PROJECTIVE CONVEXITY IN $\mathbb{P}^3$ IMPLIES GRASSMANN CONVEXITY

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Abstract. In this note we introduce the notion of Grassmann convexity analogous to the well-known notion of convexity for curves in real projective spaces. We show that the curve in $G_{2,4}$ osculating to a convex closed curve in $\mathbb{P}^3$ is Grassmann-convex. This proves that the tangent developable (i.e. the hypersurface formed by all tangents) of any convex curve in $\mathbb{P}^3$ has the 'degree' equal to 4. Here by 'degree' of a real projective hypersurface we mean the maximal total multiplicity of its intersection with a line.

§1. Preliminaries and results

Let us first recall some basic definitions.

Definition. A smooth closed curve $\gamma : S^1 \to \mathbb{P}^n$ is called *locally convex* if the local multiplicity of intersection of γ with any hyperplane $H \subset \mathbb{P}^n$ at any of their intersection points does not exceed $n = \dim \mathbb{P}^n$ and *globally convex* or simply *convex* if the above condition holds for the global multiplicity, i.e for the sum of local multiplicities.

Local convexity of γ is a simple requirement equivalent to the nondegeneracy of the osculating Frenet n-frame of γ, i.e. to the linear independence of $\gamma'(t), ..., \gamma^n(t)$ for any $t \in S^1$. Global convexity is a rather nontrivial property studied under different names (e.g. disconjugacy of linear ordinary differential equations) since the beginning of the century. (There exists a vast literature on convexity and the classical achievements are well summarized in [Co]. For one of the most recent developments see e.g. [Ar1].)

Let $G_{k,m}$ denote the usual Grassmannian of k-dimensional real planes in $\mathbb{R}^m$ (or equivalently, $(k-1)$-dimensional planes in $\mathbb{P}^{m-1}$).

Definition. Given an $(m-k)$-plane $L \subset \mathbb{R}^m$ we call by *the Grassmann hyperplane* $H_L \subset G_{k,m}$ *associated to* L the set of all k-dimensional subspaces in $\mathbb{R}^m$ nontransversal to L.

Remark. Grassmann hyperplanes is a well-known classical concept in Schubert calculus, see e.g. [Cn]. More exactly, H_L coincides with the union of all Schubert cells of positive codimension in $G_{k,m}$ constructed using some complete flag containing L as a subspace. The complement to each H_L is the open Schubert cell isomorphic to the standard affine chart in $G_{k,m}$. If $n-k \geq k$ then each H_L considered in some other standard affine chart is isomorphic

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to the $k(n-2k)$-dimensional cylinder over the set Deg_k of all degenerate $k \times k$-matrices. (By duality $H_L \subset G_{k,m}$ is isomorphic to $H_{L'} \subset G_{m-k,m}$.)

REMARK. A usual hyperplane $H \subset \mathbb{P}^n$ is a particular case of a Grassmann hyperplane if we consider H as the set of points nontransversal (i.e. belonging to) H considered as a point in $\mathbb{P}^{n*}$.

DEFINITION. A smooth closed curve $\Gamma : S^1 \to G_{k,m}$ is called *locally Grassmann-convex* if the local multiplicity of intersection of Γ with any Grassmann hyperplane $H_L \subset G_{k,m}$ does not exceed $k(n-k) = \dim G_{k,m}$ and *globally Grassmann-convex* or simply *Grassmann-convex* if the above condition holds for the global multiplicity.

A detailed account of the theory of complex algebraic curves in Grassmanianns can be found in [Pe] and references therein.

Given a locally convex curve $\gamma : S^1 \to \mathbb{P}^n$ and $1 \le k \le n-1$ we can define *its osculating Grassmann curve* $osc_k\gamma : S^1 \to G_{k+1,n+1}$ formed by the osculating k-dimensional subspaces to the initial γ. (The curves $osc_k\gamma$ are well-defined for any $k = 1, ..., n-1$ due to the local convexity of γ.)

The main result of the paper is the following

1.1. Theorem. *For any convex Grassmann curve $\gamma : S^1 \to \mathbb{P}^3$ its osculating curve $osc_2\gamma : S^1 \to G_{2,4}$ is Grassmann-convex.*

Let us reformulate the above result in other terms.

