CONFORMAL SYMMETRIES OF THE SELF-DUAL YANG-MILLS EQUATIONS

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Abstract

We describe an infinite-dimensional Kac-Moody-Virasoro algebra of new hidden symmetries for the self-dual Yang-Mills equations related to conformal transformations in the 4-dimensional base space.

1 Introduction

The self-dual Yang-Mills (SDYM) equations in the space R^4 with the metric of signature $(++++)$ or $(++--)$ are the famous example of the nonlinear integrable equations in four dimensions. These equations are invariant under the group of gauge transformations and the group of conformal transformations of the space R^4, both of which are the "obvious symmetries". It is well-known by now that the SDYM equations in R^4 have an infinite-dimensional algebra of "hidden symmetries" [1-5]. For the Yang-Mills (YM) potentials with values in a Lie algebra $\mathcal{G}$ these symmetries are an affine extension of the Lie algebra $\mathcal{G}$ of global gauge transformation to the Kac-Moody algebra $\mathcal{G}\otimes C(\lambda,\lambda^{-1})$. It is also well-known that for integrable models in two dimensions the algebra of hidden symmetries includes Virasoro-like generators (for a recent discussion and references see [6]). We shall show that the SDYM equations also have Virasoro-like and new Kac-Moody symmetries, which generate new solutions from any old one.

New Kac-Moody-Virasoro symmetries may be interesting for applications because

- they give new arguments in support of the old idea that the SDYM theory is a generalization of the $d = 2$ conformal theories to dimension $d = 4$;

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- the SDYM equations are known to arise in $N = 2$ supersymmetric open string theory and Kac-Moody-Virasoro symmetries may underlie the cancellation of almost all amplitudes in the theory of $N = 2$ self-dual strings;

- the study of symmetries is important for understanding non-pertubative properties and quantization of Yang-Mills theories and $N = 2$ self-dual strings;

- the SDYM equations are "master" integrable equations since a lot of integrable equations in $1 \leq d \leq 3$ can be obtained from them by suitable reductions (for a recent discussion and references see [7]). In particular, the matrix Ernst-type equations appearing in the study of T- and S-duality symmetries of string theory may be obtained by reduction of the SDYM equations. Their symmetries in turn may also be obtained by an appropriate reduction of symmetries of the SDYM equations.

Clearly this list of reasons is not complete and can be extended.

2 Definitions and notation

We shall consider the Euclidean space $R^{4,0}$ with the metric $g_{\mu\nu} = diag(+1, +1, +1, +1)$, where $\mu, \nu, ... = 1, ..., 4$. Let us denote by A_μ the potentials of the YM fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, with $\partial_\mu := \partial/\partial x^\mu$. The fields A_μ and $F_{\mu\nu}$ take values in a Lie algebra $\mathcal{G}$. For simplicity one may think that $\mathcal{G} = sl(n, C)$.

The SDYM equations have the form

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = F_{\mu\nu}, \tag{1}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor in $R^{4,0}$ ($\varepsilon_{1234} = 1$). In $R^{4,0}$ we introduce complex coordinates $y = x^1 + ix^2, z = x^3 - ix^4, \bar{y} = x^1 - ix^2, \bar{z} = x^3 + ix^4$ and set $A_y = \frac{1}{2}(A_1 - iA_2), A_z = \frac{1}{2}(A_3 + iA_4), A_{\bar{y}} = \frac{1}{2}(A_1 + iA_2), A_{\bar{z}} = \frac{1}{2}(A_3 - iA_4)$. The SDYM equations read then

$$F_{yz} = 0, \quad F_{\bar{y}\bar{z}} = 0, \tag{2a}$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{2b}$$

These equations can be obtained as compatibility conditions of the following linear system of equations [8]:

$$[\partial_{\bar{y}} + A_{\bar{y}} - \lambda(\partial_z + A_z)]\Psi(x, \lambda) = 0, \tag{3a}$$

$$[\partial_{\bar{z}} + A_{\bar{z}} + \lambda(\partial_y + A_y)]\Psi(x, \lambda) = 0, \tag{3b}$$

where $\Psi \in G$ is a group-valued function depending on the coordinates x^μ of $R^{4,0}$ and a complex parameter $\lambda \in CP^1$. In fact, Ψ is defined on the twistor space $\mathcal{Z} = R^{4,0} \times CP^1$ for the space $R^{4,0}$, and eqs. (3) are equivalent to the holomorphicity of the matrix-function Ψ (Ward theorem [9]).

Equation (2a) implies that the gauge potentials can be written in the form

$$A_y = h^{-1}\partial_y h, \;\; A_z = h^{-1}\partial_z h, \;\; A_{\bar{y}} = \tilde{h}^{-1}\partial_{\bar{y}}\tilde{h}, \;\; A_{\bar{z}} = \tilde{h}^{-1}\partial_{\bar{z}}\tilde{h}, \tag{4}$$

where h and $\tilde{h}$ are some group-valued functions on $R^{4,0}$. One may perform the following gauge transformation:

$$A_{\bar{y}} \to B_{\bar{y}} = \tilde{h}A_{\bar{y}}\tilde{h}^{-1} + \tilde{h}\partial_{\bar{y}}\tilde{h}^{-1} = 0, \;\; A_{\bar{z}} \to B_{\bar{z}} = \tilde{h}A_{\bar{z}}\tilde{h}^{-1} + \tilde{h}\partial_{\bar{z}}\tilde{h}^{-1} = 0, \tag{5a}$$

$$A_y \to B_y = \tilde{h} A_y \tilde{h}^{-1} + \tilde{h}\partial_y \tilde{h}^{-1} = g^{-1}\partial_y g,\ A_z \to B_z = \tilde{h} A_z \tilde{h}^{-1} + \tilde{h}\partial_z \tilde{h}^{-1} = g^{-1}\partial_z g, \quad (5b)$$

where $g := h\tilde{h}^{-1}$, and thus fix the gauge $B_{\bar{y}} = B_{\bar{z}} = 0$ [10,1-5]. Then eqs. (2) are replaced by the matrix equation

$$\partial_{\bar{y}} B_y + \partial_{\bar{z}} B_z = \partial_{\bar{y}}(g^{-1}\partial_y g) + \partial_{\bar{z}}(g^{-1}\partial_z g) = 0. \quad (6)$$

