, relations among these relations ... – in short, we compare minimal resolutions for homogeneous coordinate rings of X and C and try to learn what happens as new points of C are added to the set X. It is not hard to show that minimal resolutions of the sets X actually approximate the minimal resolution of C, and that a weakened form of this assertion is true for varieties of arbitrary dimension – cf. Theorem 1.1 below. However, computer experiments with Macaulay [2] suggest that something much more interesting happens as the subset X grows: eventually the minimal resolution for X splits into two parts, one of which is just the minimal resolution for C, while the other behaves very regularly. This suggests Conjecture 1.2 (see below), which we call "Periodicity conjecture".

I was not able to prove this conjecture. Instead, I provide some evidence for it by proving some of its consequences. I hope that the techniques we use in section 2, as well as some of the proved results, are of independent interest.

Before stating the results and conjectures, let us fix some notation.

Assume that the ground field k is of characteristic 0, algebraically closed, and uncountable.

If $\mathbb{P}^n = \operatorname{Proj} R$, where $R = k[T_0, \dots, T_n]$, and if $X \subset \mathbb{P}^n$ is a closed subscheme, we denote the homogeneous ideal of X by $I_X \subset R$ and put $R_X = R/I_X$. The minimal free graded resolution of the R-module R_X will be denoted by $L.(X)$. Thus, $L_0(X) = R$ and $L_m(X) = 0$ for $m > n$. If $L_i = \bigoplus_j R(-i-j)^{b_{ij}(X)}$, then the numbers $b_{ij}(X)$ are called *graded Betti numbers* of X. Our notation for graded Betti numbers agrees with that adopted by the program Macaulay but differs from, for example, Birkenhake's [3]

Now we can state the approximation result (see Definition 3.4 below for the precise meaning of the words "generic infinite sequence of points"):

Theorem 1.1 (Approximation). *Suppose that $V \subset \mathbb{P}^n$ is an irreducible projective variety and $\{P_t\}_{t\in\mathbb{N}}$ is a generic sequence of distinct points of V; put $X_i = \{P_1, \dots, P_i\} \subset V$. Then for any $j \geq 0$ there exists an integer $N > 0$ such that $b_{rs}(X_i) = b_{rs}(V)$ whenever $s \leq j$ and $i \geq N$.*

If $\dim V = 1$, then this property holds for any sequence of distinct points $P_i \in V$ and for any algebraically closed field of characteristic 0.

If the ground field is countable and $\dim V > 1$, this theorem asserts nothing since in this case *any* property of infinite sequences is true for a generic sequence.

In the language of Betti diagrams (cf. [5]; by definition, Betti diagram of a closed subscheme $X \subset \mathbb{P}^n$ is a rectangular table with $b_{ij}(X)$ at the intersection of the i-th column and the j-th row; rows are numbered from top to bottom and columns from left to right), this means that, for all $n \gg 0$, Betti diagram of X_n looks as follows: in the first few rows we see the Betti diagram for the curve C, an array of empty rows follows, and then a "tail", which distinguishes the Betti diagram for X_n from that of C, appears. The following conjecture asserts that the "tail" behaves very regularly:

Conjecture 1.2 (Periodicity). *Suppose that $C \subset \mathbb{P}^n$ is an irreducible projective curve of degree d and that $P = \{P_m\}_{m\in\mathbb{N}}$ is a generic sequence of distinct points of C. Put $\delta = \max\{\, j \mid \exists i\colon b_{ij}(C) \neq 0\,\}$; for each $m \in \mathbb{N}$ put $X_m = \{P_1, \dots, P_m\} \subset C$ and $t_m = \min\{\, j > \delta \mid \exists i\colon b_{ij}(X_m) \neq 0\,\}$. Then for all $m \gg 0$ we have: $t_{m+d} = t_m + 1$, and if $j \geq t_m$, then $b_{i,j+1}(X_{m+d}) = b_{ij}(X_m)$ for all i.*

In the language of Betti diagrams this means that, for all $n \gg 0$, the "tail" repeats periodically with period $d = \deg C$; after every d steps, the tail shifts one row down, leaving zeroes in the vacant places (see the appendix for an illustration).

Further Macaulay experiments suggest that even more may be true: periodicity takes place if we add points of C to an arbitrary closed $Y \subset \mathbb{P}^n$ such that $C \not\subset Y$.

Conjecture 1.3 (Periodicity, strong form). *Suppose that $Y \subset \mathbb{P}^n$ is a closed subset and $C \subset \mathbb{P}^n$ is a closed curve not contained in Y. Put $d = \deg C$ and $\delta = \max\{\, j \mid \exists i\colon b_{ij}(Y \cup C) \neq 0\,\}$; for each $m \geq 0$, put $X_m = Y \cup \{P_1, \dots, P_m\}$ and $t_m = \min\{\, j > \delta \mid \exists i\colon b_{ij}(Y \cup X_m) \neq 0\,\}$. Then for all $m \gg 0$ we have $t_{m+d} = t_m + 1$, and if $j \geq t_n$, then $b_{i,j+1}(X_{m+d}) = b_{ij}(X_m)$ for all i. Moreover, the periodic pattern* *appearing for large m in Betti diagrams of $Y \cup \{P_1, \dots, P_m\}$) depends on C, but not on Y.*

Presently there is no evidence for this conjecture besides computer experiments.

The paper is organized as follows. In section 2 we explain how to express Koszul cohomology of an embedded projective variety via its Macaulay's inverse system. This allows us to prove the following result, which is used in subsequent computations (the ground field is algebraically closed of characteristic 0):

Proposition 1.4. *Let $X \subset V \subset \mathbb{P}^n$ be Zariski closed sets. Suppose that, for an integer $m > 0$, linear spans of the m-th Veronese images of X and V coincide. Then $b_{rs}(X) = b_{rs}(V)$ whenever $s < m$, and $b_{rm}(X) \geq b_{rm}(V)$ for all m.*

This proposition means that the syzygies of degree $< m$ are completely determined by equations of degree $\leq m$.

In section 3 we make precise the notion of a generic infinite sequence and prove the approximation property (Theorem 1.1). Section 4 contains various auxiliary results on graded Betti numbers. Most of them must be known to the experts. In particular, we compute graded Betti numbers for subvarieties of minimal degree. Our method allows us to do it almost for free, without using the determinantal techniques (cf. [1, II.5.3]).

In section 5 we prove two consequences of Conjecture 1.2. First, we show that the lower right corners of Betti diagrams of subsets $X_n \subset C$ behave as predicted by the conjecture. This is our Proposition 5.1. Then we prove Conjecture 1.2 for rational normal curves. It turns out that in this case the assertion of Conjecture 1.2 is true for arbitrary sequences without repetitions – this is our Proposition 5.7.

Finally, in section 6 we compute graded Betti numbers for some finite subsets $X \subset C$, where $C \subset \mathbb{P}^n$ is a rational normal curve of degree n. Proposition 5.7 says that graded Betti numbers for finite subsets $X \subset C$ depend only on $\operatorname{card}(X)$, and we find explicit formulas for the cases $\operatorname{card}(X) \equiv 0, 1$ or $2 \pmod{n}$.

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Notation and conventions

The ground field k will be always assumed to have characteristic 0. $\mathbb{P}(E)$ means $\operatorname{Proj}\operatorname{Sym}(E^*)$; thus, over an algebraically closed field closed points of $\mathbb{P}(E)$ correspond to lines in (fibers of) E. If $X \subset \mathbb{P}^n = \mathbb{P}(E)$ is a closed subset, then $\hat{X} \subset E$ denotes the affine cone over X and $\langle X \rangle \subset \mathbb{P}^n$ denotes the linear span of X. The sheaf on $\mathbb{P}^n$ corresponding to a graded module M will be denoted by $\tilde{M}$.

