Vertex Operators in Hilbert Space

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Abstract: We introduce free vertex operators $V(\gamma, z)$ as operators in Hilbert space and prove: Vertex operators are densely defined for $|z| < 1$. Any radially ordered product of vertex operators is defined on a dense subset as an operator product. We bound $V(\gamma, z)$ by $\exp(cN)$ where N is the number operator and apply this result to screened vertex operators.

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1 Introduction

Free vertex operators are defined as expressions of the following type:

$$\tilde{V}(\gamma, z) = \exp\left(\gamma \sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^n\right) \exp\left(-\gamma \sum_{n=1}^{\infty} \frac{a_n}{n} z^{-n}\right), \tag{1}$$

where $\{a_n, a_{-n} : n \in \mathbf{N}\}$ are elements of a Heisenberg algebra with relations $[a_n, a_m] = n\delta_{n,-m}, \gamma \in \mathbf{C}$ and z is an indeterminate. Also formal expressions like

$$\begin{aligned} &\tilde{V}(\gamma_1, z_1, \ldots, \gamma_r, z_r) = \\ &= \exp\left(\sum_{n=1}^{\infty} (\gamma_1 z_1^n + \ldots + \gamma_r z_r^n) \frac{a_{-n}}{n}\right) \exp\left(-\sum_{n=1}^{\infty} \left(\gamma_1 z_1^{-n} + \ldots + \gamma_r z_r^{-n}\right) \frac{a_n}{n}\right) \end{aligned} \tag{2}$$

are usually considered. We introduce the following notations. Let $\omega = (\omega_1, \omega_2, \ldots)$ be an infinite sequence of functions all depending on the same set of variables $\gamma_1, z_1, \ldots, \gamma_r, z_r$ for some integer r, i.e. $\omega_i \equiv w_i(\gamma_1, z_1, \ldots, \gamma_r, z_r)$. Define

$$\omega_i^+(\gamma_1, z_1, \ldots, \gamma_r, z_r) = -\omega_i\left(\gamma_1, z_1^{-1}, \ldots, \gamma_r, z_r^{-1}\right). \tag{3}$$

We want to study

$$\tilde{V}(\omega) = \exp\left(\sum_{n=1}^{\infty} \omega_n \frac{a_{-n}}{\sqrt{n}}\right) \exp\left(\sum_{n=1}^{\infty} \omega_n^+ \frac{a_n}{\sqrt{n}}\right). \tag{4}$$

The two examples above are given by

$$\omega_i(\gamma, z) = \gamma \frac{z^i}{\sqrt{i}} \quad \text{and} \quad \omega_i(\gamma_1, z_1, \ldots, \gamma_r, z_r) = \frac{1}{\sqrt{i}}\left(\gamma_1 z_1^i + \ldots + \gamma_r z_r^i\right). \tag{5}$$

Remark that we denote by $\tilde{V}$ the formal object whereas V is reserved for the operators. We omit the dependence of $\tilde{V}$ on a_0 and T_γ (see [1]) with relations $[a_0, a_n] = 0$ and $[T_\gamma, a_n] = \delta_{n,0}\gamma T_\gamma$, which are unimportant from a functional analytic point of view. All properties we prove here hold even in the case when these dependencies are included.

The objects we have introduced up to now could be understood as formal Laurent series in z_i (or ω_i) with coefficients in the Heisenberg algebra. In this paper we want to define $\tilde{V}$ as an operator acting in the Hilbert space which is the Fock space $\mathcal{F}$ defined as the irreducible vacuum vector representation of the Heisenberg algebra. Moreover we will show the following properties:

(i) $\tilde{V}(\omega)$ defines an operator in $\mathcal{F}$ which is at least defined on the finite elements of $\mathcal{F}$ (and hence densely defined). In section 5 prove that in fact $D(V(\omega))$ contains a much bigger domain.

(ii) Any product of vertex operators is well defined in the operator sense on the finite elements of $\mathcal{F}$.

