Coloured graphs, Burgers equation and Jacobian conjecture.

I.V.Artamkin

Generating functions for modular graphs were considered in [1]. It was proved there out that these generating functions satisfy the one-dimensional Burgers equation. Here we present several multi-dimensional analogues of these results obtained in similar way from consideration of *coloured modular graphs* i.e. modular graphs whose edges and half-edges are coloured by r colours, corresponding to r independent variables $x_1, \ldots, x_r$. In the most simple version the generating function for connected graphs satisfies the r-dimensional Burgers equation and the generating function for all graphs (not necessarily connected) satisfies the heat equation. Other versions provide some systems of partial differential equations generalizing respectively Burgers or heat equations. The solution of the Burgers equation (or its generalizations) is obtained by the *genus expansion* of the generating function. The initial term of this expansion is the corresponding generating function for trees. The consequence of the Burgers equation for this term turns to be equivalent to the inversion problem for the gradient mapping defined by the initial condition. The connections between the inversion problem for a formal mapping $\mathbb{C}^r \to \mathbb{C}^r$ and the Burgers and heat equations was recently observed by Zhao, Meng and Wright (see [4], [2], [3]) in the study of the Jacobian conjecture. The use of generating functions enables to explain the results of [4] in rather short and natural way.

The equations for the higher terms of the genus expansion are linear. The solutions of these equations can be expressed explicitly by substitution of the initial conditions and the initial term (the tree expansion) into some universal polynomials (for $g > 1$) which are generating functions for *stable closed graphs*. (For $g = 1$ instead of polynomials appears logarithm.) The stable graph polynomials satisfy certain recurrence. In [1] some of these results were obtained for $r = 1$ by more or less direct solution of differential equations. Here we present purely combinatorial proofs.

I am grateful to Don Zagier for helpful discussions which inspired the study of counting functions in section 5. I also wish to thank the Max Planck Institute for Mathematics in Bonn whose hospitality I enjoyed while I was preparing this paper.

1 Introduction.

Consider vectors $X = (x_1, \dots, x_r)$ and $S = (s_1, \dots, s_r)$, where x_i and s_i are independent commutative variables. In this section a diagonal $r \times r$ matrix with diagonal elements s_i will be denoted $\hat{S}$. In section 3 we discuss similar constructions for arbitrary symmetric matrix $\hat{S}$. This generalization enables to give natural combinatorial proofs for all our formulas. For applications, however, it is sufficient to consider diagonal (or even scalar) matrix $\hat{S}$. So now we shall explain all the results and applications for this special case leaving most of the proofs to section 3.

For a multiindex $N = (n_1, \dots, n_r)$ we shall use the notations $X^N = x_1^{n_1} \cdot \ldots \cdot x_r^{n_r}$, $N! = n_1! \dots n_r!$ and $n = \sum n_i$. $N \geq 0$ will mean that all $n_i \geq 0$. Let us denote the multiindex $L_i = (0, \dots, 0, 1, 0, \dots, 0) = \{i\}$ (all the components are zero except $l_i = 1$). Multiindex $L_i + L_j$ will be also denoted by $\{ij\}$; multiindex $L_i + L_j + L_k$ will be denoted by $\{ijk\}$ and so on.

We shall consider modular graphs whose edges and half-edges are coloured by the variables $x_1, \dots, x_r$. To each vertex v of a modular graph we attach a nonnegative integer $g(v)$; we shall call a graph combinatorial if for all its vertices $g(v) = 0$. *Genus* of a modular graph Γ is defined by

$$g(\Gamma) = \sum g(v) + b_1(\Gamma) - b_0(\Gamma) + 1, \tag{1.1}$$

where $b_m(\Gamma)$ is the m-th Betti number of the graph (considered as a 1-dimensional simplicial complex). Thus for a connected graph Γ

$$g(\Gamma) = \sum g(v) + b_1(\Gamma)). \tag{1.2}$$

For a graph Γ let us fix multiindices $K(\Gamma) = (k_1, \dots, k_r)$ and $N(\Gamma) = (n_1, \dots, n_r)$ to denote the number of edges and, respectively, half-edges of the graph. A graph without half-edges (i.e. $N(\Gamma) = (0, \dots, 0)$) will be called *closed*. Valence of a vertex v of a coloured graph Γ is a multiindex

$N(v)=(\nu_1(v),\dots,\nu_r(v))$, where $\nu_i(v)$ is the number of outgoing edges and half-edges coloured by i. Denote by $\tilde{\mathcal{G}}^K_{g,N}$ the set of all coloured modular graphs having k_i edges of the colour i, n_i half-edges of the colour i and $g(\Gamma)=g$, and $\tilde{\mathcal{G}}^k_{g,N}=\bigcup_{\sum k_i=k}\tilde{\mathcal{G}}^K_{g,N}$. The set of corresponding connected graphs will be denoted by the same symbol without tilde.

We may also consider sets of coloured modular graphs with additional structures: ordering of its edges, denoted by $\mathcal{G}^{[K]}_{g,N}$; or ordering of its half-edges, denoted by $\mathcal{G}^K_{g,[N]}$; the set of graphs with both orderings, denoted by $\mathcal{G}^{[K]}_{g,[N]}$. Let $\{a_{g,N}\}$, $g\geq 0, N\geq 0$ be a set of (commutative) variables, Γ — a modular graph. Consider the monomial:

$$\mu(\Gamma)=\prod_{v\in V(\Gamma)}a_{g(v),\nu(v)}. \tag{1.3}$$

It is not hard to prove that there are four different ways to define the same generating series

$$\begin{aligned}\Psi(S,X,\hbar)&=\sum_{g\geq 0}\sum_{N\geq 0}\sum_{K\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^K_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N S^K\hbar^{g-1}=\\
&=\sum_{g\geq 0}\sum_{N\geq 0}\sum_{K\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^K_{g,[N]}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)\frac{X^N}{N!}S^K\hbar^{g-1}=\\
&=\sum_{g\geq 0}\sum_{N\geq 0}\sum_{K\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^{[K]}_{g,[N]}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)\frac{X^N}{N!}\frac{S^K}{K!}\hbar^{g-1}=\\
&=\sum_{g\geq 0}\sum_{N\geq 0}\sum_{K\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^{[K]}_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N\frac{S^K}{K!}\hbar^{g-1}.\end{aligned} \tag{1.4}$$

where $\operatorname{Aut}\Gamma$ is the automorphism group of the modular graph Γ. (Note that $\operatorname{Aut}\Gamma$ may be different in each of these four cases: an automorphism of a graph with fixed ordering of edges or/and half-edges should preserve these orderings.)

In fact, $\Psi(S,X,\hbar)$ depends on the variables $\{a_{g,N}\}$; so sometimes we shall write it as $\Psi(\{a_{g,N}\},S,X,\hbar)$. Generating series for not necessarily connected graphs also may be defined in the same four ways:

$$\tilde\Psi(S,X,\hbar)=\sum_{-\infty<g<+\infty}\sum_{N\ge 0}\sum_{K\ge 0}\left(\sum_{\Gamma\in\tilde{\mathcal G}^K_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^NS^K\hbar^{g-1}=$$

$$=\sum_{-\infty<g<+\infty}\sum_{N\ge 0}\sum_{K\ge 0}\left(\sum_{\Gamma\in\tilde{\mathcal G}^K_{g,[N]}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)\frac{X^N}{N!}S^K\hbar^{g-1}=$$

$$=\sum_{-\infty<g<+\infty}\sum_{N\ge 0}\sum_{K\ge 0}\left(\sum_{\Gamma\in\tilde{\mathcal G}^{[K]}_{g,[N]}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)\frac{X^N}{N!}\frac{S^K}{K!}\hbar^{g-1}=$$

$$=\sum_{-\infty<g<+\infty}\sum_{N\ge 0}\sum_{K\ge 0}\left(\sum_{\Gamma\in\tilde{\mathcal G}^{[K]}_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N\frac{S^K}{K!}\hbar^{g-1}. \quad (1.5)$$

For a variable s let us also define the series

$$\psi(s,X,\hbar)=\Psi(s,s,\dots s,X,\hbar)=\sum_{g\ge 0}\sum_{N\ge 0}\sum_{k\ge 0}\left(\sum_{\Gamma\in\mathcal G^k_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^Ns^k\hbar^{g-1} \quad (1.6)$$

and

$$\tilde\psi(s,X,\hbar)=\tilde\Psi(s,s,\dots s,X,\hbar)=\sum_{-\infty<g<+\infty}\sum_{N\ge 0}\sum_{k\ge 0}\left(\sum_{\Gamma\in\tilde{\mathcal G}^k_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^Ns^k\hbar^{g-1}. \quad (1.7)$$

Note that for $S=0$ (i. e. $s_1=\ldots=s_r=0$) these functions coincide; we shall denote this function by U:

$$U(X,\hbar)=\Psi(0,X,\hbar)=\psi(0,X,\hbar)=\sum_{g\ge 0}\sum_{N\ge 0}a_{g,N}\frac{X^N}{N!}\hbar^{g-1}, \quad (1.8)$$

These functions will play the role of the initial conditions for the Burgers equation. We shall also use the genus expansion of U

$$U(X,\hbar)=\sum_{g\ge 0}U_g(X)\hbar^{g-1}, \quad (1.9)$$

the gradient vector function

$$F(X) = \nabla_X \; U_0(X) \tag{1.10}$$

and the Hessian matrix function

$$H(X) = \nabla_X \; F(X) = \left(\frac{\partial^2 U_0}{\partial x_i \partial x_j} \right). \tag{1.11}$$

A standard combinatorial principle says that the generating function for all graphs is the exponent of the generating function for connected graphs.

Theorem 1.1.

$$\tilde{\Psi}(S, X, \hbar) = \exp(\Psi(S, X, \hbar)); \tag{1.12}$$

$$\tilde{\psi}(s, X, \hbar) = \exp(\psi(s, X, \hbar)). \tag{1.13}$$

Choose a multiindex $L = (l_1, \ldots, l_r)$, $l = \sum l_i$. Let $\tilde{\mathcal{G}}^{K,[L]}_{g,N}$ ($\mathcal{G}^{K,[L]}_{g,N}$) be the set of all genus g coloured modular (connected) graphs having $k_i + l_i$ edges of the colour i, l_i of which are marked and ordered, n_i half-edges of the colour i. In the same way, let $\tilde{\mathcal{G}}^{K}_{e,N,[L]}$ ($\mathcal{G}^{K}_{e,N,[L]}$) be the set of all genus g coloured modular (connected) graphs having k_i edges of the colour i, $n_i + l_i$ half-edges of the colour i, l_i of which are marked and ordered. Finally by $\tilde{\mathcal{G}}^{k,[k]}_{g,N}$ ($\mathcal{G}^{k,[k]}_{g,N}$) we shall denote the set of all genus g coloured modular (connected) graphs having $k + l$ edges, l of which are marked and ordered, n_i half-edges of the colour i, and by $\tilde{\mathcal{G}}^{K}_{e,N,[L]}$ ($\mathcal{G}^{K}_{e,N,[L]}$) we shall denote the set of all genus g coloured modular (connected) graphs having k edges, $n_i + l_i$ half-edges of the colour i, l_i of which are marked and ordered. It is not hard to see that the corresponding generating series are the derivatives of the ones we have introduced above.