For any multi-index $\alpha = (\alpha_0, \dots, \alpha_n)$, put

$$|\alpha| = \sum \alpha_i, \qquad \alpha! = \prod \alpha_i!,$$
$$X^\alpha = \prod X_i^{\alpha_i}, \qquad D_\alpha = \prod (\partial/\partial X_i)^{\alpha_i}.$$

For any number t, we write $\lfloor t \rfloor$ for $\max\{ m \in \mathbb{Z} \mid m < t\}$.

If $X \subset \mathbb{P}^n$ is a closed subscheme, we will say that X is projectively normal if $H^1(\mathbb{P}^n, \mathcal{I}_X(m)) = 0$ for all $m \in \mathbb{Z}$. This is equivalent to saying that $\operatorname{depth}(R_X) \geq 2$. If X is irreducible and reduced, its projective normality is equivalent to linear normality of Veronese images $v_m(X)$ for all $m \geq 0$, and finite sets containing more than one point are never projectively normal.

Finally, let us fix our notation with Veronese maps. If $(x_0 : \dots : x_n)$ are homogeneous coordinates in $\mathbb{P}^n$, consider a projective space $\mathbb{P}^{\binom{n+m}{n}-1}$ in which homogeneous coordinates y_α are numbered by multi-indices $\alpha = (\alpha_0, \dots, \alpha_n)$ such that $|\alpha| = m$. By Veronese map $v_m \colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+m}{n}-1}$ we mean the map defined by the formulas $y_\alpha = x^\alpha/\alpha!$. This map differs from the usual Veronese nap by a projective automorphism of $\mathbb{P}^{\binom{n+m}{n}-1}$, which does not affect geometric properties of the map.

2. Inverse systems and Koszul cohomology

Let $R = \sum_{i=0}^\infty R_i$ be the graded ring of polynomials in $T_0, \dots, T_n$; denote by $S = \sum_{i=0}^\infty S_i$ the graded ring of polynomials in $\xi_0, \dots, \xi_n$. If we let T_j act on S by the rule $T_j \cdot \varphi \mapsto \partial\varphi/\partial\xi_j$, S becomes a graded R-module.

It is well known (cf. [4, §1, ex. 32b]) that S is injective as an R-module; moreover, it is an injective hull of the R-module k.

Elements of the R-module S are known as Macaulay's inverse polynomials (cf. [6]).

The following definition is due to Macaulay.

Definition 2.1. A homogeneous linear subspace $J \subseteq S$ which is an R-submodule with respect to the above action of R will be called an *inverse system* in S.

For any $j \geq 0$, there is a pairing $S_j \otimes R_j \to k$ which acts by the formula

$$\langle \varphi, f \rangle \mapsto \varphi(\partial/\partial T_0, \dots, \partial/\partial T_n) \cdot f. \tag{2.1}$$

Proposition 2.2. *Pairing* (2.1) *is perfect.* □

Proposition 2.3. *If $\varphi \in S_m$, $f \in R_{m-1}$, and $j \in [0;n]$, then*

$$\langle \partial\varphi/\partial\xi_j, f\rangle = \langle \varphi, T_j f\rangle\,; \tag{2.2}$$

if $\varphi \in S_{m-1}$, $f \in R_m$, and $j \in [0;n]$, then

$$\langle \varphi, \partial f/\partial T_j\rangle = \langle \xi_j\varphi, f\rangle\,. \tag{2.3}$$

Proof. To prove (2.2) it suffices to assume that $\varphi = \xi_j^t \cdot \psi$, where ψ does not depend on ξ_j. Put $g = \psi\,(\partial/\partial T_0, \dots, \partial/\partial T_n) \cdot f$. Then the right-hand side of (2.2) equals $(\partial/\partial\xi^j)^t \cdot (T_j g)$, which by Leibnitz rule equals $T_j\,(\partial/\partial\xi^j)^t \cdot g + j\,(\partial/\partial\xi^j)^{t-1} \cdot g$. Since $\deg g = t-1 < m$, the first summand is zero, whence the desired equality. The identity (2.3) is obvious. $\square$

Definition 2.4. If $I \subset R$ is a homogeneous linear subspace, put $I^\perp = \bigoplus_j \operatorname{ann}(I_j) \subset S$. If $J \subset S$ is a homogeneous linear subspace, put $J^\perp = \bigoplus_t \operatorname{ann}(J_t) \subset R$ (in both cases ann means "annihilator with respect to the pairing (2.1)").

We will say that $I^\perp$ is the annihilator of I and $J^\perp$ is the annihilator of J.

If $I \subset R$ is a homogeneous linear subspace, then I is an ideal in R iff $I^\perp$ is an inverse system in S. The correspondence $I \mapsto I^\perp$ is an inclusion-reverting 1-1 correspondence between homogeneous ideals of R and inverse systems of S.

Proposition 2.5. *If homogeneous polynomials $F_1, \dots, F_r$ generate an ideal $I \subset R$ and $I^\perp \subset S$ is the annihilator of I, then each homogeneous component $(I^\perp)_m \subset S_m$ consists of all $\Phi \in S_m$ such that*

$$F_i(\partial/\partial\xi_0, \dots, \partial/\partial\xi_n)(\Phi) = 0 \tag{2.4}$$

for all $i \in [1;r]$.

Proof. Put $J = I^\perp$ and suppose that $\Phi \in J_m$. Then $\langle \Phi, MF_j\rangle = 0$ for any $j \in [1;r]$ and any $M \in R_{m-\deg F_j}$. If $\Phi' = F_j(\partial/\partial\xi_0, \dots, \partial/\partial\xi_n)\Phi$, then (2.2) implies that $M(\partial/\partial\xi_0, \dots, \partial/\partial\xi_n)\Phi' = 0$ for any $M \in R_{m-\deg F_j}$, whence $\Phi' = 0$. The inverse inclusion is obvious. $\square$

From now on we will consider the projective space $\mathbb{P}^n$ as $\operatorname{Proj} R$. Pairing (2.1) identifies S_1 with $(R_1)^*$; thus, we may (and from now on will) identify $\mathbb{P}^n$ with $\mathbb{P}(S_1)$. If $v = \sum c_i\xi_i \in S_1$, $f \in R_m$, then by $f(v)$ we mean $f(\xi_0, \dots, \xi_n)$. If k is algebraically closed and X is reduced, then $v \in \hat{X}$ iff $f(v) = 0$ for all homogeneous $f \in I_X$.

Definition 2.6. If $X \subset \mathbb{P}^n$ is a closed subscheme and $I_X \subset R$ is its graded ideal, then the inverse system $(I_X)^\perp \subset S$ will be denoted by J_X and called *the inverse system of X*.

Proposition 2.7. *If k is algebraically closed and $X \subset \mathbb{P}^n$ is reduced, then $(J_X)_r$ is linearly generated by the polynomials $\varphi_x = \sum_{|\alpha|=r}(x^\alpha/\alpha!)\xi^\alpha$ for all $x \in \hat{X}$.*

Proof. Take $(\dots, T^\alpha, \dots)_{|\alpha|=r}$ as a basis of R_r; then $(\dots, \xi^\alpha/\alpha!, \dots)$ will be the dual basis of S_r. If $f = \sum_{|\alpha|=r} c_\alpha T^\alpha \in R_r$, then $f \in I_r$ iff $\sum_{|\alpha|=r} c_\alpha x^\alpha = 0$ for all $x \in \hat{X}$ iff $\langle \varphi_x, f\rangle = 0$ for all $x \in \hat{X}$. Thus, $((I_X)_r)^\perp$ is generated by the φ_x's with $x \in \hat{X}$. □

Corollary 2.8. *If k is algebraically closed and $p \in \mathbb{P}^n$ is a (closed) point that corresponds to a vector $v \in \hat{X}\setminus 0 \subset S_1\setminus 0$, then $J_p = k[v] \subset S$.* □

Proposition 2.9. *In the above setting, suppose that $X \subset \mathbb{P}^n = \mathbb{P}(S_1)$ is a closed subset and $p \in \mathbb{P}^n$ is the point corresponding to a vector $v \in S_1$. Then $p \in X$ iff $v^r \in J_X$ for all $r > 0$.*