It is also possible to perform the gauge transformation

$$A_{\bar{y}} \to h A_{\bar{y}} h^{-1} + h\partial_{\bar{y}} h^{-1} = g\partial_{\bar{y}} g^{-1},\ A_{\bar{z}} \to h A_{\bar{z}} h^{-1} + h\partial_{\bar{z}} h^{-1} = g\partial_{\bar{z}} g^{-1}, \quad (7a)$$

$$A_y \to h A_y h^{-1} + h\partial_y h^{-1} = 0,\ A_z \to h A_z h^{-1} + h\partial_z h^{-1} = 0, \quad (7b)$$

then eq.(2) gets converted into the equation

$$\partial_y(g\partial_{\bar{y}} g^{-1}) + \partial_z(g\partial_{\bar{z}} g^{-1}) = 0. \quad (8)$$

Let C denote a contour in the λ-plane about the origin, C_+ denote C and the inside of C, and C_- denotes C and the outside of C ($C = C_+ \cap C_-$). Then there exist two matrix functions $\Psi_\pm(x,\lambda)$ holomorphic and nonsingular on $C_\pm$, each satisfying eqs. (3) (see e.g. [5]). From the linear system (3) it is easy to see that

$$h = \Psi_-^{-1}(\lambda = \infty), \quad \tilde{h} = \Psi_+^{-1}(\lambda = 0). \quad (9)$$

Therefore, it is obvious that eqs. (6) are the compatibility conditions of the linear system

$$\partial_{\bar{y}}\eta - \lambda(\partial_z + B_z)\eta = 0, \quad (10a)$$

$$\partial_{\bar{z}}\eta + \lambda(\partial_y + B_y)\eta = 0, \quad (10b)$$

obtained from (3) for Ψ_+ by performing the gauge transformation $\Psi_+(x,\lambda) \to \eta(x,\lambda) = \Psi_+^{-1}(x,0)\Psi_+(x,\lambda) = \tilde{h}(x)\Psi_+(x,\lambda), \lambda \in C_+$. Analogously, eqs. (8) are compatibility conditions for the linear system

$$\frac{1}{\lambda}(\partial_{\bar{y}} + g\partial_{\bar{y}} g^{-1})\hat{\eta} - \partial_z\hat{\eta} = 0, \quad (11a)$$

$$\frac{1}{\lambda}(\partial_{\bar{z}} + g\partial_{\bar{z}} g^{-1})\hat{\eta} + \partial_y\hat{\eta} = 0, \quad (11b)$$

where $\hat{\eta}(\lambda) = \Psi_-^{-1}(\infty)\Psi_-(\lambda)$ is well defined for $\lambda \in C_-$.

In the following, we shall consider the space $\mathcal{Z}_+ = R^{4,0} \times C_+ \subset \mathcal{Z} = R^{4,0} \times CP^1$, matrix-valued function η on $\mathcal{Z}_+$, the linear system (10) and the Wess-Zumino-Novikov-Witten-type eq.(6). In this short paper we describe new hidden symmetries of eqs. (6) omitting direct computations and writing out only the final formulas.

3 Space-time symmetries

Let A_μ be a solution of the SDYM equations (1). Then $\delta : A_\mu \to \delta A_\mu$ is called an infinitesimal symmetry transformation if δA_μ satisfies the linearized form of eqs.(1). It is well-known that the group of conformal transformations is the maximal group of transformations of the space $R^{4,0}$ under which the SDYM equations (1) are invariant. This group is locally isomorphic to the group $SO(5,1)$.

Let us introduce the self-dual $\eta^a_{\mu\nu}$ and the anti-self-dual $\bar\eta^a_{\mu\nu}$ 't Hooft tensor,

$$\eta^a_{\mu\nu} = \{\epsilon^a_{bc}, \mu = b, \nu = c;\ \delta^a_\mu, \nu = 4; -\delta^a_\nu, \mu = 4\}, \tag{12a}$$

$$\bar\eta^a_{\mu\nu} = \{\epsilon^a_{bc}, \mu = b, \nu = c;\ -\delta^a_\mu, \nu = 4; \delta^a_\nu, \mu = 4\}, \tag{12b}$$

where $a, b, ... = 1, 2, 3$ and ϵ^a_{bc} are the structure constants of the group $SO(3)$. Then the generators of the group $SO(5,1)$ can be realized in terms of the following vector fields on $R^{4,0}$,

$$X_a = \delta_{ab}\eta^b_{\mu\nu}x_\mu\partial_\nu,\ Y_a = \delta_{ab}\bar\eta^b_{\mu\nu}x_\mu\partial_\nu,\ P_\mu = \partial_\mu,$$
$$K_\mu = \frac{1}{2}x_\sigma x_\sigma\partial_\mu - x_\mu D, \quad D = x_\sigma\partial_\sigma, \tag{13}$$

where $\{X_a\}$ and $\{Y_a\}$ generate two commuting $SO(3)$-subgroups in $SO(4)$, P_μ are the generators of translations, K_μ are the generators of special conformal transformations and D is the generator of dilatations.

Infinitesimal transformations of the YM potentials A_μ under the action of the conformal group $SO(5,1)$ are given by

$$\delta_N A_\mu = \mathcal{L}_N A_\mu := N^\nu A_{\mu,\nu} + A_\nu N^\nu_{\ ,\mu} \quad , \tag{14}$$

where $N = N^\nu\partial_\nu$ is any generator of the conformal group, and $\mathcal{L}_N$ is the Lie derivative along the vector field N. It is not hard to show that for each $N \in so(5,1)$ the transformation (14) constitutes a symmetry of the SDYM equations (1).

The group-valued functions $\Psi_\pm$ satisfying the linear equations (3) are defined on the subspaces $\mathcal{Z}_\pm = R^{4,0} \times C_\pm$ of the twistor space $\mathcal{Z}$. Therefore, we have to define the action of $SO(5,1)$ on $\mathcal{Z}_\pm$ preserving the linear system (3). The lifted vector fields $\tilde N$ on $\mathcal{Z}_+$, which form the generators of $SO(5,1)$, are given by [7],

$$\tilde X_a = X_a,\ \tilde Y_a = Y_a + 2Z_a,\ \tilde P_\mu = P_\mu,$$
$$\tilde K_\mu = K_\mu + \bar\eta^a_{\sigma\mu}x_\sigma Z_a, \quad \tilde D = D, \tag{15}$$

with the following expression for the generators Z_a of the $SO(3)$ rotations on $C_+ \subset CP^1$:

$$Z_1 = \frac{i}{2}(\lambda^2-1)\partial_\lambda - \frac{i}{2}(\bar\lambda^2-1)\partial_{\bar\lambda},\ Z_2 = \frac{1}{2}(\lambda^2+1)\partial_\lambda + \frac{1}{2}(\bar\lambda^2+1)\partial_{\bar\lambda},\ Z_3 = i\lambda\partial_\lambda - i\bar\lambda\partial_{\bar\lambda}. \tag{16}$$