(iii) $V(\omega)$ could be estimated by what is called a field or a "Φ-Bound": let $N = \sum_{n=1}^{\infty} a_{-n}a_n$ be the Number operator in $\mathcal{F}$. Then we prove $||V(\omega)f|| \leq c_1||\exp\,(c_2N)f||$ for some constants $c_1, c_2 > 0$ and for all $f \in D(\exp(c_2N))$. $\Phi-$bounds were used in constructive field theory [2] for proving the self-adjointness of the (Minkowski) fields. In the present context they should be of interest for looking at the Hamiltonian approach to two-dimensional conformal quantum field theory.
(iv) We give an application of (iii) to screened vertex operators as introduced in [1]. We can define screened vertex operators on $D(e^{cN})$ for some constant $c > 0$ if the products $\gamma_i\gamma_j$ have positive real part for all $i < j$.

2 Definitions

Let $\{a_n, a_{-n} : n \in \mathbf{N}\} \cup I$ be the generators of the Heisenberg algebra with commutation relations $[a_n, a_m] = \delta_{n,-m}nI$ and let $\mathcal{F}$ be the unique Hilbert space on which $a_n, n > 0$ acts as annihilation and $a_{-n}, n > 0$ acts as creation operator, the relation $a_n^* = a_{-n}$ holds and which is generated by the vacuum vector Φ_0. We use the canonical basis of $\mathcal{F}$ which is given by Φ_0 and the vectors

$$\Phi_\alpha = (\alpha! I^\alpha)^{-1/2} a_{-k}^{\alpha_k} \cdots a_{-1}^{\alpha_1} \Phi_0, \tag{6}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots)$ is an infinite multiindex where all but a finite number of α_i's are zero, $\alpha! = \prod \alpha_i!$ and $I^\alpha = \prod i^{\alpha_i}$. We set $||\alpha|| = \sum i\alpha_i$ and $\mathcal{F}_0 = \mathrm{Lin}\{\Phi_\alpha : ||\alpha|| < \infty\}$. The operators $a_{\pm n}$ are uniquely determinated by their action on the basis which is given by

$$a_n\Phi_\alpha = \sqrt{n\alpha_n}\,\Phi_{\alpha-f_n}, \text{ and } a_{-n}\Phi_\alpha = \sqrt{n(\alpha_n+1)}\,\Phi_{\alpha+f_n}, \tag{7}$$

where f_n is the multiindex which is one in the $n-$th entry and zero otherwise. The domain of definition of a_n is given by

$$D(a_{\pm n}) = \left\{ \Psi = \sum_\alpha c_\alpha \Phi_\alpha : \sum_\alpha (\alpha_n + 1)|c_\alpha|^2 < \infty \right\}. \tag{8}$$

Let $(a_{i,j})$ be an infinite matrix, let H be a Hilbert space with basis $\{e_i : i \in \mathbf{N}\}$. Then (see for instance [3])

$$D(A) = \left\{ f \in H : \lim_{k\to\infty} \sum_{j=1}^{k} a_{i,j} < e_j, f > \text{ exists } \sum_{i=1}^{\infty} | \sum_{j=1}^{\infty} a_{i,j} < e_j, f > |^2 < \infty \right\} \tag{9}$$

and

$$Af = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{i,j} < e_j, f > \right) e_i \quad \text{for } f \in D(A) \tag{10}$$

defines a linear operator A in H with domain of definition $D(A)$.

3 Definition of vertex operators in $\mathcal{F}$

We want to define vertex operators via matrix elements, i.e. we assign to $\tilde{V}(\omega)$ a set of matrix elements, which defines a matrix operator in $\mathcal{F}$. The matrix elements with respect to the basis $\{\Phi_\alpha\}$ can be calculated using the following well known commutation relations for $\tilde{V}(\omega)$: For any $k \in \mathbb{N}$

$$\left[a_k, \tilde{V}(\omega)\right] = \frac{\omega_k}{\sqrt{k}}\tilde{V}(\omega) \quad \text{and} \quad \left[a_{-k}, \tilde{V}(\omega)\right] = -\frac{\omega_k^+}{\sqrt{k}}\tilde{V}(\omega). \tag{11}$$

We only state the result.