Proposition 1.1.

$$\sum_{-\infty < g < +\infty} \sum_{N \geq 0} \sum_{K \geq 0} \left(\sum_{\Gamma \in \tilde{\mathcal{G}}^{K,[L]}_{g,N}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N S^K \hbar^{g-1} = \frac{\partial^l \tilde{\Psi}(S, X, \hbar)}{\partial s_1^{l_1} \ldots \partial s_r^{l_r}} \tag{1.14}$$

$$\sum_{-\infty < g < +\infty} \sum_{N \geq 0} \sum_{K \geq 0} \left(\sum_{\Gamma \in \tilde{\mathcal{G}}^{K}_{g,N,[L]}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N S^K \hbar^{g-1} = \frac{\partial^l \tilde{\Psi}(S, X, \hbar)}{\partial x_1^{l_1} \ldots \partial x_r^{l_r}} \tag{1.15}$$

$$\sum_{-\infty<g<+\infty}\sum_{N\geq 0}\sum_{k\geq 0}\left(\sum_{\Gamma\in\tilde{\mathcal{G}}^{k,[l]}_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N S^K \hbar^{g-1}=\frac{\partial^l\tilde{\psi}(s,X,\hbar)}{\partial s^l}\tag{1.16}$$

$$\sum_{-\infty<g<+\infty}\sum_{N\geq 0}\sum_{k\geq 0}\left(\sum_{\Gamma\in\tilde{\mathcal{G}}^{k}_{g,N,[L]}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N S^K \hbar^{g-1}=\frac{\partial^l\tilde{\psi}(s,X,\hbar)}{\partial x_1^{l_1}\dots\partial x_r^{l_r}}\tag{1.17}$$

The same is true for the generating series for connected graphs (i.e. for Ψ and ψ without tilde).

$$\sum_{g\geq 0}\sum_{N\geq 0}\sum_{K\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^{K,[L]}_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N S^K \hbar^{g-1}=\frac{\partial^l\Psi(S,X,\hbar)}{\partial s_1^{l_1}\dots\partial s_r^{l_r}}\tag{1.18}$$

$$\sum_{g\geq 0}\sum_{N\geq 0}\sum_{K\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^{K}_{g,N,[L]}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N S^K \hbar^{g-1}=\frac{\partial^l\Psi(S,X,\hbar)}{\partial x_1^{l_1}\dots\partial x_r^{l_r}}\tag{1.19}$$

$$\sum_{g\geq 0}\sum_{N\geq 0}\sum_{k\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^{k,[l]}_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N S^K \hbar^{g-1}=\frac{\partial^l\psi(s,X,\hbar)}{\partial s^l}\tag{1.20}$$

$$\sum_{g\geq 0}\sum_{N\geq 0}\sum_{k\geq 0}\left(\sum_{\Gamma\in\mathcal{G}^{k}_{g,N,[L]}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N S^K \hbar^{g-1}=\frac{\partial^l\psi(s,X,\hbar)}{\partial x_1^{l_1}\dots\partial x_r^{l_r}}\tag{1.21}$$

In the next few lines we present the crucial observation for this paper. Consider the set $\tilde{\mathcal{G}}^K_{g,N,[\{ii\}]}$ of all genus g graphs having two marked (and ordered) half-edges of the same colour i. There is a natural involution ε acting on $\tilde{\mathcal{G}}^K_{g,N,[\{ii\}]}$ by changing the order of the two marked edges; clutching these two marked half-edges together provides the one-to-one correspondence

$$\tilde{\mathcal{G}}^K_{g,N,[\{ii\}]}/\varepsilon\cong\tilde{\mathcal{G}}^{K,[\{i\}]}_{g-1,N}\tag{1.22}$$

which is compatible with the automorphisms of the corresponding graphs. This proves that $\tilde{\Psi}$ satisfies the heat equation.

Theorem 1.2.

$$\frac{\partial\tilde{\Psi}}{\partial s_i} = \frac{\hbar}{2}\frac{\partial^2\tilde{\Psi}}{\partial x_i^2} \tag{1.23}$$

Corollary 1.1.

$$\frac{\partial\tilde{\psi}}{\partial s} = \frac{\hbar}{2}\,\triangle_X\,\tilde{\psi} \tag{1.24}$$

($\triangle_X = \sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator.) Using (1.12) and (1.13), for Ψ and ψ we obtain the Burgers equation.

Corollary 1.2.

$$\frac{\partial\Psi}{\partial s_i} = \frac{\hbar}{2}\left[\frac{\partial^2\Psi}{\partial x_i^2} + \left(\frac{\partial\Psi}{\partial x_i}\right)^2\right]. \tag{1.25}$$

$$\frac{\partial\psi}{\partial s} = \frac{\hbar}{2}\left[\triangle_X\,\psi + (\nabla_X\,\psi)^T\,(\nabla_X\,\psi)\right]. \tag{1.26}$$

($\nabla_X\,\psi$ is the vector $(\frac{\partial\psi}{\partial x_1},\ldots,\frac{\partial\psi}{\partial x_r})^T$)

Consider the genus expansions for Ψ and ψ:

$$\Psi(S,X,\hbar) = \sum_{g\geq 0}\Psi_g(S,X)\hbar^{g-1} \tag{1.27}$$

and

$$\psi(s,X,\hbar) = \sum_{g\geq 0}\psi_g(s,X)\hbar^{g-1} \tag{1.28}$$

For $g=0$ the equations (1.25) and (1.26) provide the equations

$$\frac{\partial\Psi}{\partial s_i} = \frac{1}{2}\left(\frac{\partial\Psi}{\partial x_i}\right)^2. \tag{1.29}$$

and

$$\frac{\partial\psi}{\partial s} = \frac{1}{2}\,(\nabla_X\,\psi)^T\,(\nabla_X\,\psi)\,. \tag{1.30}$$

Consider the vector functions

$$\Phi(S,X) = \nabla_X\,\Psi_0(S,X) \tag{1.31}$$

and

$$\phi(s,X) = \nabla_X\,\psi_0(s,X). \tag{1.32}$$

(Note that $\Phi(0,x) = \phi(0,x) = F(x)$, see (1.10).)

The equations (1.25) and (1.26) provide the following equations for $\Phi = (\Phi_1,\ldots,\Phi_r)$ and ϕ.

Corollary 1.3.

$$\frac{\partial \Phi_m}{\partial s_i} = \Phi_i \frac{\partial \Phi_m}{\partial x_i}. \tag{1.33}$$

$$\frac{\partial \phi}{\partial s} = (\nabla_X \ \phi)\, \phi. \tag{1.34}$$

(Here $\nabla_X \ \phi$ is the matrix with the components $\frac{\partial^2 \psi_0}{\partial x_i \partial x_j}$.) Consider the Hessian matrices

$$\Theta(S,X) = \nabla_X \ \Phi(S,X) = \left(\frac{\partial^2 \Psi_0}{\partial x_i \partial x_j} \right). \tag{1.35}$$

and

$$\theta(s,X) = \nabla_X \ \phi(S,X) = \left(\frac{\partial^2 \psi_0}{\partial x_i \partial x_j} \right) = \Theta(s,\dots,s,X). \tag{1.36}$$

(Note that $\Theta(0,x) = \theta(0,x) = H(X)$, see (1.11)).

Then the equation (1.34) may be written as

$$\frac{\partial \phi}{\partial s} = \theta \phi. \tag{1.37}$$

The system of equations (1.33) and the equation (1.37) may be solved explicitly.

Theorem 1.3. *Let $F(X) = (F_1(X),\dots,F_r(X))$ be any formal series vector. Denote by $\hat{S}$ the diagonal matrix whose diagonal elements are the variables s_i. Then the solution of the system (1.33) with the initial condition $\Phi(0,\dots 0,X) = F(X)$ satisfies the functional equation*

$$\Phi(S,X) = F(X + \hat{S}\Phi(S,X)). \tag{1.38}$$

It is not hard to verify that the solution of the functional equation (1.38) satisfies the Burgers equations (1.33). The partial derivatives of the m-th equation (1.38) by s_i and x_i are:

$$\frac{\partial \Phi_m}{\partial s_i} = \sum \frac{\partial F_m}{\partial x_j} \frac{\partial (x_j + s_j \Phi_j)}{\partial s_i} = \sum \frac{\partial F_m}{\partial x_j} s_j \frac{\partial \Phi_j}{\partial s_i} + \sum \frac{\partial F_m}{\partial x_i} \Phi_i; \tag{1.39}$$

$$\frac{\partial \Phi_m}{\partial x_i} = \sum \frac{\partial F_m}{\partial x_j} \frac{\partial (x_j + s_j \Phi_j)}{\partial x_i} = \sum \frac{\partial F_m}{\partial x_j} s_j \frac{\partial \Phi_j}{\partial x_i} + \sum \frac{\partial F_m}{\partial x_i}. \tag{1.40}$$

Therefore

$$\frac{\partial \Phi_m}{\partial s_i} - \Phi_i \frac{\partial \Phi_m}{\partial x_i} = \sum \frac{\partial F_m}{\partial x_j} s_j \left(\frac{\partial \Phi_j}{\partial s_i} - \Phi_i \frac{\partial \Phi_j}{\partial x_i} \right). \tag{1.41}$$

Denoting the matrix $(\frac{\partial F_m}{\partial x_j})$ by H we obtain

$$\left(E - H\hat{S}\right) \left(\frac{\partial \Phi}{\partial s_i} - \Phi_i \frac{\partial \Phi}{\partial x_i} \right) = 0, \tag{1.42}$$

where E is the unit matrix. Since $E - H\hat{S}$ is nondegenerate for generic $\hat{S}$, $\frac{\partial \Phi_m}{\partial s_i} = \Phi_i \frac{\partial \Phi_m}{\partial x_i}$ for all m. (See purely combinatorial proof in section 3.

Substituting $s_1 = \cdots = s_r = s$ we obtain the corresponding statement for the function ϕ, first observed by W.Zhao [5].

Corollary 1.4. *([5]). The solution of the system (1.34) with the initial condition $\phi(0, X) = F(X)$ satisfies the functional equation*

$$\phi(s, X) = F(X + s\phi(S, X)). \tag{1.43}$$

Corollary 1.5. *Let $\Phi(S, X)$ be the solution of (1.33). Consider the following formal mappings from $\mathbb{C}^r$ to $\mathbb{C}^r$:*

$$A(X) = X - \hat{S}F(X) \tag{1.44}$$

$$B(X) = X + \hat{S}\Phi(S, X) \tag{1.45}$$

Then these mappings are inverse to each other:

$$A(B(X)) = X \text{ and } B(A(X)) = X. \tag{1.46}$$

Differentials of inverse mappings are also inverse to each other. Note that both $\Theta(0, X)$ and $\theta(0, X)$ are the Hessian matrix of the initial condition (1.11), which was denoted in the proof of the previous theorem by H. Thus we get the following equations for Θ and θ.

Corollary 1.6.

$$E + \hat{S}\Theta(S, X) = \left(E - \hat{S}\Theta\left(0, X + \hat{S}\Phi(S, X)\right)\right)^{-1} = \\ = \left(E - \hat{S}H\left(X + \hat{S}\Phi(S, X)\right)\right)^{-1} \tag{1.47}$$

$$E + s\theta(s, X) = (E - s\theta(0, X + s\phi(s, X)))^{-1} = (E - sH(X + s\phi(s, X)))^{-1} \tag{1.48}$$

Thus the formal Cauchy problems for the systems of the Burgers equations for Φ and ϕ are equivalent to the problem of finding the inverse function for the initial conditions. Integrating Φ or ϕ we may get the first term of the expansions (1.27) or (1.28). It is remarkable that the second term of these expansions may be presented explicitly in terms of Φ (or ϕ) and H.

The equations (1.25) and (1.26) provide the following equations for Ψ_1 and ψ_1.

$$\frac{\partial \Psi_1}{\partial s_i} = \frac{1}{2}\frac{\partial \Phi_i}{\partial x_i} + \Phi_i \frac{\partial \Psi_1}{\partial x_i}. \tag{1.49}$$

$$\frac{\partial \psi_1}{\partial s} = \frac{1}{2}\operatorname{tr}\theta + \phi^T\left(\nabla_X \psi_1\right). \tag{1.50}$$

Theorem 1.4. *The solution of the equations (1.49) or (1.50) with the initial conditions* $\Psi_1(0,X) = U_1(X)$ *or* $\psi_1(0,X) = U_1(X)$ *is given by the following formulas:*

$$\Psi_1(S,X) = U_1\left(X + \hat{S}\Phi(S,X)\right) - \frac{1}{2}\operatorname{tr}\ln\left(E - \hat{S}H(X + \hat{S}\Phi(S,X))\right) \tag{1.51}$$

$$\psi_1(s,X) = U_1\left(X + s\phi(s,X)\right) - \frac{1}{2}\operatorname{tr}\ln\left(E - sH(X + s\phi(s,X))\right) \tag{1.52}$$

The shortest way to prove the theorem is to substitute the solutions (1.51) and (1.52) into the equations (1.49) and (1.50). Instead of that we postpone the proof to section 3 where we shall explain these formulas in terms of geometry of graphs for the case of general symmetric matrix $\hat{S}$.