Here $\bar\lambda$ is the complex conjugate to $\lambda \in C_+$ and $\partial_\lambda := \partial/\partial\lambda$. For $\mathcal{Z}_-$ we have analogous formulas. Using identities for the 't Hooft tensors [11], it can be easily shown that $[X_a, X_b] = -2\epsilon^c_{ab}X_c$, $[X_a, Y_b] = 0, [\tilde Y_a, \tilde Y_b] = -2\epsilon^c_{ab}\tilde Y_c$ and so on. Now one can define the infinitesimal transformation of the group-valued function Ψ_+ under the action of the conformal group [7]:

$$\delta_{\tilde N}\Psi_+ = \mathcal{L}_{\tilde N}\Psi_+ := \tilde N\Psi_+, \tag{17}$$

where $\tilde N$ is any of the generators defined in (15). It is not hard to show that the linear system (3) is invariant under the transformations (14), (17).

Obviously the gauge $B_{\bar y} = B_{\bar z} = 0$ is not invariant under conformal transformations, because in general $(\mathcal{L}_N B_{\bar y})\,|_{B_{\bar y}=0} \neq 0, (\mathcal{L}_N B_{\bar z})\,|_{B_{\bar z}=0} \neq 0$. However, as noticed by Pohlmeyer [1], conformal invariance can be restored by compensating gauge transformations. We shall write out the explicit formulas for these compensating gauge transformations. First, it can be shown that

$$\delta_{X_a}B_{\bar y} = \mathcal{L}_{X_a}B_{\bar y} = B_y X^y_{a,\bar y} + B_z X^z_{a,\bar y} = 0; \quad \delta_{X_a}B_{\bar z} = 0,$$

$$\delta_{P_\mu} B_{\bar y} = \delta_{P_\mu} B_{\bar z} = 0,\ \delta_D B_{\bar y} = \delta_D B_{\bar z} = 0,\ \delta_{Y_3} B_{\bar y} = \delta_{Y_3} B_{\bar z} = 0, \tag{18}$$

and for $N \in \{Y_1, Y_2, K_\mu\}$ we have $\delta_N B_{\bar y} \neq 0$ and $\delta_N B_{\bar z} \neq 0$. For example,

$$\delta_{Y_2} B_{\bar y} = -B_z,\ \delta_{Y_2} B_{\bar z} = B_y, \tag{19a}$$

$$\delta_{K_y} B_{\bar y} = -zB_z,\ \delta_{K_y} B_{\bar z} = zB_y, \tag{19b}$$

where B_y and B_z are defined in (5).

As a second step, for an arbitrary generator $\tilde N$ of (15) let us introduce the Lie algebra-valued function $\psi_{\tilde N}(x,\lambda)$ on $\mathcal{Z}_+$,

$$\psi_{\tilde N}(x,\lambda) = (\tilde N(\lambda)\eta(\lambda))\eta^{-1}(\lambda) \in \mathcal{G}, \tag{20}$$

and consider the function $\varphi_N = \psi^0_{\tilde N}(x) = \psi_{\tilde N}(x, \lambda = 0)$, the infinitesimal gauge transformation $\partial_\mu\varphi_N + [B_\mu, \varphi_N]$ of the YM potentials generated by $\varphi_N(x)$, and the transformation

$$\delta^0_{\tilde N} B_y = \delta_N B_y + \partial_y\varphi_N + [B_y, \varphi_N],\ \delta^0_{\tilde N} B_z = \delta_N B_z + \partial_z\varphi_N + [B_z, \varphi_N], \tag{21a}$$

which is a combination of the conformal transformations δ_N and a gauge transformation. By direct calculation one may verify that

$$\delta^0_{\tilde N} B_{\bar y} = \delta_N B_{\bar y} + \partial_{\bar y}\varphi_N = 0,\ \delta^0_{\tilde N} B_{\bar z} = \delta_N B_{\bar z} + \partial_{\bar z}\varphi_N = 0, \tag{21b}$$

for any generator $\tilde N$ of the conformal group. The identity $\eta(x, \lambda = 0) = 1$ implies that $\varphi_N = 0$ and $\delta^0_{\tilde N} = \delta_N$ for $N \in \{P_\mu, X_a, Y_3\}$ and $\varphi_N \neq 0$ if $N \in \{Y_1, Y_2, K_\mu\}$. Using eqs. (14) and (21), it is easy to show that

$$[\delta^0_{\tilde M}, \delta^0_{\tilde N}]B_y = \delta^0_{[\tilde M,\tilde N]}B_y,\ [\delta^0_{\tilde M}, \delta^0_{\tilde N}]B_z = \delta^0_{[\tilde M,\tilde N]}B_z,\ [\delta^0_{\tilde M}, \delta^0_{\tilde N}]B_{\bar y} = 0, [\delta^0_{\tilde M}, \delta^0_{\tilde N}]B_{\bar z} = 0, \tag{22}$$

i.e. the transformations $\delta^0_{\tilde N}$ define an action of the conformal algebra $so(5,1)$ on the YM potentials which preserves the gauge $B_{\bar y} = B_{\bar z} = 0$.

The action of $\delta^0_{\tilde N}$ on the group-valued function $\eta(x, \lambda)$ has the form

$$\delta^0_{\tilde N}\eta(\lambda) = \delta_{\tilde N}\eta(\lambda) - \varphi_N(x)\eta(\lambda) = \tilde N(\lambda)\eta(\lambda) - \varphi_N(x)\eta(\lambda), \tag{23}$$

where $\delta^0_{\tilde N}$ is precisely the combination of a conformal and a compensating gauge transformation. It may be shown that the linear system (10) is invariant under the transformations (21), (23). Thus, fixing a complex structure on the space $R^{4,0}$ and fixing the gauge (5) does not destroy the conformal invariance of the SDYM equations and the linear system for them.