Proof. $p \in X$ iff $J_p \subset J_X$; now Corollary 2.8 applies. □

Corollary 2.10. *If the ground field is algebraically closed and $X \subset \mathbb{P}^n$ is a closed subset, then $\mathbb{P}((J_X)_r)) \subset \mathbb{P}(S_r)$ coincides with $\langle v_r(X)\rangle$.*

Corollary 2.11. *Suppose again that k is algebraically closed. Let $X \subset \mathbb{P}^n$ be a closed subset such that $X = \bigcup_\alpha X_\alpha$, where all the X_α's are closed in $\mathbb{P}^n$. Then $J_X = \sum_\alpha J_{X_\alpha}$.* □

If $M = \bigoplus_i M_i$ is a graded R-module, one may consider a graded R-module $M^* = \sum_i (M_{-i})^*$, where the action $T_j\colon (M^*)_{-i-1} \to (M^*)_{-i}$ is the adjoint to $T_j\colon M_i \to M_{i+1}$. In particular, $R^* = \bigoplus_{i\le 0} S_{-i}$, and if $I \subset R$ is a homogeneous ideal, then $(R/I)^* = \bigoplus_{i\le 0}(I^\perp)_{-i}$. Abusing the language, we will nevertheless write $(R/I)^* = I^\perp$; this should not lead to confusion. If we put $k = R/(T_0, \dots, T_n)$, then $k^* = k$; for any finitely generated graded R-module M, the graded modules $\operatorname{Tor}_j^R(k, M)$ and $(\operatorname{Ext}^j_R(k, M^*))^*$ are naturally isomorphic.

Suppose that $J \subset S$ is an inverse system. Then $\Omega^r(J)$ will denote the vector space of r-differential forms whose coefficients are polynomials in $\xi_0, \dots, \xi_n$ lying in J. The spaces $\Omega^r(J)$ are naturally graded: namely, $\Omega^r(J) = \bigoplus \Omega^r_s(J)$, where $\Omega^r_s(J)$ consists of forms whose coefficients are homogeneous of degree s. If $J = J_X$, where $X \subset \mathbb{P}^n$ is a closed subset (or subscheme), we will write $\Omega^r_s(X)$ instead of $\Omega^r_s(J)$; this should not lead to confusion.

Exterior differentiation maps $\Omega^r_s(J)$ into $\Omega^{r+1}_{s-1}(J)$. Thus, for an arbitrary integer l, exterior differentiation d is a graded homomorphism from $\Omega^r(J)(l)$ to $\Omega^{r+1}(J)(l+1)$. Consider a graded complex

$$0 \to \Omega^0_J \to \Omega^1_J(1) \to \cdots \to \Omega^{n+1}_J(n+1) \to 0. \tag{2.5}$$

and denote it by $\mathcal{M}^\cdot_J$. Its cohomology groups are naturally graded. We will write $H^r(\mathcal{M}^\cdot_J) = \bigoplus_s H^r_s(J)$; thus, $H^r_s(J)$ is the vector space of closed r-forms whose coefficients lie in J_s, modulo exterior derivatives of forms with coefficients in J_{s+1}. If $J = J_X$, we will write $\mathcal{M}^\cdot(X)$ and $H^r_s(X)$ instead of $\mathcal{M}^\cdot(J_X)$ and $H^r_s(J_X)$.

Proposition 2.12. *Suppose that $I \subset R$ is a homogeneous ideal. Then* $(\operatorname{Tor}^R_m(k, R/I)_i)^* \cong H^m_{i-m}(I^\perp)$.

Proof. Put $J = I^\perp$. The left-hand side is isomorphic to $\operatorname{Ext}^R_m(k, J)$. If we compute this Ext with the aid of the standard Koszul resolution $\mathcal{K}.$ for k, which is

$$0 \to R(-n-1)^{\binom{n+1}{n+1}} \to R(-n)^{\binom{n+1}{n}} \to \cdots \to R(-1)^{\binom{n+1}{1}} \to R \twoheadrightarrow k,$$

then $\operatorname{Hom}(K., J)$ coincides with the complex (2.5), whence the conclusion. □

Remark 2.13. Putting $I = 0$ in the above proposition, we see that Poincaré lemma for forms with polynomial coefficients is essentially equivalent to injectivity of the module of inverse functions.

Corollary 2.14. *If $X \subset \mathbb{P}^n$ is a closed subscheme, then each graded Betti number $b_{rs}(X)$ equals* $\dim H^r_s(X)$.

Remark 2.15. For the first term of the resolution, i.e. for the minimal defining equations of a variety, the duality between $\operatorname{Tor}^S_1(k, R/I)_r$ and $H^1_{r-1}(I^\perp)$ may be made explicit. To wit, suppose that $I \subset R$ is a homogeneous ideal and that $r \geq 1$ is an integer. Then $[\operatorname{Tor}^R_1(k, R/I)]_r \cong I_r/R_1 I_{r-1}$. Let $F_1, \dots, F_s \in I_r$ form a basis of this vector space. Since the pairing $\langle\cdot,\cdot\rangle$ is perfect, there exist forms $\Phi_1, \dots, \Phi_s \in S_r$ such that $\langle \Phi_i, F\rangle = 0$ for any $F \in R_1(I_X)_{r-1}$ and $\langle \Phi_i, F_j\rangle = \delta_{ij}$. Under the above assumptions, the differential forms $\omega_i = d\Phi_i$ lie in $\Omega^1_{r-1}(I^\perp)$ and form a basis of $H^1_{r-1}(I^\perp)$ that is dual to $(F_1, \dots, F_s)$ with respect to the duality from Proposition 2.12.

Example 2.16. If $X \subset \mathbb{P}^n$ is a quadric defined by the equation $T_0^2 + \cdots + T_n^2 = 0$, then $H^m_r(I_X^\perp)$ is not zero only if $m = r = 0$ or $m = r = 1$, and $H^1_1(I_X^\perp)$ is generated by the cohomology class of any of the forms $\xi_i d\xi_i$.

Example 2.17. A homogeneous ideal I is saturated, i.e. $(I : \mathfrak{m}) = I$, where $\mathfrak{m} \subset R$ is the maximal homogeneous ideal, iff $\operatorname{depth}(R/I) > 0$; according to Proposition 2.12 and Ausalnder-Buchsbaum formula, this should be equivalent to vanishing of $H^{n+1}_{\cdot}(I^\perp)$, i.e. to the following assertion:

> For any $f \in I^\perp$, there exist $f_0, \dots, f_n \in I^\perp$ such that $f = \sum_{i=0}^n \partial f_i/\partial \xi_i$.

This means that $I^\perp = (\xi_0, \dots, \xi_n)I^\perp$, which is indeed just the dual of the condition $(I : \mathfrak{m}) = I$.

3. Approximation

Throughout this section we assume that the ground field is algebraically closed. We start with a very simple observation.