Lemma 1

$$\begin{aligned} < \Phi_\alpha, \tilde{V}(\omega)\Phi_\beta > &= \frac{1}{\sqrt{\alpha!\beta!}} \prod_{i=1}^{\infty} \sum_{j=0}^{min(\alpha_i,\beta_i)} \binom{\alpha_i}{j}\binom{\beta_i}{j} j! w_i^{\alpha_i - j}\left(\omega_i^+\right)^{\beta_i - j} \\ &= \frac{1}{\sqrt{\alpha!\beta!}} \prod_{i=1}^{\infty} f_{\alpha_i,\beta_i}\left(\omega_i, \omega_i^+\right) =: v_{\alpha,\beta}(\omega) \end{aligned} \tag{12}$$

with

$$f_{n,m}(x,y) = \sum_{j=0}^{min(n,m)} \binom{n}{j}\binom{m}{j} j! x^{n-j} y^{m-j}.$$

The infinite product above is well defined since only a finite number of α_i's and β_i's are different from zero and hence only a finite number of factors are different from one. Remark that $f_{n,m}(x,y) = f_{m,n}(y,x)$ and $f_{0,m}(x,y) = y^m$.

We denote by $V(\omega)$ the operator defined by these matrix elements in $\mathcal{F}$ via (9) and (10). As a first step we prove:

Theorem 1 *Let* $\sum_{i=1}^{\infty} |\omega_i|^2 < \infty$. *Then we have:*

$\mathcal{F}_0 \subset D(V(\omega))$, *hence,* $V(\omega)$ *is a densely defined operator.*

To prove this, we make use of the following helpful identity, that encodes the properties of products of vertex operators (which should be clearer later on).

Lemma 2 *For all* $x, y, z, w \in \mathbb{C}, i, j \in \mathbb{N}_0$ *the following identity holds (in the sense that both sides define the same entire function of four variables).*

$$\sum_{k=0}^{\infty} \frac{1}{k!} f_{i,k}(x,y) f_{k,j}(z,w) = f_{i,j}(x+z, y+w)e^{zy}. \tag{13}$$

A proof of Lemma 2 will be given in the appendix.

Proof of Theorem 1: We have to show $\Phi_\beta \in D(V(\omega))$ for all $||\beta|| < \infty$, i.e.

$$\sum_{\alpha:||\alpha||<\infty} |v_{\alpha,\beta}(\omega)|^2 < \infty \quad \text{for all} \quad ||\beta|| < \infty.$$

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Now:

$$\begin{aligned}\sum_\alpha |v_{\alpha,\beta}(\omega)|^2 &= \frac{1}{\beta!}\sum_{\alpha:\|\alpha\|<\infty}\frac{1}{\alpha!}\prod_{i=1}^{\infty}|f_{\alpha_i,\beta_i}(\omega_i,\omega_i^+)|^2 \\ &\leq \frac{1}{\beta!}\prod_{i=1}^{\infty}\sum_{\alpha_i=0}^{\infty}\frac{1}{\alpha!}f_{\beta_i,\alpha_i}(|\omega_i^+|,|\omega_i|)\,f_{\alpha_i,\beta_i}(|\omega_i|,|\omega_i^+|),\end{aligned} \tag{14}$$

where we have used $|f_{n,m}(x,y)| \leq f_{n,m}(|x|,|y|)$ and the above mentioned symmetry of f. The application of Lemma 2 gives

$$\begin{aligned}\sum_\alpha |v_{\alpha,\beta}(\omega)|^2 &\leq \frac{1}{\beta!}\prod_{i=1}^{\infty}\left(f_{\beta_i,\beta_i}(|\omega_i|+|\omega_i^+|,|\omega_i|+|w_i^+|)\exp\left(|\omega_i|^2\right)\right) \\ &= \frac{1}{\beta!}\prod_{i=1}^{\infty}\left(f_{\beta_i,\beta_i}(|\omega_i|+|\omega_i^+|,|\omega_i|+|w_i^+|)\right)\exp\left(\sum_{i=1}^{\infty}|\omega_i|^2\right) < \infty.\end{aligned} \tag{15}$$

In the last step once again we have used the fact that for any β with $\|\beta\| < \infty$ only finitely many factors in the product are different from one, since $f_{0,0}(x,y) = 1$. q.e.d.