For $g > 1$ the equations (1.25) and (1.26) provide the following recurrent equations for Ψ_g and ψ_g.

$$\frac{\partial \Psi_g}{\partial s_i} = \frac{1}{2}\frac{\partial^2 \Psi_{g-1}}{\partial x_i^2} + \Phi_i \frac{\partial \Psi_g}{\partial x_i} + \frac{1}{2}\sum_{m=1}^{g-1}\frac{\partial \Psi_m}{\partial x_i}\frac{\partial \Psi_{g-m}}{\partial x_i}. \tag{1.53}$$

$$\frac{\partial \psi_g}{\partial s} = \frac{1}{2}\,\triangle_X\ \psi_{g-1} + \phi^T \nabla_X\ \psi_g + \frac{1}{2}\sum_{m=1}^{g-1}\left(\nabla_X\ \psi_m\right)^T\left(\nabla_X\ \psi_{g-m}\right). \tag{1.54}$$

In section 3 we introduce for $g > 1$ *stable graph polynomials* $P_g(\{a_{g,N}\}, \hat{S})$ (see (3.56)) depending on independent variables $a_{g,N}$ for all $g \geq 0$ $|N| \geq 3$ and symmetric matrix $\hat{S}$. The stable graph polynomials satisfy certain recurrence (see theorem 4.1). The solution of (1.53) and (1.54) are expressed by the stable graph polynomials as follows (see theorem 3.4):

$$\Psi_g(\hat{S},X) = \\ = P_g\left(\left\{a_{g,N} := \frac{\partial^{|N|}U_g\left(X+\hat{S}\Phi(\hat{S},X)\right)}{\partial X^N}\right\}, \left(E-\hat{S}H\left(X+\hat{S}\Phi(\hat{S},X)\right)\right)^{-1}\hat{S}\right); \tag{1.55}$$

$$\psi_g(s,X) = \\ = P_g\left(\left\{a_{g,N} := \frac{\partial^{|N|}U_g\left(X+s\phi(s,X)\right)}{\partial X^N}\right\}, s\left(E-sH\left(X+s\phi(s,X)\right)\right)^{-1}\right). \tag{1.56}$$

By definition *stable graph* is a connected graph having no 1-valent and 2-valent genus 0 vertices. Stable graph polynomials are simply generating functions for stable closed graphs; we postpone formal definition for section 3 because $P_g(\{a_{g,N}\},\hat{S})$ essentially depend on arbitrary symmetric matrix $\hat{S}$.

Right now we can discuss the case $r=1$ which is far from being trivial. For this case we have one variable s corresponding to edges of a graph and two-index variables $a_{g,n}$. Denoting by $\mathcal{A}_g^k$ the set of genus g stable closed graphs we define

$$P_g(\{a_{m,N}\},s) = \sum_{k=0}^{3g-3}\sum_{\Gamma\in\mathcal{A}_g^k}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}s^k. \tag{1.57}$$

(Stable genus g graph without half-edges has at most $3g-3$ edges.) For instance for $g=2$

$$P_2 \;=\; a_{2,0} + \frac{1}{2}a_{1,1}^2 s + \frac{1}{2}a_{1,2}s + \frac{1}{2}a_{1,1}a_{0,3}s^2 + \frac{1}{8}a_{0,4}s^2 + \frac{5}{24}a_{0,3}^2 s^3. \tag{1.58}$$

There are two interesting specializations of the variables $\{a_{g,n}\}$: counting functions for all combinatorial graphs and counting functions for all stable combinatorial graphs and

For the counting functions for all combinatorial graphs we put

$$a_{g,n}^{\text{comb}} = \begin{cases} 1 & \text{if } g=0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.59}$$

and for the counting functions for all stable combinatorial graphs we put

$$a^{\mathrm{st}}_{g,n} = \begin{cases} 1 & \text{if } g = 0 \quad \text{and } n \geq 3 \\ 0 & \text{otherwise.} \end{cases} \tag{1.60}$$

The function Φ satisfies the functional equation

$$\Phi^{\mathrm{comb}}(s,x) = e^{x+s\Phi^{\mathrm{comb}}(s,x)} \tag{1.61}$$

for the counting functions for all combinatorial graphs[1] and the functional equation

$$e^{x+s\Phi^{\mathrm{st}}(s,x)} = 1 + x + (s+1)\Phi^{\mathrm{st}}(s,x) \tag{1.62}$$

for the counting functions for all stable combinatorial graphs.

Stable graph polynomials for both cases coincide; we denote

$$P_g^{\mathrm{comb}}(s) = P_g\left(\left\{a_{m,N} := a^{\mathrm{comb}}_{m,N}\right\}\right) = P_g\left(\left\{a_{m,N} := a^{\mathrm{st}}_{m,N}\right\}, s\right). \tag{1.63}$$

For instance for $g = 2$

$$P_2^{\mathrm{comb}} = \frac{1}{8}s^2 + \frac{5}{24}s^3. \tag{1.64}$$

In section 5 we prove the following theorem.

Theorem 1.5. *1) Combinatorial stable graph polynomials P_g^{comb} for $g > 2$ satisfy the recurrence*

$$\frac{dP_g^{\mathrm{comb}}}{ds} = \frac{1}{2}\left[D^2_{comb}(P^{\mathrm{comb}}_{g-1}) + 2sD_{comb}(P^{\mathrm{comb}}_{g-1}) + \sum_{m=2}^{g-2} D_{comb}(P^{\mathrm{comb}}_m)D_{comb}(P^{\mathrm{comb}}_{g-m})\right], \tag{1.65}$$

where

$$D_{comb} = s(s+1)\frac{d}{ds} - (g-1). \tag{1.66}$$

[1]Note that $\Phi^{\mathrm{comb}}(s,0)$ is the classical generating function for rooted trees (without half-edges) whose coefficients are given by the well-known Caley formula:

$$\Phi^{\mathrm{comb}}(s,0) = \sum_{k=0}^{\infty} \frac{(k+1)^k}{(k+1)!}s^k.$$

2) For $g \geq 2$ the counting function for all combinatorial graphs

$$\Psi_g^{\mathrm{comb}}(s,x) = \frac{1}{\Phi^{\mathrm{comb}}(s,x)^{g-1}} P_g^{\mathrm{comb}}\left(\frac{s\Phi^{\mathrm{comb}}(s,x)}{1 - s\Phi^{\mathrm{comb}}(s,x)}\right). \tag{1.67}$$

3) For $g \geq 2$ the counting function for all stable combinatorial graphs

$$\Psi_g^{\mathrm{st}}(s,x) = \frac{1}{(1+x+(s+1)\Phi^{\mathrm{st}}(s,x))^{g-1}} P_g^{\mathrm{comb}}\left(\frac{s\,(1+x+(s+1)\Phi^{\mathrm{st}}(s,x))}{1 - s(x+(s+1)\Phi^{\mathrm{st}}(s,x))}\right). \tag{1.68}$$

Formula (1.68) was derived in [1] by direct solution of the equations (1.53).

2 The Hessian conjecture.

The *Jacobian conjecture* states that for a polynomial mappings $A : \mathbb{C}^r \to \mathbb{C}^r$ with constant determinant of the Jacobian matrix the inverse mapping is also polynomial. It is well-known (see for instance [3]) that it is sufficient to prove the Jacobian conjecture for all $r > 1$ for the mapping $A(X) = X - F(X)$ where $F(X)$ is the gradient vector of a homogenous polynomial[2] $U_0(X)$. In this case the determinant of the Jacobian matrix is constant if and only if the Hessian matrix $H(X) = \nabla_X F(X)$ is nilpotent. This form of statement is called the *Hessian conjecture*. We use the notations of section 1 to emphasize that the Hessian conjecture may be stated in the language of generating functions. Corollary 1.5 for $\hat{S} = E$ provides a formula for the inverse mapping (however we don't need F to be homogenous). Thus we may state the Hessian conjecture in the language of generating functions. Since now we are interested only in in the inversion problem we may consider only combinatorial graphs, i.e. put $a_{g,N} = 0$ for $g > 0$. Then the initial condition (1.8) has only zero term, i.e. $U_g = 0$ for $g > 0$. In the Hessian conjecture $U_0(X)$ is a polynomial of some degree d. This means that we consider graphs with $a_{0,N} = 0$ for $|N| > d$. Homogeneous polynomial $U_0(X)$ implies generating functions of d-valent graphs. Thus the Hessian conjecture looks as follows.

The Hessian Conjecture. *Let $a_{g,N} = 0$ for $g > 0$ $a_{0,N} = 0$ for $|N| > d$. Suppose that the matrix $H(X)$ (1.11) is nilpotent. Then the generating function for modular trees $\Psi_0(E,X)$ is a polynomial.*

[2]In fact it is sufficient to prove the Jacobian conjecture only for the case $\deg U_0 = 4$.

The proof is still unknown, but we can present short and natural proofs for several statements observed by W.Zhao in [4].

Proposition 2.1. *Let $a_{g,N} = 0$ for $g > 0$ and $H(X)$ is nilpotent. Then $\psi_g(s, X) = 0$ for $g > 0$.*

First of all, formula (1.52) of theorem 1.4 shows that $\psi_1(s, X) = 0$. Suppose that for some $g > 1$ $\psi_m(s, X) = 0$ for all positive $m < g$. Then (1.54) provides linear equation

$$\frac{\partial \psi_g}{\partial s} = \phi^T \nabla_X \ \psi_g \tag{2.1}$$

with the initial condition $\psi_g(0, X) = U_g(X) = 0$. Therefore $\psi_g(s, X) = 0$.

Corollary 2.1. *Let $a_{g,N} = 0$ for $g > 0$ and $H(X)$ is nilpotent. Then $\psi(s, X) = \frac{\psi_0(s,X)}{\hbar}$.*

Thus for $\hbar = 1$ these functions coincide and so we may apply corollaries 1.1 and 1.2.

Corollary 2.2. *Let $a_{g,N} = 0$ for $g > 0$ and $H(X)$ is nilpotent. Then $\psi_0(s, X)$ satisfies the Burgers equation*

$$\frac{\partial \psi_0}{\partial s} = \frac{1}{2}\left[\triangle_X \ \psi_0 + (\nabla_X \ \psi_0)^T (\nabla_X \ \psi_0)\right]. \tag{2.2}$$

and the function $\tilde{\psi}_0(s, X) = e^{\psi_0(s,X)}$ satisfies the heat equation

$$\frac{\partial \tilde{\psi}_0}{\partial s} = \frac{1}{2} \ \triangle_X \ \tilde{\psi}_0. \tag{2.3}$$

Comparing (1.30) and (2.2) we see that ψ_0 is a harmonic function.

Corollary 2.3. *Let $a_{g,N} = 0$ for $g > 0$ and $H(X)$ is nilpotent. Then*

$$\triangle_X \ \psi_0(s, X) = 0. \tag{2.4}$$

In conclusion let us compare two inversion formulas.

Proposition 2.2. *Corollary 1.5 for $\hat{S} = E$ is equivalent to the Bass-Connell-Wright Tree Inversion Formula (see [3]).*

Here we present (in our notations) the Bass-Connell-Wright Tree Inversion Formula from [3] (Theorem 2.3). In [3] $\mathbb{T}$ denotes the set of isomorphism classes of finite trees (without half-edges).

