DIRICHLET'S ENERGY AND THE NIELSEN REALIZATION PROBLEM

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Abstract

Dirichlet's energy function on Teichmüller space is used to give a solution to the Nielsen realization problem. In particular we show that Dirichlet's energy is convex along Weil-Petersson geodesics.

We shall prove the following result originally due to Kerckoff [7].

Theorem (Main) Let $\mathcal{G}$ be any finite subgroup of a group $\mathcal{D}/\mathcal{D}_0$, the surface modular group of Teichmüller space $\mathcal{T}(M)$. Then the action of $\mathcal{G}$ on $\mathcal{T}(M)$ has a fixed point.

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Let M be an oriented compact surface without boundary and with genus greater than one. Let $\mathcal{A}$ be the space of almost complex structures on M compatible with its orientation and let $\mathcal{D}_0$ be the space of all diffeomorpisms of M homotopic to the identity. Then [3], [4], [5] Teichmüller space is defined to be the quotient $\mathcal{A}/\mathcal{D}_0$, where $\mathcal{D}_0$ acts on $\mathcal{A}$ by pull back. In [3] it is shown that $\mathcal{T}(M)$ has the structure of a 6(genus M) - 6 C^∞ smooth manifold. If $\mathcal{M}_{-1}$ denotes the infinite dimensional Fréchet manifold of Riemannian metrics of constant curvature -1, then $\mathcal{D}_0$ acts naturally on $\mathcal{M}_{-1}$ and $\mathcal{T}(M)$ is diffeomorphic to $\mathcal{M}_{-1}/\mathcal{D}_0$.

This diffeomorphism is described as follows (for details see [3], [8]: There is a natural $\mathcal{D}$-invariant diffeomorphism $\Phi : \mathcal{M}_{-1} \to \mathcal{A}$ given by

$$\Phi(g) = -g^{-1}\mu_g$$

where $\mathcal{M}_g$ is the volume element of g. Φ then passes to a diffeomorphism $\bar{\Phi}$ from $\mathcal{M}_{-1}/\mathcal{D}_0$ to $\mathcal{A}/\mathcal{D}_0$. Let $\theta : \mathcal{A} \to M_{-1}$ be the inverse of Φ. For $J \in \mathcal{A}$, $\theta(J)$ is the unique Poincaré metric associated to J. Denote by $\bar{\theta}$ the induced diffeomorphism from $\mathcal{A}/\mathcal{D}_0$ to $\mathcal{M}_{-1}/\mathcal{D}_0$. We also have a natural $\mathcal{D}_0$ invariant metric on A given by

$$\langle\langle H,K\rangle\rangle = \frac{1}{2}\int_M \operatorname{tr}(HK)d\mu_{\Phi(J)}$$

and a natural L_2 splitting [8] of $T_J\mathcal{A}$, namely each $H \in T_J\mathcal{A}$ can be uniquely decomposed as

$$(1) \qquad H = H^{TT} + L_X J$$

where $L_X J$ is the Lie derivative of J w.r.t. the vector field X on M, and H^{TT} denotes a (1,1) tensor which is trace free and divergence free w.r.t. $\theta(J)$. The decomposition (1.1) is L_2-orthogonal. Since $\mathcal{D}_0$ acts as a group of isometries $\langle\langle,\rangle\rangle$ passes to a metric $\langle,\rangle$ on $\mathcal{T}(M) = \mathcal{A}/\mathcal{D}_0$ described as follows. The term $L_X J$ is always tangent to the orbit of $\mathcal{D}_0$ through J. We say that $L_X J$ is the <u>vertical</u> part of $H \in T_J A$ in the decomposition (1.1). Similarly we say that H^{TT} represents the <u>horizontal</u> part of H. Let $\pi : \mathcal{A} \to \mathcal{A}/D_0$ be the natural projection map. Given $H,K \in T_{[J]}\mathcal{A}/\mathcal{D}_0$ there are unique horizontal vectors $\tilde{H},\tilde{K} \in T_J\mathcal{A}$ such that $D\pi(J)K = K$. Then

$$(2) \qquad \langle H,K\rangle_{[J]} = \langle\langle\tilde{H},\tilde{K}\rangle\rangle_J .$$