The condition of Theorem 1 is fulfilled in our examples (5) if $|z_k| < 1$ for all $1 \leq k \leq r$.

4 Products of vertex operators

The next question we ask ourself is about products of vertex operators, since in QFT we want to take products of the field operators to build the $n-$point functions. We will show that any product of vertex operators is well defined on $\mathcal{F}_0$.

On the formal side we know that for two sequences $\omega^1 = (\omega_1^1,\omega_2^1,\ldots)$ and $\omega^2 = (\omega_1^2,\omega_2^2,\ldots)$

$$\tilde{V}(\omega^1)\tilde{V}(\omega^2) = \tilde{V}(\omega^1+\omega^2)\exp\left(\sum_{i=1}^{\infty}\omega_i^{1+}w_i^2\right) \tag{16}$$

by the CBH-formula. Usually one claims that $\sum_{i=1}^{\infty}\omega_i^{1+}\omega_i^2 < \infty$ which gives the time ordering condition for products of fields (but this is unnecessary in order to treat both sides as formal Laurent series). Surely we also require this for giving both sides of (16) a definite meaning. We prove that (16) holds for the corrresponding operators in the strong sense on $\mathcal{F}_0$.

Theorem 2 *Let $\sum_{i=1}^{\infty}|\omega_i^l|^2 < \infty$ for $l = 1,2$ and $\sum_{i=1}^{\infty}\omega_i^{1+}\omega_i^2$ be convergent. Then the following holds:*

$$V(\omega^2)(\mathcal{F}_0) \subset D(V(\omega^1)) \tag{17}$$

and

$$V(\omega^1)V(\omega^2)|_{\mathcal{F}_0} = V(\omega^1+\omega^2)\exp\left(\sum_{i=1}^{\infty}\omega_i^{1+}\omega_i^2\right)|_{\mathcal{F}_0}. \tag{18}$$

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Proof: First we show that the matrix product of the matrices $v_{\alpha,\beta}(\omega^1)$ and $v_{\alpha,\beta}(\omega^2)$ exists if $\sum \omega_i^{1+}\omega_i^2$ is convergent. If we interchange the product and the infinite number of infinite sums we can calculate

$$\begin{aligned}\sum_{\delta:||\delta||<\infty} v_{\alpha,\delta}(\omega^1)v_{\delta,\beta}(\omega^2) &= \frac{1}{\sqrt{\alpha!\beta!}}\sum_{\delta:||\delta||<\infty}\prod_{i=1}^{\infty}\left(\tfrac{1}{\delta_i!}f_{\alpha_i,\delta_i}(\omega_i^1,\omega_i^{1+})f_{\delta_i,\beta_i}(\omega_i^2,\omega_i^{2+})\right)\\ &= \frac{1}{\sqrt{\alpha!\beta!}}\prod_{i=1}^{\infty}\left(\sum_{\delta_i=0}^{\infty}\tfrac{1}{\delta_i!}f_{\alpha_i,\delta_i}(\omega_i^1,\omega_i^{1+})f_{\delta_i,\beta_i}(\omega_i^2,\omega_i^{2+})\right)\\ &= \frac{1}{\sqrt{\alpha!\beta!}}\prod_{i=1}^{\infty}\left(f_{\alpha_i,\delta_i}(\omega_i^1+\omega_i^2,\omega_i^{1+}+\omega_i^{2+})\right)\exp\left(\sum_{i=1}^{\infty}\omega_i^{1+}\omega_i^2\right)\\ &= v_{\alpha,\beta}(\omega^1+\omega^2)\exp\left(\sum_{i=1}^{\infty}\omega_i^{1+}\omega_i^2\right),\end{aligned} \tag{19}$$

where we have used Lemma 2 and the convergence of the sum.