Theorem. *Consider the formal map $A(X) = X - F(X)$ where $F(X) = \nabla_X\, U_0(X)$, and let $B(X)$ be its inverse. Then $B(X) = X + \nabla_X\, Q(X)$ with*

$$Q = \sum_{T\in\mathbb{T}} \frac{1}{\operatorname{Aut} T} \sum_{l\colon E(T)\to\{1,\ldots,r\}} \prod_{v\in V(T)} \frac{\partial^{|N(v)|}U_0}{\partial X^{N(v)}} \tag{2.5}$$

First of all, mapping $l\colon E(T)\to\{1,\ldots,r\}$ is just colouring of the edges of the tree T so in fact we have sum over the set $\mathcal{G}_{0,0}$ all closed coloured trees (i.e. trees without half-edges). Denote the tree T with edges coloured by the mapping $: E(T)\to\{1,\ldots,r\}$ by Γ. Thus we can rewrite (2.5) as

$$Q = \sum_{\Gamma\in\mathcal{G}_{0,0}} \frac{1}{|\operatorname{Aut}\Gamma|} \prod_{v\in V(\Gamma)} \frac{\partial^{|N(v)|}U_0}{\partial X^{N(v)}} \tag{2.6}$$

Consider a modular tree $\Delta\in\mathcal{G}^K_{0,N}$. Removal of all its half-edges defines a mapping

$$c\colon \bigcup_{N\geq 0}\mathcal{G}^K_{0,N}\to\mathcal{G}^K_{0,0}. \tag{2.7}$$

It is not hard to see that for a closed tree $\Gamma\in\mathcal{G}^K_{0,0}$

$$\sum_{\Delta\in c^{-1}(\Gamma)} \frac{1}{|\operatorname{Aut}\Delta|}\mu(\Delta)X^{N(\Delta)} = \frac{1}{|\operatorname{Aut}\Gamma|}\prod_{v\in V(\Gamma)}\frac{\partial^{|N(v)|}U_0}{\partial X^{N(v)}}. \tag{2.8}$$

Therefore $Q(X) = \Psi_0(E, X)$.

3 Bipartite graphs.

Now our aim is to prove the results of section 1 using similar equations for arbitrary symmetric matrix $\hat{S}$. Instead of coloured modular graphs we shall consider *bipartite coloured modular graphs.* By definition a bipartite coloured modular graph has the following structure:

(1) the set of vertices $V(\Gamma)$ of a bipartite coloured modular graph Γ is a disjoint union of two partite sets $V(\Gamma) = V_a(\Gamma)\sqcup V_s(\Gamma)$;

(2) two vertices from the same partite sets are not connected by an edge;

(3) to each vertex of $v \in V_a(\Gamma)$ we attach a nonnegative integer $g(v)$;

(4) vertices from $V_s(\Gamma)$ should be only two-valent and should have no adjacent half-edges, we shall call such vertices s-vertices.

Informally speaking we may say that a new bipartite graph is obtained from a usual modular graph by inserting new two-valent s-vertices into the middle of each edge. Therefore we attach the variables $\{a_{g,N}\}$ to the vertices from $V_a(\Gamma)$, and the variables $\{s_{ij}\}$ will correspond to the two-valent s-vertices from $V_s(\Gamma)$. As before we define

$$g(\Gamma) = \sum_{v \in V_a(\Gamma)} g(v) + b_1(\Gamma) - b_0(\Gamma) + 1. \tag{3.1}$$

Denote by $\tilde{\mathcal{B}}_{g,N}$ the set of all such bipartite graphs having n_i half-edges of the colour i and $g(\Gamma) = g$ and by $\mathcal{B}_{g,N}$ the set of all such connected bipartite graphs. Put $\tilde{\mathcal{B}}_g = \bigcup_{N \geq 0} \tilde{\mathcal{B}}_{g,N}$ and $\mathcal{B}_g = \bigcup_{N \geq 0} \mathcal{B}_{g,N}$.

For a bipartite coloured modular graph we define

$$\mu(\Gamma) = \prod_{v \in V_s(\Gamma)} s_{ij} \prod_{v \in V_a(\Gamma)} a_{g(v),\nu(v)}. \tag{3.2}$$

Next we define the generating functions

$$\Psi(\hat{S}, X, \hbar) = \sum_{g \geq 0} \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N \hbar^{g-1} \tag{3.3}$$

and

$$\tilde{\Psi}(\hat{S}, X, \hbar) = \sum_{-\infty < g < +\infty} \sum_{N \geq 0} \left(\sum_{\Gamma \in \tilde{\mathcal{B}}_{g,N}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N \hbar^{g-1}. \tag{3.4}$$

Here $\hat{S}$ is the matrix (s_{ij}); Ψ and $\tilde{\Psi}$ depend on $\hat{S}$ via $\mu(\Gamma)$ (see 3.2)). Note that for diagonal matrix $\hat{S}$ (i.e. for $s_{ij} = 0$ for $i \neq j$) the definitions (3.3) and (3.4) coincide with (1.4) and (1.5). For the new functions Ψ and $\tilde{\Psi}$ it is not hard to prove the same theorems as for the old ones.

Theorem 3.1.

$$\tilde{\Psi}(\hat{S}, X, \hbar) = \exp(\Psi(\hat{S}, X, \hbar)); \tag{3.5}$$

For the analogue of the proposition 1.1 we define the sets of bipartite coloured modular (connected) graphs with one marked ij-valent s-vertex by $\tilde{\mathcal{B}}^{[ij]}_{g,N}$ ($\mathcal{B}^{[ij]}_{g,N}$). For a graph $\Gamma \in \tilde{\mathcal{B}}^{[ij]}_{g,N}$ ($\mathcal{B}^{[ij]}_{g,N}$) definition of $\mu(\Gamma)$ should be improved: we put

$$\mu(\Gamma) = \prod_{\substack{\text{nonmarked} \\ v \in V_s(\Gamma)}} s_{ij} \prod_{v \in V_a(\Gamma)} a_{g(v),\nu(v)}. \tag{3.6}$$

The set of bipartite coloured modular (connected) graphs with n_i+l_i marked half-edges of the colour i, l_i of them marked and ordered, will be denoted by $\tilde{\mathcal{B}}_{g,N,[L]}$ ($\mathcal{B}_{g,N,[L]}$). ($N=(n_1,\dots,n_r)$ and $L=(l_1,\dots,l_r)$ are multiindices.) Then

$$\sum_{-\infty<g<+\infty} \sum_{N\ge 0} \left(\sum_{\Gamma \in \tilde{\mathcal{B}}^{[ij]}_{g,N}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial \tilde{\Psi}(S,X,\hbar)}{\partial s_{ij}}, \tag{3.7}$$

$$\sum_{-\infty<g<+\infty} \sum_{N\ge 0} \left(\sum_{\Gamma \in \tilde{\mathcal{B}}^{K}_{g,N,[L]}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial^l \tilde{\Psi}(S,X,\hbar)}{\partial x_1^{l_1} \dots \partial x_r^{l_r}}, \tag{3.8}$$

and the same is true for the generating series for connected graphs (i.e. for Ψ without tilde):

$$\sum_{g\ge 0} \sum_{N\ge 0} \left(\sum_{\Gamma \in \mathcal{B}^{[ij]}_{g,N}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial \Psi(S,X,\hbar)}{\partial s_{ij}}, \tag{3.9}$$

$$\sum_{g\ge 0} \sum_{N\ge 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N,[L]}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N \hbar^{g-1} = \frac{\partial^l \Psi(S,X,\hbar)}{\partial x_1^{l_1} \dots \partial x_r^{l_r}}, \tag{3.10}$$

Now we are able to clutch together half-edges of different colours: clutching together two half-edges of the colours i and j we insert a ij-valent s-vertex between them. Pick a graph from $\tilde{\mathcal{B}}^{[ij]}_{g,N}$. Deletion of the marked s-vertex provides a graph in $\tilde{\mathcal{B}}_{g,N,[\{ij\}]}$, which proves the following analogue of the heat equation (1.23):

Theorem 3.2. *For $i \neq j$*

$$\frac{\partial \tilde{\Psi}}{\partial s_{ij}} = \hbar \frac{\partial^2 \tilde{\Psi}}{\partial x_i \partial x_j}; \tag{3.11}$$

for $i = j$

$$\frac{\partial \tilde{\Psi}}{\partial s_{ii}} = \frac{\hbar}{2} \frac{\partial^2 \tilde{\Psi}}{\partial x_i^2} \tag{3.12}$$

Note that using the notations

$$\{ij\}! = (L_i + L_j)! = \begin{cases} 1! \cdot 1! = 1 & \text{for } i \neq j \\ 2! = 2 & \text{for } i = j \end{cases} \tag{3.13}$$

we may rewrite (3.11) and (3.12) in the following uniform way

$$\frac{\partial \tilde{\Psi}}{\partial s_{ij}} = \frac{\hbar}{\{ij\}!} \frac{\partial^2 \tilde{\Psi}}{\partial x_i \partial x_j}. \tag{3.14}$$

For Ψ the corresponding generalization of the Burgers equations (1.23) looks as follows:

Corollary 3.1.

$$\frac{\partial \Psi}{\partial s_{ij}} = \frac{\hbar}{\{ij\}!} \left[\frac{\partial^2 \Psi}{\partial x_i \partial x_j} + \left(\frac{\partial \Psi}{\partial x_i} \right) \left(\frac{\partial \Psi}{\partial x_j} \right) \right]. \tag{3.15}$$

Next let us consider the genus expansion of Ψ

$$\Psi(\hat{S}, X, \hbar) = \sum_{g \geq 0} \Psi_g(\hat{S}, X) \hbar^{g-1}. \tag{3.16}$$

For $g = 0$ (3.15) provides the equation

$$\frac{\partial \Psi_0}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left(\frac{\partial \Psi_0}{\partial x_i} \right) \left(\frac{\partial \Psi_0}{\partial x_j} \right). \tag{3.17}$$

For $g > 0$ (3.15) provides recursive equations

$$\frac{\partial \Psi_g}{\partial s_{ij}} = \frac{1}{\{ij\}!} \left[\frac{\partial^2 \Psi_{g-1}}{\partial x_i \partial x_j} + \sum_{m=0}^{g} \left(\frac{\partial \Psi_m}{\partial x_i} \right) \left(\frac{\partial \Psi_{g-m}}{\partial x_j} \right) \right]. \tag{3.18}$$

The equation (3.17) looks better for the gradient vector function

$$\Phi(\hat{S}, X) = \nabla_X \ \Psi_0(\hat{S}, X) = \left(\Phi_1(\hat{S}, X), \ldots, \Phi_r(\hat{S}, X) \right). \tag{3.19}$$

Corollary 3.2. *For $i \neq j$*

$$\frac{\partial \Phi_m}{\partial s_{ij}} = \Phi_i \frac{\partial \Phi_m}{\partial x_j} + \Phi_j \frac{\partial \Phi}{\partial x_i}; \tag{3.20}$$

for $i = j$

$$\frac{\partial \Phi_m}{\partial s_{ii}} = \Phi_i \frac{\partial \Phi_m}{\partial x_i}. \tag{3.21}$$

Theorem 3.3. *Let $F(X) = (F_1(X), \dots, F_r(X))$ be any formal series vector. Then the solution of the system (3.20) and (3.21) with the initial condition $\Phi(\hat{0}, X) = F(X)$ satisfies the functional equation*

$$\Phi(\hat{S}, X) = F(X + \hat{S}\Phi(\hat{S}, X)). \tag{3.22}$$

The proof repeats the proof of theorem 1.3. Define the Hessian matrix function

$$\Theta(\hat{S}, X) = \nabla_X \ \Phi_0(\hat{S}, X) = \left(\frac{\partial^2 \Psi_0}{\partial x_i \partial x_j} \right) \tag{3.23}$$

and put $H = \Theta(\hat{0}, X)$. Combining the partial derivatives of the m-th equation (3.22) by s_{ij}, x_j and x_i we obtain

$$\left(E - H\hat{S} \right) \left(\frac{\partial \Phi_m}{\partial s_{ij}} - \Phi_i \frac{\partial \Phi_m}{\partial x_j} - \Phi_j \frac{\partial \Phi}{\partial x_i} \right) = 0, \tag{3.24}$$

which provides (3.20).