Proposition 3.1. *Suppose that $X \subset \mathbb{P}^n$ is a closed subset and that $X = \bigcup_\alpha Y_{\alpha\in A}$, where Y_α is closed in $\mathbb{P}^n$ for all α (the index set A may be infinite). For any finite $B \subset A$ put $Y_B = \bigcup_{\alpha\in B} Y_\alpha$. Then, for any r and m, $H^r_s(X) = \varinjlim_{B\,\subset\, Y,\ B \text{ is finite}} H^r_s(J_{Y_B})$.*

Proof. Corollary 2.11 implies that $J_X = \sum_{\alpha\in A} J_{Y_\alpha} = \varinjlim_{B\,\subset\, Y,\ B \text{ is finite}} J_{Y_B}$. Since, for any inverse system $J \subset S$, $H^r_s(J)$ is just the degree s component of $\operatorname{Ext}^r_S(k, J)$, and since the partially ordered set of finite subsets of Y is directed, Proposition 5 from [4, § 6, n^0 4] shows that

$$\operatorname{Ext}^r_S(k, J_X) = \varinjlim_B \operatorname{Ext}^r_S(k, J_{Y_B}),$$

whence the conclusion. □

If the family A consists of all (closed) points of X, the assertion of the above proposition may be made more precise. We begin with proving Proposition 1.4. It follows immediately from

Proposition 3.2. *Let $X \subset V \subset \mathbb{P}^n$ be Zariski closed subsets. Suppose that $\langle v_m(X)\rangle = \langle v_m(V)\rangle$ for an integer $m > 0$. Then the natural homomorphisms $H^r_s(X) \to H^r_s(V)$ are isomorphisms for $s < m$ and epimorphisms for $s = m$.*

Proof. The condition $\langle v_m(X)\rangle = \langle v_m(V)\rangle$ means that, for any hypersurface W of degree m such that $W \supset X$, we have $W \supset V$. If this condition holds, then it holds with m replaced by an arbitrary $l < m$ as well. Hence, it follows from the hypothesis that $\langle v_l(X)\rangle = \langle v_l(V)\rangle$ for any $l \le m$; Corollary 2.10 implies that $\Omega^r_l(X) = \Omega^r_l(V)$ whenever $l \le m$. Since $d(\Omega^{\cdot}_{\cdot}(V) \cap \Omega^r_l(V)) = d(\Omega^{r-1}_{l+1}(V))$, the proposition is proved. □

Corollary 3.3. *If a closed subscheme $X \subset \mathbb{P}^n$ is not contained in a hyperplane, then $b_{i0}(X) = 0$ for $i > 0$.*

Proof. Put $V = \mathbb{P}^n$ and $m = 1$ in the above proposition. □

Now we can prove Theorem 1.1 for the case $\dim V = 1$. Recall that in this case $V \subset \mathbb{P}^n$ is an irreducible curve, $P = \{P_m\}_{m\in\mathbb{N}}$ is a sequence of distinct points of the curve V, and we are to prove that for any $j > 0$ there exists a number N such that $b_{rs}(V) = b_{rs}(X_m)$ whenever $s \le j$ and $m \ge N$, where $X_m = \{ P_1, \dots, P_m \}$.

Proof. Any curve that is not a line can meet a proper linear subspace only in a finite number of points. Hence, for any $j > 1$ there exists an N such that $\langle X_m\rangle = \langle v_{j+1}(V)\rangle$ whenever $m \ge N$. Now Proposition 3.2 applies. □

If $\dim V > 1$, the approximation property can not hold for all sequences $\{P_j\}$. For example, all the P_i's may lie on a proper subvariety of V, in which case the X_i's cannot know about graded Betti numbers of V. However, this property holds for generic sequences of points.

Let us give a precise definition of "genericity" for infinite sequences. Suppose that V is an irreducible quasiprojective variety of positive dimension, and let $\mathcal{V}$ denote the set of all sequences $p = \{P_i \in V\}_{i\in\mathbb{N}}$.

Identifying $\mathcal{V}$ with $V^{\mathbb{N}}$, we may consider a natural projection $p_S\colon V^{\mathbb{N}} \to V^S$, where $S \subset \mathbb{N}$ is a finite subset (if $S = \{n_1, \dots, n_l\}$, then V^S is just the product of l copies of the variety V and p_S maps the sequence $\{P_i\}$ to the point $(P_{i_1}, \dots, P_{i_l}) \in V^S$).

Definition 3.4. A subset $\mathcal{W} \subset \mathcal{V}$ is called *negligible* if it is contained in a countable union of sets of the form $p_S^{-1}(Z)$, where $S \subset \mathbb{N}$ is a finite non-empty subset and $Z \subset V^S$ is a Zariski closed subset such that $Z \neq V^S$.

We will say that a property holds for a generic sequence of points of V if the set of sequences $P \in \mathcal{V}$ for which this property fails, is negligible.

Informally speaking, a property holds for a generic sequence $P = \{P_1, P_2, \dots\} \in \mathcal{V}$ if to ensure it it suffices to perform the following procedure countably many times: take a finite subsequence of hte sequence P and impose on it a Zariski open condition.

Of course, me must check whether negligible sets deserve their name. If the base field is countable, this is not the case, since any subset of $\mathcal{V}$ is negligible ($\mathcal{V} = \bigcup_{x\in V} p_{\{1\}}^{-1}(x)$, and V is a countable set). The situation is better over uncountable fields:

Proposition 3.5. *Any countable union of negligible sets is negligible, and $\mathcal{V}$ is not negligible provided that the base field k is uncountable.*

Proof. Only the second assertion deserves proof. For the sake of brevity, let us give a proof only for a projective V over the complex numbers. Endow $\mathcal{V} = V^{\mathbb{N}}$ with the Tikhonov topology of direct product of classical topologies on V's. With this topology $\mathcal{V}$ becomes a compact Hausdorff space; each of $p_S^{-1}(Z)$'s is a closed nowhere dense subset of $\mathcal{V}$, and the assertion follows from the Baire theorem. $\square$

Now we can prove the second part of Theorem 1.1.

Suppose that $V \subset \mathbb{P}^n$ is an irreducible projective variety and that $\{P_t\}_{t\in\mathbb{N}}$ is a generic sequence of distinct points of V; put $X_i = \{P_1, \dots, P_i\} \subset V$. We are to show that for any $j \geq 0$ there exists an integer $N > 0$ such that $b_{rs}(X_i) = b_{rs}(V)$ whenever $s \leq j$ and $i \geq N$.

Proof. It suffices to prove that, for a fixed $j > 0$ and a generic sequence $P \in \mathcal{V}$, we have

$$\lim_{n\to\infty} \dim \langle v_j(X_n)\rangle = \dim \langle v_j(X)\rangle .$$

It is clear that, for any projective variety $X \subset \mathbb{P}^N$ and any $l > \dim \langle X\rangle$, the set

$$\{ (P_1, \dots, P_l) \in X^l \mid \langle P_1, \dots, P_l\rangle \neq \langle X\rangle \}$$

is contained in a Zariski closed proper subset of X^l, which yields our assertion. $\square$

Remark 3.6. Our definition of negligible subsets is admittedly coarse: we allow pretty large sets to be negligible. However, Theorem 1.1 is not likely to hold in higher dimensions if we replace Definition 3.4 by a more economical one. Let us, for example, say that a set $\mathcal{W} \subset \mathcal{V}$ is "strongly negligible" if it is a countable union of subsets of the form $\prod_{i\in\mathbb{N}} Z_i$, where each Z_i is Zariski closed in V and $Z_i \neq V$ for almost all $i \in N$. Then the set of sequences $P \in \mathbb{V}$ for which the assertion of Theorem 1.1 fails is *not* strongly negligible if $\dim V > 1$. Indeed, suppose that the ground field is uncountable and (for the sake of brevity) that X is swept out by lines. Then for any sequence $Z_i \subset V$ there exists a line $L \subset V$ and an $N \in \mathbb{N}$ such that $L \setminus \bigcup_{i\geq N} Z_i$ is infinite. Hence, one may pick a sequence $P = \{P_i\}$ such that $P_i \in L \setminus W_i$ provided that $i \geq N$. For this sequence P, graded Betti numbers of X_i will converge to graded Betti numbers of L rather than of X.

4. Lemmas on graded Betti numbers

In this section some auxiliary results on graded Betti numbers of (chiefly finite) subsets of $\mathbb{P}^n$ are gathered. I believe that most of these facts are known to the experts and belong to the folklore.

We begin with the following fact.

Lemma 4.1. *Suppose that $X \subset \mathbb{P}^n$ is a closed subscheme, $\mathcal{I}_X$ is the ideal sheaf of X, and $c \in \mathbb{Z}$. Then $H^t(\mathbb{P}^n, \mathcal{I}_X(c))$ is isomorphic to $H_{n+1-t}(K.)$, where $K_i = H^n(\mathbb{P}^n, \widetilde{L_i(X)}(c))$ for $i \in [1;n]$, all the other K_j's are zero, and differentials $K_i \to K_{i-1}$ are induced by differentials in the resolution $L.(X)$.* $\square$

The next proposition relates the homogeneous coordinate ring of a projectively normal variety to that of its hypersurface section.