Why can we interchange the summations and the product? Of course there exists a number $i_0 = i_0(\alpha,\beta)$ such that for any $i \geq i_0$ we have $f_{\alpha_i,\delta_i}(\omega_i^1,\omega_i^{1+})f_{\delta_i,\beta_i}(\omega_i^2,\omega_i^{2+}) = (\omega_i^{1+}\omega_i^2)^{\delta_i} = X_i^{\delta_i}$. Because for a finite number of factors we can interchange, it remains to show

$$\sum_{\delta:||\delta||<\infty}\prod_{i=1}^{\infty}\frac{X_i^{\delta_i}}{\delta_i!} = \prod_{i=1}^{\infty}\left(\sum_{i=0}^{\infty}\frac{X_i^{\delta_i}}{\delta_i!}\right). \tag{20}$$

Since $\sum_{i=1}^{\infty} X_i < \infty$ we have $\prod_{i=1}^{\infty} X_i = 0$ and hence $\prod_{i=1}^{\infty} X_i^{\delta_i}/\delta_i! = 0$ for any multiindex δ with $||\delta|| = \infty$. Therefore we can forget about the restriction $||\delta|| < \infty$ on the l.h.s. of (20). Hence

$$\sum_{\delta:||\delta||<\infty}\prod_{i=1}^{\infty}\frac{X_i^{\delta_i}}{\delta_i!} = \sum_{\delta}\prod_{i=1}^{\infty}\frac{X_i^{\delta_i}}{\delta_i!}. \tag{21}$$

Now it is easy to see that

$$\sum_{\delta}\prod_{i=1}^{\infty}\frac{X_i^{\delta_i}}{\delta_i!} = \prod_{i=1}^{\infty}\left(\sum_{i=1}^{\infty}\frac{X_i^{\delta_i}}{\delta_i!}\right) = \prod_{i=1}^{\infty}e^{X_i} \tag{22}$$

The proof is complete by the following observation: eq. (19) tells us that we can multiply the infinite matrices and get as result a matrix operator which is also defined on $\mathcal{F}_0$, since the matrix elements on the r.h.s. of (19) and hence the r.h.s. of (18) fulfill the condition of Theorem 1 too. By definition of a matrix operator this implies that also the l.h.s. of (18) is well defined on $\mathcal{F}_0$ (this could easily be proved using (9) and (10)), which immediately gives our assertions. q.e.d.

Theorem 2 shows, that any product of radial ordered vertex operators is well defined on $\mathcal{F}_0$ as a product of operators in $\mathcal{F}$. Moreover, Theorem 2 gives information about the mapping properties of the vertex operators if we consider triple products of vertices which via Theorem 2 are well defined.

5 Φ-Bounds for vertex operators

The Hamilton operator in Fock space is given by the selfadjoint operator $N = \sum_{k=1}^{\infty} a_{-k}a_k$ on the domain of definition $D(N) = \left\{ \Psi = \sum_\alpha c_\alpha \Phi_\alpha : \sum_\alpha ||\alpha|||c_\alpha|^2 < \infty \right\}$. We want to prove $||V(\omega)\Psi|| \leq c_1||e^{c_2 N}\Psi||$ for some constants $c_1, c_2 > 0$. To do so, we have to impose a further condition on ω :

$$\begin{array}{lll} |\omega_i| \leq KR^i & \text{for some constants } & K > 0, 0 < R < 1 \quad \text{and} \\ |\omega_i^+| \leq \bar{K}\bar{R}^i & \text{for some constants } & \bar{K}, \bar{R} > 0. \end{array} \tag{23}$$

Surely (5) fulfills this condition if $|z_i| < 1$, i.e. there is no additional condition for the standard vertex operators.

Theorem 3 *Let ω obey (23). Then there exists $c \in \mathbf{R}_+$ such that*

$$B(\omega) = V(\omega)c^N \tag{24}$$

is a Hilbert - Schmidt - operator. Hence $V(\omega)$ can be defined on $D(c^{-N})$ using the equality

$$V(\omega) = B(\omega)c^{-N} \tag{25}$$

which is valid a priori on $\mathcal{F}_0$.