Let us now consider the model $\mathcal{M}_{-1}/\mathcal{D}_0$ of $\mathcal{T}(M)$. The tangent space of $\mathcal{M}_{-1}$ at a metric, $g \in T_g\mathcal{M}_{-1}$ consists of those (0,2) tensors h on M satisfying the equation

$$(3) \qquad -\Delta(\operatorname{tr}_g h) + \delta_g\delta_g h + \frac{1}{2}(\operatorname{tr}_g h) = 0$$

where $tr_g h = g^{ij}h_{ij}$ is the trace of h w.r.t. the metric tensor g_{ij}, $\delta_g \delta_g h$ is the double covariant divergence of h w.r.t. g and Δ is the Laplace-Beltrami operator on functions. For example see [8] for details. The L_2-metric on $\mathcal{M}_{-1}$ is given by the inner product

$$(4) \qquad \langle\langle h,k\rangle\rangle_g = \frac{1}{2}\int_M \text{trace } (HK)d\mu_g$$

where $H = g^{-1}h$, $K = g^{-1}k$ are the (1,1) tensors on M obtained from h and k via the metric g, or "by raising an index", i.e.

$$H^i_j = g^{ik}h_{kj}$$

and similarly for K.

The inner product (1.4) is $\mathcal{D}_0$ invariant. Thus $\mathcal{D}_0$ acts smoothly on $\mathcal{M}_{-1}$ as a group of isometries with respect to this metric, and consequently we have an induced metric on $\mathcal{T}(M)$ in such a way that the projection map $\pi : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$ becomes a Riemannian submersion [3]. In [4] it is shown that this induced metric is precisely the metric originally introduced by Weil, now called the Weil-Petersson metric.

Let $\langle,\rangle$ be the induced metric on $\mathcal{T}(M)$. We can characterize $\langle,\rangle$ as follows. From [3] we can show that given $g \in \mathcal{M}_{-1}$ every

$$(5) \qquad h = h^{TT} + L_X g$$

where $L_X g$ is the Lie derivative of g w.r.t. some (unique X) and h^{TT} is a trace free, divergence free, symmetric tensor. Moreover the decomposition (1.5) is L_2-orthogonal. Recall that a conformal coordinate system (where $g_{ij} = \lambda\delta_{ij}$, λ some smooth positive function) is also a complex holomorphic coordinate system. In this system

$$h^{TT} = \text{Re}(\xi(z)dz^2)$$

where Re is "real part" and $\xi(z)dz^2$ is a holomorphic quadratic differential. In fact, trace free, divergence free symmetric two tensors are precisely the real parts of holomorphic quadratic differentials.

Now $L_X g$ is always tangent to the orbit of $\mathcal{D}_0$ through g. We say that $L_X g$ is the <u>vertical</u> part of h in decomposition 1.4. Similarly we say that h^{TT} represents the <u>horizontal</u> part of h. Given $h, k \in T_{[g]}\mathcal{T}(M)$ there are unique horizontal vectors $\tilde{h}, \tilde{k} \in T_g \mathcal{M}_{-1}$ such that $D\pi(g)\tilde{h} = h$ and $D\pi(g)\tilde{k} = k$. Then

$$\langle h, k \rangle_{[g]} = \langle\langle \tilde{h}, \tilde{k} \rangle\rangle_g .$$

Suppose now that $g_0 \in \mathcal{M}_{-1}$ is fixed and that $s : (M,g) \to (M,g_0)$ is a smooth C^1 map homotopic to the identity and is viewed as a map from M with some aribtrary metric $g \in \mathcal{M}_{-1}$ to M with its g_0 metric.

Define the Dirichlet energy of s by the formula

$$E_g(s) = \frac{1}{2} \int_M |ds|^2 d\mu_g \tag{6}$$

where $\to ds \to^2 = \text{trace } ds \otimes ds$ depends on both g and g_0.