Proposition 4.2. *Suppose that $X \subset \mathbb{P}^n$ is a projectively normal closed subscheme and $V \subset \mathbb{P}^n$ is a hypersurface of degree d that does not meet* $\operatorname{Ass}(X)$*. Then $R_{X\cap V} \cong R_X/(F)$, where F is the equation of V.*

Proof. The set $\operatorname{Ass}(R_X)$ consists of homogeneous graded ideals corresponding to the points of $\operatorname{Ass}(X)$, plus possibly the maximal homogeneous ideal $\mathfrak{m} = (T_0, \dots, T_n)$. Since the homogeneous ideal $I(X) \subset R$ is saturated, we see that $\mathfrak{m} \notin \operatorname{Ass}(R_X)$, whence F is not a zerodivisor in R_X. Hence, $\operatorname{depth}(R_X/(F)) = \operatorname{depth}(R_X) - 1$; since X is projectively normal, the right-hand side of this equality is positive, whence the ideal $I_X + (F) \subset R$ is saturated and coincides with the homogeneous ideal of the subscheme $X \cap V$. The proposition is proved. $\square$

Corollary 4.3. *Under the assumptions of the above proposition, the minimal resolution $L.(X\cap V)$ is isomorphic to the cone of the morphism*

$L.(X)(-d) \xrightarrow{F} L.(X)$, *and* $b_{ij}(X \cap V) = b_{ij}(X) + b_{i-1,j-d+1}(X)$, *where* $d = \deg F$.

Proof. Both assertions follow immediately from the exact sequence

$$0 \to R_X(-d) \xrightarrow{F} R_X \to R_{X\cap V} \to 0,$$

which exists by virtue of Proposition 4.3 . □

In the sequel we will need to know graded Betti numbers of degenerate subvarieties of $\mathbb{P}^n$. These numbers are easily computable with the aid of Proposition 4.2. Subsequently we will use the following notation: if $X \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a closed subset, then $b_{ij}(X, \mathbb{P}^{n-1})$ (resp. $b_{ij}(X, \mathbb{P}^n)$) denotes a graded Betti number of X regarded as a closed subset of $\mathbb{P}^{n-1}$ (resp. $\mathbb{P}^n$).

Proposition 4.4. *If* $X \subset \mathbb{P}^{n-1}$ *is a closed subset and* $\mathbb{P}^{n-1}$ *is a hyperplane in* $\mathbb{P}^n$, *then* $b_{ij}(X, \mathbb{P}^n) = b_{ij}(X, \mathbb{P}^{n-1}) + b_{i-1,j}(X, \mathbb{P}^{n-1})$.

Proof. We may and will assume that $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is defined by the equation $T_n = 0$; let R'_X denote the homogeneous coordinate ring of R as subvariety of $\mathbb{P}^{n-1}$. If $p \in \mathbb{P}^n$ is a point with homogeneous coordinates $(0 : \cdots : 1)$ and $\bar{X} \subset \mathbb{P}^n$ is the cone over X with the vertex p, then $R_{\bar{X}} \cong R'_X \otimes_k k[T_n]$. Hence, the minimal resolution for the R-module $R_{\bar{X}}$ may be obtained from the minimal resolution for the $k[T_0, \ldots, T_{n-1}]$-module R'_X by tensoring with $k[T_n]$. In particular, $b_{ij}(\bar{X}) = b_{ij}(X)$ for all i, j. Since X is a hyperplane section of $\bar{X}$, everything follows now from Proposition 4.2. □

Remark 4.5. Since the minimal resolution for $R_{\bar{X}}$ over R has the same length as the minimal resolution for R'_X over $k[T_0, \ldots, T_{n-1}]$, this length is at most $n-1$, whence $\bar{X}$ is projectively normal. Thus, all cones are projectively normal. I do not know a geometric proof of this simple fact. The following argument, which is due to F.L. Zak, shows at least that $\bar{X}$ is linearly normal. Suppose that $\pi \colon \tilde{Y} \to \bar{X}$ is an isomorphic projection from $\mathbb{P}^{n+1}$. Since $\bar{X}$ is swept out by lines passing through p and lines are linearly normal, we infer that Y is swept out by lines passing through $\tilde{p} = \pi^{-1}(p)$. Hence, the embedded Zariski tangent space to $\bar{X}$ at p (resp. to Y at $\tilde{p}$) coincides with the linear span of $\bar{X}$ (resp. Y). Since π induces an isomorphism of Zariski tangent spaces, we see that linear spans of $\bar{X}$ and Y are of the same dimension, which proves linear normality.

Propositions 4.2 and 4.4 have an amusing consequence:

Proposition 4.6. *Suppose that* $Y \subset \mathbb{P}^{n+1}$ *is a projective variety and that* $X = Y \cap \mathbb{P}^n$ *is its hyperplane section. If either* Y *or* X *is projectively normal, then* $b_{ij}(X, \mathbb{P}^n) = b_{ij}(Y, \mathbb{P}^{n+1})$ *for all* i *and* j.

Proof. Since any extension of a projectively normal variety is projectively normal, we may and will assume that Y is projectively normal. Hence, the following equalities hold for all i and j:

$$\begin{aligned} b_{ij}(X,\mathbb{P}^{n+1}) =& b_{ij}(Y) + b_{i-1,j}(Y) && \text{(Corollary 4.3);}\\ b_{ij}(X,\mathbb{P}^{n+1}) =& b_{ij}(X,\mathbb{P}^{n}) + b_{i-1,j}(X,\mathbb{P}^{n}) && \text{(Proposition 4.4).}\end{aligned}$$

Since $b_{0j}(X,\mathbb{P}^n) = b_{0j}(Y,\mathbb{P}^{n+1})$ for all j, the proposition follows from the above equalities by induction on j. □

Proposition 4.7. *Let $Y \subset \mathbb{P}^n$ be a closed subset and $p \in \mathbb{P}^n$ a point outside X. Suppose that a number $r \in \mathbb{N}$ is such that $v_{r-1}(p) \in \langle v_{r-1}(Y)\rangle$ and $v_r(p) \notin \langle v_r(Y)\rangle$. Then the natural homomorphisms $H^l_s(Y) \to H^l_s(Y\cup p)$ are isomorphisms whenever s does not equal r or $r-1$, and for each $l \geq 0$ there is an exact sequence*

(4.1)
$$0 \to H^l_r(Y) \to H^l_r(Y\cup p) \to H^l_0(p) \to H^{l+1}_{r-1}(Y) \to H^{l+1}_{r-1}(Y\cup p) \to 0.$$

Proof. The construction we use is present (in a dualized form and without mentioning the geometric meaning of the number r) in [10, section 2].

It follows immediately from the hypothesis and Corollaries 2.10 and 2.11 that there exists an exact sequence of graded modules

$$0 \to J(Y) \to J(Y\cup p) \to J(p)(-r) \to 0.$$

The long exact sequence of Ext's yields exact sequences

$$\cdots \to H^l_{j-r+1}(p) \to H^l_j(Y) \to H^l_j(Y\cup p) \to \\ H^l_{j-r}(p) \to H^{l+1}_{j-1}(Y) \to H^{l+1}_{j-1}(Y\cup p) \to \dots.$$

Since $H^m_j(p) = 0$ for all $j \neq 0$, the proposition follows. □

Corollary 4.8. *If a closed set $Y \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ spans the hyperplane $\mathbb{P}^n$ and if $p \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$, then $b_{m1}(Y\cup p) = b_{m1}(Y,\mathbb{P}^{n-1}) + b_{m-1,1}(Y,\mathbb{P}^{n-1}) + \binom{n}{m}$, and $b_{mj}(Y\cup p) = b_{mj}(Y,\mathbb{P}^n)$ if $j \neq 1$.*

Proof. We may apply Proposition 4.7 with $r = 1$. Since $b_{i0}(Y\cup p) = 0$ for $i > 0$ by virtue of Corollary 3.3, the exact sequence (4.1) with $j = 1$ boils down to

$$0 \to H^i_1(Y) \to H^i_1(Y\cup p) \to H^i_0(p) \to 0.$$

Applying Proposition 4.4 and recalling that $b_{i0}(p,\mathbb{P}^n) = \binom{n}{i}$, we obtain the desired formula. □

Corollary 4.9. *Under the assumptions of Proposition 4.7, $b_{nr}(Y \cup p) = b_{nr}(Y) + 1$.*

Proof. Put $l = n$ in the exact sequence (4.1) and recall that $b_{n0}(p) = 1$. □

I have not found a reference for the next proposition, but I believe that it is known to the experts.