4 Hidden symmetries

In Sect.3 we assigned to each generator $\tilde N$ of the conformal group $SO(5,1)$ a function $\psi_{\tilde N}(x,\lambda)$ on $\mathcal{Z}_+$ with values in the Lie algebra $\mathcal{G}$ (see (20)). Using eqs. (10) on $\eta(x,\lambda)$, one can verify that the function $\psi_{\tilde N}$ satisfies the equations

$$\partial_{\bar y}\psi_{\tilde N}(\lambda) + \mathcal{L}_N B_{\bar y} - \lambda(\partial_z\psi_{\tilde N}(\lambda) + [B_z, \psi_{\tilde N}(\lambda)] + \mathcal{L}_N B_z) = 0, \tag{24a}$$

$$\partial_{\bar z}\psi_{\tilde N}(\lambda) + \mathcal{L}_N B_{\bar z} + \lambda(\partial_y\psi_{\tilde N}(\lambda) + [B_y, \psi_{\tilde N}(\lambda)] + \mathcal{L}_N B_y) = 0, \tag{24b}$$

where $\mathcal{L}_N$ is the Lie derivative along the vector field N (see Sect.3).

Let us recall that the group-valued function $\eta(x,\lambda)$ is holomorphic and nonsingular for $\lambda \in C_+$. Hence $\psi_{\tilde{N}}$ can be expanded in powers of λ:

$$\psi_{\tilde{N}}(x,\lambda) = \sum_{n=0}^{\infty} \lambda^n \psi^n_{\tilde{N}}(x), \tag{25}$$

where the coefficients $\psi^n_{\tilde{N}}$ depend only on $x^\mu \in R^{4,0}$ and are conserved nonlocal charges (cf. [1-4]). After substituting (25) into (24) we obtain the following recurrence relations:

$$\mathcal{L}_N B_{\bar{y}} + \partial_{\bar{y}}\psi^0_{\tilde{N}} = 0,\ \ \mathcal{L}_N B_{\bar{z}} + \partial_{\bar{z}}\psi^0_{\tilde{N}} = 0, \tag{26a}$$

$$\partial_{\bar{y}}\psi^1_{\tilde{N}} - \partial_z\psi^0_{\tilde{N}} - [B_z,\psi^0_{\tilde{N}}] - \mathcal{L}_N B_z = 0,\ \ \partial_{\bar{z}}\psi^1_{\tilde{N}} + \partial_y\psi^0_{\tilde{N}} + [B_y,\psi^0_{\tilde{N}}] + \mathcal{L}_N B_y = 0, \tag{26b}$$

$$\partial_{\bar{y}}\psi^{n+1}_{\tilde{N}} - \partial_z\psi^n_{\tilde{N}} - [B_z,\psi^n_{\tilde{N}}] = 0,\ \ \partial_{\bar{z}}\psi^{n+1}_{\tilde{N}} + \partial_y\psi^n_{\tilde{N}} + [B_y,\psi^n_{\tilde{N}}] = 0,\ \ n \geq 1. \tag{26c}$$

The starting point (26a) is true by construction of $\delta^0_{\tilde{N}}$:

$$\delta^0_{\tilde{N}} B_{\bar{y}} = 0,\ \ \delta^0_{\tilde{N}} B_{\bar{z}} = 0. \tag{27a}$$

For B_y and B_z we obtain

$$\delta^0_{\tilde{N}} B_y := \partial_y\psi^0_{\tilde{N}} + [B_y,\psi^0_{\tilde{N}}] + \mathcal{L}_N B_y = -\partial_{\bar{z}}\psi^1_{\tilde{N}}, \tag{27b}$$

$$\delta^0_{\tilde{N}} B_z := \partial_z\psi^0_{\tilde{N}} + [B_z,\psi^0_{\tilde{N}}] + \mathcal{L}_N B_z = \partial_{\bar{y}}\psi^1_{\tilde{N}}. \tag{27c}$$

Using eqs. (27) it is trivial to deduce the invariance of the SDYM equations (6) under conformal transformations:

$$\partial_{\bar{y}}(\delta^0_{\tilde{N}} B_y) + \partial_{\bar{z}}(\delta^0_{\tilde{N}} B_z) = -\partial_{\bar{y}}\partial_{\bar{z}}\psi^1_{\tilde{N}} + \partial_{\bar{z}}\partial_{\bar{y}}\psi^1_{\tilde{N}} = 0. \tag{28}$$

Eqs. (26) are straightforward to generalize eqs.(27) to an infinite number of infinitesimal transformations $\delta^n_{\tilde{N}}$ with $n \geq 0$:

$$\delta^n_{\tilde{N}} B_{\bar{y}} := 0,\quad \delta^n_{\tilde{N}} B_{\bar{z}} := 0,\quad n \geq 1, \tag{29a}$$

$$\delta^n_{\tilde{N}} B_y := \partial_y\psi^n_{\tilde{N}} + [B_y,\psi^n_{\tilde{N}}] = -\partial_{\bar{z}}\psi^{n+1}_{\tilde{N}},\ \ n \geq 1, \tag{29b}$$

$$\delta^n_{\tilde{N}} B_z := \partial_z\psi^n_{\tilde{N}} + [B_z,\psi^n_{\tilde{N}}] = \partial_{\bar{y}}\psi^{n+1}_{\tilde{N}},\ \ n \geq 1. \tag{29c}$$

Introducing $\delta_{\tilde{N}}(\zeta) = \sum_{n=0}^{\infty}\zeta^n\delta^n_{\tilde{N}}$, $\zeta \in C_+$, we obtain a one-parameter family of infinitesimal transformations

$$\delta_{\tilde{N}}(\zeta) B_{\bar{y}} = 0,\quad \delta_{\tilde{N}}(\zeta) B_{\bar{z}} = 0, \tag{30a}$$

$$\delta_{\tilde{N}}(\zeta) B_y := \partial_y\psi_{\tilde{N}}(\zeta) + [B_y,\psi_{\tilde{N}}(\zeta)] + \mathcal{L}_N B_y = -\frac{1}{\zeta}(\partial_{\bar{z}}\psi_{\tilde{N}}(\zeta) + \mathcal{L}_N B_{\bar{z}}), \tag{30b}$$

$$\delta_{\tilde{N}}(\zeta) B_z := \partial_z\psi_{\tilde{N}}(\zeta) + [B_z,\psi_{\tilde{N}}(\zeta)] + \mathcal{L}_N B_z = \frac{1}{\zeta}(\partial_{\bar{y}}\psi_{\tilde{N}}(\zeta) + \mathcal{L}_N B_{\bar{y}}). \tag{30c}$$