DEFINITION. Given a generic $\gamma : S^1 \to \mathbb{P}^n$ we call by its *standard discriminant* or *tangent developable* $D_\gamma \subset \mathbb{P}^n$ the hypersurface consisting of all codimension 2 osculating subspaces to γ. (Here 'generic' means having a nondegenerate osculating $(n-1)$-frame at every point.)

Notice that certain singularities of the standard discriminant coming from inflection points were recently used by V. Arnold [Ar2] in a construction of a Sturm-type theory for space curves.

The above theorem is equivalent to the following

Reformulation. *For any convex curve $\gamma : S^1 \to \mathbb{P}^3$ the 'degree' of its discriminant $D_\gamma \subset \mathbb{P}^3$ equals 4. (Here the 'degree' of a real projective hypersurface is the maximal total multiplicity of its intersection with projective lines.)*

A linear homogeneous ode of order $n+1$ (defined on some interval $I = [a, b]$ or S^1) is called *disconjugate* if any its solution does not vanish on I more than n times counted with multiplicities (in the periodic case one considers only zeros on a period).

1.2. Corollary. *Consider the Wronskian $W_{\phi_1,\phi_2}(t)$ of any 2 linearly independent solutions ϕ_1, ϕ_2 to some disconjuate ode of order 4 given on some interval I. Then $W_{\phi_1,\phi_2}(t)$ vanishes on I at most 4 times counted with multiplicities.*

Let $\pi_p : \mathbb{P}^3 \setminus \{p\} \to \mathbb{P}^2$ denote the projection of $\mathbb{P}^3 \setminus \{p\}$ to $\mathbb{P}^2$ from the point $p \in \mathbb{P}^3$.

1.3. Corollary. *A generic projection $\pi_p\gamma$ of a convex curve $\gamma \in \mathbb{P}^3$ to $\mathbb{P}^2$ from some point $p \in \mathbb{P}^3$ has at most 4 tangents passing through any given point on $\mathbb{P}^2$.*

In particular, the curve $y(x) = -\frac{x^2}{x^3+x}$ on fig 1a) is one of the projections of the standard rational normal curve. On the other side, a similarly looking curve on fig 1b) can not be obtained as a projection of any convex curve in $\mathbb{P}^3$ since it has 6 parallell tangents. Two other obvious restrictions for a projection of a convex curve are a) the 'degree' of such a plane curve is at most 3; b) such a plane curve has at most 3 inflection points. (Notice that both restrictions are satisfied on fig. 1b.)

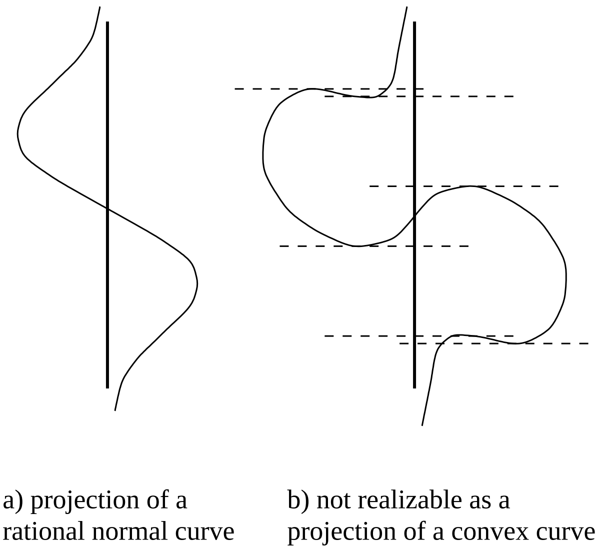

Fig.1. Examples of plane curve related to projections of convex curves.

Conjecture. *The restriction that the local 'degree' is at most 3 is the necessary and sufficient condition for a plane curve to be representable as a projection of a locally convex curve in* $\mathbb{P}^3$.

Problem. Find necessary and sufficient conditions for a plane curve to be representable as a projection of a convex curve in $\mathbb{P}^3$.

The general hope is as follows.