For a gradient initial condition $F(X) = \nabla_X U_0(X)$ we may give a different proof of this theorem which is purely combinatorial. Note that according to (3.10) the series $\Phi_i(\hat{S}, X)$ is the generating function for the trees from $\mathcal{B}_{0,[i]} = \bigcup_{N \geq 0} \mathcal{B}_{0,N,[i]}$ (trees with one marked half-edge of the colour i). Consider the set of edgeless genus 0 connected trees with one marked half-edge of the colour i $\mathcal{B}^0_{0,[i]}$ (i. e. single vertices). Attaching to each graph $\Gamma \in \mathcal{B}_{0,[i]}$ the vertex adjacent to the marked half-edge provides the mapping

$$c_0 \colon \mathcal{B}_{0,[i]} \to \mathcal{B}^0_{0,[i]}. \tag{3.25}$$

For any $\Delta \in \mathcal{B}^{(0)}_{0,[i]}$ (Δ consists of a single vertex with several half-edges and one marked half-edge of the colour i) all the graphs in $c_1^{-1}(\Delta)$ are constructed from Δ by clutching arbitrary genus 0 trees with one marked half-edge to some of the half-edges of Δ (inserting a two-valent s-vertex between the corresponding half-edge of Γ and the marked edge of the tree). Therefore

$$\sum_{\Gamma\in c_0^{-1}(\Delta)} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} X^{N(\Gamma)} = \frac{\mu(\Delta)}{|\operatorname{Aut}\Delta|} \left(X + \hat{S}\Phi(\hat{S}, X)\right)^{N(\Delta)}. \tag{3.26}$$

But the generating function for $\mathcal{B}^{(0)}_{0,[i]}$ is $\Phi_i(0, X) = F_i(X)$ and therefore taking the sum over all $\Delta \in \mathcal{B}^{(0)}_{0,[i]}$ we get

$$\Phi_i(\hat{S}, X) = \sum_{\Gamma\in\mathcal{B}_{0,[i]}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} X^{N(\Gamma)} = \sum_{\Delta\in\mathcal{B}^{(0)}_{0,[i]}} \sum_{\Gamma\in c_0^{-1}(\Delta)} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} X^{N(\Gamma)} =$$

$$= \sum_{\Delta\in\mathcal{B}^{(0)}_{0,[i]}} \frac{\mu(\Delta)}{|\operatorname{Aut}\Delta|} \left(X + \hat{S}\Phi(\hat{S}, X)\right)^{N(\Delta)} = F_i\left(X + \hat{S}\Phi(\hat{S}, X)\right). \tag{3.27}$$

An s-vertex $v' \in V_s(\Gamma)$ of a coloured bipartite connected modular graph $\Gamma \in \mathcal{B}_{g,N}$ will be called 1-*cut* if deletion of v' disconnects the graph and at least one of the two new connected components has genus zero. A graph without 1-cuts will be called 2-*connected*. Pick a graph $\Gamma \in \mathcal{B}_{g,N}$. Let us mark all the vertices $v'' \in V_a(\Gamma)$ which are connected by an edge with at least one vertex $v' \in V_s(\Gamma)$ which is not a 1-cut and all the the vertices $v'' \in V_a(\Gamma)$ with $g(v'') > 0$. (Note that a genus 0 graph will have no marked vertices.) Next let us delete all the 0-cuts connected by an edge with at least one marked vertex. As the result we shall obtain a number of genus 0 connected component Γ_i, $i > 0$ and one genus g connected component Γ_1 without 1-cuts (the one having marked vertices). Let us for $g > 0$ denote the set of all coloured bipartite connected and 2-connected (i.e. without 1-cuts) modular graphs $\Gamma \in \mathcal{B}_{g,N}$ by $\mathcal{B}^{(1)}_{g,N}$. The above construction provides the mapping

$$c_1 \colon \mathcal{B}_{g,N} \to \mathcal{B}^{(1)}_{g,N}. \tag{3.28}$$

Consider for $g > 0$ the corresponding generating function

$$\Psi^{(1)}_g(\hat{S}, X) = \sum_{N\geq 0} \left(\sum_{\Gamma\in\mathcal{B}^{(1)}_{g,N}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \right) X^N \tag{3.29}$$

For any $\Gamma \in \mathcal{B}^{(1)}_{g,N}$ all the graphs in $c_1^{-1}(\Gamma)$ may be constructed from Γ by clutching arbitrary genus 0 trees with one marked half-edge to some of the half-edges of Γ (inserting a two-valent s-vertex between the corresponding half-edge of Γ and the marked edge of the tree). Therefore

$$\sum_{\Delta \in c_1^{-1}(\Gamma)} \frac{\mu(\Delta)}{|\operatorname{Aut}\Delta|} X^{N(\Delta)} = \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\left(X + \hat{S}\Phi(\hat{S}, X)\right)^N. \tag{3.30}$$

Summing over all genus g 2-connected graphs we obtain the following expression of Ψ_g via $\Psi_g^{(1)}$

Proposition 3.1.

$$\Psi_g(\hat{S}, X) = \Psi_g^{(1)}(\hat{S}, X + \hat{S}\Phi(\hat{S}, X)) \tag{3.31}$$

Consider the set $\mathcal{L}^k_{N,[ij]}$ of coloured bipartite connected modular trees Γ consisting of a chain of k genus 0 vertices $V_a(\Gamma)$ interleaving with $k-1$ two-valent s-vertices $v' \in V_s(\Gamma)$ having two marked ordered half-edges i and j incident to the farthest vertices of Γ. Put $\mathcal{L}_{N,[ij]} = \bigcup_{k\geq 0} \mathcal{L}^k_{N,[ij]}$ and $\mathcal{L}_{[ij]} = \bigcup_{N\geq 0} \mathcal{L}_{N,[ij]}$. Note that such graphs have no nontrivial automorphisms (at least for the case of ordered half-edges); reading all the vertices of Γ along the chain starting from the vertex incident to the first marked half-edge matches $\mu(\Gamma)\frac{X^N}{N!}$ to a certain summand of the ij element of the matrix

$$\underbrace{H\hat{S}H\hat{S}H\ldots\hat{S}H}_{k \text{ times } H}, \tag{3.32}$$

where $H = H(X)$ is the the Hessian matrix $(\frac{\partial^2 U_0(X)}{\partial x_i \partial x_j})$. Consider the matrix $\Upsilon_k(X)$ of generating series defined by

$$\Upsilon_k(X)_{ij} = \sum_{N\geq 0} \sum_{\Gamma \in \mathcal{L}^k_{N,[ij]}} \mu(\Gamma)\frac{X^N}{N!}. \tag{3.33}$$

and the generating series

$$\Upsilon(X)_{ij} = \sum_{\Gamma \in \mathcal{L}_{[ij]}} \mu(\Gamma)\frac{X^{N(\Gamma)}}{N(\Gamma)!}. \tag{3.34}$$

Summing over all the trees in $\mathcal{L}^k_{N,[ij]}$ we shall get all the summands of the corresponding term of (3.32). Therefore we have obtained the following formula.

Proposition 3.2.

$$\Upsilon_k(X) = \underbrace{H\hat{S}H\hat{S}H\dots\hat{S}H}_{k \textit{ times } H}. \tag{3.35}$$

Corollary 3.3.

$$\Upsilon(X) = H + H\hat{S}H + H\hat{S}H\hat{S}H + \dots = H\left(E - \hat{S}H\right)^{-1} = \left(E - H\hat{S}\right)^{-1} H. \tag{3.36}$$

Now it is very easy to give a purely combinatorial proof of the formula (1.47). According to (3.10) the ij component of the matrix $\Theta(\hat{S}, X)$ is the generating function for the trees from $\bigcup_{N\geq 0} \mathcal{B}_{0,N,[ij]}$. In any tree $\Gamma \in \mathcal{B}_{0,N,[ij]}$ there is a unique chain connecting the two marked edges; this chain we may consider as an element of $\mathcal{L}^k_{N,[ij]}$ (k is the length of this chain). Thus we have defined a mapping

$$c_1 \colon \mathcal{B}_{0,N,[ij]} \to \mathcal{L}_{N,[ij]}. \tag{3.37}$$

As in the proof of the proposition 3.1 for any chain $\Lambda \in \mathcal{L}_{N,[ij]}$ all the graphs in $c_1^{-1}(\Lambda)$ are constructed from Λ by clutching arbitrary genus 0 trees with one marked half-edge to some of the half-edges of Λ (inserting a two-valent s-vertex between the corresponding half-edge of Γ and the marked edge of the tree). Therefore

$$\sum_{\Gamma\in c_1^{-1}(\Lambda)} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} X^{N(\Gamma)} = \frac{\mu(\Lambda)}{|\operatorname{Aut}\Lambda|}\left(X + \hat{S}\Phi(\hat{S}, X)\right)^{N(\Lambda)} \tag{3.38}$$

and

$$\begin{aligned}\Theta(\hat{S}, X)_{ij} &= \sum_{\Gamma\in\mathcal{B}_{0,[ij]}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} X^{N(\Gamma)} = \sum_{\Lambda\in\mathcal{L}_{[ij]}}\left(\sum_{\Gamma\in c_1^{-1}(\Lambda)} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} X^{N(\Gamma)}\right) = \\ &= \sum_{\Lambda\in\mathcal{L}_{[ij]}} \frac{\mu(\Lambda)}{|\operatorname{Aut}\Lambda|}\left(X + \hat{S}\Phi(\hat{S}, X)\right)^{N(\Lambda)} = \Upsilon\left(X + \hat{S}\Phi(\hat{S}, X)\right)_{ij}.\end{aligned} \tag{3.39}$$

Using (3.36) and multiplying by $\hat{S}$ we get

$$\begin{aligned}\hat{S}\Theta(\hat{S}, X) &= \hat{S}H\left(X + \hat{S}\Phi(\hat{S}, X)\right) + \hat{S}H\left(X + \hat{S}\Phi(\hat{S}, X)\right)\hat{S}H\left(X + \hat{S}\Phi(\hat{S}, X)\right) + \\ &+ \hat{S}H\left(X + \hat{S}\Phi(\hat{S}, X)\right)\hat{S}H\left(X + \hat{S}\Phi(\hat{S}, X)\right)\hat{S}H\left(X + \hat{S}\Phi(\hat{S}, X)\right) + \dots\end{aligned} \tag{3.40}$$

and finally adding E we obtain (1.47):

$$E+\hat{S}\Theta(\hat{S},X)=\left(E-\hat{S}H\left(X+\hat{S}\Phi(\hat{S},X)\right)\right)^{-1}. \tag{3.41}$$

Now we are able to describe $\Psi_g^{(1)}$. First let us study the case $g=1$. The set of 2-connected genus 1 graphs splits into two parts $\mathcal{B}_{1,N}^{(1)}=\mathcal{B}'^{(1)}_{1,N}\sqcup\mathcal{B}''^{(1)}_{1,N}$, where $\mathcal{B}'^{(1)}_{1,N}$ is the set of all 2-connected genus 1 graphs having only genus 0 vertices. A connected genus 1 graph may have at most one vertex of positive genus; if such a vertex exists it should have genus 1. So a graph $\Gamma\in\mathcal{B}''^{(1)}_{1,N}$ has no cycles, therefore it has no edges. Hence for each N $\mathcal{B}''^{(1)}_{1,N}$ consists of one graph, namely single genus 1 vertex with $|N|$ half-edges coloured by N. Therefore

$$\sum_{N\geq 0}\left(\sum_{\Gamma\in\mathcal{B}''^{(1)}_{1,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N=\Psi_1(\hat{0},X)=U_1(X). \tag{3.42}$$

If a genus 1 graph has only genus 0 vertices then it must have exactly one cycle. Therefore a graph $\Gamma\in\mathcal{B}'^{(1)}_{1,N}$ consists of one cycle having $k>0$ vertices $v''\in V_a(\Gamma)$ interleaving with k two-valent s-vertices $v'\in V_s(\Gamma)$. Denote the set of such graphs by $\mathcal{B}^{k\ (1)}_{1,N}$; the set of such graphs with the additional choice of one two-valent vertex $v_0''\in V_s(\Gamma)$ and of an orientation of the cycle will be denoted by $\overline{\mathcal{B}^{k\ (1)}_{1,N}}$. The $2k$-sheet covering $\overline{\mathcal{B}^{k\ (1)}_{1,N}}\to\mathcal{B}^{k\ (1)}_{1,N}$ is compatible with the automorphisms of the corresponding graphs. Therefore $\sum_{\Gamma\in\mathcal{B}^{k\ (1)}_{1,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}X^N=\frac{1}{2k}\sum_{\overline{\Gamma}\in\overline{\mathcal{B}^{k\ (1)}_{1,N}}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}X^N$ where $\overline{\Gamma}$ denotes a graph Γ together with the described additional structure. Deletion of the vertex v_0'' defines a bijection

$$\mathcal{B}^{k\ (1)}_{1,N}\to\bigcup_{ij}\mathcal{L}^k_{N,[ij]}, \tag{3.43}$$

therefore

$$\sum_{\Gamma\in\mathcal{B}^{k\ (1)}_{1,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}X^N=\frac{1}{2k}\operatorname{tr}\left(\hat{S}\ H(X)\right)^k. \tag{3.44}$$

Summing for all k we obtain the formula for Ψ_1.