Clearly $\delta_{\tilde{N}}(\zeta)$ generates a symmetry of the SDYM equations (6), since if B_y and B_z satisfy eqs. (6) then

$$\partial_{\bar{y}}(\delta_{\tilde{N}}(\zeta) B_y) + \partial_{\bar{z}}(\delta_{\tilde{N}}(\zeta) B_z) = -\frac{1}{\zeta}\{\partial_{\bar{y}}(\mathcal{L}_N B_{\bar{z}}) - \partial_{\bar{z}}(\mathcal{L}_N B_{\bar{y}})\} = 0, \tag{31}$$

for any generator $\tilde{N}$ of the conformal group. For example, for Y_2 from (19a) we have

$$\partial_{\bar{y}}(\delta_{\tilde{Y}_2}(\zeta)B_y) + \partial_{\bar{z}}(\delta_{\tilde{Y}_2}(\zeta)B_z) = -\frac{1}{\zeta}\{\partial_{\bar{y}}B_y + \partial_{\bar{z}}B_z\} = 0.$$

To find the infinitesimal transformation $\eta(\lambda) \to \delta_{\tilde{N}}(\zeta)\eta(\lambda)$ corresponding to the transformations (30) let us consider the variation of eqs. (10). We obtain

$$\partial_{\bar{y}}\chi_{\tilde{N}}(\lambda,\zeta) - \lambda(\partial_z\chi_{\tilde{N}}(\lambda,\zeta) + [B_z, \chi_{\tilde{N}}(\lambda,\zeta)]) - \lambda\delta_{\tilde{N}}(\zeta)B_z = 0, \tag{32a}$$

$$\partial_{\bar{z}}\chi_{\tilde{N}}(\lambda,\zeta) + \lambda(\partial_y\chi_{\tilde{N}}(\lambda,\zeta) + [B_y, \chi_{\tilde{N}}(\lambda,\zeta)]) + \lambda\delta_{\tilde{N}}(\zeta)B_y = 0, \tag{32b}$$

where $\chi_{\tilde{N}}(\lambda,\zeta) = \{\delta_{\tilde{N}}(\zeta)\eta(\lambda)\}\eta^{-1}(\lambda)$. One can verify that the function $\chi_{\tilde{N}}(\lambda,\zeta)$ satisfies eqs. (32) (cf. [1-6]):

$$\chi_{\tilde{N}}(\lambda,\zeta) = \frac{\lambda}{\lambda-\zeta}\{\psi_{\tilde{N}}(\lambda) - \psi_{\tilde{N}}(\zeta)\} \Rightarrow$$

$$\delta_{\tilde{N}}(\zeta)\eta(\lambda) = \frac{\lambda}{\lambda-\zeta}\{\psi_{\tilde{N}}(\lambda) - \psi_{\tilde{N}}(\zeta)\}\eta(\lambda) \tag{33}$$

For $\zeta = 0$ formula (33) coincides with eq. (23). Thus, we succeeded to assign to each generator $\tilde{N}$ of the conformal group a one-parameter family $\delta_{\tilde{N}}(\zeta)$, $\zeta \in C_+$, of infinitesimal transformations of solutions of the SDYM equations and of the solution η of the associated linear system. For each $\tilde{N} \in so(5,1)$ these transformations are new "hidden symmetries" of the SDYM equations.

5 Algebraic structure of the symmetries

In Sect.4 we described the infinitesimal symmetry transformations, the exponentiation of which acts on the set $\mathcal{M}$ of solutions of the SDYM equations (6). Let us consider any solution $\{B_\mu\} = \{B_y, B_z, B_{\bar{y}} = 0, B_{\bar{z}} = 0\}$ of these equations. Then the solutions $\delta_{\tilde{N}}(\zeta)B_\mu$ of the linearized SDYM equations describe the vector space tangent to the manifold $\mathcal{M}$ of solutions at the point $\{B_\mu\}$. So, the infinitesimal symmetries $\delta_{\tilde{N}}(\zeta)$ are vector fields on the manifold $\mathcal{M}$ and they define a map $B_\mu \to \delta_{\tilde{N}}(\zeta)B_\mu$.

Notice that we consider $\lambda, \zeta \in C_+$. We restrict our attention to only half of the symmetries. The rest will be obtained when we focus on $\mathcal{Z}_- = R^{4,0} \times C_-$, the linear system (11) for $\hat{\eta}$ on $\mathcal{Z}_-$ and the function $\hat{\psi}_{\tilde{N}}(x,\zeta) = \{\tilde{N}(\zeta)\hat{\eta}(\zeta)\}\hat{\eta}^{-1}(\zeta), \zeta \in C_-$. We recall that the function $\hat{\eta}$ is holomorphic and nonsingular for $\zeta \in C_-$ and therefore $\hat{\psi}_{\tilde{N}}(\zeta) = \sum_{n=0}^{\infty}\zeta^{-n}\hat{\psi}^n_{\tilde{N}}$. Then we can derive for $\hat{\psi}_{\tilde{N}}$ an equation analogous to Eq.(24) and introduce a second set of symmetry transformations $\hat{\delta}_{\tilde{N}}(\zeta)$ similar to (30) and (33) with $\hat{\psi}_{\tilde{N}}$ replacing $\psi_{\tilde{N}}$.

Let us discuss now the algebraic properties of the "on-shell" symmetry transformations $\delta_{\tilde{N}}(\zeta)$ that preserve the equations of motion. After a lengthy computation using the formulas of Sect.3 and Sect.4, we obtain the following expression for the commutator between two successive infinitesimal transformations:

$$[\delta_{\tilde{M}}(\lambda), \delta_{\tilde{N}}(\zeta)]B_y = \frac{1}{\lambda-\zeta}\{\lambda\delta_{[\tilde{M},\tilde{N}]}(\lambda) - \zeta\delta_{[\tilde{M},\tilde{N}]}(\zeta)\}B_y - \tag{34a}$$

$$-\frac{1}{\lambda\zeta(\lambda-\zeta)^2}\left\{\zeta^2\tilde{M}^\lambda\,(\lambda\delta_{\tilde{N}}(\lambda) - \zeta\delta_{\tilde{N}}(\zeta)) + \lambda^2\tilde{N}^\zeta\,(\lambda\delta_{\tilde{M}}(\lambda) - \zeta\delta_{\tilde{M}}(\zeta))\right\}B_y + \tag{34b}$$

$$+\frac{1}{\lambda-\zeta}\{\tilde{M}^{\zeta}\partial_{\zeta}(\zeta\delta_{\tilde{N}}(\zeta))+\tilde{N}^{\lambda}\delta_{\lambda}(\lambda\delta_{\tilde{M}}(\lambda)\}B_{y}+ \tag{34c}$$