1.4. Main conjecture. *For any convex curve* $\gamma : S^1 \to \mathbb{P}^n$ *and any* $1 \le k \le n-1$ *its osculating curve* $osc_k\gamma : S^1 \to G_{k+1,n+1}$ *is Grassmann-convex. In particular, the 'degree' of the standard discriminant* $D_\gamma \subset \mathbb{P}^n$ *for any convex curve* γ *in* $\mathbb{P}^n$ *equals* $2(n-1)$.

The above result (and conjecture) should be related to the main theorem of [BSh2] formulated below. They are still another illustration of the principle *'convex curves are qualitatively equivalent'*. (On the other hand, some deep results of Kazarian [Ka] suggest that this qualitative equivalency of convex curves might fail for more complicated associated discriminants in the space of complete flags.)

Theorem, see [BSh2]**.** *For any two convex curves* $\gamma_1 : S^1 \to \mathbb{P}^n$ *and* $\gamma_2 : S^1 \to \mathbb{P}^n$ *the pairs* $(\mathbb{P}^n, D_{\gamma_1})$ *and* $(\mathbb{P}^n, D_{\gamma_2})$ *are diffeomorphic, i.e. there exists a diffemorphism of* $\mathbb{P}^n$ *sending* D_{γ_1} *onto* D_{γ_2}.

Remark. The above main conjecture was stated as a theorem in [BSh1] with an incomplete proof mainly consisting of a reference to certain results on dimensional transversality in [EH]. Later the authors realized that [EH] (dealing with some family of rational complex curves) is insufficient for our purposes. The present paper is an attempt to 'repair the damage' at least partially and to find an argument independent of [EH].

Finally, notice that the detailed description of π_0 in the space of closed locally convex curves in S^n and $\mathbb{P}^n$ was obtained in [ShSh] and [MSh]. (Strangely enough no information is available about other homology or homotopy groups of this space even for $n = 2$.) The analogous question for Grassmannians seems to be more complicated since such curves can be singular.

Problem. Calculate π_0 in the space of all closed locally Grassmann-convex curves in $G_{k,m}$.

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§2. Proofs

Grassmann discriminants $D_\gamma^{k+1,n+1}$.

Analogously to the standard discriminant $D_\gamma \subset \mathbb{P}^n$ one can associate to each γ (under the assumption that its $(n-1)$-dimensional osculating subspace is well-defined at almost every moment $t \in S^1$) a discriminantal hypersurface $D_\gamma^{k+1,n+1}$ in every Grassmannian $G_{k+1,n+1}$ as follows. Given a complete flag $f \in \mathbb{P}^n$ and chosen $G_{k+1,n+1}$ one gets the standard Schubert cell decomposition $\mathfrak{Sch}_f$ of $G_{k+1,n+1}$ with respect to f. Namely, each cell of $\mathfrak{Sch}_f$ consists of all subspaces in $G_{k+1,n+1}$ which have a given set of dimensions of intersections with the subspaces of f. Let $\mathfrak{D}_f$ denote the union of all cells in $\mathfrak{Sch}_f$ whose codimension at least 2. (Obviously, codim $\mathfrak{D}_f = 2$.)

Definition. For a given curve $\gamma : S^1 \to \mathbb{P}^n$ we denote by the Grassmann discriminant $D_\gamma^{k+1,n+1} \subset G_{k+1,n+1}$ the union $\bigcup_{t\in S^1} \mathfrak{D}_{f\gamma(t)}$ where $f\gamma(t)$ is the complete osculating flag to γ at the moment t. (Under our assumptions $f\gamma(t)$ is well-defined almost everywhere.)

An alternative (but equivalent) way to define $D_\gamma^{k+1,n+1}$ is as follows.

A k-dimensional projective subspace L *belongs* to $D_\gamma^{k+1,n+1}$ if the Grassmann hyperplane $H_L \subset G_{n-k,n+1}$ determined by L is nontransversal to the osculating curve $osc_{n-k}\gamma : S^1 \to G_{n-k,n+1}$.

Direct local calculations show that for locally convex curves the local multiplicity of the intersection of H_L with $osc_{n-k}\gamma$ at some moment t equals to the codimension of the Schubert cell in $\mathfrak{Sch}_{f\gamma(t)}$ to which L belongs, see e.g. [BSh1]. An example of such calculations is given in lemma 2.1.