Proposition 4.10. *If $X_n \subset \mathbb{P}^{n-1}$ is a set of n points spanning $\mathbb{P}^{n-1}$, then its graded Betti numbers are as follows: $b_{00}(X_n) = 0$, $b_{m1}(X_n) = m\binom{n}{m+1}$ for $m \in [1; n-1]$, and all the other b_{ij}'s are zero.*

Proof. We proceed by induction on n. For $n = 2$ the proposition is obvious. To make the induction step observe that $X_{n+1} \subset \mathbb{P}^n$ is the union of $X_n \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and a point outside $\mathbb{P}^{n-1}$. Applying Corollary 4.8 and Proposition 4.4, we obtain the equalities

$$b_{m1}(X_{n+1}) = b_{m1}(X_n) + b_{m-1,1}(X_n) + \binom{n}{m} = \\ m\left(\binom{n}{m+1} + \binom{n}{m}\right) = m\binom{n+1}{m+1}.$$

This completes the induction and the proof. □

Corollary 4.11. *If $X \subset \mathbb{P}^N$ is a non-degenerate variety of degree n and dimension $N - n + 1$, then $b_{00}(X) = 1$, $b_{m1}(X) = m\binom{n}{m+1}$ for $m \in [1; n-1]$, and all the other b_{ij}'s are zero.*

Proof. If X is a rational normal curve, then all follows from Proposition 4.6 since rational normal curves are projectively normal. If $\dim X > 1$, then X may be obtained from a rational normal curve by a sequence of extensions, and the result follows by induction from Proposition 4.6 and the case $\dim X = 1$. □

The next proposition will be used in section 6. For the case of finite X it is essentially contained in [7, Lemma 1.2].

Proposition 4.12. *For any closed subscheme $X \subset \mathbb{P}^n$ and any $i \in [1; n]$, put $u_i(X) = \max\{\, j \mid b_{ij}(X) \neq 0\,\}$. If $H^l(\mathcal{I}_X(t)) = 0$ for all $l > 1$ and all $t \geq -n$, then the sequence $u_1, u_2, \ldots u_n$ is non-decreasing.*

Proof. Put $v_0 = 0$ and $v_j = u_j + j$ for any $j \in [1; n]$. To prove the lemma, we are to show that $v_0 < v_1 \cdots < v_n$. Assume the converse. Then there exists a number $s \in [1; n-1]$ such that $v_s \geq v_{s+1}$ and $v_s \geq v_{s-1}$. Put $c = v_s - n - 1$ in Lemma 4.1. Since v_j equals the maximal number m for which $R(-m)$ is present (as a direct summand) in $L_j(X)$, $K_j \neq 0$ and that the differentials $\partial_{j+1}\colon K_{j+1} \to K_j$ and $\partial_j\colon K_j \to K_{j-1}$ are zero (we use the notation from Lemma 4.1). Indeed, if $R(-v_s)$ is not present in L_{s+1} or L_{s-1}, then the vanishing of ∂_{j+1} and ∂_j is obvious, and if it is present, we observe that minimality of the resolution implies that the induced homomorphisms $R(-v) \to R(-v)$, where the two $R(-v)$'s are direct summands of adjacent terms of the resolution, should be zero.

Hence, $H^{n+1-s}(\mathcal{I}_X((v_s - n - 1)) = H_s(K_.) \neq 0$. Since $v_s \geq 1$ and $s \leq n - 1$, this contradicts hypothesis of the proposition. □

5. Some evidence for the periodicity conjecture

Suppose that $C \subset \mathbb{P}^n$ is a curve and that $P = \{P_m \in C\}_{m\in\mathbb{N}}$ is a generic sequence of points. Proposition 5.1 below shows that at least lower right corners of Betti diagrams for the sets $X_m = \{P_1, \dots, P_m\}$ behave as predicted by Conjecture 1.2.

Proposition 5.1. *Suppose that $C \subset \mathbb{P}^n$ is an irreducible curve of degree d and that $\{P_i\}_{i\in\mathbb{N}}$ is a generic sequence of distinct points of C; put $X_i = \{P_1, \dots, P_i\} \subset C$. For any i, put $a_i = \max\{l \mid b_{nl}(X_i) \neq 0\}$. Then there exists a number N such that for any $i \geq N$ we have:*

$$\begin{aligned} a_{i+d} &= a_i + 1; \\ b_{n,a_i}(X_i) &= b_{n,a_i+1}(X_{i+d}). \end{aligned} \tag{5.1}$$

Remark 5.2. Proposition 4.12 shows that $b_{n,a_i}(X_i)$ indeed stands in the lower right corner of the Betti diagram of X.

Proof. We start with some notation.

Notation 5.3. If $X \subset \mathbb{P}^n$ is a closed subset and $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n}$ is its sheaf of ideals, put $\beta(X) = \max\{t \in \mathbb{Z} \mid H^1(\mathbb{P}^n, \mathcal{I}_X(t)) \neq 0\}$.

Lemma 5.4. *Suppose that $X \subset \mathbb{P}^n$ is a closed subset. Then:*

(a) *If $b_{nj}(X) \neq 0$, then $H^1(\mathbb{P}^n, \mathcal{I}_X(j-1)) \neq 0$.*
(b) *If X is not projectively normal, then $\beta(X) = l - 1$, where $l = \max\{t \in \mathbb{Z} \mid b_{nt}(X) \neq 0\}$; moreover, $h^1(\mathbb{P}^n, \mathcal{I}_X(\beta(X))) = b_{nl}$.*

Proof of the lemma. We start with part (a). Lemma 4.1 shows that

$$H^1(\mathcal{I}_X(t)) \cong \operatorname{Ker}(H^n(\tilde{L}_n(t)) \to H^n(\tilde{L}_{n-1}(t))). \tag{5.2}$$

Observe that the module L_n contains $R(-n-j)$ as a direct summand and that, if $\partial\colon L_n \to L_{n-1}$ is the homomorphism from the resolution, then minimality of the resolution implies that $\partial(R(-n-j))$ is contained in a submodule of L_{n-1} that is a sum of $R(-m_i)$'s with $m_i > -n-j$. Hence, the induced homomorphism $H^n(\tilde{L}_n(s)) \to H^n(\tilde{L}_{n-1}(s))$ is zero when restricted to $H^n(\tilde{R}(-n-1))$. This provides a non-zero element in the kernel in the right-hand side of (5.2) and proves part (a) of the proposition.

To prove part (b), observe that it follows from (a) that $\beta(X) \geq l-1$; on the other hand, if $t \geq l$, then $\tilde{L}_n(t)$ is a direct sum of $\mathcal{O}(m_i)$'s with $m_i \geq -n$, and it follows from (5.2) that $h^1(\mathcal{I}_X(t)) = 0$. To compute $h^1(\mathcal{I}_X(l-1))$, observe that $\tilde{L}_n(l-1)$ equals $(\mathcal{O}(-n-1))^{b_{nl}}$ plus a sum of $\mathcal{O}(m_i)$'s with $m_i \geq -n$. The same argument as in the proof of the assertion (a) shows that ∂ maps $H^n((\mathcal{O}(-n-1)))^{b_{nl}}$ to zero, which finishes the proof. ∎

We proceed with the proof of the proposition. If $X \subset \mathbb{P}^n$ is a finite subset, then $\beta(X)$ equals the maximal t for which the set $v_t(X)$ does not consist of linearly independent points, and $h^1(\mathcal{I}_X(\beta(X))) = \operatorname{card}(X) - 1 - \dim\left\langle v_{\beta(X)}(X)\right\rangle$.