Proposition 3.3.

$$\Psi_1^{(1)}(\hat{S},X)=U_1(X)-\frac{1}{2}\operatorname{tr}\ln(E-\hat{S}\ H(X)). \tag{3.45}$$

Corollary 3.4.

$$\Psi_1(\hat{S}, X) = U_1\left(X + \hat{S}\Phi(\hat{S}, X)\right) - \frac{1}{2}\operatorname{tr}\ln\left(E - \hat{S}H\left(X + \hat{S}\Phi(\hat{S}, X)\right)\right). \tag{3.46}$$

A pair of two-valent s-vertices $v_1', v_2' \in V_s(\Gamma)$ of a coloured bipartite connected and 2-connected modular graph $\Gamma \in \mathcal{B}^{(1)}_{g,N}$ with $g \geq 1$ will be called a 2-*cut* if deleting of v_1' and v_2' disconnects the graph and at least one of the two new connected components has genus zero. Note that for $g > 1$ at most one of the two components may have genus zero and that the genus 0 component is a tree from $\mathcal{L}^{k'}_{N',[ij]}$ for some N', k'. A graph without 2-cuts will be called 3-*connected*; the set of 3-connected genus g graphs having n_i half-edges of the colour i will be denoted by $\mathcal{B}^{(2)}_{g,N}$; $\mathcal{B}^{(2)}_g = \bigcup_{N\geq 0}\mathcal{B}^{(2)}_{g,N}$. The 2-cuts of a given graph $\Gamma \in \mathcal{B}^{(1)}_{g,N}$ are partially ordered by the inclusion relation of the corresponding genus 0 components. Consider all the maximal 2-cuts. Replacing each corresponding maximal genus 0 component by a new two-valent s-vertex we obtain a 3-connected graph $\bar{\Gamma} \in \mathcal{B}^{(2)}_{g,N}$. This provides the mapping

$$c_2\colon \mathcal{B}^{(1)}_g \to \mathcal{B}^{(2)}_g. \tag{3.47}$$

Pick a graph $\Gamma \in \mathcal{B}^{(2)}_g$. The preimage $c_2^{-1}(\bar{\Gamma})$ consists of all graphs obtained from Γ by replacing some of the two-valent s-vertices by arbitrary trees from $\mathcal{L}^{k}_{N,[ij]}$ (bounded by two two-valent s-vertices on the clutching positions). Therefore

$$\left(\sum_{\Gamma\in c_2^{-1}(\Gamma)}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^{N(\Gamma)} \tag{3.48}$$

is obtained from $\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}X^N$ by substituting

$$\left(\hat{S} + \hat{S}H\hat{S} + \hat{S}H\hat{S}H\hat{S} + \ldots\right)_{ij} \tag{3.49}$$

instead of all s_{ij}. Note that the matrix in (3.49) may be expressed as

$$\hat{S} + \hat{S}H\hat{S} + \hat{S}H\hat{S}H\hat{S} + \ldots = \hat{S}\left(E - H\hat{S}\right)^{-1} = \left(E - \hat{S}H\right)^{-1}\hat{S}. \tag{3.50}$$

This enables to express the function Ψ in terms of the generating function for 3-connected graphs

$$\Psi_g^{(2)}(\hat{S}, X) = \sum_{N \geq 0} \left(\sum_{\Gamma \in \mathcal{B}_{g,N}^{(2)}} \frac{\mu(\Gamma)}{|\operatorname{Aut} \Gamma|} \right) X^N. \tag{3.51}$$

Proposition 3.4. *For $g > 1$*

$$\Psi_g^{(1)}(\hat{S}, X) = \Psi_g^{(2)} \left(\left(E - \hat{S} H(X) \right)^{-1} \hat{S}, X \right). \tag{3.52}$$

Corollary 3.5. *For $g > 1$*

$$\Psi_g(\hat{S}, X) = \Psi_g^{(2)} \left(\left(E - \hat{S} H \left(X + \hat{S}\Phi(\hat{S}, X) \right) \right)^{-1} \hat{S}, X + \hat{S}\Phi(\hat{S}, X) \right). \tag{3.53}$$

Now we are left to describe $\Psi_g^{(2)}$. Deletion of all the half-edges defines the mapping

$$c_3 \colon \mathcal{B}_g^{(2)} \to \mathcal{B}_{g,0}^{(2)}, \tag{3.54}$$

where $\mathcal{B}_{g,0}^{(2)}$ is the set of genus $g > 1$ 3-connected graphs without half-edges. Note that a graph is 3-connected if and only if it is stable. We shall denote the set of stable closed graphs $\mathcal{B}_{g,0}^{(2)}$ by $\mathcal{A}_g$. Pick a stable closed graph $\Gamma \in \mathcal{A}_g$ and a vertex $v \in V_a(\Gamma)$; let $N(v)$ ($|N(v)| \geq 3$ for $g = 0$) be the multiindex of its valences. Note that $a_{g(v),N(v)} = \frac{\partial^{|N(v)|} U_{g(v)}}{\partial X^{N(v)}}(0)$, and the same vertex v in different graphs from $c_3^{-1}(\Gamma)$ corresponds to certain terms of the expansion of $\frac{\partial^{|N(v)|} U_{g(v)}(X)}{\partial X^{N(v)}}$. Thus it is not hard to verify that

$$\left(\sum_{\Delta \in c_3^{-1}(\Gamma)} \frac{\mu(\Delta)}{|\operatorname{Aut} \Delta|} \right) X^{N(\Delta)} = \frac{1}{|\operatorname{Aut} \Gamma|} \prod_{v \in V_a(\Gamma)} \frac{\partial^{|N(v)|} U_{g(v)}}{\partial X^{N(v)}}(X). \tag{3.55}$$

The product on the right side of (3.55) looks like the second product in the definition of $\mu(\Gamma)$ (3.2) with $\frac{\partial^{|N|} U_g(X)}{\partial X^N}$ substituted instead of the variables $a_{g,N}$. Therefore, defining for $g \geq 1$ the generating functions P_g for stable closed graphs by

$$P_g(\{a_{m,N}\}, \hat{S}) = \sum_{\Gamma \in \mathcal{A}_g} \frac{\mu(\Gamma)}{|\operatorname{Aut} \Gamma|} = \Psi_g^{(2)}(\{a_{m,N}\}, \hat{S}, 0) \tag{3.56}$$

we obtain the following expression for $\Psi_g^{(2)}$:

$$\Psi_g^{(2)}\left(\{a_{m,N}\},\hat{S},X\right)=\sum_{N\geq 0}\left(\sum_{\Gamma\in\mathcal{B}^{(2)}_{g,N}}\frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}\right)X^N=$$
$$=P_g\left(\left\{a_{m,N}:=\frac{\partial^{|N|}U_m}{\partial X^N}(X)\right\},\hat{S}\right). \quad (3.57)$$

Thus we are able to express the generating functions Ψ_g in terms of the generating functions for stable closed graphs P_g.

Theorem 3.4. *For $g>1$*

$$\Psi_g(\hat{S},X)=$$
$$=P_g\left(\left\{a_{g,N}:=\frac{\partial^{|N|}U_g}{\partial X^N}\left(X+\hat{S}\Phi(\hat{S},X)\right)\right\},\left(E-\hat{S}H\left(X+\hat{S}\Phi(\hat{S},X)\right)\right)^{-1}\hat{S}\right). \quad (3.58)$$

Note that for each $g>1$ the set of stable closed graphs $\mathcal{A}_g$ is finite: for $g>1$ a stable closed genus g graph has at most $3g-3$ two-valent s-vertices. Hence $P_g(\{a_{m,N}\},\hat{S})$ is a polynomial in s_{ij} and $\{a_{m,N}\}$ for $|N|\leq 2g-2$ ($|N|\leq 2$ for $g=1$) and $0\leq m\leq g$. It has degree $3g-3$ as a polynomial in s_{ij}; for combinatorial case the degree of all terms is at least g. We shall call the polynomials $P_g(\{a_{m,N}\},\hat{S})$ *stable graph polynomials.* For instance the first stable graph polynomial for $g=2$

$$P_2=a_{2,0}+\sum_{i,j}\frac{1}{\{ij\}!}a_{1,\{i\}}a_{1,\{j\}}s_{ij}+\sum_{i,j}\frac{1}{\{ij\}!}a_{1,\{ij\}}s_{ij}+$$
$$+\sum_{i,j,k,l}\frac{1}{\{kl\}!}a_{1,\{i\}}a_{0,\{jkl\}}s_{ij}s_{kl}+\sum_{i,j,k,l,p,q}\frac{1}{\operatorname{Aut}\Gamma_5}a_{0,\{ijk\}}a_{0,\{lpq\}}s_{ij}s_{kl}s_{pq}+$$
$$+\sum_{i,j,k,l,p,q}\frac{1}{\operatorname{Aut}\Gamma_6}a_{0,\{ijk\}}a_{0,\{lpq\}}s_{ip}s_{kl}s_{jq}+\sum_{i,j,k,l}\frac{1}{\operatorname{Aut}\Gamma_7}a_{0,\{ijkl\}}s_{ij}s_{kl}. \quad (3.59)$$

has seven groups of terms corresponding to the seven possible genus 2 graphs; different terms in each group correspond to different ways of colouring edges of the given graph. All the coefficients are the inverses to the number of automorphisms of the corresponding graph; in the last three terms we do not indicate an explicit expression of dependence of these numbers on the way of colouring. For combinatorial case P_2 has only three last terms.

It is not hard to verify that the stable graph polynomials are homogeneous in the following sense.

Proposition 3.5. *Define the grading of the polynomial ring* $\mathbb{C}\left[\{s_{ij}\}, \{a_{m,N}\}\right]$ *by*

$$\deg a_{g,N} = 1 - |N| - g \quad \textit{and} \quad \deg s_{ij} = 1. \tag{3.60}$$

Then stable graph polynomial $P_g(\{a_{g,N}\}, \hat{S})$ *is homogeneous polynomial of degree* $1 - g$.

4 Stable graph polynomials

Next let us derive the recurrence for the stable graph polynomials. The idea of it is quite similar to the proof of theorem 1.23 (or theorem 3.2): we delete one s-vertex of a given stable closed genus g graph and obtain a genus $g-1$ graph with two half-edges. Unlike the cases considered in the theorems 1.23 and 3.2 the new graph does not correspond to the same generating function for genus $g-1$, because it is not closed and may be not connected. First let us study the latter case. Let Γ be a stable closed genus $g>1$ graph and assume that deleting of some s-vertex disconnects it into the disjoint union of two connected graphs Γ' and Γ''. Then both Γ' and Γ'' have positive genus and each of the two has exactly one half-edge. Let us denote by $\mathcal{C}_{g,\{i\}}$ the set of all genus $g \geq 1$ graphs with the only half-edge of colour i obtained from stable closed graphs in the described way. Pick a graph $\Gamma \in \mathcal{C}_{g,\{i\}}$. For $g>1$ deletion of its only half-edge provides a stable closed graph unless the vertex v_0 incident to the half-edge was a trivalent genus 0 vertex. But in the latter case we obtain a stable closed graph by substituting an s-vertex instead of the subgraph consisting of the vertex v_0 together with the two s-vertices adjacent to v_0 (see Fig. 2). Thus for $g>1$ we have defined a mapping

$$c_5 \colon \mathcal{C}_{g,\{i\}} \to \mathcal{B}^{(2)}_{g,0}. \tag{4.1}$$

The generating function ($\mu(\Gamma)$ is defined in (3.2))

$$Q_g^{(i)}(\{a_{m,N}\}, \hat{S}) = \sum_{\Gamma \in \mathcal{C}_{g,\{i\}}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \tag{4.2}$$

is a derivative of P_g in the following sense. For $1 \le k \le r$ define the differentiation D_k of the ring of polynomials in all $a_{m,N}$ and s_{ij} by its action on its generators:

$$D_k(a_{m,N}) = a_{m,N+\{k\}} \tag{4.3}$$

$$D_k(s_{ij}) = \sum_{p,q} s_{ip} s_{jq} a_{0,\{pqk\}} \tag{4.4}$$

Proposition 4.1. *The differentiation D_k is homogeneous and has degree -1.*

Using the mapping (4.1) it is not hard to verify the following statement.