Detailed information about Grassmann and (in)complete flag discriminants can be found in [Ka].

Problem. Describe the set of all 'good' pairs $(k+1, n+1)$, i.e. those for which $D_\gamma^{k+1,n+1}$ has no local moduli.

The answer to this question is very important for the development of the qualitative theory for linear ode of high order.

More details about Grassmann discriminant $D_\gamma^{2,4}$.

For curves in $\mathbb{P}^3$ we are mostly interested in the discriminant $D_\gamma^{2,4}$. Let us first consider a normal form of a germ of a locally convex γ in $\mathbb{P}^3$ and its osculating curve $osc_2\gamma$ in $G_{2,4}$. Since the osculating frame to γ is everywhere nondegenerate then at each point of γ we can choose affine coordinates x, y, z such that γ is given by $x = t, y = \frac{t^2}{2} + \dots, z = \frac{t^3}{6} + \dots$. (In the homogeneous coordinates we add $u = 1$ as the 1st coordinate.) The tangent line to γ (or the corresponding 2-dimensional plane) is spanned by γ and γ'. i.e. by the columns of

$$\begin{pmatrix} 1 & x = t & y = \frac{t^2}{2} + \dots & z = \frac{t^3}{6} + \dots \\ 0 & x' = 1 & y' = t + \dots & z' = \frac{t^2}{2} + \dots \end{pmatrix}. \qquad (1)$$

Recall that fixing any complete flag f in $\mathbb{R}^4$ (or $\mathbb{P}^3$) one gets the standard Schubert cell decomposition of $G_{2,4}$ with 6 cells enumerated by the Young diagrams $(\emptyset), (1), (2), (1,1), (2,1), (2,2)$. Let us denote these cells as $\mathcal{C}^f_{(\emptyset)}, \mathcal{C}^f_{(1)}, \mathcal{C}^f_{(2)}, \mathcal{C}^f_{(1,1)}, \mathcal{C}^f_{(2,1)}, \mathcal{C}^f_{(2,2)}$ and their closures, i.e. the corresponding Schubert varieties by $\bar{\mathcal{C}}^f_{(\emptyset)}, \bar{\mathcal{C}}^f_{(1)}, \bar{\mathcal{C}}^f_{(2)}, \bar{\mathcal{C}}^f_{(1,1)}, \bar{\mathcal{C}}^f_{(2,1)}, \bar{\mathcal{C}}^f_{(2,2)}$ resp. The dimension of each cell equals to the area of the corresponding diagram.

The above germ of γ determines its complete osculating flag $f\gamma(0)$ at $t=0$. In the homogeneous coordinates (s,x,y,z) of $\mathbb{P}^3$ the flag $f\gamma(0)$ coincides with the standard coordinate flag, i.e. its line coincides with the s-axis, its plane coincides with sx-plane and its 3-space coincides with the sxy-space.

Notice that the standard affine chart on $G_{2,4}$ consisting of all 2-planes transversal to the yz-plane in the homogeneous coordinates on $\mathbb{P}^3$ is represented by all matrices of the form $\begin{pmatrix} 1 & 0 & \alpha & \beta \\ 0 & 1 & \gamma & \delta \end{pmatrix}$ and the hyperplane H_L of all 2-planes nontransversal to a given 2-plane presented by $\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$ is determined by the equation

$$\begin{vmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 1 & 0 & \alpha & \beta \\ 0 & 1 & \gamma & \delta \end{vmatrix} = 0. \tag{2}$$

Expression (1) gives the following normal form for a germ of $osc_2\gamma$ in the above affine chart on $G_{2,4}$

$$osc_2\gamma = \begin{pmatrix} 1 & 0 & y - ty' = -\frac{t^2}{2} + \dots & z - tz' = -\frac{t^3}{3} + \dots \\ 0 & 1 & y' = t + \dots & z' = \frac{t^2}{2} + \dots \end{pmatrix}. \tag{3}$$