Lemma 5.5. *If $V \subset \mathbb{P}^n$ is a projective variety, then for a generic sequence $P = \{P_i\}_{j\in\mathbb{N}}$ of distinct points of V one has*

$$\dim\langle X_m\rangle = \min(m-1, \dim\langle V\rangle) \tag{5.3}$$

for all m, where $X_m = \{\, P_1, \dots, P_m\}$.

Proof of the lemma. For any fixed m, the set

$$Z = \{\,(P_1, \dots, P_m) \in V^m \mid \text{equality (5.3) does not hold}\,\}$$

is closed in V^m, and $Z \neq V^m$. Hence, to ensure that (5.3) holds for all m it suffices to impose a countable number of algebraic inequalities on finite subsequences of the sequence P. This is just Definition 3.4. □

Applying the above lemma to all Veronese images of the curve C we see that, for a generic sequence P and for any $l, m \in \mathbb{N}$,

$$\dim\langle v_l(X_m)\rangle = \min(m-1, \dim\langle v_m(C)\rangle).$$

Choose a number $l \in \mathbb{N}$ such that, for all $t \geq l$, the equality $\dim\langle v_t(C)\rangle = td - g$ holds, where g denotes arithmetical genus of C. I claim that equalities (5.1) hold for a generic P and for all $i > ld + 1$. Indeed, $\dim\langle v_m(C)\rangle \leq md$ for all m. Hence, for a generic P and for $i > ld+1$, the set $v_m(X_i)$ consists of linearly dependent points whenever $m < l$, and $\beta(X_i) \geq l$. Thus, $\beta(X_i) = \lfloor (g + i - 1)/d\rfloor$, and $h^1(\mathcal{I}_{X_i}(\beta(X_i))) = i - 1 + g - \beta(X_i)\cdot d$. This proves (5.1) and the proposition. □

Remark 5.6. The above proof, which used linearity of Hilbert functions of curves, indicates that Conjecture 1.2 should not be true for higher-dimensional varieties.

Another evidence for Conjecture 1.2 is provided by normal rational curves. The following proposition shows that for these curves the conjecture is true for arbitrary sequences of distinct points.

Proposition 5.7. *Suppose that $C \subset \mathbb{P}^n$ is a rational normal curve of degree n and that $p = \{P_m\}_{m\in\mathbb{N}}$ is an arbitrary sequence of distinct points of C. Put $X_m = \{\, P_1, \dots, P_m\,\}$. Then $b_{ij}(X_m) = b_{i,j+1}(X_{m+n})$ whenever $j \geq 2$.*

Proof. We use the notation introduced in the statement of Conjecture 1.2. Let $I_l \subset R_C$ stand for the homogeneous ideal of the closed subset $X_l \subset C$. If H is the hyperplane spanned by $P_{m+1}, \dots, P_{m+n}$ and $L \in (R_C)_1$ is the image of its equation in R_C, then $I_{m+n} = I_m \cap (L)$. Since the points are distinct, L is not a zerodivisor in R/I_m. Hence, multiplication by L induces an isomorphism $R_C/I_m \cong (L)/I_m \cap L$. Taking grading into account, we obtain an exact sequence

$$0 \to (R_C/I_m)(-1) \xrightarrow{L} R_C/I_{m+n} \to R_C/(L) \to 0,$$

which gives rise to the long exact sequence

$$(5.4)\quad \dots \to \mathrm{Tor}_i^R(k, R_{X_m})(-1) \to \mathrm{Tor}_i^R(k, R_{X_{m+n}}) \to \\ \mathrm{Tor}_i^R(k, R_C/(L)) \to \mathrm{Tor}_{i-1}^R(k, R_{X_m})(-1) \to \dots.$$

Put $Y = \{P_{m+1}, \dots, P_{m+n}\}$. Since C is projectively normal, Proposition 4.2 says that $R_C/(L) \cong R_Y$. Hence, (5.4) yields the following exact sequences:

$$\dots \to [\mathrm{Tor}_i^R(k, R_{X_m})]_{i+j-1} \to [\mathrm{Tor}_i^R(k, R_{X_{m+n}})]_{i+j} \to \\ [\mathrm{Tor}_i^R(k, R_C/(L))]_{i+j} \to [\mathrm{Tor}_{i-1}^R(k, R_{X_m})]_{i+j-1} \to \dots.$$

Propositions 4.10 and 4.4 show that $[\mathrm{Tor}_a^R(k, R_Y)]_{a+b} = 0$ for $b \neq 0, 1$. Hence, $b_{ij}(X_m) = b_{i,j+1}(X_{m+n})$ for $j \geq 2$. This proves our proposition. □

6. Some explicit computations

In this section we compute graded Betti numbers of some finite subsets $X \subset C$, where C is a rational normal curve of degree n in $\mathbb{P}^n$ and $\mathrm{card}(X) \equiv 0, 1$ or $2 \pmod{n}$.

The case $\mathrm{card}(X) \equiv 0 \pmod{n}$ is the easiest:

Proposition 6.1. *Suppose that $C \subset \mathbb{P}^n$ is a rational normal curve of degree n and that $X \subset C$ is a set of mn points, where $m > 1$. Then the graded Betti numbers of X are as follows: $b_{i1}(X) = b_{i1}(C) = i\binom{n}{i+1}$ for any $i \in [1; n-1]$, $b_{1,m-1}(X) = 1$, $b_{im}(X) = (i-1)\binom{n}{i}$ for all $i \in [2; n]$, $b_{00}(X) = 1$, and all the other b_{ij}'s are zero.*

Proof. Decompose X into a disjoint union of m subsets containing n points each. If H_i is the linear span of the i-th subset, then X is a scheme-theoretic intersection of C with the hypersurface $H_1 \cup \dots \cup H_m$. Now everything follows from Corollary 4.3 and Corollary 4.11. □

In the sequel P will always denote Hilbert polynomial of $\mathbb{P}^n$, i.e. $P(t) = \binom{n+t}{t}$.

Proposition 6.2. *Let $m > 1$ be an integer and $C \subset \mathbb{P}^n$ be a rational curve of degree n in $\mathbb{P}^n$. Suppose that $X \subset C$ is a set of $mn+1$ points. Then the graded Betti numbers of X are as follows: $b_{i1}(X) = b_{i1}(C) = i\binom{n}{i+1}$ for any $i \in [1; n-1]$, $b_{sm}(X) = n \cdot \binom{n-1}{s-1}$ for any $s \in [1; n]$, and all the other b_{ij}'s with $i > 0$ are zero.*

Proof. Since $v_m(X)$ is a set of $mn+1$ linearly independent points that span $\mathbb{P}^{mn} = \langle v_m(C) \rangle$, Proposition 3.2 shows that $b_{ij}(X) = b_{ij}(C)$ whenever $j < m$, and Lemma 5.4 shows that $b_{nj}(X) = 0$ for $j > m$. Since any finite subset of $\mathbb{P}^n$ satisfies the hypotheses of Proposition 4.12, this proposition shows that $b_{ij}(X) = 0$ whenever $j > m$.