Theorem 3 gives trivially:

Corollary 1 *Let ω, c be as above. Then with $c_1 = ||B(\omega)||, c_2 = -\ln c$:*

$$||V(\omega)\Psi|| \leq c_1||e^{c_2 N}\Psi|| \quad \textit{for all} \quad \Psi \in D\left(e^{c_2 N}\right) \tag{26}$$

Theorem 3 has also an application to screened vertex operators. In [1] they were introduced as follows:

Let $\omega_i = \omega_i(\gamma, z, \gamma_+, z_1, \ldots, \gamma_+, z_r, \gamma_-, z_{r+1}, \ldots, \gamma_-, z_{r+r'})$ for all i where $\gamma_\pm$ is given e.g. by the Kac determinant formula. Let z be fixed and denote by C_z the path, which starts in z, encircles zero once and ends in z. Then a screened vertex operators is defined by

$$\tilde{V}_{r,r'}(\gamma, z) = \int_{C_z^{r+r'}} V(\omega) dz_1 \wedge \ldots \wedge dz_{r+r'}. \tag{27}$$

We can give a meaning to (27) if Re $(\gamma_i\gamma_j) \geq 0$ and $|z| < 1$. By Theorem 3 there exists $0 < c < |z|$ with $B(\omega)$ defined by (24) is Hilbert - Schmidt. The constant c could be chosen independent from $(z_1, \ldots, z_{r+r'}) \in C_z^{r+r'}$. Moreover, the Hilbert - Schmidt norm of $B(\omega)$ is uniformly bounded if $(z_1, \ldots z_{r+r'})$ vary in $C_z^{r+r'}$. Hence we have a uniformly bounded holomorphic function $B(\omega) : C_z^{r+r'} \to B_2(\mathcal{F})$ (with $B_2(\mathcal{F})$ the (Hilbert -) space of Hilbert - Schmidt operators) which integrated gives again a Hilbert - Schmidt operator. Consequently we have:

Corollary 2 *Let ω be as above and $|z| < 1$. Then there exists $c > 0$ such that $V_{r,r'}(\gamma, z)$ can be defined on $D(c^{-N})$ by*

$$V_{r,r'}(\omega) = \int_{C_z^{r+r'}} B(\omega) dz_1 \wedge \ldots \wedge dz_{r+r'} \quad c^{-N}. \tag{28}$$

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Proof of Theorem 3: we have to show

$$\sum_{\alpha,\beta} |v_{\alpha,\beta}(\omega)c^{||\beta||}|^2 < \infty \quad \text{for an admissible } c. \tag{29}$$

First we evaluate the sum over α. This is already done in the proof of Theorem 1 and with a trivial modification we get (see (15))

$$\begin{aligned}
\sum_\alpha |v_{\alpha,\beta}(\omega)c^{||\beta||}|^2 &= c^{2||\beta||} \sum_\alpha |v_{\alpha,\beta}(\omega)|^2 \\
&\leq \frac{c^{2||\beta||}}{\beta!} \prod_{i=1}^{\infty} \left(f_{\beta_i,\beta_i}(|\omega_i| + |\omega_i^+|, |\omega_i| + |\omega_i^+|) \right) \exp\left(\sum_{i=1}^{\infty} |\omega_i|^2 \right) \\
&= c^{2||\beta||} \prod_{i=1}^{\infty} \frac{1}{\beta_i!} \sum_{j=0}^{\beta_i} \binom{\beta_i}{j}^2 j! \left(|\omega_i| + |\omega_i^+|\right)^{2(\beta_i - j)} \exp\left(\sum_{i=1}^{\infty} |\omega_i|^2 \right) \\
&\leq c^{2||\beta||} \prod_{i=1}^{\infty} \sum_{j=0}^{\beta_i} \binom{\beta_i}{j} \left(|\omega_i| + |\omega_i^+|\right)^{2(\beta_i - j)} \exp\left(\sum_{i=1}^{\infty} |\omega_i|^2 \right) \\
&= c^{2||\beta||} \prod_{i=1}^{\infty} \left(1 + \left(|\omega_i| + |\omega_i^+|\right)^2 \right)^{\beta_i} \exp\left(\sum_{i=1}^{\infty} |\omega_i|^2 \right).
\end{aligned} \tag{30}$$