The restriction of the divisor (2) to the germ (3) is given by

$$\begin{vmatrix} a + \frac{t^2}{2} + \dots & b + \frac{t^3}{3} + \dots \\ c - t + \dots & d - \frac{t^2}{2} + \dots \end{vmatrix} = ad - bc + bt + \frac{(d-a)t^2}{2} - \frac{ct^3}{3} + \frac{t^4}{12} + \dots. \tag{4}$$

As was mentioned above the Grassmann discriminant $D^{2,4}_\gamma$ is a stratified hypersurface ruled by $\bar{\mathcal{C}}^f_{(1,1)} \cup \bar{\mathcal{C}}^f_{(2)}$ where f runs over the curve $f\gamma : S^1 \to F_{n+1}$ of all osculating complete flags to γ.

Proposition, see [Ka]**, pp. 124-125.** *In a neighborhood of any point $p \in osc_2\gamma$ the discriminant $D^{2,4}_\gamma$ has no moduli and is diffeomorphic to a one-dimensional cylinder over the discriminant $\mathcal{D}$ for the family $P(t,u,v,w) = t^4 + 3(u+v)t^2 + wt - 3uv$, i.e. to the hypersurface of all triples (u,v,w) for which $P(t,u,v,w)$ considered as a polynomial in t has a multiple zero, see Fig.2.*

Remark. The above result is due to V.Arnold who discovered it for different purposes. $D^{2,4}_\gamma$ (and $\mathcal{D}$) has two irreducible components since it is spanned by two families of 2-planes. Irreducible components of $\mathcal{D}$ are diffeomorphic to Whitney umbrellas which are cubically tangent. The explicit equation defining $\mathcal{D}$ is

$$(4u^3 + 24u^2v + 36uv^2 + w^2)(4v^3 + 24uv^2 + 36u^2v + w^2) = 0.$$

It is quasihomogeneous with the weights $2, 3, 2$ for u, v, w resp. (An appropriate name for this discriminant seems to be *the* $G_{2,4}$*-swallowtail*.)

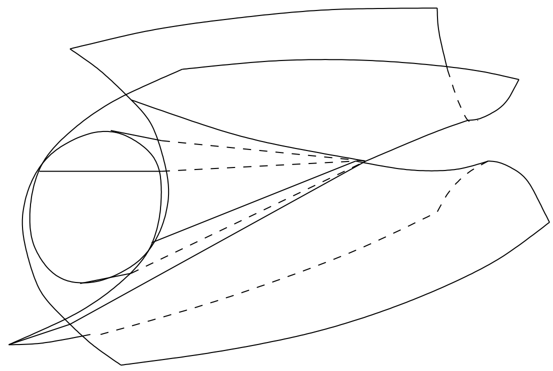

FIG.2. $G_{2,4}$-SWALLOWTAIL.

2.1. Lemma. *The local multiplicity* $\sharp_{\gamma,L}(0)$ *of the intersection of the germ* $osc_2\gamma(t)$ *with* H_L *at* $t=0$ *does not exceed 4 and*

$$\begin{cases} \sharp_{\gamma,L}(0)=4 & \textit{iff } a=b=c=d=0 \textit{ which coincides with } \bar{\mathcal{C}}^{f\gamma(0)}_{(\emptyset)}, \\ \sharp_{\gamma,L}(0)\geq 3 & \textit{iff } a=b=d=0 \textit{ which coincides with } \bar{\mathcal{C}}^{f\gamma(0)}_{(1)}, \\ \sharp_{\gamma,L}(0)\geq 2 & \textit{iff either } a=b=0 \textit{ which coincides with } \bar{\mathcal{C}}^{f\gamma(0)}_{(2)}, \\ & \textit{or } b=d=0 \textit{ which coincides with } \bar{\mathcal{C}}^{f\gamma(0)}_{(1,1)}, \\ \sharp_{\gamma,L}(0)\geq 1 & \textit{iff } ad-bc=0 \textit{ which coincides with } \bar{\mathcal{C}}^{fl_\gamma(0)}_{(2,1)}. \end{cases}$$

Proof. See (4). All possible arrangements of a line and a flag in $\mathbb{P}^3$ are given on Fig.3. □