Proposition 4.2. *For $g > 1$*

$$Q_g^{(i)}(\{a_{m,N}\}, \hat{S}) = D_i(P_g(\{a_{m,N}\}, \hat{S}) \tag{4.5}$$

For $g = 1$ the mapping (4.1) is not well-defined but it is not hard to list all the graphs of $\mathcal{C}_{1,\{i\}}$ explicitly. In fact there are only two such graphs: single genus 1 vertex with one colour i half-edge and a length one cycle with one s-vertex and one genus 0 trivalent vertex having one half-edge of the colour i. Therefore

$$Q_1^{(i)}(\{a_{m,N}\}, \hat{S}) = a_{1,\{i\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{ipq\}}, \tag{4.6}$$

which may be considered as a formal definition of $D_i(P_1)$ (whereas P_1 does not exist).

Next let us consider the second possibility. Pick a stable closed genus $g > 2$ graph and assume that deleting of some s-vertex does not disconnect it. Let us denote by $\mathcal{C}_{g,\{ij\}}$ the set of all genus $g > 1$ graphs having exactly two half-edges of colours i and j obtained from stable closed graphs in the described way. Our next purpose is to define for $g > 1$ a mapping

$$c_4 \colon \mathcal{C}_{g,\{ij\}} \to \mathcal{B}_{g,0}^{(2)}. \tag{4.7}$$

First let us assume that the two half-edges are not attached to the same trivalent genus 0 vertex. In this case the mapping (4.7) may be described as the result of twice repeated operations used in the definition of the mapping c_5 (4.1). This corresponds to double differentiation $\frac{1}{\{ij\}!} D_i D_j$. Next consider a graph $\Gamma \in \mathcal{C}_{g,\{ij\}}$ having two half-edges attached to the same trivalent genus 0 vertex v_0. Then 0 is adjacent to exactly one s-vertex which connects it to the remaining part of the graph. Removal of this s-vertex (together with v_0) provides a graph $\bar{\Gamma} \in \mathcal{C}_{g,\{m\}}$ for some m. This enables to describe the generating function

$$R_g^{(ij)}(\{a_{m,N}\}, \hat{S}) = \sum_{\Gamma \in \mathcal{C}_{g,\{ij\}}} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|} \tag{4.8}$$

using the mapping c_4 (4.7).

Proposition 4.3. *For $g > 1$*

$$R_g^{(ij)}(\{a_{m,N}\}, \hat{S}) = \frac{1}{\{ij\}!}\left[D_i D_i(P_g(\{a_{m,N}\}, \hat{S}) + \sum_{p,q} D_p\left(P_g(\{a_{m,N}\}, \hat{S})\right) s_{pq} a_{0,\{ijq\}}\right]. \tag{4.9}$$

For $g = 1$ the mapping (4.7) is not well-defined but it is not hard to list all the graphs of $\mathcal{C}_{1,\{ij\}}$ explicitly. In fact there are only three such graphs corresponding to the three terms of the following expression

$$\begin{aligned} R_1^{(ij)}(\{a_{m,N}\}, \hat{S}) &= \frac{1}{\{ij\}!}\left[\sum_{p,q,u,t} \frac{1}{\{\{pq\}\{ut\}\}!} s_{pu} s_{qt} a_{0,\{ipq\}} a_{0,\{jut\}} + \right. \\ &\left. + \sum_{p,q}\left(a_{1,\{p\}} s_{pq} a_{0,\{ijq\}} + \sum_{u,t} \frac{1}{\{ut\}!} s_{ut} a_{0,\{utp\}} s_{pq} a_{0,\{ijq\}}\right)\right] = \\ &= \frac{1}{\{ij\}!}\left[\sum_{p,q,u,t} \frac{1}{\{\{pq\}\{ut\}\}!} s_{pu} s_{qt} a_{0,\{ipq\}} a_{0,\{jut\}} + \sum_{p,q} D_p(P_1) s_{pq} a_{0,\{ijq\}}\right], \end{aligned} \tag{4.10}$$

where $\{\{pq\}\{ut\}\}!$ means 2 for $p = q$ and $u = t$ and 1 otherwise. Note that in the last expression the term $\sum_{p,q,u,t} \frac{1}{\{\{pq\}\{ut\}\}!} s_{pu} s_{qt} a_{0,\{ipq\}} a_{0,\{jut\}}$ may be considered as a formal definition of $D_i D_j(P_1)$ (different from $D_i(D_j(P_1))$ and $D_j(D_i(P_1))$ which are not equal).

Now we are prepared to present the recurrence for stable graph polynomials. Deletion of an s-vertex from a given stable closed genus g graph provides either a connected graph from $\mathcal{C}_{g-1,\{ij\}}$ or a pair of connected graph from $\mathcal{C}_{m,\{i\}}$ and $\mathcal{C}_{g-m,\{j\}}$ for some $1 \le m \le g-1$. Therefore

$$\frac{\partial Pg}{\partial s_{ij}} = \frac{1}{\{ij\}!}\left[R_{g-1}^{(ij)}(\{a_{m,N}\}, \hat{S}) + \sum_{m=1}^{g-1} Q_m^{(i)}(\{a_{m,N}\}, \hat{S}) Q_{g-m}^{(j)}(\{a_{m,N}\}, \hat{S})\right]. \tag{4.11}$$

Substituting (4.5), (4.6) and (4.9) we get the desired recurrence.

Theorem 4.1. *For $g > 2$ stable graph polynomials $P_g(\{a_{m,N}\}, \hat{S})$ for coloured bipartite graphs are given by the recurrences (for each pair ij)*

$$\frac{\partial Pg}{\partial s_{ij}} = \frac{1}{\{ij\}!}\Bigg[D_i D_j(P_{g-1}) + \sum_{p,q} D_p(P_{g-1}) s_{pq} a_{0,\{ijq\}} + \\ + D_i(P_{g-1})\left(a_{1,\{j\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{jpq\}}\right) + \\ + D_j(P_{g-1})\left(a_{1,\{i\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{ipq\}}\right) + \sum_{m=2}^{g-2} D_i(P_m) D_j(P_{g-m})\Bigg]. \tag{4.12}$$

and by the initial condition $P_g(\{a_{m,N}\}, \hat{0}) = a_{g,0}$, where the differentiations D_i are given by (4.3) and (4.4) and P_2 is defined in (3.59) .

Regardless of the absence of genus 1 stable closed graphs we may start the recurrence (4.12) from $g = 1$ formally putting

$$D_i(P_1) = a_{1,\{i\}} + \sum_{p,q} \frac{1}{\{pq\}!} s_{p,q} a_{0,\{ipq\}}. \tag{4.13}$$

Then for any $g > 1$

$$\frac{\partial Pg}{\partial s_{ij}} = \frac{1}{\{ij\}!}\Bigg[D_i D_j(P_{g-1}) + \sum_{p,q} D_p(P_{g-1}) s_{pq} a_{0,\{ijq\}} + \\ + \sum_{m=1}^{g-1} D_i(P_m) D_j(P_{g-m})\Bigg]. \tag{4.14}$$

5 Counting functions for combinatorial graphs.

For $r = 1$ we have one variable s and two-index variables $a_{g,n}$. The differentiation D of the ring of polynomials in s and $a_{g,n}$ is defined by

$$D(a_{g,n}) = a_{g,n+1} \tag{5.1}$$

$$D(s) = s^2 a_{0,3} \tag{5.2}$$

The polynomial P_2 is given by (3.59):

$$P_2 \;=\; a_{2,0} + \frac{1}{2}a_{1,1}^2 s + \frac{1}{2}a_{1,2}s + \frac{1}{2}a_{1,1}a_{0,3}s^2 + \frac{1}{8}a_{0,4}s^2 + \left(\frac{1}{12}+\frac{1}{8}\right)a_{0,3}^2 s^3, \tag{5.3}$$

and for combinatorial case ($a_{g,n} = 0$ for $g > 0$)

$$P_2 = \frac{1}{8}a_{0,4}s^2 + \frac{5}{24}a_{0,3}^2 s^3. \tag{5.4}$$

The recurrence (4.12) becomes

$$\frac{\partial Pg}{\partial s} \;=\; \frac{1}{2}\left[D^2(P_{g-1}) + 2D(P_{g-1})\left(a_{1,1} + sa_{0,3}\right) + \sum_{m=2}^{g-2} D(P_m)D(P_{g-m})\right] \tag{5.5}$$

and the formula (3.58) looks like

$$\begin{aligned}&\Psi_g(s,x) = \\ &\quad = P_g\left(\left\{a_{g,n} := \frac{d^n U_g}{dx^n}\left(x + s\Phi(s,x)\right)\right\}, \frac{s}{1 - sH\left(x + s\Phi(s,x)\right)}\right).\end{aligned} \tag{5.6}$$

For the counting functions for combinatorial graphs we put

$$a_{g,n}^{\mathrm{comb}} = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

which we may write uniformly as $a_{g,n}^{\mathrm{comb}} = \delta_{g0}$. Then the initial conditions (see (1.8) — (1.10)) are

$$\begin{gathered} U_g^{\mathrm{comb}}(x) = 0 \quad \text{for} \quad g > 0, \\ U_0^{\mathrm{comb}}(x) = e^x, \end{gathered} \tag{5.8}$$

and hence $F^{\mathrm{comb}}(x) = H^{\mathrm{comb}}(x) = e^x$. The counting series for all trees satisfies the functional equation (1.38):

$$\Phi^{\mathrm{comb}}(s,x) = e^{x + s\Phi^{\mathrm{comb}}(s,x)} \tag{5.9}$$

and therefore

$$\frac{d^n U_g^{\mathrm{comb}}}{dx^n}\big(x + s\Phi^{\mathrm{comb}}(s,x)\big) = 0 \quad \text{for} \quad g > 0, \tag{5.10}$$

and

$$\frac{d^n U_0^{\mathrm{comb}}}{dx^n}\big(x + s\Phi^{\mathrm{comb}}(s,x)\big) = e^{x+s\Phi^{\mathrm{comb}}(s,x)} = \Phi^{\mathrm{comb}}(s,x) \quad \text{for} \quad g = 0, \tag{5.11}$$

which we may write in a uniform way as $a_{g,n} := \Phi^{\mathrm{comb}}(s,x)\delta_{g0}$. Denote the second argument of (5.6) by Y:

$$Y = \frac{s}{1 - sH\left(x + s\Phi(s,x)\right)} := \frac{s}{1 - se^{x+s\Phi^{\mathrm{comb}}(s,x)}} = \frac{s}{1 - s\Phi^{\mathrm{comb}}(s,x)}. \tag{5.12}$$

Therefore for the counting function for combinatorial graphs

$$\begin{aligned}\Psi_g^{\mathrm{comb}}(s,x) = \Psi_g(\{a_{g,n} := \delta_{g0}\}, s, x) = \\ = P_g\left(\left\{a_{g,n} := \Phi^{\mathrm{comb}}(s,x)\delta_{g0}\right\}, \frac{s}{1 - s\Phi^{\mathrm{comb}}(s,x)}\right).\end{aligned} \tag{5.13}$$