By the embedding theorem of Nash-Moser we may assume that (M,g_0) is isometrically embedded in some Euclidean $\mathbf{R}^K$. Thus we can think of $s : (M,g) \to (M,g_0)$ as a map into $\mathbf{R}^K$ and Dirichlet's functional takes the equivalent form

$$E_g(s) = \frac{1}{2} \sum_{i=1}^{k} \int g(x) < \nabla_g s^i(x), \nabla_g s^i(x) > d\mu_g . \tag{7}$$

There is another, equivalent, and useful way to express (1.5) and (1.8) using local conformal cordinate systems $g_{ij} = \lambda\delta_{ij}$ and $(g_0)_{ij} = \rho\delta_{ij}$ on (M,g) and (M,g_0) respectively, namely

(8) $$E_g(s) = \frac{1}{4}\int_M [\rho(s(z))|s_z|^2 + \rho(s(z))|s_{\bar z}|^2]dzd\bar z$$

For fixed g, the critical points of E_g are then said to be harmonic maps. The following result is due to Eells-Sampson, Hartman and Schoen-Yau [2], [10].

Theorem 10. Given metrics g and g_0, with $g_0 \in \mathcal{M}_{-1}$ there exists a unqiue harmonic map $s(g) : (M,g) \to (M,g_0)$ which is homotopic to the identity, and is the absolute minimum for E_g. Moreover $s(g)$ depends differentialy on g in any H^r topology, $r > 2$, and is a C^∞ diffeomorphism.

Consider now the function

$$g \to E_g(s(g)).$$

This function on $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant and thus can be viewed as a function on Teichmüller space. To see this one must show that

$$E_{f^*g}(s(f^*(g))) = E_g(s(g)).$$

Let $c(g)$ be the complex structure associated to g, and induced by a conformal coordinate system for g. For $f \in \mathcal{D}_0$, $f : (M,f^*c(g)) \to (M,c(g))$ is holomorphic and consequently since the composition of harmonic maps and holomorphic maps is still harmonic we may conclude, by uniqueness that

$$s(f^*g) = s(g) \circ f .$$

Since Dirichlet's functional is invariant under complex holomorphic changes of coordinates it follows immediately that

$$E_{f^*(g)}(s(g) \circ f) = E_g(s(g)).$$

Consequently for $[g] \in \mathcal{M}_{-1}/\mathcal{D}_0$ define the C^∞ smooth function

$$\tilde{E} : \mathcal{M}_{-1}/\mathcal{D}_0 \to R$$

by

$$\tilde{E}[g] = E_g(s(g)).$$

In [9] we prove the following

<u>Theorem 11</u> If $s : (M,g) \to (M,g_0)$ is <u>harmonic</u> the form $\xi(z)dz^2$ is a holomorphic quadratic differential on the complex curve $(M,c(g_0))$, and thus $\mathrm{Re}\ \xi(z)dz^2$ represents a trace free, divergence free symmetric two tensor on (M,g). Hence $\mathrm{Re}\ \xi(z)dz^2$ is a horizontal tangent vector to $\mathcal{M}_{-1}$ at g. In addition

$$(12) \qquad D\tilde{E}[g]h = -\frac{1}{2}\langle\langle \mathrm{Re}\ \xi(z)dz^2, \tilde{h}\rangle\rangle_g = -\frac{1}{2}\sum_{\ell}\int_M g(x)(\tilde{H}\nabla_g s^\ell, \nabla_g s^\ell)d\mu_g$$

where $\tilde{h}$ is the horizontal lift of $h = T_{(g)}\mathcal{T}(M)$ and $\tilde{H} = (\tilde{h})^\#$ is obtained from h by raising an index via g.

Finally $[g_0]$ is the only critical point of $\tilde{E}$. The Hessian of $\tilde{E}$ at $[g_0]$ is given by

$$(13) \qquad D^2\tilde{E}[g_0](h,k) = \langle h,k\rangle$$

$h,k \in T_{[g_0]}\mathcal{T}(M)$. That is, the second variation of Dirichlet's energy function is the Weil-Petersson metric.

Suppose we look at the first derivative 1.12 in conformal coordinates $(g)_{ij} = \lambda\delta_{ij}$. Then if $\tilde{h}$ is horizontal

$$2\,\frac{\partial E}{\partial g}(g,s)\tilde{h} = -\int \langle h^{\#}\nabla s^{\ell}, \nabla s^{\ell}\rangle_{R^2} dxdy$$

$$= -\int \frac{1}{\lambda}(\tilde{h}_{11}(\frac{\partial s^{\ell}}{\partial x})^2 + 2\tilde{h}_{12}(\frac{\partial s^{\ell}}{\partial x})(\frac{\partial s^{\ell}}{\partial y}) + \tilde{h}_{22}(\frac{\partial s^{\ell}}{\partial x})^2) dxdy \quad .$$

where $h^{\#} = \frac{1}{\lambda}(h_{ij})$. Since $\tilde{h}_{11} = -\tilde{h}_{22}$ this is equal to

$$-\int \frac{1}{\lambda}(\tilde{h}_{11}[(\frac{\partial s^{\ell}}{\partial x})^2 - (\frac{\partial s^{\ell}}{\partial y})^2 + 2\tilde{h}_{12}(\frac{\partial s^{\ell}}{\partial x})\ (\frac{\partial s^{\ell}}{\partial y}))dxdy.$$