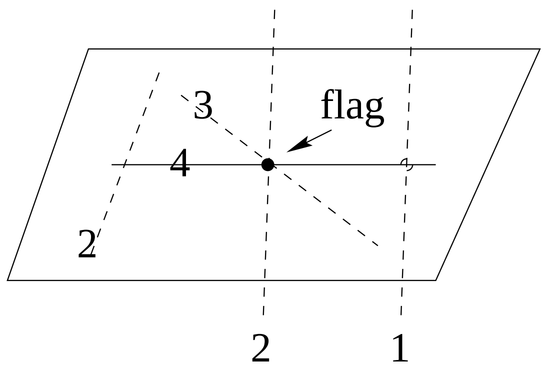

FIG.3. RELATIVE POSITIONS OF LINES IN $\mathbb{P}^3$ W.R.T A GIVEN FLAG AND CORRESPONDING MULTIPLICITIES.

Our aim is to prove that the total multiplicity of intersection of $osc_2\gamma$ with any Grassmann hyperplane $H_L \subset G_{2,4}$ is at most 4. By definition H_L is nontransversal to $osc_2\gamma$ if and only if $L \in D^{2,4}_\gamma$. Thus by semicontinuity of multiplicity it suffices to prove the statement only for $L \in D^{2,4}_\gamma$. Since $D^{2,4}_\gamma$ is ruled by $\bar{\mathcal{C}}^f_{(1,1)} \cup \bar{\mathcal{C}}^f_{(2)}$ we have to consider 2 different types of lines $L \subset \mathbb{P}^3$, namely, lines intersecting γ (type 1) and lines lying in the osculating hyperplanes to γ (type 2).

2.2. Lemma. *For any line L intersecting γ the total multiplicity $\sharp_{\gamma,L}$ of the intersection of H_L and $osc_2\gamma$ is at most 4.*

Proof. Let $\pi_p : \mathbb{P}^3 \setminus \{p\} \to \mathbb{P}^2$ be the standard projection where p is the intersection point of L and γ. Let us additionally assume that L does not belong to the tangent space to γ at $\gamma(p)$ since this case is treated in Lemma 2.3. Then one has $\sharp_{\gamma,L}(p) = 2$, see Lemma 2.1. Let $q \in \mathbb{P}^2$ denote the image of L under π_p. Notice that the image $\pi_p\gamma \subset \mathbb{P}^2$ is a smooth convex plane curve. Indeed, for any line $l \subset \mathbb{P}^2$ one gets $\sharp_{\pi_p\gamma,l} = \sharp_{\gamma,\pi^{-1}l} - \sharp_{\gamma,\pi^{-1}l}(p)$. The local multiplicity $\sharp_{\gamma,\pi^{-1}l}(p)$ being positive implies that $\sharp_{\pi_p\gamma,l}$ is at most 2. Thus $\pi_p\gamma$ is convex. The assumption that L does not belong to the tangent space to γ at $\gamma(p)$ translates into the condition that q does not lie on the tangent line to $\pi_p\gamma$. Further, any tangent line to γ crossing L is mapped to the tangent line to $\pi_p\gamma$ passing through q. Note that the local multiplicity $\sharp_{\gamma,L}(t)$ is at most 2 if $t \neq p$ and equals 2 if and only if L crosses γ at the point $\gamma(t)$. Indeed, L can not lie in the tangent plane to γ at $\gamma(t)$ since such a plane would intersect γ with multiplicity 4 at $\gamma(t)$ and with some positive multiplicity at $\gamma(p)$ which contradicts to the convexity of γ. Thus $\sharp_{\gamma,L}(t)$ is at most 2 and equals 2 only if L goes through $\gamma(t)$, see Fig.3. Finally, if q does not lie on $\pi_p\gamma$ one has at most two tangent lines to $\pi_p\gamma$ passing through q each contributing the local multiplicity 1 to $\sharp_{\gamma,L}$. Thus $\sharp_{\gamma,l} \leq 4$. If q belongs to π_p then the only tangent to γ passing through q is the tangent to $\pi_p\gamma(t)$. It contributes the local multiplicity 2 and we still have $\sharp_{\gamma,l} \leq 4$. $\square$