$$+\{\frac{1}{\zeta}\tilde{N}^{y}_{,\bar{z}}\delta_{\tilde{M}}(\lambda)-\frac{1}{\lambda}\tilde{M}^{y}_{,\bar{z}}\delta_{\tilde{N}}(\zeta)\}B_{y}- \tag{34d}$$

$$-\frac{1}{\lambda-\zeta}\{\tilde{M}^{\zeta}_{,\bar{z}}\partial_{\zeta}\psi_{\tilde{N}}(\zeta)+\tilde{N}^{\zeta}_{,\bar{z}}\partial_{\lambda}\psi_{\tilde{M}}(\lambda)\}+ \tag{34e}$$

$$+\frac{\lambda\tilde{N}^{\zeta}_{,\bar{z}}}{\zeta(\lambda-\zeta)^{2}}\{\psi_{\tilde{M}}(\lambda)-\psi_{\tilde{M}}(\zeta)\}+\frac{\zeta\tilde{M}^{\lambda}_{,\bar{z}}}{\lambda(\lambda-\zeta)^{2}}\{\psi_{\tilde{N}}(\lambda)-\psi_{\tilde{N}}(\zeta)\}. \tag{34f}$$

Here $\tilde{N}^{y}, \tilde{N}^{\lambda}, \ldots$ are the components of any generator of $SO(5,1)$ of (15): $\tilde{N}=\tilde{N}^{y}\partial_{y}+\tilde{N}^{\bar{y}}\partial_{\bar{y}}+\tilde{N}^{z}\partial_{z}+\tilde{N}^{\bar{z}}\partial_{\bar{z}}+\tilde{N}^{\lambda}\partial_{\lambda}+\tilde{N}^{\bar{\lambda}}\partial_{\bar{\lambda}}$. The commutator $[\delta_{\tilde{M}}(\lambda),\delta_{\tilde{N}}(\zeta)]B_{z}$ looks similar to (34), except $\tilde{N}^{z}_{,\bar{y}}, \tilde{M}^{z}_{,\bar{y}}, \tilde{N}^{\zeta}_{,\bar{y}}$ and $\tilde{M}^{\zeta}_{,\bar{y}}$ replace $\tilde{N}^{y}_{,\bar{z}}$ etc. in lines (34d)-(34f).

It is obvious that because of the terms (34e) and (34f) the commutator (34) does not close in general. The terms (34e) and (34f) are nonzero when $\tilde{N}^{\lambda}_{,\bar{y}}\neq 0$ and $\tilde{N}^{\lambda}_{,\bar{z}}\neq 0$, but this holds only for the generators $\tilde{K}_{\mu}$ of special conformal transformations. Therefore, if at least one of the vector fields $\tilde{M}$ or $\tilde{N}$ coincides with one of the generators $\tilde{K}_{\mu}$, then the commutator of two symmetries is no longer a symmetry.

Now we consider the 8-dimensional algebra $\mathcal{A}$ with generators $\{P_{\mu},X_{a},D\}$ and the 11-dimensional algebra $\mathcal{B}$ with generators $\{P_{\mu},X_{a},D,\tilde{Y}_{a}\}$. Both algebras $\mathcal{A}$ and $\mathcal{B}$ are subalgebras of $so(5,1)$. For each generator N of the algebra $\mathcal{A}\subset\mathcal{B}$ we have $\tilde{N}=N$ ($\tilde{P}_{\mu}=P_{\mu}, \tilde{X}_{a}=X_{a}, \tilde{D}=D$), i.e. these generators have no components along the vector fields ∂_{λ} and $\partial_{\bar{\lambda}}$. Moreover, for all of them $\tilde{N}^{y}_{,\bar{z}}=\tilde{N}^{z}_{,\bar{y}}=0$ and hence all terms (34b)-(34f) are zero. Thus, for $\tilde{M},\tilde{N}\in\mathcal{A}$ eq. (34) reduces to

$$[\delta_{M}(\lambda),\delta_{N}(\zeta)]=\frac{1}{(\lambda-\zeta)}\{\lambda\delta_{[M,N]}(\lambda)-\zeta\delta_{[M,N]}(\zeta)\}, \tag{35}$$

which defines the analytic half of the Kac-Moody algebra $\mathcal{A}\otimes C(\lambda,\lambda^{-1})$, the affine extension of the algebra $\mathcal{A}\in so(5,1)$. Let us define the variations δ^{n}_{N} for all $n\geq 0$ by the contour integral

$$\delta^{n}_{N}=\oint_{C'}\frac{d\lambda}{2\pi i}\lambda^{-n-1}\delta_{N}(\lambda), \tag{36}$$

where the contour C' circles once around $\lambda=0$. We may choose $C'=C=C_{+}\cap C_{-}$. Using the definition (36) and the commutator (35), Cauchy's theorem allows us to deduce the commutators between half of the generators of the affine Lie algebra $\mathcal{A}\otimes C(\lambda,\lambda^{-1})$:

$$[\delta^{m}_{M},\delta^{n}_{N}]=\delta^{m+n}_{[M,N]},\quad m,n\geq 0. \tag{37}$$

For the generators $\{\tilde{Y}_{a}\}$ of the $so(3)$-subalgebra of the algebra $\mathcal{B}$ we have $\tilde{Y}^{\lambda}_{a,\bar{y}}=\tilde{Y}^{\lambda}_{a,\bar{z}}=0$, $\tilde{Y}^{z}_{a,\bar{y}}=const$, $\tilde{Y}^{y}_{a,\bar{z}}=const$ and thus the terms (34e) and (34f) are zero. Instead of $\{\tilde{Y}_{a}\}$ it is convenient to rewrite the generators as follows:

$$\tilde{Y}_{+}:=\tilde{Y}_{2}-i\tilde{Y}_{1}=2z\partial_{\bar{y}}-2y\partial_{\bar{z}}+2(Z_{2}-iZ_{1}),\ \tilde{Y}_{-}:=2\bar{z}\partial_{y}-2\bar{y}\partial_{z}+2(Z_{2}+iZ_{1}), \tag{38a}$$

$$\tilde{Y}_{0}:=-i\tilde{Y}_{3}=y\partial_{y}+z\partial_{z}-\bar{y}\partial_{\bar{y}}-\bar{z}\partial_{\bar{z}}-2iZ_{3}, \tag{38b}$$

$$[\tilde{Y}_{0},\tilde{Y}_{+}]=2\tilde{Y}_{+},\ [\tilde{Y}_{0},\tilde{Y}_{-}]=-2\tilde{Y}_{-},\ [\tilde{Y}_{+},\tilde{Y}_{-}]=-4\tilde{Y}_{0}. \tag{38c}$$