Now

$$(\frac{\partial s^{\ell}}{\partial y} - i\,\frac{\partial s^{\ell}}{\partial y})^2(dx + dy)^2 = \xi(z)dz^2$$

is a quadratic differential. But

$$\mathrm{Re}(\xi(z)dz^2) = [(\frac{\partial s^{\ell}}{\partial x})^2 - (\frac{\partial s^{\ell}}{\partial y})^2]dx^2 + [(\frac{\partial s^{\ell}}{\partial y})^2 - (\frac{\partial s^{\ell}}{\partial y})^2]dy^2 + 4(\frac{\partial s^{\ell}}{\partial x})(\frac{\partial s^{\ell}}{\partial y})dxdy$$

If s is harmonic $\mathrm{Re}(s(z)dz^2)$ is a trace free divergence free tensor. In general the second derivative of $\tilde{E}$ at an arbitrary $[g]$ will not be intrinsic. However we can ask for the second derivative of the function $g \mapsto E_g(s(g)) = \hat{E}(g)$. (For $g \in \mathcal{M}$, the space of all Riemannian metrics it still follows from [2], [11] that E_g has a unique minimum $s(g)$ which depends differentiably on g). This was computed in [9]. Thus we have

<u>Theorem 14</u> For arbitrary k

$$D^2\hat{E}(g)k = -\frac{1}{2}\sum_{\ell}\int_M g(x)(K_T\nabla_g s^{\ell}, \nabla_g s^{\ell})d\mu_g$$

where $K = (k)^{\#}$ and K_T is the trace free parts of K. For h and k trace free we have

$$D^2\hat{E}(g)(h,k) = \frac{1}{2}\sum_{\ell}\int_M (h \cdot k)g(x)(\nabla_g s^\ell, \nabla_g s^\ell)d\mu_g$$

$$- \sum_{\ell}\int_M g(x)(h^\# \cdot \nabla_g s^\ell, \nabla w^\ell(k))d\mu_g$$

where

$$h \cdot k = g^{ab}g^{cd}h_{ac}k_{bd} \tag{5}$$

$$= tr(HK)$$

$H = h^\#$, $K = k^\#$ the (1.1) tensors obtained from h and k by raising an index and

$w^\ell(k) = Ds^\ell(g)k$, the derivative of $s(g)$ in the direction k.

The following lemma whose proof can be found in [10] will be of importance to us..

Lemma 15

For h trace free, $D^2\hat{E}(g)(h,h) > 0$.

§2 Weil-Petersson Geodesics and the Nielsen Realizaition Problem

Let $\sigma(t)$ be a geodesic on Teichmüller space $\mathcal{T}(M)$. We can lift $\sigma(t)$ be a smooth path $\tilde{\sigma}(t)$ in $\mathcal{M}_{-1}$ with the property that $\tilde{\sigma}'(t)$ is horizontal for each t.

We know that $\mathcal{M}_{-1} \subset \mathcal{M}$ the space of all metrics which itself is an open subset of the space of all symmetric tensors S_2. Thus every second derivative $\sigma''(t)$ can be thought of as an element of S_2. Let $S_2^{TT}(\sigma)$ be the space of trace free divergence free symmetric two tensors and let

$$\Pi_{\tilde{\sigma}} : S_2 \to S_2^{TT}(\tilde{\sigma})$$

be the L_2-orthogonal projection.

Then as usual we see that σ is a geodesic iff $\Pi_{\tilde{\sigma}}\tilde{\sigma}''(t) = 0$.