Note that we have used that the condition (23) in Theorem 3 is stronger than the requirements for Theorem 1. The constant c is now fixed by the condition of convergence of the β_i–sums. We have to sum

$$\sum_{\beta_i} c^{2i\beta_i} \left(1 + \left(|\omega_i| + |\omega_i^+|\right)^2 \right)^{\beta_i} \leq \sum_{\beta_i} c^{2i\beta_i} \left(1 + \left(KR^i + \bar{K}\bar{R}^i\right)^2 \right)^{\beta_i}, \tag{31}$$

which is summable for all i if

$$|c^2 \left(1 + \left(KR + \bar{K}\bar{R}\right)^2 \right)| < 1 \quad \text{or} \quad |c|^2 < \frac{1}{1 + \left(KR + \bar{K}\bar{R}\right)^2}. \tag{32}$$

We choose a c which fulfills (32) and set $X_i = c^{2i}\left(1 + \left(KR^i + \bar{K}\bar{R}^i\right)^2 \right)$ and $\Omega = \exp\left(\sum_{i=1}^{\infty} |\omega_i|^2 \right)$. We obtain

$$\begin{aligned}
\sum_{\alpha,\beta} |v_{\alpha,\beta}(\omega)c^{||\beta||}|^2 &\leq \prod_{i=1}^{\infty} \left(\sum_{\beta_i=0}^{\infty} c^{2i\beta_i} \left(1 + \left(|\omega_i| + |\omega_i^+|\right)^2 \right)^{\beta_i} \right) \exp\left(\sum_{i=1}^{\infty} |\omega_i|^2 \right) \\
&\leq \Omega \prod_{i=1}^{\infty} \left(\sum_{\beta_i=0}^{\infty} X_i^{\beta_i} \right) = \Omega \prod_{i=1}^{\infty} \frac{1}{1-X_i} < \infty
\end{aligned} \tag{33}$$

since $\sum_{i=1}^{\infty} X_i < \infty$. q.e.d.

Remarks:

(i) If $\omega_i = \frac{\gamma}{\sqrt{i}} z^i$ and $|z| < 1$, Theorem 3 shows that $B(\omega)$ is Hilbert - Schmidt if

$$c < \left[1 + |\gamma|^2 \left(|z|^2 + |z|^{-2} + 2 \right) \right]^{-\frac{1}{2}}.$$

6 APPENDIX

A similar expression could be obtained in the second case of (5).

(ii) Another question which naturally arises is if $V(\omega)$ is a closed operator. Equivalently one ask if $V(\omega)^*$ is densely defined. Since formally $V(\omega)^* = V(\bar{\omega}^+)$ we see, that the problem is to define $V(\omega)$ for $|z| > 1$. In this case the basis Φ_α and hence $\mathcal{F}_0$ is not in the domain of definition of $V(\omega)$ because of the divergence of $\sum_{i=0}^{\infty} |\omega_i^+|^2$. This problem could be interesting for perturbation theory of CFT. A minimal requirement for any kind of perturbation is it's closability.

6 Appendix

Here we give a proof of Lemma 2.

We expand both sides of the identity (13) in a power series in four variable in $(0,0,0,0)$, i.e. we write (13) in the form

$$\sum_{\alpha,\beta,\gamma,\delta} c^{(n,m)}_{\alpha,\beta,\gamma,\delta} x^\alpha y^\beta z^\gamma \omega^\delta \stackrel{!}{=} \sum_{\alpha,\beta,\gamma,\delta} \tilde{c}^{(n,m)}_{\alpha,\beta,\gamma,\delta} x^\alpha y^\beta z^\gamma w^\delta$$

and proof that $c^{(n,m)}_{\alpha,\beta,\gamma,\delta}$ and $\tilde{c}^{(n,m)}_{\alpha,\beta,\gamma,\delta}$ coincide.