2.3. Lemma. *For any line L lying in the osculating hyperplane to γ the total multiplicity $\sharp_{\gamma,L}$ of intersection of H_L and $osc_2\gamma$ is at most 4.*

Proof. To show this we will use a dual type of projection. Let T containing L denote the osculating hyperplane to γ at $\gamma(p)$. As before, $p = \gamma \cap L$. We will project γ on T along its pencil of tangent lines. The projected curve $\mu\gamma$ is again a convex curve on T, see [BSh2]. Tangent lines to γ intersecting L are mapped to the intersection points of $\mu\gamma$ with L. If L coincides with the tangent line to γ at $\gamma(p)$ then it projects to the tangent line to $\mu\gamma$ and has no other intersection points. Thus $\sharp_{\gamma,l} = 4$ in this case. If L passes through $\gamma(p)$ then L passes through $\mu\gamma(p)$ as well and has one more intersection point with $\mu\gamma$ where the local multiplicity is 1. Indeed, otherwise (as in Lemma 2.2) the hyperplane T should cross γ at some point different from $\gamma(p)$ which is forbidden by its convexity. Thus $\sharp_{\gamma,l} = 4$ in this case. Finally, if L does not pass through $\gamma(p)$ then L is transversal to $\mu\gamma$ and has at most two intersection points with it. Each such point contributes the local multiplicity 1 to $\sharp_{\gamma,L}$. The statement follows. $\square$

Proof of corollary 1.2. As it is explained in e.g. [BSh1] any linear ode of order $n+1$ is associated with a canonical locally convex curve in the projectivization of the dual space to the linear $(n+1)$-dimensional space of its solutions. Moreover disconjugate linear ode correspond to convex curves. Namely, to each moment t we associate the hyperplane of all solutions of the considered ode vanishing at t. This curve determines the original ode up to the multiplication of its fundamental system by a nonvanishing scalar function and therefore captures all the information about its zeros. Standard arguments of algebraic geometry show that zeros of Wronskians are in $1-1$-correspondence with the intersection points of the corresponding osculating curve with the corresponding Grassmann hyperplane. The corollary follows. □

Proof of corollary 1.3. Follows the argument in lemma 2.2. Let us project $\mathbb{P}^3 \setminus \{p\}$ with a convex curve γ to $\mathbb{P}^2$ from some point p. If L is any line through p then the tangents to γ intersecting L will be mapped to the tangents to $\pi_p\gamma$ passing through the point corresponding to L after projection. Thus by Theorem 1.1 there are at most 4 of them. □

§3. Final remarks

Apperently the result and conjectures of the present paper fit into a much more general perspective of a (still nonexisting) qualitative theory for linear ode of higher order. It is a common knowledge that for higher order one can not hope to get a direct analog of the classical Sturm separation theorem (saying that two consecutive zeros of one solution to a given linear ode of order 2 are separated by a zero of any other solution). Around 89 the first author made an attempt to generalize the separation theorem to higher order equations by substituting different individual solutions by different fundamental solutions, see [BSh1]. This ideology is borrowed from linear Hamiltonian systems. Geometrically speaking, this means that instead of comparing the intersection points of the projective curve related to the considered ode (more exactly, obtained by the projectivization of its fundamental solution) with different projective hyperplanes we take its osculating curve in the space of complete flags and compare its intersection points with different hyperplanes in the later space. But at the moment the situation with this theory is far from being satisfactory. It is worth mentioning that if one considers some linear ode on an expanding time interval then the moment when the equation looses its disconjugacy is in many ways similar to the appearence of the first conjugate point along a curve in Riemannian (or, probably subRiemannian) geometry, comp. [Shr].

Methods of the present paper, in principle, are applicable to the main conjecture, see §1. The case of $G_{2,n}$ seems to be especially doable. There are also 2 types of lines to be considered, namely lines intersecting the codimension 2 osculating subspace to γ at some moment t (type 1) and lines lying in osculating hyperplanes to γ (type 2). For type 2) the argument to the present paper works similarly and gives the necessary estimation. However for type 1) the projection from the intersection point is singular and one needs to prove that the 'degree' of this singular projection is at most $2(n-2)$. The authors plan to return to the general case later.

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