Using the explicit form (38) of the vector fields $\tilde{Y}_0, \tilde{Y}_\pm$ and eqs. (34) we obtain:

$$[\delta^m_{\tilde{Y}_0}, \delta^n_{\tilde{Y}_0}] = -4(m-n)\delta^{m+n}_{\tilde{Y}_0}, \ \delta^m_{\tilde{Y}_0} := \oint_C \frac{d\lambda}{2\pi i}\lambda^{-m-1}\delta_{\tilde{Y}_0}(\lambda), \tag{39a}$$

$$[\delta^m_{\tilde{Y}_+}, \delta^n_{\tilde{Y}_+}] = -4(m-n)\delta^{m+n-1}_{\tilde{Y}_+}, \ \delta^m_{\tilde{Y}_+} := \oint_C \frac{d\lambda}{2\pi i}\lambda^{-m-1}\delta_{\tilde{Y}_+}(\lambda), \tag{39b}$$

$$[\delta^m_{\tilde{Y}_-}, \delta^n_{\tilde{Y}_-}] = -4(m-n)\delta^{m+n+1}_{\tilde{Y}_-}, \ \delta^m_{\tilde{Y}_-} := \oint_C \frac{d\lambda}{2\pi i}\lambda^{-m-1}\delta_{\tilde{Y}_-}(\lambda),. \tag{39c}$$

From (39) one can see that $\delta^m_{\tilde{Y}_0}$, $\delta^m_{\tilde{Y}_+}$ and $\delta^m_{\tilde{Y}_-}$ generate three different Virasoro-like subalgebras of the symmetry algebra. Obviously these Virasoro-like subalgebras do not commute with each other. Using eq.(34), the contour integral definitions and Cauchy's theorem one finds

$$[\delta^m_{\tilde{Y}_0}, \delta^n_N] = \delta^{m+n}_{[\tilde{Y}_0,N]} + 4n\delta^{m+n}_N, \tag{40a}$$

$$[\delta^m_{\tilde{Y}_+}, \delta^n_N] = \delta^{m+n}_{[\tilde{Y}_+,N]} + 4n\delta^{m+n-1}_N, \tag{40b}$$

$$[\delta^m_{\tilde{Y}_-}, \delta^n_N] = \delta^{m+n}_{[\tilde{Y}_-,N]} + 4n\delta^{m+n+1}_N, \tag{40c}$$

$$[\delta^m_{\tilde{Y}_0}, \delta^n_{\tilde{Y}_+}] = \delta^{m+n}_{[\tilde{Y}_0,\tilde{Y}_+]} + 4n\delta^{m+n}_{\tilde{Y}_+} - 4m\delta^{m+n-1}_{\tilde{Y}_0}, \tag{40d}$$

$$[\delta^m_{\tilde{Y}_0}, \delta^n_{\tilde{Y}_-}] = \delta^{m+n}_{[\tilde{Y}_0,\tilde{Y}_-]} + 4n\delta^{m+n}_{\tilde{Y}_-} - 4m\delta^{m+n+1}_{\tilde{Y}_0}, \tag{40e}$$

$$[\delta^m_{\tilde{Y}_+}, \delta^n_{\tilde{Y}_-}] = \delta^{m+n}_{[\tilde{Y}_+,\tilde{Y}_-]} + 4n\delta^{m+n-1}_{\tilde{Y}_-} - 4m\delta^{m+n+1}_{\tilde{Y}_+}. \tag{40f}$$

Thus, the subset of symmetries of the SDYM equations with generators $\delta^m_{P_\mu}, \delta^m_{X_a}, \delta^m_D$ and $\delta^m_{\tilde{Y}_a}$ forms a Kac-Moody-Virasoro algebra with commutation relations (37), (39) and (40).

In [7] it has been shown that the linear system for the SDYM equations will be invariant under the action of the conformal group only if we add the combinations of the Virasoro generators $L_0 = -\lambda\partial_\lambda, L_1 = -\lambda^2\partial_\lambda$ and $L_{-1} = -\partial_\lambda$ to the generators Y_a and K_μ . Therefore, the appearance of a Virasoro-like algebra as an algebra of symmetries is not surprising. Beyond expectation the symmetry algebra contains *three* different Virasoro-like subalgebras with the generators $\delta^m_{\tilde{Y}_0}, \delta^m_{\tilde{Y}_+}$ and $\delta^m_{\tilde{Y}_-}$. In comparison with the previously known symmetries of the SDYM equations, these new symmetries are the affine extension not of gauge symmetries, but of space-time symmetries of the SDYM equations.

6 Off-shell Kac-Moody and Virasoro algebras

We will now proceed to define the "off-shell" action of the graded Lie algebra $so(5,1) \otimes C(\lambda)$ and the Virasoro algebra on the space of YM potentials and group-valued functions η. The action of the Virasoro algebra does not preserve the SDYM equations. As to the algebra $so(5,1) \otimes C(\lambda)$, only the action of the subalgebra $\mathcal{A} \otimes C(\lambda)$ of this algebra preserves eq. (6).

As usual, we assign to each vector field N the function $\psi_N(x,\zeta) = \{N\eta(\zeta)\}\eta^{-1}(\zeta)$ on $\mathcal{Z}_+$ and consider the following transformation of the YM potentials:

$$\delta_N(\zeta)B_y := \partial_y\psi_N(\zeta) + [B_y, \psi_N(\zeta)] + \mathcal{L}_N B_y, \quad \delta_N(\zeta)B_{\bar{y}} := \mathcal{L}_N B_{\bar{y}}, \tag{41a}$$

$$\delta_N(\zeta)B_z := \partial_z\psi_N(\zeta) + [B_z, \psi_N(\zeta)] + \mathcal{L}_N B_z, \quad \delta_N(\zeta)B_{\bar{z}} := \mathcal{L}_N B_{\bar{z}}. \tag{41b}$$

For $\eta(x,\lambda)$ we postulate the following transformation rule:

$$\delta_N(\zeta)\eta(\lambda) := \frac{\lambda}{\lambda-\zeta}\{\psi_N(\lambda)-\psi_N(\zeta)\}\eta(\lambda). \tag{41c}$$