(i) The l.h.s. of (13):

$$\begin{aligned}\sum_{k=0}^{\infty} \tfrac{1}{k!} f_{n,k}(x,y) f_{k,m}(z,w) &= \sum_{k=0}^{\infty} \tfrac{1}{k!} \sum_{i=0}^{\min(n,k)} \sum_{j=0}^{\min(k,m)} \binom{n}{i}\binom{k}{i} i! \binom{k}{j}\binom{m}{j} j! \times \\ &\qquad \times x^{n-i} y^{k-i} z^{k-j} w^{m-j} \\ &= \sum_{\alpha,\beta,\gamma,\delta} c^{(n,m)}_{\alpha,\beta,\gamma,\delta} x^\alpha y^\beta z^\gamma w^\delta\end{aligned}$$

with

$$\begin{aligned}c_{\alpha,\beta,\gamma,\delta} &= \tfrac{1}{(n-\alpha+\beta)!}\binom{n}{n-\alpha}\binom{n-\alpha+\beta}{n-\alpha}(n-\alpha)!\binom{n-\alpha+\beta}{m-\delta} \times \\ &\qquad \times \binom{m}{m-\delta}(m-\delta)!\,\delta_{\gamma,n-m-\alpha+\beta+\delta} \\ &= \binom{n}{\alpha}\binom{m}{\delta}\tfrac{(n-\alpha+\beta)!}{\beta!(n-m-\alpha+\beta+\delta)!}\delta_{\gamma,n-m-\alpha+\beta+\delta}.\end{aligned} \tag{35}$$

(ii) The r.h.s. of (13):

$$\begin{aligned}&f_{n,m}(x+z, y+w)e^{zy} = \\ &= \sum_{k=0}^{\min(n,m)} \binom{n}{k}\binom{m}{k} k!(x+z)^{n-k}(y+w)^{m-k} e^{zy} \\ &= \sum_{l=0}^{\infty} \sum_{k=0}^{\min(n,m)} \sum_{i=0}^{n-k} \sum_{j=0}^{m-k} \tfrac{k!}{l!}\binom{n}{k}\binom{m}{k}\binom{n-k}{i}\binom{m-k}{j} x^{n-k-i} y^{j+l} z^{i+l} w^{m-k-j} \\ &= \sum_{\alpha,\beta,\gamma,\delta} \tilde{c}^{(n,m)}_{\alpha,\beta,\gamma,\delta} x^\alpha y^\beta z^\gamma w^\delta\end{aligned}$$

REFERENCES

A tedious calculation (eliminating γ) gives

$$\begin{aligned}
\tilde{c}^{(n,m)}_{\alpha,\beta,\gamma,\delta} &= \sum_{l=0}^{\beta} \frac{(m+l-\beta-\delta)!}{l!} \binom{n}{m+l-\beta-\delta} \binom{m}{m+l-\beta-\delta} \times \\
&\times \binom{n-m-l+\beta+\delta}{n-m-l-\alpha+\beta+\delta} \binom{-l+\beta+\delta}{-l+\beta} \delta_{\gamma,n-m-\alpha+\beta+\delta} \\
&= \frac{m!}{\beta!\delta!} \binom{n}{\alpha} \sum_{l=0}^{\beta} \binom{\beta}{l} \binom{n-\alpha}{n-m-l-\alpha+\beta+\delta} \delta_{\gamma,n-m-\alpha+\beta+\delta} \\
&= \frac{m!}{\beta!\delta!} \binom{n}{\alpha} \binom{n-\alpha}{n-m-l-\alpha+\beta+\delta} \delta_{\gamma,n-m-\alpha+\beta+\delta} \\
&= c^{(n,m)}_{\alpha,\beta,\gamma,\delta}.
\end{aligned} \tag{37}$$

Since both sides are convergent power series for any $x, y, z, w \in \mathbb{C}, n, m \in \mathbb{N}_0$ the equality of the functions follows from the equality of the coefficients.

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