Recall that stable graph polynomials are homogenous and have degree $1-g$ (see proposition 3.5) with respect to the grading (3.60). By definition of stable graph polynomials (3.56):

$$P_g(\{a_{m,N}\}, s) = \sum_{\Gamma \in \mathcal{A}_g} \frac{\mu(\Gamma)}{|\operatorname{Aut}\Gamma|}, \tag{5.14}$$

where $\mathcal{A}_g^k$ is the set of genus g stable closed graphs. Denote the set of genus g stable closed graphs with k edges by $\mathcal{A}_g^k$. A graph $\Gamma \in \mathcal{A}_g^k$ has $k-g+1$ vertices (combinatorial graphs have no vertices of higher genus), so $\mu(\Gamma) = s^k \prod_{i=1}^{k-g+1} a_{0,n_i}$, where n_i are the valences of the vertices. Therefore the counting function for combinatorial graphs

$$
\begin{aligned}
&\Psi_g^{\mathrm{comb}}(s,x) = P_g\left(\left\{a_{m,N} := \Phi^{\mathrm{comb}}(s,x)\delta_{m0}\right\}, Y\right) = \\
&= \sum_k Y^k \left(\sum_{\Gamma\in\mathcal{A}_g^k} \frac{1}{|\operatorname{Aut}\Gamma|} \prod_{i=1}^{k-g+1} a_{0,n_i} \right) = \sum_k Y^k \left(\sum_{\Gamma\in\mathcal{A}_g^k} \frac{1}{|\operatorname{Aut}\Gamma|} \Phi^{\mathrm{comb}}(s,x)^{k-g+1} \right) = \\
&\qquad = \frac{1}{\Phi(s,x)^{g-1}} \sum_k \left(\Phi^{\mathrm{comb}}(s,x)Y\right)^k \left(\sum_{\Gamma\in\mathcal{A}_g^k} \frac{1}{|\operatorname{Aut}\Gamma|} \right) = \\
&\qquad = \frac{1}{\Phi^{\mathrm{comb}}(s,x)^{g-1}} P_g\left(\left\{a_{m,N} := \delta_{m0}\right\}, \frac{s\Phi^{\mathrm{comb}}(s,x)}{1 - s\Phi^{\mathrm{comb}}(s,x)} \right). \qquad (5.15)
\end{aligned}
$$

Note that the polynomials

$$P_g^{\mathrm{comb}}(s) = P_g\left(\left\{a_{m,N} := \delta_{m0}\right\}, s\right) \qquad (5.16)$$

are the generating functions for combinatorial stable closed graphs. Thus we have proved the second part of the theorem 1.5.

Next let us prove the first part of this theorem. The only problem in deriving an explicit recurrence for $P_g^{\mathrm{comb}}(s)$ from (4.12) is how to express the result of substitution $a_{m,N} := \delta_{m0}$ into $D(P_g)$ and $D^2(P_g)$ in terms of $P_g^{\mathrm{comb}}(s)$.

Consider the polynomial ring $\mathbb{C}[\alpha, Y]$ with the grading

$$\deg Y = 1 \quad \text{and} \quad \deg \alpha = -1 \qquad (5.17)$$

and differentiation δ^{comb} of this ring defined by its action on the generators

$$\delta_{\mathrm{comb}}(\alpha) = \alpha \quad \text{and} \quad \delta_{\mathrm{comb}}(Y) = Y^2\alpha. \qquad (5.18)$$

Then δ_{comb} is homogeneous of degree 0. Define a ring homomorphism

$$f^{\mathrm{comb}} \colon \mathbb{C}\left[\{a_{g,n}\}, s_{ij})\right] \to \mathbb{C}[\alpha, Y] \qquad (5.19)$$

by its action on the generators:

$$
\begin{aligned}
&f^{\mathrm{comb}}(a_{g,n}) = 0 \quad \text{for} \quad g > 0, \\
&f^{\mathrm{comb}}(a_{0,n}) = \alpha \\
&f^{\mathrm{comb}}(s_{ij}) = Y.
\end{aligned} \qquad (5.20)
$$

Evidently $D\circ f^{\text{comb}} = f^{\text{comb}}\circ\delta_{\text{comb}}$, therefore $D^2\circ f^{\text{comb}} = f^{\text{comb}}\circ\delta^2_{\text{comb}}$. The calculation (5.15) shows that $f^{\text{comb}}(P_g)$ is homogeneous polynomial of degree $g-1$, hence the same is true about $f^{\text{comb}}(D(P_g))) = \delta_{\text{comb}}(f^{\text{comb}}(P_g))$. But for any degree $g-1$ homogeneous polynomial $W\in\mathbb{C}[\alpha,Y]$

$$\begin{aligned}\delta_{\text{comb}}(W) = Y^2\alpha\frac{\partial W}{\partial Y} + \alpha\frac{\partial W}{\partial\alpha} =& \\ = Y^2\alpha\frac{\partial W}{\partial Y} + Y\frac{\partial W}{\partial Y} - Y\frac{\partial W}{\partial Y} + \alpha\frac{\partial W}{\partial\alpha} =& \\ = Y(Y\alpha+1)\frac{\partial W}{\partial Y} - (g-1)W.& \quad (5.21)\end{aligned}$$

(On the last step we use the Euler formula for homogeneous polynomials: $Y\frac{\partial W}{\partial Y} - \alpha\frac{\partial W}{\partial\alpha} = (g-1)W$.) Now we can apply (5.21) to the combinatorial stable graph polynomials

$$P_g^{\text{comb}}(s) = (f^{\text{comb}}P_g)(1,s) \quad (5.22)$$

and get the following recurrence for P_g^{comb}.

Proposition 5.1. *The combinatorial stable graph polynomials P_g^{comb} (see (5.16)) satisfy the following recurrence*

$$\frac{dP_g^{\text{comb}}}{ds} = \frac{1}{2}\left[D^2_{comb}(P^{\text{comb}}_{g-1}) + 2sD_{comb}(P^{\text{comb}}_{g-1}) + \sum_{m=2}^{g-2} D_{comb}(P^{\text{comb}}_m)D_{comb}(P^{\text{comb}}_{g-m})\right], \quad (5.23)$$

where

$$D_{comb} = s(s+1)\frac{d}{ds} - (g-1). \quad (5.24)$$

Here we present the explicit form of the recurrence 5.23 and the polynomials P_g for $g\le 6$:

$$\begin{aligned}\frac{d}{ds}P_g^{\text{comb}}(s) = \frac{1}{2}\Bigg[&s^2(s+1)^2\frac{d^2}{ds^2}P^{\text{comb}}_{g-1}(s) - \\ &- s(s+1)(2g-4s-5)\frac{d}{ds}P^{\text{comb}}_{g-1}(s) + (g-1)P^{\text{comb}}_{g-1}(s) + \\ &+ \sum_{m=2}^{g-2}\Bigg(\left(s(s+1)\frac{d}{ds}P^{\text{comb}}_m(s) - (m-1)P^{\text{comb}}_m(s)\right)\times \\ &\times\left(s(s+1)\frac{d}{ds}P^{\text{comb}}_{g-m}(s) - (g-m-1)P^{\text{comb}}_{g-m}(s)\right)D(P_m)D(P_{g-m})\Bigg)\Bigg] \quad (5.25)\end{aligned}$$

Here we present the polynomials $P_g^{\rm comb}$ for $g\le 6$:

$$P_2^{\rm comb}=\frac{5}{24}\,s^3+\frac{1}{8}s^2$$

$$P_3^{\rm comb}=\frac{5}{16}\,s^6+\frac{25}{48}\,s^5+\frac{11}{48}\,s^4+\frac{1}{48}s^3$$

$$P_4^{\rm comb}=\frac{1105}{1152}\,s^9+\frac{985}{384}\,s^8+\frac{1373}{576}\,s^7+\frac{515}{576}\,s^6+\frac{223}{1920}\,s^5+\frac{1}{384}s^4$$

$$P_5^{\rm comb}=\frac{565}{128}\,s^{12}+\frac{12455}{768}\,s^{11}+\frac{26581}{1152}\,s^{10}+\frac{12227}{768}\,s^9+\frac{2089}{384}\,s^8+\frac{9583}{11520}\,s^7+\frac{27}{640}\,s^6+\frac{1}{3840}s^5$$

$$P_6^{\rm comb}=\frac{82825}{3072}\,s^{15}+\frac{387005}{3072}\,s^{14}+\frac{371195}{1536}\,s^{13}+\frac{10154003}{41472}\,s^{12}+\frac{121207}{864}\,s^{11}+$$
$$+\frac{519883}{11520}\,s^{10}+\frac{1573507}{207360}\,s^9+\frac{2597}{4608}\,s^8+\frac{803}{64512}\,s^7+\frac{1}{46080}s^6$$

Similar formulas describe counting functions for combinatorial stable graphs. For this case

$$a_{g,n}^{\rm st}=\begin{cases}1 & \text{if } g=0 \text{ and } n\ge 3\\ 0 & \text{otherwise,}\end{cases}\tag{5.26}$$

so the initial conditions (see (1.8) — (1.10)) are

$$\begin{array}{l}U_g^{\rm st}(x)=0 \quad \text{for} \quad g>0,\\ U_0^{\rm st}(x)=e^x-1-x-\frac{x^2}{2},\end{array}\tag{5.27}$$

and hence $F^{\rm st}(x)=e^x-1-x$ and $H^{\rm st}(x)=e^x-1$. The counting series for all stable trees satisfies the functional equation (1.38):

$$\Phi^{\rm st}(s,x)=e^{x+s\Phi^{\rm st}(s,x)}-1-(x+s\Phi^{\rm st}(s,x))\tag{5.28}$$

and therefore

$$\frac{d^nU_g^{\rm st}}{dx^n}\big(x+s\Phi^{\rm st}(s,x)\big)=0 \quad \text{for} \quad g>0,\tag{5.29}$$

and

$$\frac{d^nU_0^{\rm st}}{dx^n}\big(x+s\Phi^{\rm st}(s,x)\big)=e^{x+s\Phi^{\rm st}(s,x)}=1+x+(s+1)\Phi^{\rm st}(s,x) \quad \text{for} \quad g=0,\ n\ge 3.\tag{5.30}$$

Since the terms $\frac{d^n U_0^{\mathrm{st}}}{dx^n}\left(x + s\Phi^{\mathrm{st}}(s,x)\right)$ for $n<3$ are not involved in the stable graph polynomials for the use in formula (5.6) we may write in a uniform way $a_{g,n} := (1 + x + (s+1)\Phi^{\mathrm{st}}(s,x))\delta_{g0}$.

The second argument of (5.6) for this case is :

$$\frac{s}{1 - sH^{\mathrm{st}}\left(x + s\Phi^{\mathrm{st}}(s,x)\right)} = \frac{s}{1 - s(e^{x+s\Phi^{\mathrm{st}}(s,x)} - 1)} = \frac{s}{1 - s(x + (s+1)\Phi^{\mathrm{st}}(s,x))}. \tag{5.31}$$

Thus using the same argument as in the proof of (5.15) we get the formula for counting function for combinatorial stable graphs:

$$\begin{aligned} &\Psi_g^{\mathrm{st}}(s,x) = \\ &= P_g\left(\left\{a_{m,N} := \left(1 + x + (s+1)\Phi^{\mathrm{st}}(s,x)\right)\delta_{m0}\right\}, \frac{s}{1 - s(x + (s+1)\Phi^{\mathrm{st}}(s,x))}\right) = \\ &= \frac{1}{(1 + x + (s+1)\Phi^{\mathrm{st}}(s,x))^{g-1}} P_g\left(\{a_{m,N} := \delta_{m0}\}, \frac{s\,(1 + x + (s+1)\Phi^{\mathrm{st}}(s,x))}{1 - s(x + (s+1)\Phi^{\mathrm{st}}(s,x))}\right) = \\ &\qquad = \frac{1}{(1 + x + (s+1)\Phi^{\mathrm{st}}(s,x))^{g-1}} P_g^{\mathrm{comb}}\left(\frac{s\,(1 + x + (s+1)\Phi^{\mathrm{st}}(s,x))}{1 - s(x + (s+1)\Phi^{\mathrm{st}}(s,x))}\right). \end{aligned} \tag{5.32}$$

The third part of the theorem 1.5 is proved.

References

[1] Artamkin I.V. Generating functions for modular graphs and Burgers equation, Max Planck Institute formathematics, Preprint MPI-05-108 (2005).

[2] Meng G. Legendre transforme, Hessian conjecture and tree formula, math-ph,/0308035.

[3] Wright D. The Jacobian conjecture as a problem in combinatorics, math.CO/0511214.

[4] Zhao W. Hessian nilpotent polynomials and the Jacobian conjecture, math. CV/0409534.

[5] Zhao W. Legendre transforme and the invicid Burgers equation, math. CV/0403020.