Now it is not hard to show that

$$\begin{aligned}[\delta_M(\lambda),\delta_N(\zeta)]B_y = \ & \partial_y\{\delta_N(\zeta)\psi_M(\lambda)-\delta_M(\lambda)\psi_N(\zeta)\}+ \\ & +[B_y,\delta_N(\zeta)\psi_M(\lambda)-\delta_M(\lambda)\psi_N(\zeta)] \\ & +[\delta_N(\zeta)B_y,\psi_M(\lambda)]-[\delta_M(\lambda)B_y,\psi_N(\zeta)] \\ & +\mathcal{L}_M\{\delta_N(\zeta)B_y\}-\mathcal{L}_N\{\delta_M(\zeta)B_y\}\end{aligned} \tag{42}$$

and we have the same formula for B_z. From (41), (42) and

$$\delta_N(\zeta)\psi_M(\lambda)=\{M(\delta_N(\zeta)\eta(\lambda))\}\eta^{-1}(\lambda)+\{M\eta(\lambda))\}\delta_N(\zeta)\eta^{-1}(\lambda)$$

it follows that

$$[\delta_M(\lambda),\delta_N(\zeta)]=\frac{1}{(\lambda-\zeta)}\{\lambda\delta_{[M,N]}(\lambda)-\zeta\delta_{[M,N]}(\zeta)\}, \tag{43}$$

when we consider the action on $B_y, B_z, B_{\bar{y}}$ and $B_{\bar{z}}$, as well as η. By (41) we have for $B_{\bar{y}}$ or $B_{\bar{z}}$ simply

$$[\delta_M(\lambda),\delta_N(\zeta)]B_{\bar{y}}=\frac{1}{(\lambda-\zeta)}\{\lambda\delta_{[M,N]}(\lambda)-\zeta\delta_{[M,N]}(\zeta)\}B_{\bar{y}}=\mathcal{L}_{[M,N]}B_{\bar{y}}.$$

Using eq. (43), it is not difficult to deduce the commutators of half of the affine Lie algebra $\widehat{so}(5,1)$:

$$[\delta_M^m,\delta_N^n]=\delta_{[M,N]}^{m+n},\quad \delta_N^n=\oint_C\frac{d\lambda}{2\pi i}\lambda^{-n-1}\delta_N(\lambda),\quad m,n\geq 0. \tag{45}$$

If we consider also $\hat{\eta}$, $\hat{\psi}_N$ and generators $\hat{\delta}_N$, then we obtain (45) with $-\infty\leq m,n\leq+\infty$, i.e. the full affine extention $\widehat{so}(5,1)=so(5,1)\otimes C(\lambda,\lambda^{-1})$ of the conformal algebra $so(5,1)$.

Consider now the vector field $V=\lambda\partial_\lambda$ on C_+ and the Lie algebra-valued function

$$\psi_V(x,\lambda)=\{V(\lambda)\eta(\lambda)\}\eta^{-1}(\lambda). \tag{46}$$

Let us define for $\{B_\mu\}$ and η the following transformation rules

$$\delta(\zeta)B_y:=\partial_y\psi_V(\zeta)+[B_y,\psi_V(\zeta)],\quad \delta(\zeta)B_{\bar{y}}:=0, \tag{47a}$$

$$\delta(\zeta)B_z:=\partial_z\psi_V(\zeta)+[B_z,\psi_V(\zeta)],\quad \delta(\zeta)B_{\bar{z}}:=0, \tag{47b}$$

$$\delta(\zeta)\eta(\lambda):=\frac{\lambda}{\lambda-\zeta}\{\psi_V(\lambda)-\psi_V(\zeta)\}\eta(\lambda). \tag{47c}$$

One can verify that

$$[\delta(\lambda),\delta(\zeta)]B_y=\partial_y\{\delta(\zeta)\psi_V(\lambda)-\delta(\lambda)\psi_V(\zeta)\}+$$

$$+[B_y,\delta(\zeta)\psi_V(\lambda)-\delta(\lambda)\psi_V(\zeta)]+[\delta(\zeta)B_y,\psi_V(\lambda)]-[\delta(\lambda)B_y,\psi_V(\zeta)], \tag{48}$$

and similarly for $B_{\bar{z}}$. Using (47), (48) and

$$\delta(\zeta)\psi_V(\lambda) = \frac{\lambda}{\lambda-\zeta}\{V(\lambda)\psi_V(\lambda) + [\psi_V(\lambda), \psi_V(\zeta)]\} - \frac{\lambda\zeta}{(\lambda-\zeta)^2}\{\psi_V(\lambda) - \psi_V(\zeta)\}$$

we obtain the commutator

$$[\delta(\lambda), \delta(\zeta)] = \frac{1}{(\lambda-\zeta)}\{\lambda^2\partial_\lambda\delta(\lambda) + \zeta^2\partial_\zeta\delta(\zeta)\} - \frac{2\lambda\zeta}{(\lambda-\zeta)^2}\{\delta(\lambda) - \delta(\zeta)\} \tag{49}$$

when we consider the action on η or on any component of B_μ.

Let us define

$$L^n = -\frac{1}{2}\delta^n = -\frac{1}{2}\oint_C \frac{d\lambda}{2\pi i}\lambda^{-n-1}\delta(\lambda). \tag{50}$$

Now, using (49) and (50), it is not hard to show that we obtain half of the Witt algebra:

$$[L^m, L^n] = (m-n)L^{m+n}, \quad m, n \geq 0. \tag{51}$$

We shall obtain the full centerless Virasoro algebra, i.e. the Witt algebra, if we consider $\hat{\eta}(\zeta)$ with $\zeta \in C_-$ and extend all the calculations appropriately.

Finally, we write out the formula for the commutator between the generators L^m of the Virasoro algebra and the generators δ_N^n of the Kac-Moody algebra $\widehat{so}(5,1)$:

$$[\delta(\lambda), \delta_N(\zeta)] = \frac{\zeta^2}{(\lambda-\zeta)}\partial_\zeta\delta_N(\zeta) - \frac{\lambda\zeta}{(\lambda-\zeta)^2}\{\delta_N(\lambda) - \delta_N(\zeta)\} \ \Rightarrow \tag{52a}$$

$$[L^m, \delta_N^n] = -n\delta_N^{m+n}. \tag{52b}$$

Thus we have defined the off-shell action of the Kac-Moody algebra $\widehat{so}(5,1)$ and of the Virasoro algebra on the space of YM potentials and on the group-valued functions $\eta(\lambda)$.

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