We are now ready to prove:

Theorem 17 (Geodesic convexity of $\tilde{E}$) Let $\tilde{E} : \mathcal{T}(M) \to R$ be Dirichlet's energy, and $\sigma(t)$ be a geodesic with respect to the Weil-Petersson metric. Then

$$\frac{d^2\tilde{E}}{dt^2}(\sigma(t)) > 0$$

Proof. It clearly suffices to show that

$$\frac{d^2\hat{E}}{dt^2}(\tilde{\sigma}(t)) > 0.$$

But

$$\frac{d^2\hat{E}}{dt^2}(\tilde{\sigma}(t)) = D\hat{E}_{\sigma(t)}\tilde{\sigma}''(t) + D^2\hat{E}_{\sigma(t)}(\sigma',\sigma').$$

By formula (14) and lemma (15) the second term is strictly positive. By (12), it follows that the first term is equal to

$$(18)\qquad -\frac{1}{2}\langle\langle \mathrm{Re}\ \xi(z)dz^2, \{\tilde{\sigma}''(t)\}_T\rangle\rangle$$

$$= -\frac{1}{2}\langle\langle \mathrm{Re}\ \xi(z)dz^2, \tilde{\sigma}''(t)\rangle\rangle$$

$$+\frac{1}{2}\langle\langle \mathrm{Re}\ \xi(z)dz^2, \mu g\rangle\rangle$$

$g = \tilde{\sigma}(t)$, $\mu = \frac{1}{2}\,\mathrm{tr}_g\{\tilde{\sigma}''(t)\}$, $\{\tilde{\sigma}''(t)\}_T$ is the trace free part of $\{\tilde{\sigma}''(t)\}$, and $\xi(z)dz^2$ is the holomorphic quadratic differential associated to $\tilde{\sigma}'(t)$. Since $\Pi_\sigma \sigma'' = 0$,

$$\langle\langle \mathrm{Re}\ \xi(z)dz^2, \tilde{\sigma}''(t)\rangle\rangle = 0.$$

Furthermore since $\mathrm{Re}\ \xi(z)dz^2$ is trace free it is pointwise orthogonal to μg which implies that

$$\langle\langle \mathrm{Re}\ \xi(z)dz^2, \mu g\rangle\rangle = 0$$

This concludes 17.

We are now ready to prove our main

<u>Theorem (Main).</u> Let $\mathcal{G}$ be any finite subgroup of the surface modular group $\mathfrak{D}/\mathfrak{D}_0$. Then the action of $\mathcal{G}$ on $\mathcal{T}(M)$ has a fixed point.

<u>Proof</u> Since $\mathfrak{D}$ acts on $\mathcal{M}_{-1}$ as a group of isometries with respect to the L_2-metric it follows that $\mathcal{G}$ acts on $\mathcal{T}(M)$ as a group of isometries with respect to the Weil-Petersson metric.

$\mathcal{Y}$ also acts on Dirichlet's functional in the obvious way, namely if $f \in \mathcal{Y}$

$$f^{\#}\tilde{E}(g) = \tilde{E}(f^{\#}g) = E_{f^{\#}g}(s(f^{\#}g)).$$

Since the action of $\mathcal{D}_0$ leaves E invariant we may view this action as an action of a finite subset of $\mathcal{D}$. Define a new functional

$$\mathcal{F} : \mathcal{T}(M) \to R$$

by

$$\mathcal{F}(g) = \frac{1}{|\mathcal{Y}|} \sum_{f \in \mathcal{Y}} f^{\#}\tilde{E}(g)$$

where $|\mathcal{Y}|$ is the order of $\mathcal{Y}$. $\mathcal{F}$ is clearly $\mathcal{Y}$ invariant. Since $\tilde{E} : \mathcal{T}(M) \to R^+$ is proper it follows that $\mathcal{F} : \mathcal{T}(M) \to R^+$ is also proper. Thus $\mathcal{F}$ has a minimum point. The action of $\mathcal{Y}$ clearly permutes the minima of $\mathcal{F}$. By the geodesic convexity of $\tilde{E}$ it follows that $\mathcal{F}$ is geodesically convex, i.e.

$$\frac{d^2\mathcal{F}}{dt^2}(\sigma(t)) > 0.$$

Thus any critical point of $\mathcal{F}$ must be a non-degenerate minimum. Since Teichmüller space is a cell this implies that there is a unique minimum for $\mathcal{F}$ which must therefore be fixed by $\mathcal{Y}$.

Q.E.D

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