It remains to calculate b_{sm}'s for $s < n$. To that end observe that, denoting Hilbert polynomial of $X \subset \mathbb{P}^n$ by $P_X(\cdot)$, we have the following formula:

$$P_X(t) - P_C(t) = \sum_{s=1}^{n} (-1)^s b_{sm} P(t-m-s). \tag{6.1}$$

The left-hand side equals $mn - nt$; since $\deg P = n$, the n polynomials $P(\cdot - m - s)$, where $s \in [1; n]$, are linearly independent and the equation (6.1) has a unique solution in $b_{1m}, \dots, b_{nm}$. Thus, to finish the proof it suffices to verify that equation (6.1) holds for $b_{sm} = \binom{n-1}{s-1}$. Dividing both sides of (6.1) by n, we see that this is equivalent to saying that

$$\sum_{s=1}^{n} (-1)^s \binom{n-1}{s-1} P(t-m-s) = m - t;$$

changing signs in both sides and substituting $t = m + v + 1$, we see that the above equality is equivalent to

$$\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} P(v-j) = v + 1. \tag{6.2}$$

The right-hand side of (6.2) coincides with the Hilbert polynomial $P_{\mathbb{P}^1}(v)$; if we calculate the same Hilbert polynomial via the Koszul resolution that corresponds to the representation of $\mathbb{P}^1$ as a complete intersection of $n - 1$ hyperplanes in $\mathbb{P}^n$, we arrive at the left-hand side of (6.2). This completes the proof. □

Remark 6.3. Putting this proposition together with Proposition 4.10, we see that we know graded Betti numbers of all finite subsets $X \subset C$ with $\operatorname{card}(X) \equiv 1 \pmod{n}$.

Proposition 6.4. *Suppose that $C \subset \mathbb{P}^n$ is a normal rational curve of degree n and $X \subset C$ is a finite set consisting of $mn + 2$ points, where $m > 1$. Then $b_{i1}(X) = b_{i1}(C) = i\binom{n}{i+1}$ for $i \in [1; n-1]$, $b_{im}(X) = b_{n-i,1}(X)$ for all $i \in [1; n-1]$, $b_{n,m+1}(X) = b_{00}(X) = 1$, and all the other graded Betti numbers of X are zero.*

Proof. Fix a point $p \in X$ and put $Y = X \setminus p$. Since $\langle v_m(X)\rangle = \langle v_m(Y)\rangle$ and $\langle v_{m+1}(X)\rangle \neq \langle v_{m+1}(Y)\rangle$, Proposition 4.7 applies with $r = m + 1$. Taking into account Proposition 6.2 we see that $b_{lj}(X) = b_{lj}(C)$ for $j < m$ and that $b_{lj}(X) = 0$ for $j > m + 1$; Corollary 4.9 shows that $b_{n,m+1}(X) = 1$; hence, only $b_{jm}(X)$ for $j \in [1; n]$ and $b_{j,m+1}(X)$ for $j \in [1; n-1]$ remain unknown.

To find them, put

$$a_j = b_{jm}(X) + b_{j-1,m+1}(X).$$

If $P(t)$ denotes Hilbert polynomial of $\mathbb{P}^n$, i. e. $\binom{n+t}{n}$, then Hilbert polynomial of X equals

$$P_C(t) + \sum_{i=1}^{n} (-1)^j a_j P(t-j-m), \tag{6.3}$$

where P_C denotes Hilbert polynomial of C. Since $a_{n+1} = b_{n,m+1}(X) = 1$, we have an identity

$$\sum_{i=1}^{n} (-1)^i a_j P(\cdot - j - m) = (-1)^n P(\cdot - n - m - 1) + mn + 2 - P_C(\cdot). \tag{6.4}$$

Since $\deg P = n$, the n polynomials $P(\cdot - j - m)$, where $j \in [1; n]$, are linearly independent. Hence, equation (6.4) has a unique solution in $a_1, \ldots, a_n$ (if any). I claim that the numbers $a_j = (n-j)\binom{n}{n-j+1} = b_{n-j,1}(C)$ provide such a solution. Indeed, substituting these numbers in (6.3) one obtains

$$\underbrace{P(t) + \sum_{i=1}^{n-1} (-1)^i b_{i1}(C) P(t-i-1)}_{P_C(t)}$$
$$+ \underbrace{(-1)^n \Bigl(P(t-n-1-m) + \sum_{i=1}^{n-1} b_{i1}(C) P(t-n-1-(m-1-i))\Bigr)}_{Q(t)}.$$

Since P satisfies the identity $(-1)^n P(-n-1-u) = P(u)$, we see that $Q(t) = P_C(m-t)$ and $P_C(t) + Q(t) = nt - 1 + n(m-t) + 1 = mn + 2$, which coincides with Hilbert polynomial of X. Since equation (6.4) is equivalent to the statement that (6.3) equals Hilbert polynomial of X, our claim is proved. In particular, we proved that $a_n = 0 = b_{nm}(X) + b_{n-1,m+1}(X)$. Hence, both of the summands are zero. Now Proposition 4.12 implies that $b_{j,m+1}(X) = 0$ for all $j < n$. Hence, $b_{jm}(X) = a_j$ for all j, which completes our computation. □

Remark 6.5. The previous proposition does not account for the case $\operatorname{card}(X) = n+2$. The same method applies in this case too, yielding the formulas $b_{i1}(X) = b_{i1}(C) + b_{n-i,1}(C)$ for all $i \in [1; n-1]$, $b_{00}(X) = b_{n2}(X) = 1$, with all the other b_{ij}'s being zero. These formulas are contained in [9] and [8]. Applying Proposition 4.6, one obtains another proof of Theorem 1 from [8].

Appendix

In this appendix we reproduce a fragment from a log of a Macaulay session. The program computed (over the field of 31991 elements) minimal resolutions of finite subsets $X_m \subset C$, where C is a monomial curve $C \subset \mathbb{P}^3$ given parametrically by $t \mapsto (1 : t : t^4 : t^5)$ and $X_m = \{ P_1, \ldots, P_m \}$. The points $P_1, \ldots P_{32}$ correspond to the following values of the parameter t: 1, 74, 47, 20, 15, 18, 44, 93, 64, 39, 52, 70, 76, 77, 92, 46, 11, 41, 68, 62, 84, 5, 22, 32, 2, -6, 419, 606, 891, 300, 119, 684. In the tables that follow, `betti res`m means "Betti table of X_m". Periodicity begins with $m = 21$.

```
1% betti res18
total:      1     8    12     5
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     7    10     3
    4:      -     -     2     2
```

```
1% betti res19
total:      1     8    12     5
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     6     8     2
    4:      -     1     4     3
```

```
1% betti res20
total:      1     8    13     6
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     5     6     2
    4:      -     2     7     4
```

```
1% betti res21
total:      1    10    16     7
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     5    10     5
```

```
1% betti res22
total:      1     9    13     5
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
```

```
    3:      -     4     6     2
    4:      -     4     7     2
    5:      -     -     -     1

1% betti res23
total:      1     8    12     5
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     3     4     1
    5:      -     -     2     2

1% betti res24
total:      1     8    12     5
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     2     2     -
    5:      -     1     4     3

1% betti res25
total:      1     8    13     6
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     1     -     -
    5:      -     2     7     4

1% betti res26
total:      1    10    16     7
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     -     -     -
    5:      -     5    10     5

1% betti res27
total:      1     9    13     5
--------------------------------
    0:      1     -     -     -
```

```
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     -     -     -
    5:      -     4     7     2
    6:      -     -     -     1

1% betti res28
total:      1     8    12     5
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     -     -     -
    5:      -     3     4     1
    6:      -     -     2     2


1% betti res29
total:      1     8    12     5
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     -     -     -
    5:      -     2     2     -
    6:      -     1     4     3

1% betti res30
total:      1     8    13     6
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     -     -     -
    5:      -     1     -     -
    6:      -     2     7     4

1% betti res31
total:      1    10    16     7
--------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
```

```
    4:      -     -     -     -
    5:      -     -     -     -
    6:      -     5    10     5

1% betti res32
total:      1     9    13     5
-------------------------------
    0:      1     -     -     -
    1:      -     1     -     -
    2:      -     -     -     -
    3:      -     4     6     2
    4:      -     -     -     -
    5:      -     -     -     -
    6:      -     4     7     2
    7:      -     -     -     1
```

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